Chapter 2
Limits and Derivatives

The idea of a limit is central to all of calculus. Throughout the rest of your calculus classes it will be behind everything you learn. Most people do not understand this concept the first time they see it, but they can get a feeling for the basic idea with some effort.

After looking at limits, this chapter moves on to the idea of a derivative. This idea is more natural and many more people understand the concept of a derivative. In the context of physics it is often used as the instantaneous rate change of position, velocity, or the instantaneous rate of change of velocity, acceleration. Even though physical motion is where Newton originally defined the concept of a derivative, the derivative has found uses in many other areas, including economics and biology.

2.1 Sequences and Limits

The idea of a limit was approached for thousands of years without ever being reached. The concept, when finally realized, helped to revolutionize mathematics.

One of the problems that leads toward the concept of a limit is what mathematicians call Zeno's paradox. A person is walking at a constant speed across a room. At some time the person is half way across the room. At another time they will again have halved their distance to the opposite wall. At a third time they will, again, have halved their distance from the opposite wall. This process continues forever. Does the person ever reach the opposite wall?

The problem here is that the distances left to the far wall are, assuming the original distance was 1,

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$$

If we consider only these distances, they are never zero. Does this mean that the person does not reach the wall?

On the other hand, if the person is traveling at one distance unit per minute, what are the times when the person has half the distance left to the far wall, $1/4$ the distance left, $1/8$ the distance left, etc. These times are $1/2$ min at $1/2$ the distance left, $3/4$ min at $1/4$ the distance left, $7/8$ min at $1/8$ the distance left, etc. Does the fact that these times are less than one make a difference?

Using the concept of a limit these questions will be answered. First the definition of a sequence is presented. The definition may seem strange, but the examples should make the definition fairly clear.

J.S. Treiman, *Calculus with Vectors*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-3-319-09438-0_2

Definition 4 (Infinite sequence). An *infinite sequence* is a function $\mathbf{a}$ from the natural numbers to a set, usually $\mathbb{R}$ or $\mathbb{R}^n$. The values of this function are often denoted by $\{\mathbf{a}_n\}_{n=1}^{\infty}$ instead of writing $\mathbf{a}(n)$.

The image space of $\mathbf{a}(n)$ does not need to be $\mathbb{R}$ or $\mathbb{R}^n$, but those are the only sets we will use.

A *finite sequence* is a function $\mathbf{a}$ from $1,2,3,\ldots,N$ to a set. They are very important, but in calculus we deal mostly with infinite processes, so finite sequences are not used in this section on limits.

Here are some examples of infinite sequences.

Example 32. The natural numbers are a sequence with $a(n) = a_n = n$. This is written as $\{n\}_{n=1}^{\infty}$ or, being less precise, $\{1,2,3,\ldots\}$.

Often we write down only the first few terms of a sequence to get a feeling for the pattern. The first few terms do not, however, give a precise definition of any sequence.

Example 33. The function $a(n) = a_n = 1/n^2$ defines a sequence. This is written as $\left\{1/n^2\right\}_{n=1}^{\infty}$ or, being less precise, $\{1, 1/4, 1/9, \ldots\}$.

Example 34. If we take the distances to the wall and the time traveled from Zeno's paradox we get a sequence of vectors $\{(1/2^n, 1-1/2^n)\}_{n=1}^{\infty}$.

The question dealt with in this section is what happens to the sequence $\{a_n\}$ as n heads toward infinity. This is one of the questions raised by Zeno's paradox. We can look at a simpler version of the question, does a_n approach a given value, number, point or vector, as n gets large. The following definition codifies this idea.

Definition 5 (Infinite sequence). A sequence $\{\mathbf{a}_n\}_{n=1}^{\infty}$ of vectors in $\mathbb{R}^k$ *converges* to $\mathbf{L}$ if, given any distance $r > 0$, there is an N, such that when $n \geq N$, $\mathbf{a}_n$ is within the distance r of $\mathbf{L}$. If this condition is not satisfied, the sequence *diverges*.

If a sequence $\{\mathbf{a}_n\}_{n=1}^{\infty}$ converges to $\mathbf{L}$ we say that $\mathbf{L}$ is the *limit* of the sequence. We use the notation

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{L}$$

to denote that $\mathbf{L}$ is the limit of $\{\mathbf{a}_n\}_{n=1}^{\infty}$.

Example 35. Consider the sequence $\left\{1/n^2\right\}$. This sequence converges to 0. For example, if we want to be within $r = 0.000001$ of 0, we ask which n's will make $|1/n^2 - 0| < 10^{-6}$? Rewriting this inequality gives

$$n^2 > 10^6.$$

See Fig. 2.1 on page 34 where a log scale is used on the n-axis.

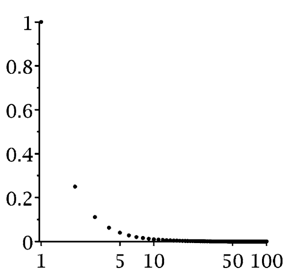
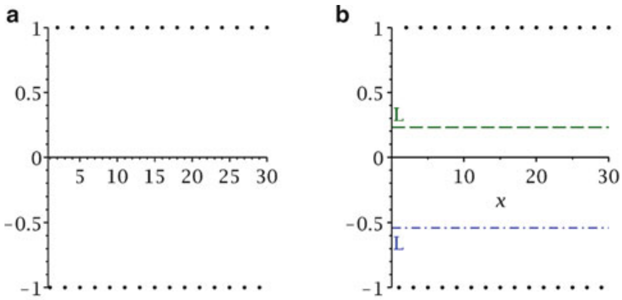


Fig. 2.1 The points $\left(n, 1/n^2\right)$



The points $(n, (-1)^n)$

The points $(n, (-1)^n)$ with potential limits

Fig. 2.2

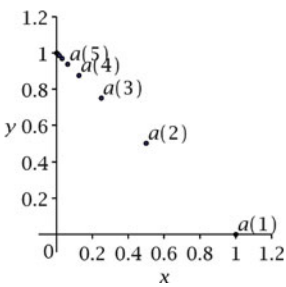


Fig. 2.3 The sequence $\{(1/2^n, 1-1/2^n)\}$

Solving for n leads to $n > 1{,}000$. Thus an N greater than $1{,}000$ is what we need in the definition of a limit for a distance of 10^{-6}.

The next example shows that limits may not exist for some sequences.

Example 36. Consider the sequence $a_n = (-1)^n$. See Fig. 2.2a on page 35. If L is the limit of this sequence, then either $L \geq 0$ or $L \leq 0$. Since r is arbitrary, we can fix the distance to consider at $r = 1/2$. If $L \geq 0$ then, for odd n the inequality $|a_n - L| = |-1-L| \geq 1 > 1/2$ holds. Similarly, if $L < 0$, for even n we have $|a_n - L| \geq 1 > 1/2$. See Fig. 2.2a on page 35.

This means that the sequence never stays near a given L, and the limit does not exist.

Now that the idea of a limit for sequence has been defined, the sequence in Zeno's paradox can be examined more closely. Recall that the sequence is $\{(1/2^n, 1-1/2^n)\}$. See Fig. 2.3 on page 35.

The distance of the nth term, $\mathbf{a}(n)$, in the sequence from $(0,1)$ is

$$\sqrt{1/((2^n)^2) + 1/((2^n)^2)} = \sqrt{2}/2^n \,.$$

Since $2^n \geq 2n$ for $n \geq 1$, we have $\sqrt{2}/2^n < 1/n$. This implies that for any fixed r, if $N > 1/r$ and $n > N$, the distance from $(1/2^n, 1-1/2^n)$ to $(0,1)$ is less than r. Thus, the sequence converges to $(0,1)$. Does this mean that at time 1 the person has reached the opposite wall? What does this say about Zeno's paradox?

To effectively use this definition of the limit for a sequence, several basic rules are needed. They are fairly simple, so they are grouped together.

Theorem 9 (Computations with sequences). *Let $\{\mathbf{a}_n\}$, $\{\mathbf{b}_n\}$, and $\{c_n\}$ be sequences that converge to $\mathbf{K}$, $\mathbf{L}$, and M respectively. Also, let s be a number and assume that $M \neq 0$. Then the following hold:*

(i) $\lim_{n\to\infty} (\mathbf{a}_n + \mathbf{b}_n) = \mathbf{K} + \mathbf{L}$
(ii) $\lim_{n\to\infty} (s\mathbf{a}_n) = s\mathbf{K}$
(iii) $\lim_{n\to\infty} (\mathbf{a}_n \cdot \mathbf{b}_n) = \mathbf{K} \cdot \mathbf{L}$
(iv) $\lim_{n\to\infty} \dfrac{\mathbf{a}_n}{c_n} = \dfrac{\mathbf{K}}{M}$

Proof. Only the first equality is proven in the scalar case. The proofs of the other results are similar and are left to the reader.

Fix an $r > 0$. Choose N and P such that if $n > N$ and $p > P$ then $|a_n - L| < r/2$ and $|b_p - M| < r/2$. Let R be the larger of N and P. If $n > R$, then

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\leq |(a_n - L)| + |(b_n - M)| \\ &< r/2 + r/2 \\ &< r. \end{aligned}$$

By our definition, $\{a_n + b_n\}$ converges to $L + M$.

We can use these rules to extend the sequences we can consider.

Example 37. Consider the sequence $\left\{\left(n^2+n\right)/n^2\right\}$. We have $\left(n^2+n\right)/n^2 = 1+n/n^2 = 1+1/n$. Since the sequence $\{1\}_{n=1}^{\infty}$ has limit 1 and $\{1/n\}_{n=1}^{\infty}$ converges to 0, the original sequence converges to $1+0=1$.

There are more complicated ways of doing this.

Example 38. Let

$$a_n = \frac{n + 6n^3}{n^3 + n^2 - 10n - 5}.$$

Assuming that a_n exists, as it does for all large n, we can divide the numerator and denominator by n to the largest power of n in the denominator. This is n^3 for this example. Using this gives

$$a_n = \frac{\frac{1}{n^2} + 6}{1 + \frac{1}{n} - \frac{10}{n^2} - \frac{5}{n^3}}.$$

Since $1/n$, $1/n^2$, and $1/n^3$ all go to zero, we have

$$\lim_{n\to\infty} \frac{1}{n^2} + 6 = 6 \qquad \text{and} \qquad \lim_{n\to\infty} 1 + \frac{1}{n} - \frac{10}{n^2} - \frac{5}{n^3} = 1.$$

This means that $\lim_{n\to\infty} a_n = 6$.

The next result is used to show a sequence converges by comparing it to a convergent sequence. The proof is not difficult, but it is a little technical. The proof is omitted.

Theorem 10. *Let $\{\mathbf{a}_n\}_{n=1}^{\infty}$ be a sequence of vectors converging to $\mathbf{L}$ and let $\{\mathbf{b}_n\}_{n=1}^{\infty}$ be a sequence such that, for n after some N,*

$$\|\mathbf{b}_n - \mathbf{L}\| \leq \|\mathbf{a}_n - \mathbf{L}\|.$$

Then

$$\lim_{n\to\infty} \mathbf{b}_n = \mathbf{L}.$$

A couple examples will illustrate the uses of this result.

Example 39. It has been demonstrated that the sequence $\left\{1/n^2\right\}_{n=1}^{\infty}$ converges to 0. Since $n^2+2n-1 > n^2$ if $n > 1$,

$$0 < \frac{1}{n^2+2n-1} < \frac{1}{n^2}$$

if $n > 1$.

This means that

$$\left|\frac{1}{n^2+2n-1} - 0\right| < \left|\frac{1}{n^2} - 0\right|$$

when $n > 1$ and the sequence $\left\{1/\left(n^2+2n-1\right)\right\}_{n=1}^{\infty}$ converges to 0.

The result can also be used for vector valued sequences.

Example 40. Consider the sequence $\mathbf{b}_n = (1/n, \sin(n)/n)$. The distance from $\mathbf{b}_n$ to $\mathbf{0}$ is

$$\begin{aligned}
\left\|\left(\frac{1}{n}, \frac{\sin(n)}{n}\right) - (0,0)\right\| &= \left\|\left(\frac{1}{n}, \frac{\sin(n)}{n}\right)\right\| \\
&= \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{\sin(n)}{n}\right)^2} \\
&= \frac{\sqrt{1+\sin^2(n)}}{n} \\
&\leq \frac{\sqrt{2}}{n}.
\end{aligned}$$

Since $\left\{\sqrt{2}/n\right\}_{n=1}^{\infty}$ converges to 0, the sequence of distances from $\mathbf{b}_n$ to $\mathbf{0}$ converges to 0. This means that $\mathbf{b}_n$ converges to $\mathbf{0}$.

We can state a slightly more general result that allows us to show convergence by “squeezing” a sequence between two convergent series. For scalar sequences, Theorem 10 on page 37 can be viewed a squeezing the sequence of b_n’s between the constant sequence with terms L and a sequence with terms a_n that converges to L.

Theorem 11 (Squeeze). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_m\}_{m=0}^{\infty}$ be two sequences that converge to some L. If $\{c_k\}_{k=0}^{\infty}$ is a sequence such that c_n is between a_n and b_n for all integers n greater than some N, then $\{c_k\}_{k=0}^{\infty}$ converges to L.*

Proof. Fix an $r > 0$. For some N, we have both $|a_n - L| < r$ and $|b_n - L| < r$ when $n > N$. Since c_n is between a_n and b_n, we have $|c_n - L| < r$ if $n > N$ and the sequence $\{c_k\}_{k=0}^{\infty}$ converges to L.

Example 41. Consider the sequences with terms $a_n = 1 - 1/\sqrt{n}$ and $b_n = 1 + 1/\sqrt{n}$. Both of these series converge to 1. Since $\sin(n)/\sqrt{n} \in [-1/\sqrt{n}, 1/\sqrt{n}]$ for all n, Theorem 11 tells us that the sequence $c_n = 1 + \sin(n)/\sqrt{n}$ converges to 1. This is illustrated in Fig. 2.4 on page 38.

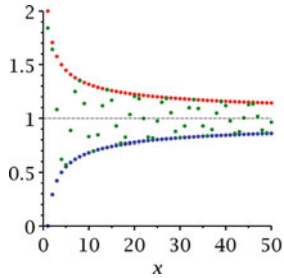


Fig. 2.4 Squeezing a sequence to a limit

A useful fact for showing that the limit of a sequence does not exist is the following result. The idea is that if the sequence always "moves" at least a minimum distance after every N, then the distance of the sequence from a fixed $\mathbf{L}$ cannot go to 0. The proof is omitted, but an example of its use is supplied.

Theorem 12. *Let* $\{\mathbf{a}_n\}_{n=1}^{\infty}$ *be a sequence of vectors. The sequence does not converge if and only if there is an* $r > 0$ *such that for every* N*, there are* $n, m > N$ *with*

$$|\mathbf{a}_n - \mathbf{a}_m| > r.$$

Example 42. Consider the sequence $a_n = \sqrt[3]{n}$. If we can show that after any $N \in \mathbb{N}$ there are k and m such that $|a_k - a_m| = 1$, this will mean that $\{a_n\}$ does not converge. Fix an integer N and let i be an integer with $i > \sqrt[3]{N}$. Then $k = i^3 > N$ and $m = (i+1)^3 > N$. We also have

$$\begin{aligned}
|a_k - a_m| &= \left|a_{i^3} - a_{(i+1)^3}\right| \\
&= \left|\sqrt[3]{i^3} - \sqrt[3]{(i+1)^3}\right| \\
&= |i - (i+1)| \\
&= 1.
\end{aligned}$$

This means that the sequence does not converge.

A final, but very useful result for this section concerns calculating limits of vector valued sequences. As was done with the example concerning Zeno's paradox, we can calculate the distance to the limit value using the vector norm. It is much easier to calculate the limits for each component separately. This result justifies that technique. The proof is omitted.

Theorem 13 (Sequence convergence by components). *Let* $\{\mathbf{a}_n\}$ *be a sequence with* $\mathbf{a}_n$ *having* m *components. Then* $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{L}$ *if and only if each of the components of* $\mathbf{a}_n$ *converges to the corresponding component of* $\mathbf{L}$*. Writing this out gives* $\lim_{n\to\infty} (\mathbf{a}_n)_i = \mathbf{L}_i$.

Example 43. Consider the sequence with $\mathbf{a}_n = \big(1/n,\ n^2 - 1/n^2,\ n^3 - 64/\big(n^4 - 1{,}000\big)\big)$. We can calculate that

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$
$$\lim_{n\to\infty} \frac{n^2-1}{n^2} = 1, \text{ and}$$
$$\lim_{n\to\infty} \frac{n^3-64}{n^4-1{,}000} = 0,$$

This means that

$$\lim_{n\to\infty} \mathbf{a}_n = (0,1,0).$$

Exercises

1. Find the limits of the following sequences as n goes to infinity.

(a) $a_n = \frac{2}{n}$

(b) $a_n = \frac{2}{n^2}$

(c) $b_n = \frac{n^{3/2}}{n}$

(d) $b_n = \frac{n^{3/2}}{n^2}$

(e) $c_n = \frac{2}{\sqrt{n}}$

(f) $c_n = \frac{(-1)^n}{n^3}$

2. Find the limits of the following sequences as n goes to infinity.

(a) $a_n = \frac{2}{n+5}$

(b) $a_n = \frac{n}{n-5}$

(c) $a_n = \frac{n-10}{n+6}$

(d) $a_n = \frac{n^2+n-1}{n^3}$

(e) $a_n = \frac{n^2+n-1}{10-n+6n^2}$

(f) $a_n = \frac{n^2+n-1}{6-n}$

3. Find the limits of the following sequences as n goes to infinity.

(a) $a_n = \frac{2}{n+5} + \frac{1}{n^2}$

(b) $a_n = \frac{n}{n-5} + \frac{3n}{2n+2}$

(c) $a_n = \left(\frac{n-10}{n+6}\right)\left(\frac{4}{n}\right)$

(d) $a_n = \left(\frac{n^2+n-1}{2n^2}\right)\left(\frac{6n+1}{n-5}\right)$

(e) $a_n = \frac{n^2+n-1}{10-n+6n^2} + \frac{n^2}{100n-1}$

(f) $a_n = \frac{n^2+n-1}{6-n}\ \frac{10n}{3n^2+4}$

4. Explain why the following sequences do not converge as n goes to infinity.

(a) $a_n = \frac{n}{1{,}000}$

(b) $a_n = \frac{n^2}{2}$

(c) $b_n = \frac{n^{3/2}}{n}$

(d) $b_n = \frac{\sqrt{n}}{\sqrt[3]{n}}$

(e) $c_n = \frac{n}{\sqrt{n}}$

(f) $c_n = \frac{(-1)^n n^2}{6n-1}$

5. Find the limits of the following sequences as n goes to infinity.

(a) $\mathbf{a}_n = \left(1, \frac{2}{n}\right)$

(b) $\mathbf{a}_n = \left(\frac{2n+1}{n}, \frac{2n+1}{n^2}\right)$

(c) $\mathbf{b}_n = \left(\frac{n+1}{\sqrt{n}}, \frac{2\sqrt{n}}{n}\right)$

(d) $\mathbf{b}_n = \left(\frac{n}{n^{3/2}}, 3, \frac{5n}{10+2n}\right)$

(e) $\mathbf{c}_n = \left(\frac{n^5}{4+n^6}, 1 - \frac{2}{n}, \frac{n^{1/2}}{n+2}\right)$

(f) $\mathbf{a}_n = \left(\frac{n^{3/2}}{2n+2n^2}, \frac{\sqrt{n}-1}{4n+6}\right)$

6. Use the fact that $\left|\sin\left(\frac{1}{n}\right)\right| < 1$ and the squeeze theorem, Theorem 11, to show that the sequence defined by

$$a_n = \frac{1}{n}\sin\left(\frac{1}{n}\right)$$

converges.

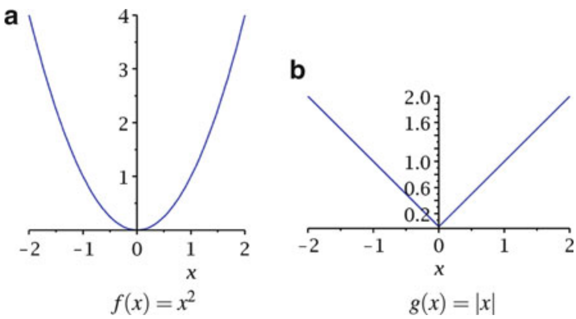


Fig. 2.5

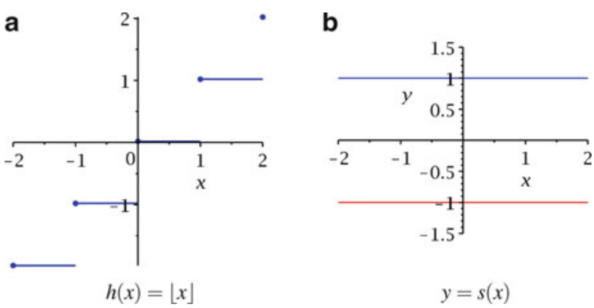


Fig. 2.6

2.2 Limits of Functions and Continuity

In these notes most of the material concerns functions from the real numbers to the real numbers or functions from the real numbers to vectors of real numbers. These functions can be nice, like $f(x) = x^2$, see Fig. 2.5a on page 40. Some are not quite as nice, $g(x) = |x|$, Fig. 2.5b on page 40,

Some have jumps. The *floor* or *greatest integer* function, $h(x) = \lfloor x \rfloor$, is the function that assigns to x the largest integer less than or equal to x, see Fig. 2.6a on page 40.

There are even rather ugly functions, see Fig. 2.6b on page 40,

$$s(x) = \begin{cases} 1 & x \text{ is rational} \\ -1 & x \text{ is irrational}\,. \end{cases}$$

It is important to note that the lines at $s = -1$ and $s = 1$ in Fig. 2.6b are not solid lines. Each has an infinite number of holes in it. These holes are offset so that each line $x = c$ hits one and only one of $s = -1$ and $s = 1$.

In this section the ideas of limits of functions and continuity of functions are used to help sort functions into reasonable and unreasonable classes of functions for calculus.

When dealing with sequences the limit was defined for a_n approaching L as $n \to \infty$. If we have a function $\mathbf{f}(x)$ defined around some point a, the question is what happens to $\mathbf{f}(x)$ as x approaches a? If we have a sequence $a_n \to a$, we can look at what happens to the sequence of function values $\mathbf{b}_n = \mathbf{f}(a_n)$. Looking at the functions above we can see that in the first two functions, no matter what sequence $\{a_n\}$ converging to $a = 0$ that we choose, the sequence $f(a_n)$ converges to 0. For the third function, what happens depends on the sequence $\{a_n\}$ chosen that approaches 0.

Example 44. Consider $f(x) = x^2$ and any sequence $a_n \to 0$. If $0 < |a_n| < 1$, then $0 < (a_n)^2 < |a_n|$. Thus $f(a_n) \to 0$.

Example 45. Consider $g(x) = |x|$ and any sequence $a_n \to 0$. Since $\big||a_n| - 0\big| = |a_n|$, the sequence $g(a_n) \to 0$.

Example 46. Consider $h(x) = \lfloor x \rfloor$ and let $a_n = (-1)^n/n$. Here $\lim_{n\to\infty} a_n = 0$, $h(a_n) = -1$ if n is odd and $h(a_n) = 0$ if n is even. This means that $\lim_{n\to\infty} h(a_n)$ does not exist.

Example 47. Consider the sequences $a_n = 1/n$, $b_n = \pi/n$, and $c_n = 1/\sqrt{n}$ that all converge to 0. All of the a_n's are rational. This means that $s(a_n) = 1$ for all n and $s(a_n) \to 1$. Similarly, all of the b_n's are irrational and $s(b_n) = -1$ for all n. This means that $s(b_n) \to -1$.

Consider the c_n's. If n is a perfect square, $n = m^2$ for some integer m, then c_n is rational. Otherwise, c_n is irrational. This means that for almost all n, $c_n = -1$. But, as n goes toward infinity, n will occasionally be a perfect square and c_n will be 1. This means that $s(c_n)$ does not converge since no matter how large N is, there are $n, m > N$ such that $g(c_n) = 1$ and $g(c_m) = -1$.

These four examples show what we want is for the function values to approach a single value. The problems with $s(x)$ mean that we need to consider all sequences, not just one, or a few, sequences. Because of this, the following is the definition of a limit used in these notes.

Definition 6 (Function limit). Let $\mathbf{f}$ be a function on an interval around a. *The interval may exclude a.* We say that the limit as x approaches a of $\mathbf{f}(x)$ equals $\mathbf{L}$ if for **all** sequences $a_n \to a$, with $a_n \neq a$ for all n, we have

$$\lim_{n\to\infty} \mathbf{f}(a_n) = \mathbf{L}.$$

A common notation for the limit is

$$\lim_{x\to a} \mathbf{f}(x) = \mathbf{L}.$$

The four examples before this definition show that $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ exist whereas $\lim_{x\to 0} h(x)$ and $\lim_{x\to 0} s(x)$ do not exist. Showing that $f(x)$ and $g(x)$ have limits at $x = 0$ was fairly difficult. Demonstrating that $h(x)$ and $s(x)$ do not have limits was much easier since all that was required was finding *a* sequence $a_n \to 0$ where $h(a_n)$ or $s(a_n)$ does not converge.

Remark 4. An intuitive way of interpreting this definition is that the limit of $\mathbf{f}$ as x approaches a is $\mathbf{L}$, if, no matter how we approach a, the function values always go to $\mathbf{L}$.

For the function $h(x)$ from above, if we approach 0 from the negative side, the function values go to -1. If we approach 0 from the positive side, the values of $h(x)$ go to 0. Since the values of $h(x)$ can approach -1 or 0, the limit does not exist.

The condition for a limit of a function to not exist is stated as the following theorem. The proof is omitted.

Theorem 14. *Limit does not exist by sequence Let $\mathbf{f}(x)$ be a function from $\mathbb{R}$ to $\mathbb{R}^n$. The limit $\lim_{x\to a}\mathbf{f}(x)$ does not exist if and only if there is a sequence $x_m \to a$ such that the sequence $\mathbf{f}(x_m)$ does not converge.*

As an example to illustrate this consider the floor function.

Example 48. Take $h(x) = \lfloor x \rfloor$ and let $a_n = (-1)^n/n$ for $n \geq 2$. Then $\lim_{n\to\infty} a_n = 0$. If n is odd $a_n = -1/n$ is between -1 and 0. This means that $-1 \leq a_n < 0$ and $h(a_n) = -1$.

Similarly, if n is even $0 < a_n < 1$. This implies that $h(a_n) = 0$. Combining the last two conclusions we have

$$h(a_n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

This sequence cannot converge since $|h(a_{n+1}) - h(a_n)| = 1$ for all n. This means that $\lim_{x\to 0} h(x)$ does not exist.

Working directly from the definition to find a limit is not easy. However, with a few rules, many limits are easy to calculate. The rules are very similar to the rules for limits of sequences.

Theorem 15 (Function limit rules). *Let $\mathbf{f}(x)$ and $\mathbf{g}(x)$ be functions from $\mathbb{R}$ to $\mathbb{R}^m$ with limits $\mathbf{K}$ and $\mathbf{L}$ at $x = a$, and let $h(x)$ be a function from $\mathbb{R}$ to $\mathbb{R}$ with limit M at a. Then the following hold:*

(i) $\lim\limits_{x\to a}(\mathbf{f}+\mathbf{g})(x) = \mathbf{K}+\mathbf{L}$

(ii) $\lim\limits_{x\to a} h(x)\mathbf{f}(x) = M\mathbf{K}$

(iii) $\lim\limits_{x\to a}(\mathbf{f}\cdot\mathbf{g})(x) = \mathbf{K}\cdot\mathbf{L}$

(iv) *If $M \neq 0$, then* $\lim\limits_{x\to a}\dfrac{\mathbf{f}(x)}{h(x)} = \mathbf{K}/M$

Proof. All of the proofs are similar, so only (i) is considered. Let a_n be any sequence converging to a. Then the sequences $\mathbf{f}(a_n)$ and $\mathbf{g}(a_n)$ converge to $\mathbf{K}$ and $\mathbf{L}$ respectively. Fix an $r > 0$. For some N_1 and N_2, if $n_1 > N_1$ and $n_2 > N_2$, we have $\|\mathbf{f}(a_{n_1}) - \mathbf{K}\| < r/2$ and $\|\mathbf{g}(a_{n_2}) - \mathbf{L}\| < r/2$.

Assume that $N = \max\{N_1, N_2\}$ and let $n > N$. Then, if $n > N$,

$$\begin{aligned} \|(\mathbf{f}+\mathbf{g})(a_n) - (\mathbf{K}+\mathbf{L})\| &= \|(\mathbf{f}(a_n) - \mathbf{K}) + (\mathbf{g}(a_n) - \mathbf{L})\| \\ &\leq \|\mathbf{f}(a_n) - \mathbf{K}\| + \|\mathbf{g}(a_n) - \mathbf{L}\| \\ &< r/2 + r/2 = r. \end{aligned}$$

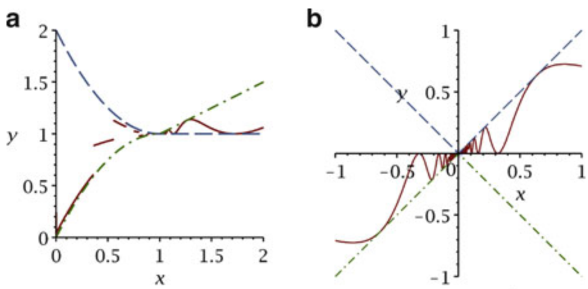
By the definition of convergence of sequences, $(\mathbf{f}+\mathbf{g})(a_n)$ converges to $\mathbf{K}+\mathbf{L}$. Since the sequence was arbitrary, this holds for all sequences, and the limit is $\mathbf{K}+\mathbf{L}$.

It should be fairly clear that a constant function $f(x) = c$ has limit c at any point a. Somewhat harder, but not too hard, is the fact that $f(x) = x$ has limit a at any point a. We can use this in a simple application of Theorem 15.

Example 49. Using the just stated facts about constant functions and $f(x) = x$, we can show that $\lim_{x\to a} cx^2 = ca^2$. First, since $x^2 = x\cdot x$, the product of $f(x)$ with itself, we have

$$\lim_{x\to a} cx^2 = (\lim_{x\to a} c)(\lim_{x\to a} x)(\lim_{x\to a} x) = c\cdot a\cdot a = ca^2.$$

We also have an analog of Theorem 11 on page 37. Assume we have functions $f(x)$, $g(x)$, and $h(x)$ defined around some point $x = a$ such that $h(x)$ is between $f(x)$ and $g(x)$ near a and such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$. For any sequence $x_n \to a$, after some N, the values of the sequence $h(x_n)$ are between $f(x_n)$ and $g(x_n)$. Since $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = L$, Theorem 11 applies and the sequence of values $h(x_n)$ converges to L. This implies that $\lim_{x\to a} h(x_n) = L$. An illustration of this idea is in Fig. 2.7a on page 43. We state this as the following theorem.



Idea of the squeeze theorem

Squeezing $x\sin^2(1/x)$

Fig. 2.7

Theorem 16 (Squeeze). *Assume the functions $f(x)$, $g(x)$, and $h(x)$ are defined for all points with $|x-a| \in (0,\delta)$ for a fixed $\delta > 0$. Also assume that $h(x)$ is between $f(x)$ and $g(x)$ for all x with $|x-a| \in (0,\delta)$ and assume that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$. Then $\lim_{x\to a} h(x) = L$.*

A simple example illustrates how this works.

Example 50. Let $h(x) = x\sin^2(1/x)$, $f(x) = |x|$, and $g(x) = -|x|$. Here, since $|\sin^2(y)| \in [-1,1]$ for all $y \in \mathbb{R}$, we have $h(x)$ is between $f(x)$ and $g(x)$ for all $x \neq 0$. See Fig. 2.7b on page 43 for a graph of the functions. This means that, since $\lim_{x\to 0} |x| = \lim_{x\to 0} -|x| = 0$, $\lim_{x\to 0} h(x) = 0$.

It is also useful to evaluate limits of compositions of functions. The next result shows how this is done.

Theorem 17 (Limit of Composition). *Assume $f(x)$ has a limit L as $x \to a$ and assume $g(y)$ is defined on an interval around L and has a limit $M = g(L)$ as $y \to L$, then $g\circ f$ has a limit as $x \to a$ and*

$$\lim_{x\to a} (g\circ f)(x) = M\ .$$

Proof. Assume that x_n is any sequence such that $x_n \to a$ and no $x_m = a$. Then the sequence $y_n = f(x_n)$ is defined after some N and y_n converges to L. Thus, the sequence $g(y_n) = g(f(x_n))$ converges to M for any sequence $x_n \to a$ with $x_n \neq a$ for all m. This means that $\lim_{x\to a}(g\circ f)(x) = M$.

A couple examples will help understand this result.

Example 51. Let $g(y) = y^2 + 4$ and let $f(x) = \left(x^2 - 1\right)/(x+1)$. Here $\lim_{x \to -1} f(x) = -2$ and $\lim_{y \to -2} g(y) = 8$. Using Theorem 17, we have

$$\lim_{x \to -1} \left(\left(\frac{x^2 - 1}{x+1} \right)^2 + 4 \right) = 8 .$$

Example 52. Here the limit exists without all of the criteria being met. The assumptions of Theorem 17 are not met at $x = 0$ if $g(y) = \cos(y)$ and $f(x) = 4\pi\,(\lfloor x \rfloor - 1/2)$ since $\lim_{x \to 0} f(x)$ does not exist. However, since $f(x) = 2\pi$ if $x \in [0,1)$ and $f(x) = -2\pi$ is $x \in [-1,0)$, $\cos(f(x)) = 1$ on $[-1,1)$ and

$$\lim_{x \to 0} (g \circ f)(x) = 1 .$$

Example 53. Here the limit does not exist when the criteria of the theorem are not met. Consider the function $g(y) = \left(y^2 - 1\right)/(y+1)$ with $\lim_{y \to -1} g(y) = -2$ where $g(-1)$ does not exist and consider the function $f(x) = -1$ that has $\lim_{x \to -a} f(x) = -1$ for any real number a. Since $g(f(x))$ does not exist anywhere, the composition $g \circ f$ cannot have a limit as $x \to a$ for any a.

As you will see in a couple sections, avoiding the point a in the definition of $\lim_{x \to a} \mathbf{f}(x) = \mathbf{L}$ is necessary. However, the nice functions $f(x) = x^2$ and $g(x) = |x|$ satisfy the property that $\lim_{x \to 0} f(x) = 0 = f(0)$ and $\lim_{x \to 0} g(x) = 0 = g(0)$. This nice property is captured in the definition of continuity.

Definition 7 (Continuous function). A function $\mathbf{f}(x)$ is *continuous* at a point a if $\lim_{x \to a} \mathbf{f}(x) = \mathbf{f}(a)$. If the function is continuous at every point of an interval, $\mathbf{f}$ is said to be continuous on that interval.

A function that is not continuous at a point a is said to be *discontinuous* at a.

Example 54. Again consider the floor function, see Fig. 2.6a on page 40. At every integer n, $\lim_{x \to n} \lfloor x \rfloor$ does not exist. This means that the floor is discontinuous at every integer. On the other hand, since $\lfloor x \rfloor$ is constant between consecutive integers, the floor function is continuous at every point that is not an integer.

The idea that a limit of a sequence $\mathbf{b}_n$ is taken coordinate by coordinate is used to prove the following theorem. This theorem is used for evaluating limits of vector valued functions and for deciding on the continuity of vector-valued functions.

Theorem 18 (Continuous vector valued functions).
Let $\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ be defined on an interval around a. The limit $\lim_{t \to a} \mathbf{f}(t) = \mathbf{L}$ if and only if $\lim_{t \to a} f_i(t) = L_i$ for $i = 1, 2, \ldots, m$. The function $\mathbf{f}(x)$ is continuous at a if and only if $f_i(x)$ is continuous at a for $i = 1, 2, \ldots, m$.

Example 55. Let $\mathbf{f}(t) = (4t^2, 1, t)$. Then

$$\begin{aligned} \lim_{t \to 2} \mathbf{f}(t) &= \left(\lim_{t \to 2} 4t^2,\ \lim_{t \to 2} 1,\ \lim_{t \to 2} t \right) \\ &= (16, 1, 2). \end{aligned}$$

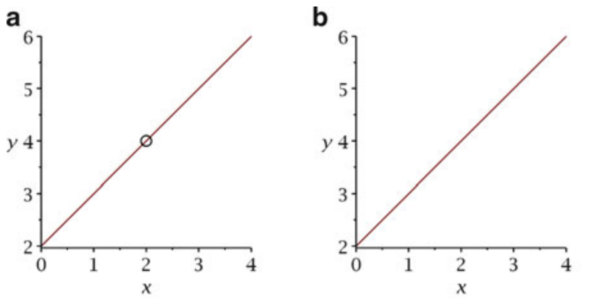
Many elementary functions are continuous at every point in their domains. It is not demonstrated now that these are continuous, but any polynomial is continuous on $\mathbb{R}$, $\sin(x)$ and $\cos(x)$ are continuous on $\mathbb{R}$, e^x is continuous on $\mathbb{R}$, and $\ln(x)$ is continuous on $(0, \infty)$.

Using the theorem on limits above, we can prove the following result. The proof is left to the ambitious reader.

Theorem 19 (Continuous vector valued functions, rules). *Let* $\mathbf{f}(x)$*,* $\mathbf{g}(x)$*,* $r(x)$ *and* $h(x)$ *be functions that are continuous at a. Assume that* $h(a) \neq 0$*. Then the following hold:*

(i) $\mathbf{f}+\mathbf{g}$ *is continuous at a,*
(ii) $r\mathbf{f}$ *is continuous at a,*
(iii) $\mathbf{f}\cdot\mathbf{g}$ *is continuous at a, and*
(iv) $\dfrac{\mathbf{f}}{h}$ *is continuous at a.*

This theorem can be used to demonstrate that many elementary functions are continuous on their domains. For example, the polynomials are continuous on $\mathbb{R}$, any rational function is continuous on its domain, and $\tan(x)$, $\sec(x)$, $\cot(x)$, and $\csc(x)$ are continuous on their domains.



The discontinuous function $y=(x^2-4)/(x-2)$

The continuous function $y=x+2$

Fig. 2.8

Example 56. Consider $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$. Since both $\sin(x)$ and $\cos(x)$ are continuous everywhere, $\tan(x)$ is continuous wherever $\cos(x) \neq 0$. This is exactly where $\tan(x)$ is defined, $\{x|x \neq n\pi/2 \text{ where n is an odd integer.}\}$.

Example 57. Consider the functions $\mathbf{f}(x) = (\sin(x), x^2, 4)$ and $h(x) = x^2+1$. Using the theorem above we have that the function $\mathbf{f}(x)/h(x)$ is continuous at all x.

There are also many functions that have points where they are discontinuous.

Example 58. The function

$$s(x) = \begin{cases} 1 & x \text{ is rational} \\ -1 & x \text{ is irrational}\,. \end{cases}$$

is discontinuous everywhere. We will not prove it here, but every open interval in the real line contains both rational and irrational numbers. This means that $s(x)$ takes on the values -1 and 1 in every interval and the limit $\lim_{x\to a} s(x)$ does not exist for any a.

A more common case in elementary calculus is the case where a function is not continuous since it does not exist at a point. Sometimes we can make a function continuous at a point where it is discontinuous by changing the value of the function or adding a value for the function at the point. Such a discontinuity is called a *removable discontinuity*.

Example 59. Consider the function $f(x) = \left(x^2-4\right)/(x-2)$. This function is discontinuous at $x=2$ since $f(x)$ is not defined at $x=2$. However, if $x \neq 2$

$$\begin{aligned} f(x) &= \frac{(x-2)(x+2)}{x-2} \\ &= x+2. \end{aligned}$$

By adding the point $(2,4)$ to the graph of $f(x)$ we get a new function $g(x) = x+2$ that is continuous at $x=2$. This means the discontinuity is a removable discontinuity. See Fig. 2.8 on page 45

A discontinuity that is not removable is called an *essential discontinuity*. Two simple examples of essential discontinuities are the floor function at every integer and the function $f(x) = 1/x$ at $x=0$.

The following result allows us to show that many more functions are continuous.

Theorem 20 (Continuity of compositions). *Let $f(x)$ be defined on an interval around a and assume that $f(x)$ is continuous at $x=a$. Also let $\mathbf{g}(y)$ be defined on an interval around $f(a)$ and be continuous at $y=f(a)$. Then the composition $(\mathbf{g}\circ f)(x)$ is continuous at $x=a$.*

Proof. Let a_n be a sequence of points converging to a. After some N_1, $b_n = f(a_n)$ is defined and $b_n \to f(a)$ by the definition of continuity. Since $b_n \to f(a)$, $\mathbf{c}_n = \mathbf{g}(b_n) = (\mathbf{g}\circ f)(a_n)$ is defined after some N and c_n converges to $\mathbf{g}(f(a))$. Therefore, $\mathbf{g}\circ f$ is continuous at a.

This is very useful when combining functions. It will be used without stating the result throughout this chapter.

Example 60. The function $f(x) = \sin(2x+10)$ is continuous everywhere. This follows from Theorem 20 since both $g(y) = \sin(y)$ and $h(x) = 2x+10$ are continuous everywhere.

There are several important applications of continuity that will be considered later. Here is one consequence of continuity that will be used when derivatives are discussed.

Theorem 21. *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous at a and assume that $f(a) > 0$ (or $f(a) < 0$). Then there is an interval (b,c) containing a such $f(x) > 0$ (or $f(x) < 0$) on (b,c).*

Proof. Assume that, to the contrary, that $f(x)$ is continuous at a, $f(a) > 0$, and there is a sequence $a_n \to a$ such that $f(a_n) \leq 0$. Then $|f(a_n) - f(a)| \geq f(a) > 0$. This means that $\lim_{x\to a} f(x)$ either does not exist or is not equal to $f(a)$. Since $f(x)$ is continuous at a, neither of these conclusions is true and $f(x) > 0$ on some open interval containing a.

Example 61. Let $f(x) = \sin(x)$. Since this function is continuous and $f(\pi/2) > 0$, $\sin(x) > 0$ on an interval containing $\pi/2$. We can use the interval $(0,\pi)$.

Example 62. Consider the function

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases},$$

See Fig. 2.9 on page 47

This function has $f(0) > 0$ but is not continuous at 0. It is clear that there are points x close to 0 where $f(x) = -1 < 0$.

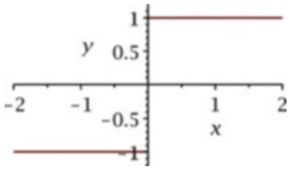


Fig. 2.9 $y = f(x)$ from Example 62

This section concludes with several examples concerning numerical computation of limits and concerning discontinuities and graphs. It is common to use a table of values to estimate the limit of a function $f(x)$ as $x \to a$. This is similar to saying the limit of a sequence exists by considering only the first few terms of a sequence of a_n's. The next two examples show that this may give the impression that there is a limit, even when no limit exists.

First consider a sequence with no limit.

Example 63. Consider the sequence $\{a_n\}_{n=0}^{\infty}$ with $a_n = 1$ if $n \leq 10{,}000$ and with $a_n = (-1)^n$ if $n > 10{,}000$. If we look only at the first 1,000 elements, the limit appears to be 1. However, the limit does not exist since the a_n's alternate between -1 and 1 after $n = 10{,}000$.

For functions the situation is more complex. Some sequences of function values at points converging to the a of interest may converge when others diverge or converge to different values.

Example 64. Consider the function $f(x) = \sin(\pi/x)$, Fig. 2.10 on page 47.

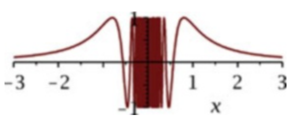


Fig. 2.10 $f(x) = \sin(\pi/x)$

If we choose any sequence of the form $a_n = 1/m(n)$ where $m(n)$ is an increasing sequence of integers that goes to infinity, for example $a_n = 10^{-n}$, then $f(a_n) = 0$ for all n. From the graph, this does not capture the behavior of the function. In fact, if we take $a_n = 1/(2n+1/2)$, we $f(a_n) = 1$ for all n. This is a different possible limit.

Since the sample sequences do not converge to the same value, the function does not have a limit as x goes to 0.

The previous example shows that using only one sequence to show that a limit exists does not work. If we know that the of a limit function exists, we can use any sequence to estimate the limit.

Theorem 22. *If* $\lim_{x\to a} \mathbf{f}(x) = \mathbf{L}$*, then for any sequence* $a_n \to a$*, we have*

$$\lim_{n\to\infty} \mathbf{f}(a_n) = \mathbf{L}.$$

Here is how we can use this theorem.

Example 65. Let $f(x) = \cos(\pi x)$ and consider the sequence $a_n = (4^{n-1} - 1)/4^n$. We can show that $|a_n - 1/4| = 1/4^n$ and that $a_n \to 1/4$. Since $f(x)$ is continuous everywhere, $f(a_n) \to f(1/4) = \cos(\pi/4) = \sqrt{2}/2$. Here The values for the first 8 terms of the sequence are in Table 2.1 on page 48. From the table we can guess that the limit is approximately 0.707. Since this is $\sqrt{2}/2$ to three decimal places, it is correct.

x	0	$\frac{3}{16}$	$\frac{15}{64}$	$\frac{63}{256}$	$\frac{255}{1{,}024}$	$\frac{1{,}023}{4{,}096}$	$\frac{4{,}095}{16{,}384}$	$\frac{16{,}383}{65{,}536}$
$f(x)$	1	0.831470	0.740951	0.715731	0.709273	0.707649	0.707242	0.707141

Table 2.1 Values of $\cos(\pi x)$ as x approaches $1/4$

t	$1/2$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
$\sin(t)/t$	0.958851	0.989616	0.997398	0.999349	0.999837	0.999959	0.999990	0.999997
$(\cos(t)-1)/t^2$	−0.4896698	−0.497401	−0.499349	−0.499837	−0.499959	−0.5000	−0.499990	−0.500

Table 2.2 Values of $\left(\sin(t)/t, (\cos(t)-1)/t^2\right)$ as t approaches 0

Example 66. Consider the case of the function $\mathbf{r}(t) = \left(\sin(t)/t, (\cos(t)-1)/t^2\right)$. It is known that both $\lim_{t\to 0} \sin(t)/t$ and $\lim_{t\to 0} (\cos(t)-1)/(t^2)$ exist. (These limits are considered in Sect. 7.4.) To get an estimate for the values of the limits we can create Table 2.2 on 48.

From this table we can conclude that $\lim_{t\to 0} \sin(t)/t$ is probably 1 and that $\lim_{t\to 0} (\cos(t)-1)/t^2$ is probably $-1/2$. This means that $\lim_{t\to 0} \mathbf{r}(t)$ is probably $(1, -1/2)$.

Frequently we can see discontinuities and possible discontinuities from the graph of a function. The three basic things that can cause a function $f(x)$ to be discontinuous at a are that the function may not be defined at a, the limit of $f(x)$ does not exist at a, and the limit exists but does not equal $f(a)$.

It is important to remember that graphs are only partial representations of a function. As such, not all the features of a function may be visible. Therefore we must do more than look at a graph to guarantee that a function is continuous at a point.

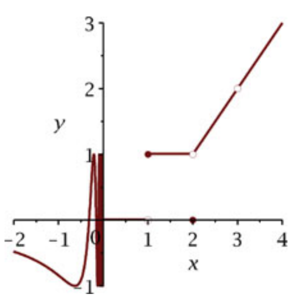


Fig. 2.11 The graph of a function with different types of discontinuities

Example 67. Figure 2.11 on page 48 is the graph of a function with several discontinuities.

The apparent discontinuities are at $x = 0, 1, 2,$ and 3. *If we assume that the graph represents all of the behavior of $f(x)$ we can make some conclusions.* At both $x = 0$ and $x = 1$ the limit of $f(x)$ does not exist. This means that $f(x)$ is not continuous at either point.

At both $x = 2$ and $x = 3$ the limit of $f(x)$ exists The value $f(2) = 1$ does not equal $\lim_{x\to 2} f(x) = 0$. Since $f(x)$ is not defined at $x = 3$, it is not continuous there. The discontinuities at $x = 2$ and $x = 3$ are removable discontinuities.

Exercises

1. Find the following limits of functions.

(a) $\lim\limits_{x\to 3} 4$

(b) $\lim\limits_{x\to -2} \dfrac{\pi}{4}$

(c) $\lim\limits_{x\to 7} \dfrac{x}{2}$

(d) $\lim\limits_{z\to -1} 2z$

(e) $\lim\limits_{z\to \frac{3}{2}} 3z-2$

(f) $\lim\limits_{z\to 0} \dfrac{3-2z}{5}$

(g) $\lim\limits_{w\to 0} 10w-6$

(h) $\lim\limits_{w\to 1} (3w+2)^2$

(i) $\lim\limits_{w\to -1} w^2-3w+2$

(j) $\lim\limits_{y\to 0} y^3+7y-2$

(k) $\lim\limits_{y\to -2} y^4-y^6$

(l) $\lim\limits_{y\to 0} \dfrac{y^2-2}{3-y}$

(m) $\lim\limits_{t\to \pi} \dfrac{\cos(t)}{t^2+2}$

(n) $\lim\limits_{w\to 1} (w^1-3, w+1, w^2)$

(o) $\lim\limits_{y\to 0} (y^2-6y^3, 3y-5, 4y^2-1)$

(p) $\lim\limits_{y\to 1} (y^4-y^6+1, -6, 5y+10)$

(q) $\lim\limits_{y\to 2} \left(\dfrac{y^2-2}{3-y}, \dfrac{6y^2-1}{y^3+2}\right)$

2. Find the limits of the following functions.

(a) $\lim\limits_{x\to 1} \ln(3x-2)$

(b) $\lim\limits_{t\to 2} \cos(t^2+t)$

(c) $\lim\limits_{y\to \pi} \tan\left(\dfrac{y}{2}-\dfrac{\pi}{4}\right)$

(d) $\lim\limits_{x\to -1} e^{2x+2}$

(e) $\lim\limits_{w\to 0} \sec\left(w^2+\dfrac{\pi}{6}\right)$

(f) $\lim\limits_{t\to 6} \left(t^2-7t+1\right)^{110}$

(g) $\lim\limits_{x\to -1} \left(e^{2x+2}, \cos(\pi x), \ln(x+2)\right)$

(h) $\lim\limits_{w\to 0} \left(e^{w+1}, \sec\left(w^2+\dfrac{\pi}{6}\right), w^3+3\right)$

(i) $\lim\limits_{t\to 6} \left(\dfrac{t+2}{t-3}, \left(t^2-7t+1\right)^{110}\right)$

3. Show that the following limits do not exist.

(a) $\lim\limits_{x\to 0} \cos\left(\dfrac{1}{x}\right)$

(b) $\lim\limits_{x\to 0} \dfrac{1}{x}$

(c) $\lim\limits_{x\to \frac{\pi}{2}} \tan(x)$

(d) $\lim\limits_{x\to -1} \dfrac{1}{1-e^{-(2x+2)}}$

(e) $\lim\limits_{x\to 0} \cot(x)$

(f) $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2}\right)$

(g) $\lim\limits_{x\to 0} \left(x^2+1, \cot(x)\right)$

(h) $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2}, \cos(x^2)\right)$

4. Estimate the following limits numerically.

(a) $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x^2+x-6}$

(b) $\displaystyle\lim_{x\to 0} \frac{\cos(x)}{x^2}$

(c) $\displaystyle\lim_{x\to \pi} \frac{\sin(x)}{x-\pi}$

(d) $\displaystyle\lim_{x\to 3} \frac{x^3-27}{x^2-9}$

(e) $\displaystyle\lim_{x\to \pi} \left(\frac{\sin(x)}{x-\pi}, \sqrt{x}\right)$

(f) $\displaystyle\lim_{x\to 3} \left(\sin\left(\frac{\pi x}{18}\right), \frac{x^3-27}{x^2-9}\right)$

5. Where are the following functions continuous?

(a) $f(x) = x^2+3x-6$

(b) $g(x) = \cot(x)$

(c) $h(x) = \dfrac{x+2}{x^2-1}$

(d) $f(x) = \dfrac{x}{\cos(x)}$

(e) $g(x) = \left\lfloor \dfrac{x}{2} \right\rfloor$

(f) $\mathbf{h}(w) = \left(w^5-w^2, \sin(w)+\cos(2w)\right)$

(g) $\mathbf{g}(w) = \left(\left\lfloor \dfrac{w}{2} \right\rfloor, \tan(w)\right)$

(h) $\mathbf{h}(r) = \left(\dfrac{1}{r^2-4}, |r|\right)$

6. Are the discontinuities of the following functions at the given points removable discontinuities?

(a) $f(x) = \lfloor x \rfloor$, $x = 2$

(b) $g(h) = \dfrac{(3+h)^2-9}{h}$, $h = 0$

(c) $h(x) = \dfrac{1}{x}$, $x = 0$

(d) $f(w) = \dfrac{w}{\cos(w)}$, $w = \pi/2$

(e) $g(w) = \dfrac{(w-1)^3+1}{w}$, $w = 0$

(f) $h(x) = \dfrac{x+1}{x^2-1}$, $x = 1$

(g) $h(x) = \dfrac{x^2-5x+6}{x^2-4x+4}$, $x = 2$

(h) $h(x) = \dfrac{x^2+3x-4}{x^3-3x^2+3x-1}$, $x = 1$

7. Where are the following functions continuous?

(a) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

(b) $g(t) = \begin{cases} 4t-2 & \text{if } t \leq 1 \\ t+2 & \text{if } t > 1 \end{cases}$

(c) $h(z) = \begin{cases} 4-z^2 & \text{if } |z| < 2 \\ z^2-4 & \text{if } |z| \geq 2 \end{cases}$

(d) $s(w) = \begin{cases} w^3-4w & \text{if } w < 2 \\ w-1 & \text{if } w \geq 2 \end{cases}$

8. Use the squeeze theorem, Theorem 16 on page 43, to show that if

$$g(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational}, \end{cases}$$

then $g(x)$ is continuous at $x = 0$. How does this relate to the idea of a function being continuous at $x = a$ "if you can draw the graph of the function without taking your pencil off the paper?"

2.3 Rates of Change and Derivatives

Differential calculus can be considered to be a study of how things change. As the first part of this study of change in this book, the idea of instantaneous rate of change is considered. The preliminary example here is that of a car driving along a straight east-west road. At a time, say t_0, how fast is the car going?

Assume that the position of the car is know as a function of time, say $r(t)$. Here r is the signed distance from a fixed point on the road, positive for east and negative for west. Consider the distance traveled from time t_0 to time t_1, $r(t_1) - r(t_0)$. If the car is traveling at a constant velocity, that velocity is

$$v = \frac{r(t_1) - r(t_0)}{t_1 - t_0}.$$

The *velocity* is the speed of the car with a sign to determine direction. In this case, if the car is moving east in the positive direction, the velocity is positive, and if the car is moving west in the negative direction, the velocity is negative.

If the car is not traveling at a constant velocity, we say that the *average velocity* of the car from t_0 to t_1 is

$$v_{ave} = \frac{r(t_1) - r(t_0)}{t_1 - t_0}.$$

This is the constant velocity the car would need to travel from $r(t_0)$ to $r(t_1)$ in the same length of time, $t_1 - t_0$. The graph in Fig. 2.12 on page 51 illustrates this idea.

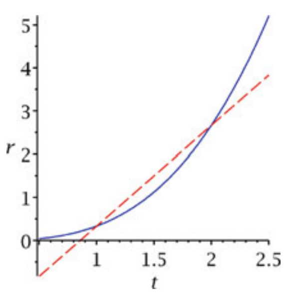


Fig. 2.12 A secant line representing average velocity

The solid curve represents the position $r(t)$ and the dashed line represents position with a constant velocity that has the same distance values as $r(t)$ at $t = 1$ and $t = 2$. The line including the points $(t_0, r(t_0))$ and $(t_1, r(t_1))$ is called the *secant line* to the graph of $r(t)$ through $(t_0, r(t_0))$ and $(t_1, r(t_1))$.

The ratio used to find an average rate of change for any function

$$\frac{\mathbf{f}(y)-\mathbf{f}(x)}{y-x}$$

is called a *difference quotient*. It is often written in the form

$$\frac{\mathbf{f}(x+h)-\mathbf{f}(x)}{h} \tag{2.1}$$

by replacing y with $x+h$. This quotient is fundamental for differential calculus.

Example 68. Assume the position of a car is given by $r(t) = 50t + 3\cos(t/5)$. The average velocity of the car from $t = 0$ to $t = 2$ is given by

$$\begin{aligned} v_{\text{avg}} &= \frac{r(2)-r(0)}{2-0} \\ &= \frac{50\cdot 2+3\cos(2/5)-(50\cdot 0+3\cos(0))}{2} \\ &= 50+\frac{3\cos(2)-3}{2} \\ &\approx 47.88. \end{aligned}$$

If this is a real car, we assume that the average velocity over the interval from t_0 to t_1 gets closer to the instantaneous velocity *at* t_0 as t_1 gets closer to t_0. Assuming that this is true, we define the *instantaneous velocity* of the car at t_0 as

$$v(t_0) = \lim_{t_1\to t_0} \frac{r(t_1)-r(t_0)}{t_1-t_0}.$$

The same definition is used for vector valued position functions to define velocity and to define acceleration for a vector valued velocity function. *Acceleration* is the rate of change, derivative, of velocity.

Example 69. Let the position function of a mass be given by $r(t) = -16t^2 + 20t + 6$. The velocity at $t = 1$ is given by

$$\begin{aligned} v(1) &= \lim_{t\to 1} \frac{r(t)-r(1)}{t-1} \\ &= \lim_{t\to 1} \frac{-16t^2+20t+6-(-16+20+6)}{t-1} \\ &= \lim_{t\to 1} \frac{-16(t^2-1)+20(t-1)}{t-1} \\ &= \lim_{t\to 1} -16(t+1)+20 \\ &= -16\,(1+1)+20 \\ &= -12. \end{aligned}$$

As you will see in the next section, the velocity of the mass is $v(t) = -32t + 20$. This allows us to calculate the acceleration of the mass at $t = 1$, $a(1)$:

$$\begin{aligned} a(1) &= \lim_{t\to 1} \frac{v(t)-v(1)}{t-1} \\ &= \lim_{t\to 1} \frac{-32t+20-(-32+20)}{t-1} \\ &= \lim_{t\to 1} \frac{-32(t-1)}{t-1} \\ &= -32. \end{aligned}$$

We can do the same type of example with vector valued motion.

Example 70. Assuming there is no friction, the position of a projectile fired from $(0,0)$ at an angle θ above horizontal with an initial speed of $v\,\mathrm{m/sec}$ in the positive x direction can be written as

$$\mathbf{r}(t) = \big(v\,t\cos(\theta), -9.8t^t + v\,t\sin(\theta)\big) \ .$$

The average velocity over the interval from $t=0$ to $0+h$ is

$$\begin{aligned}\mathbf{v}_{ave} &= \frac{\big(v\,h\cos(\theta), -9.8h^2 + v\,h\sin(\theta)\big) - (0,0)}{h} \\ &= \frac{v\,h\big(\cos(\theta),\sin(\theta)\big) - 9.8\,h^2\,(0,-1)}{h} \\ &= v\,\big(\cos(\theta),\sin(\theta)\big) - 9.8\,h\,(0,-1)\ .\end{aligned}$$

Taking the limit as $h \to 0$ of $\mathbf{v}_{ave}$ we get

$$\mathbf{v}(0) = v\,\big(\cos(\theta),\sin(\theta)\big)\ .$$

This gives us that the velocity is a vector of length v, the speed, in the direction of travel, $\big(\cos(\theta),\sin(\theta)\big)$.

Following this idea for general functions we define the derivative of a vector (scalar) valued function.

Definition 8 (Derivative). Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ be a function that is defined on an interval $(a-r, a+r)$ with $r > 0$. If the limit exists, the *derivative* of f at a is defined as

$$\frac{d\,\mathbf{f}(x)}{dx}(a) = \lim_{x\to a} \frac{\mathbf{f}(x) - \mathbf{f}(a)}{x-a}.$$

The derivative of $\mathbf{f}$ at a is also denoted by

$$D\mathbf{f}(a), \quad \mathbf{f}'(a), \quad \text{and} \quad \frac{d\,\mathbf{f}(a)}{dx}.$$

For a few functions this is easy to calculate.

Example 71. Consider any function of the form $\mathbf{f}(x) = \mathbf{m}x + \mathbf{b}$. The derivative at any point a is given by

$$\begin{aligned}\mathbf{f}'(a) &= \lim_{x\to a} \frac{(\mathbf{m}x + \mathbf{b}) - (\mathbf{m}a + \mathbf{b})}{x-a} \\ &= \lim_{x\to a} \frac{(\mathbf{m}\,(x-a))}{x-a} \\ &= \mathbf{m}.\end{aligned}$$

This is the slope of the line $y = mx + b$ if m and b are scalars and it is the velocity vector that includes speed and direction of travel if $\mathbf{f}(x)$ is a position function.

This gives a common interpretation of the derivative of a scalar valued function f at a. It is the slope of the tangent line to the graph of f at $(a, f(a))$. This can also be seen by looking at the “limit” of secant lines.

Example 72. Consider the function $f(x) = x^3 - x^2/2 - x/2$ at the point $x_0 = 1$. We can show that $f'(1) = 3/2$. We can look at the slopes of the secant lines for the f between the points $(1, f(1))$ and $(1+(1/2)^n, f(1+(1/2)^n))$. The picture indicates that the lines "approach" the tangent line to the graph of f at $(1,0)$. Figure 2.13 on page 54 is a graph that includes that tangent line and the first five secant lines.
A table of slopes, Table 2.3 on page 54, also indicates that the slopes of the secant lines approach $3/2$.

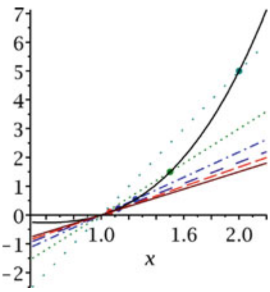


Fig. 2.13 Secant lines with slopes decreasing to the slope of a tangent line

$x+h$	2.	1.50000	1.25000	1.12500	1.06250	1.03125	1.01562	1.00781	1.00391	1.00195
Slope	5.	3.	2.18750	1.82812	1.66016	1.57910	1.53931	1.51959	1.50978	1.50489

Table 2.3 Slopes of secant lines for $f(x) = x^3 - x^2/2 - x/2$

The way in which the slope of the *tangent line* was introduced above gives an easy definition in the case of functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 9 (Tangent line). Assume the function $f : \mathbb{R} \to \mathbb{R}$ has a derivative at a. The *tangent line* to the graph of $f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$.
Assume the function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ has a derivative at a. The *tangent line* to the curve parametrized by $\mathbf{f}(x)$ at $\mathbf{f}(a)$ is the line through $\mathbf{f}(a)$ with direction $\mathbf{f}'(a)$.

Example 73. Consider the function $r(t) = -16t^2 + 20t + 6$ from Example 69 on page 52. The derivative of $r(t)$ at $t = 1$ is $r'(1) = -12$. The point where the tangent line meets the graph is $(1, r(1)) = (1, 10)$. The the equation of the tangent line is $y - 10 = -12(t-1)$ or $y = -12t + 22$, see Fig. 2.14 on page 55.

The derivative can also be interpreted as the rate of change of a quantity with respect to a variable. Examples of this are the rate of change of the surface of a sphere with respect to change in the volume and the rate of change of the profit for a company with respect to the price the company charges for a product.

An important result that is often used without statement is the relationship between derivatives and continuity. If the derivative of f exists at a, then the numerator in the limit of the definition of the derivative must go to 0 as $x \to a$. (Why is this true?). This means that

$$0 = \lim_{x \to a} \big(f(x) - f(a)\big) = \lim_{x \to a} \big(f(x)\big) - f(a).$$

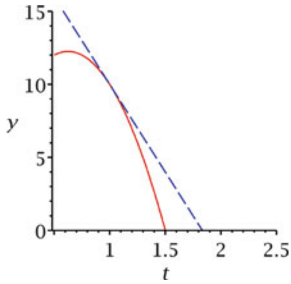


Fig. 2.14 A tangent line to $r(t) = -16t^2 + 20t + 6$

We then conclude that

$$\lim_{x \to a} f(x) = f(a),$$

and f is continuous at a. Restated, this says:

Theorem 23 (Differentiability implies continuity). *If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.*

Example 74. Consider the function $f(x) = mx + b$. Since it is differentiable everywhere, it is continuous everywhere.

There are cases when $f(x)$ is continuous at a point, but not differentiable at that point.

Example 75. Let $f(x) = |x|$. Then

$$\lim_{x \to 0} |x| = 0.$$

On the other hand, if $x < 0$, then

$$\frac{|x|}{x} = -1,$$

and if $x > 0$, then

$$\frac{|x|}{x} = 1.$$

Combined, these equations mean that $f'(0)$ does not exist.

Working with derivatives of vector-valued functions is very similar to working with limits of vector-valued functions. It is done coordinate by coordinate.

Theorem 24 (Differentiability of vector valued functions).
Let $\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ be defined on an interval around a. The derivative of $\mathbf{f}(x)$ at a exists if and only if $f_i'(a)$ exist for $i = 1, 2, \ldots, n$. In that case we have

$$\frac{d\,\mathbf{f}(x)}{dx}(a) = (f_1'(a), f_2'(a), \ldots, f_n'(a)).$$

Example 76. The derivative of $\mathbf{f}(t) = (6t + 2, 2 - t, 3)$ is $\mathbf{f}'(t) = (6, -1, 0)$ since

$$\frac{d\,(6t+2)}{dt} = 6, \qquad \frac{d\,(2-t)}{dt} = -1, \qquad \text{and} \qquad \frac{d\,(3)}{dt} = 0.$$

This example is important in physical situations. Interpreting $\mathbf{f}'(t)$ as velocity and $\mathbf{f}(t)$ as position, we can say that an object moving with constant velocity, $(6,-1,0)$ starting at $\mathbf{f}(0) = (2,2,3)$ moves along the straight line $\mathbf{f}(t) = (6t+2, 2-t, 3)$. This characterizes the motion of a mass in the absence of any forces.

We can visualize this in a manner similar to that done for functions of one variable. Here the secant vectors,

$$\mathbf{v} = \frac{\mathbf{f}(x) - \mathbf{f}(t)}{x - t},$$

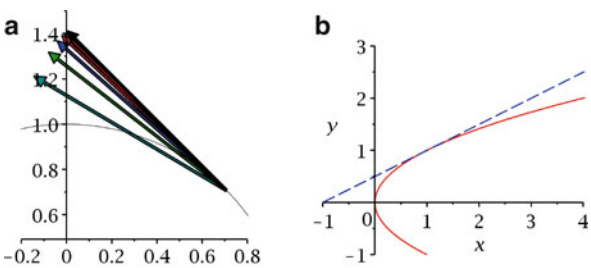
are used instead of slopes. (Why are slopes not appropriate for vector-valued functions with more than one output?) For vector-valued functions the idea is that the secant vectors converge to a tangent vector.

Example 77. Figure 2.15 on page 56 is the graph of $\mathbf{W}(t) = (\cos(t), \sin(t))$ around $\mathbf{W}(\pi/4) \approx (0.707, 0.707)$. Recall that the tangent to a circle at a point is perpendicular to the radius of the circle. In this case the tangent vector in the picture is $\mathbf{v} = (-\sqrt{2}/2, \sqrt{2}/2)$, a unit vector that is showing counter clockwise motion around the circle.

The secant vectors are taken between the point $\mathbf{W}(\pi/4)$ and the points $\mathbf{W}(\pi/4 + (1/2)^n)$ for $n = 1, 2, \ldots, 5$ with length

$$\left\| \frac{\mathbf{W}(\pi/4 + (1/2)^n) - \mathbf{W}(\pi/4)}{(1/2)^n} \right\| .$$

The graph in Fig. 2.15a on page 56 shows how the secant vectors approach this tangent vector.



Secant vectors going to a tangent vector

A tangent line to $\mathbf{r}(t) = (t^2, t)$

Fig. 2.15

Example 78. Consider the function $\mathbf{r}(t) = (t^2, t)$ that has derivative $\mathbf{r}' = (2t, 1)$. The value of $\mathbf{r}(t)$ at $t = 1$ is $\mathbf{r}(1) = (1,1)$ and derivative of $\mathbf{r}(t)$ at $t = 1$ is $\mathbf{r}'(1) = (2,1)$. This means that a parametrization of the tangent line to the curve parametrized by $\mathbf{r}(t)$ at $t = 1$ is $\ell(s) = (1,1) + s(2,1)$, see Fig. 2.15b on page 56.

Example 79. Consider the position curve of the projectile and velocity at $t = 0$ in Example 70.

$$\mathbf{r}(t) = \left(v\,t\cos(\theta), -9.8t^t + v\,t\sin(\theta)\right)$$
$$\mathbf{v}(0) = v\left(\cos(\theta), \sin(\theta)\right) .$$

In this case the velocity vector at $t = 0$ is tangent to the curve of motion of the projectile. Its direction is the direction of travel of the projectile at $t = 0$ and its length is equal to the speed of the projectile, see Fig. 2.16 on page 57.

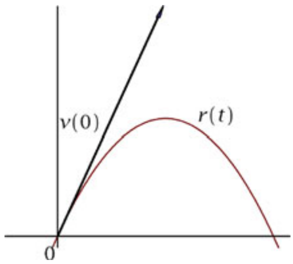


Fig. 2.16 The initial velocity vector and path of motion for Example 78

Exercises

1. What is the average rate of change of the function from $x = a$ to $x = b$ as given below?

(a) $f(x) = x^2 + 2$, $a = 1$, $b = 4$
(b) $f(x) = \sin(\pi x)$, $a = 0$, $b = \frac{1}{4}$
(c) $f(x) = e^{x-1}$, $a = 1$, $b = 2$
(d) $f(x) = x^3 + \frac{1}{x}$, $a = -3$, $b = -1$
(e) $g(x) = \frac{x^2+1}{4+x^3}$, $a = -1$, $b = 5$
(f) $g(x) = x^2 + 2x$, $a = -2$, $b = 0$
(g) $g(x) = \ln(x+1)$, $a = 0$, $b = 3$
(h) $g(x) = \cos(\pi(x+1)/3)$, $a = -2$, $b = -1$

2. Use the definition of the derivative to find the derivatives of the following functions at the given point.

(a) $f(x) = 0$, $a = 3$
(b) $f(x) = -3$, $a = 1$
(c) $g(x) = x + 1$, $a = 0$
(d) $g(x) = 2x - 1$, $a = -1$
(e) $h(z) = 3z + 2$, $a = 2$
(f) $h(z) = -4z - 4$, $a = 6$
(g) $h(z) = z^2 + 2$, $a = 1$
(h) $h(z) = -z^2 - 4$, $a = 0$
(i) $h(z) = z^2 - 2z + 1$, $a = 0$
(j) $h(z) = 2z^2 + z - 3$, $a = -1$

3. Find the average velocity of a mass whose position is given by $\mathbf{r}(t)$ from a to b.

(a) $\mathbf{r}(t) = (t, 3t - 1)$, $a = 0$, $b = 3$
(b) $\mathbf{r}(t) = (e^t, \ln(3t - 1))$, $a = 1$, $b = 2$
(c) $\mathbf{r}(t) = (t^2, \cos(\pi t/4))$, $a = 1$, $b = 7$
(d) $\mathbf{r}(t) = (3t, t^2, 1 - 2t)$, $a = -1$, $b = 1$
(e) $\mathbf{r}(t) = (e^{3t}, \ln(t^2 + 1), 1 - 2t^3)$, $a = 3$, $b = 5$
(f) $\mathbf{r}(t) = (\sin(\pi t/6), \cos(\pi t/6), t - 1)$, $a = 2$, $b = 4$

4. Use the definition of the derivative to find the derivatives of the following functions at the given point.

(a) $\mathbf{f}(t) = (0, 1, 3)$, $a = 2$
(b) $\mathbf{f}(t) = (-3, t, t)$, $a = 1$
(c) $\mathbf{g}(t) = (2t, t^2)$, $a = -2$
(d) $\mathbf{g}(t) = (t^2 + 1, \frac{1}{2}t)$, $a = -1$
(e) $\mathbf{h}(s) = (3s + 2, -3s - 2)$, $a = 2$
(f) $\mathbf{h}(s) = (-s^2, s^2 + 2)$, $a = 6$

5. Explain why the following functions do not have derivatives at the designated points.

(a) $f(x) = \frac{1}{x}$, $x = 0$
(b) $f(x) = |x-1|$, $x = 1$
(c) $g(x) = x - |x|$, $x = 0$
(d) $g(x) = \frac{\sin(x)}{x}$, $x = 0$
(e) $h(x) = x^{2/3}$, $x = 0$
(f) $h(x) = x^{1/3}$, $x = 0$
(g) $\mathbf{r}(t) = \left(3t^2, |t+2|, \cos(t)\right)$, $t = -2$
(h) $\mathbf{r}(t) = (\tan(t), \sec(t))$, $t = \frac{\pi}{2}$

6. The following functions have derivatives at the given points. Estimate the derivative to two decimal places using a numerical technique.

(a) $f(x) = x^{1/3}$, $x = 4$
(b) $f(x) = \cos(x)$, $x = \frac{\pi}{3}$
(c) $g(x) = \ln(x)$, $x = 1$
(d) $r(t) = \tan(x^3 - x)$, $x = 1$
(e) $g(z) = e^z$, $z = 0$
(f) $h(z) = z^{-1/3}$, $z = 4$
(g) $h(z) = \sqrt{z+1}$, $x = 6$
(h) $\mathbf{r}(t) = (t^{1/3}, t^6)$, $t = 1$
(i) $\mathbf{r}(t) = (\sin(\pi t/6), t^{-5}, t^2+1)$, $t = 2$
(j) $\mathbf{s}(t) = \left(\exp\left(\frac{t}{3}\right), \cos\left(t^{-2}\right), \ln(t^2+1)\right)$, $t = 2$

7. Explain why the following functions do not have a derivative at the given point.

(a) $f(x) = \lfloor x \rfloor$, $x = 4$
(b) $f(x) = \sqrt{x}$, $x = 0$
(c) $g(x) = \frac{1}{x^2}$, $x = 0$
(d) $g(z) = \frac{z^2-4}{z-2}$, $z = 2$
(e) $h(z) = z^{-1/3}$, $z = 0$
(f) $h(z) = \sqrt{z^2 - 2z + 1}$, $x = 1$
(g) $\mathbf{r}(t) = \left(3t^2+2, \cos(t), \lfloor \frac{t}{2} \rfloor\right)$, $z = 0$
(h) $\mathbf{h}(s) = \left(\frac{1}{s+1}, \frac{s+1}{s^2+3s+2}\right)$, $s = -1$

8. Use the definition of the derivative and the squeeze theorem, Theorem 16 on page 43, to show that the function

$$h(w) = \begin{cases} w^2 & \text{if } w \text{ is rational} \\ -w^2 & \text{if } w \text{ is irrational} \end{cases}$$

has a derivative at $w = 0$.

9. Why does the function

$$h(w) = \begin{cases} w^2 & \text{if } w \text{ is rational} \\ -w^2 & \text{if } w \text{ is irrational} \end{cases}$$

not have a derivative at any point besides $w = 0$.

10. Assume that $g(y)$ has a derivative at $y = 3.5$ with $g'(-3.5) = -3$. Using the definition of the derivative, explain why the function $f(y) = g(y) - 10$ has a derivative at $y = 3.5$ with $f'(3.5) = -3$.

11. Assume that $g(y)$ has a derivative at $y = 3.5$ with $g'(-3.5) = -3$. Use geometry to explain why the function $f(y) = g(y) - 10$ has a derivative at $y = 3.5$ with $f'(3.5) = -3$.

2.4 Derivatives of a Few Common Functions

As you should be able to tell from the last section, only having the derivatives of polynomials is very limiting. Even though not all of the derivatives of the functions in this section will be derived here, having them available for the rest of this chapter allows us to consider many more problems and applications.

The first functions we consider are $\sin(\theta)$ and $\cos(\theta)$. Recall the definition of $\sin(\theta)$ and $\cos(\theta)$ in terms of radians. The $\sin(\theta)$ and $\cos(\theta)$ are the x and y coordinates of the point on the unit circle centered at $(0,0)$ obtained by going θ units counter clockwise around the circle from the point $(1,0)$. This means that the function $\mathbf{W}(\theta) = (\cos(\theta), \sin(\theta))$ has us traveling counter clockwise around the circle at unit speed.

The above means that the derivative of $\mathbf{W}$ must be tangent to the unit circle and have length 1. The speed is 1 since the rotation goes at one radian per time unit. The only unit vectors tangent to the unit circle at $\mathbf{W}(\theta)$ are $(-\sin(\theta), \cos(\theta))$ and $(\sin(\theta), -\cos(\theta))$. From Fig. 2.17 on page 59, the choice should be
$(-\sin(\theta), \cos(\theta))$. It matches direction of travel around the unit circle.

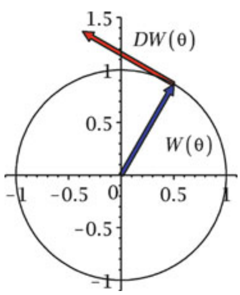


Fig. 2.17 $\mathbf{W}(\theta)$ and $D\mathbf{W}(\theta) = \mathbf{W}'(\theta)$

In particular, this means that

$$\frac{d}{d\theta}\cos(\theta) = -\sin(\theta)$$

and

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta).$$

It is easy to use these formulas.

Example 80. Find the derivative of $f(\theta) = 4\cos(\theta)$.

$$\begin{aligned}\frac{d}{d\theta} f(\theta) &= \lim_{h\to 0} \frac{4\cos(\theta+h) - 4\cos(\theta)}{h} \\ &= 4 \lim_{h\to 0} \frac{\cos(\theta+h) - \cos(\theta)}{h} \\ &= -4\sin(\theta).\end{aligned}$$

Another two functions considered in this section are the natural exponential and natural logarithm functions. You should be familiar with the graphs of $1/x$ and $\ln(x)$, Fig. 2.18 on page 60.

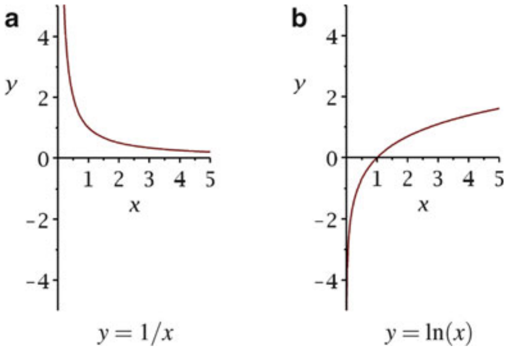


Fig. 2.18

The graph of $\ln(x)$ indicates that as x goes to 0 from the right that the derivative of $\ln(x)$ goes to infinity. As x goes to infinity the derivative of $\ln(x)$ goes to 0. Pictorially, $1/x$ looks like the derivative of $\ln(x)$. In fact, the derivative of $\ln(x)$ is $1/x$.

The final function in this section is the natural exponential, $\exp(x) = e^x$. Plotting the slopes of the tangent lines gives an indication of the function. Figure 2.19 on page 60 contains the plots of e^x and approximations of its derivative at intervals of 0.2. The approximations were done with the difference quotient

$$\frac{e^{x+0.001} - e^x}{0.001}.$$

Given the similarity in the graphs, we can guess that the derivative of e^x is e^x. In fact, that is true.

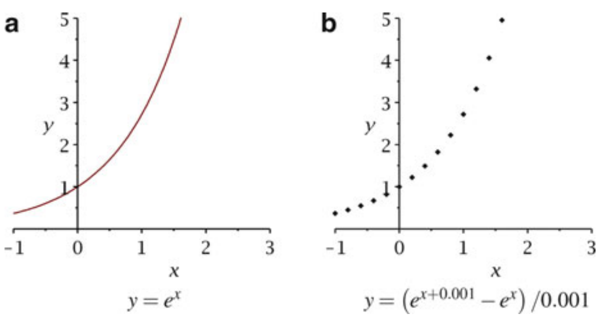


Fig. 2.19

Example 81. The derivative of $g(x) = \ln(x) + e^x$ is given by

$$\begin{aligned} g'(a) &= \lim_{x \to a} \frac{\ln(x) + e^x - (\ln(a) + e^a)}{x - a} \\ &= \lim_{x \to a} \frac{\ln(x) - \ln(a)}{x - a} + \lim_{x \to a} \frac{e^x - e^a}{x - a} \\ &= \frac{1}{a} + e^a. \end{aligned}$$

The natural exponential function has many uses in applications.

Example 82. A common assumption for the growth of a simple population is that the rate of change of the population, $P(t)$, is proportional to the population. If the proportionality constant is one, the equation describing this is

$$\frac{d\,P(t)}{dt} = P(t).$$

Since

$$\begin{aligned} \frac{d}{dt} Ce^t &= \lim_{x \to t} \frac{Ce^x - Ce^t}{x - t} \\ &= C \lim_{x \to t} \frac{e^x - e^t}{x - t} \\ &= C \frac{d}{dt} e^t \\ &= Ce^t, \end{aligned}$$

every function of the form $P(t) = Ce^t$ is a solution for this mathematical model of population growth.

Another class of functions that is frequently used is the class of polynomials. In the next chapter you will see how the rule for finding the derivative of a polynomial is derived. For now, the rule is simply stated.

Theorem 25 (Derivatives of polynomials). *Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be any polynomial of degree n. The derivative of $p(x)$ is given by*

$$\frac{d\,p(x)}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + ma_m x^{m-1} + \cdots + na_n x^{n-1}.$$

This is easy to use.

Example 83. The derivative of $p(x) = 3 - 2x + 4x^2 + 6x^3$ is

$$\begin{aligned} p'(x) &= -2 + 2 \cdot 4x^{2-1} + 3 \cdot 6x^{3-1} \\ &= -2 + 8x + 18x^2. \end{aligned}$$

Example 84. The derivative of $q(x) = x^{101}$ is

$$\frac{d\,q(x)}{dx} = 101x^{101-1} = 101x^{100}.$$

Example 85. If there was no air resistance and no wind, the position of a projectile near the surface of the earth would be

$$\mathbf{s}(t) = \left(v_x t + x_0, -4.9t^2 + v_y t + y_0\right).$$

Here (v_y, v_x) is the initial velocity at $t = 0$ and (x_0, y_0) is the initial position. Since the only force acting on the projectile is gravity, the horizontal velocity should be constant.

The velocity is

$$\begin{aligned} v(t) &= \frac{d\,s(t)}{dt} \\ &= \left(\frac{d}{dt}\left(v_x t + x_0\right)\right), \frac{d}{dt}\left(-4.9t^2 + v_y t + y_0\right) \\ &= (v_x, -9.8\,t + v_y). \end{aligned}$$

The result matches what is taught in elementary physics classes.

The section ends with a function whose derivative can be found using a little skill and the definition of the derivative. It will be used for problems involving various differentiation rules. There are mathematical models where it is used.

Example 86. The derivative of $\sqrt{x}$ is given by

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

The key to the derivation of this rule is that, if the square roots are defined, $(\sqrt{a} - \sqrt{b})\,(\sqrt{a} + \sqrt{b}) = a - b$.

The following is the derivation of this formula. Assuming that x and $x + h$ are both positive,

$$\begin{aligned} \frac{d}{dx}\sqrt{x} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})\,(\sqrt{x+h} + \sqrt{x})}{h\,(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{x + h - x}{h\,(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

Exercises

1. Find the derivatives of the following functions at the given point a.

(a) $f(x) = x^2 + 2$, $a = 1$

(b) $f(x) = x^3 - 3x^2 + 6x - 2$, $a = 0$

(c) $f(x) = 10x + 6 - 3x^2 - x^5$, $a = 2$

(d) $f(x) = 101x^{201} + 55x^{50} - 1{,}000x$, $a = -1$

(e) $f(x) = \cos(x)$, $a = \frac{\pi}{4}$

(f) $f(x) = e^x$, $a = 1$

(g) $f(x) = \sin(x)$, $a = \frac{\pi}{2}$

(h) $f(x) = e^x$, $a = -1$

(i) $\mathbf{r}(t) = \left(t^2 - 2t + 3, \cos(t), e^t\right)$, $a = -2$

(j) $\mathbf{r}(t) = \left(t^2 - 2t + 3, \cos(t), e^t\right)$, $a = -2$

(k) $\mathbf{r}(t) = \left(t^5 - 5t^3, \ln(t), \sin(t)\right)$, $a = 2$

(l) $\mathbf{r}(t) = \left(t, t^2, t^3, t^4\right)$, $a = 1$

(m) $\mathbf{r}(t) = (e^t, \ln(t), \cos(t))$, $a = 4$

2. Use the same type of reasoning as was used to find the derivatives of $\sin(\theta)$ and $\cos(\theta)$ to find the derivatives of the following functions.

(a) $\sin(-\theta)$

(b) $\cos(2\theta)$

(c) $\cos\left(\dfrac{\theta}{3}\right)$

(d) $\sin(\theta + \frac{\pi}{2})$

3. Each of the following is the velocity of an object. Find its acceleration.

(a) $\mathbf{v}(t) = (5t - 6, -16t + 32)$

(b) $\mathbf{v}(t) = (-\cos(t), \sin(t), 2)$

(c) $\mathbf{v}(t) = \left(e^t, \ln(t), 4t^2 - 6t\right)$

(d) $\mathbf{v}(t) = \left(5t^2 - 26t + 10, -16t^2 + 150\right)$

2.5 Derivatives, Graphs, and Approximations

Recall that the derivative of $\mathbf{f}(x)$ at a, if it exists, is defined as

$$\mathbf{f}'(a) = \lim_{x \to a} \frac{\mathbf{f}(x) - \mathbf{f}(a)}{x - a}.$$

Letting $h = x - a$, this can be rewritten as

$$\mathbf{f}'(a) = \lim_{h \to 0} \frac{\mathbf{f}(a+h) - \mathbf{f}(a)}{h}.$$

In many cases this form of the definition of the derivative is easier to use.

Since the limit is the derivative we have that if h is close to 0, then

$$\mathbf{f}'(a) \approx \frac{\mathbf{f}(a+h) - \mathbf{f}(a)}{h}.$$

Multiplying both sides by h and rearranging gives

$$\mathbf{f}(a+h) \approx \mathbf{f}(a) + \mathbf{f}'(a)\, h. \tag{2.2}$$

Given a function value and a derivative at a, this enables us to approximate values of the function near a. This idea is used throughout calculus.

Example 87. Given that $f(1) = 4$ and $f'(1) = -0.5$, approximate $f(1.2)$. Here we have $h = 1.2 - 1 = 0.2$. Thus

$$\begin{aligned} f(1.2) &\approx f(1) + f'(1)\, h \\ &\approx 4 + (-0.5)\,(0.2) \\ &\approx 3.9 \end{aligned}$$

This can be a good approximation for points close to a. As the following example shows, the approximation may not be good for points away from a.

Example 88. Consider the function $f(x) = \cos(x) - 4x^2$ around $a = 0$, see Fig. 2.20 on page 64. Since $f'(x) = -\sin(x) - 8x$, we have $f'(0) = 0$. The linear approximation is $f(0+h) \approx \ell(h) = 1 + 0h = 1$. This means that the error at $h = -0.01$ is $f(-0.01) - \ell(-0.01) \approx -0.00045$ and at $h = 0.2$ the error is $f(0.2) - \ell(0.2) \approx -0.17993$. The error is growing very fast in comparison with the size of h.

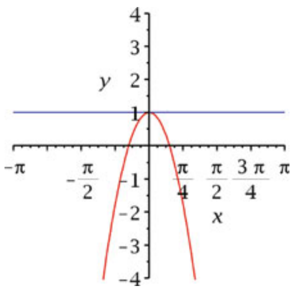


Fig. 2.20 Tangent line to $f(x) = \cos(x) - 4x^2$ at $x = 0$

This idea of approximation is also used in differential notation. This notation is commonly used in the sciences. If we take Eq. (2.2) and rewrite it we get

$$\mathbf{f}(a+h) - \mathbf{f}(a) \approx \mathbf{f}'(a)\,h.$$

Setting $\Delta\mathbf{f} = \mathbf{f}(a+h) - \mathbf{f}(a)$ and $\Delta x = h$, this approximation becomes

$$\Delta\mathbf{f} \approx \mathbf{f}'(a)\,\Delta x. \tag{2.3}$$

The Δ's are small changes in the values of $\mathbf{f}$ and x.

If we take the limit in the sense of making the distances $\Delta\mathbf{f}$ and Δx infinitely small, infinitesimals, we get the equation

$$d\mathbf{f} = \mathbf{f}'(x)\,dx. \tag{2.4}$$

We can use the equation in the same way that we use Eq. (2.3).

Example 89. Assume that the position of a mass is $\mathbf{s}(t) = \left(t^2 - t, 3t + 5, \cos(t)\right)$ when written as a function of time. Approximate the position of the mass at $t = -0.1$.

The derivative of $\mathbf{s}(t)$ is

$$\mathbf{s}'(t) = (2t - 1, 3, -\sin(t)),$$

$s(0) = (0, 5, 1)$ and $s'(0) = (-1, 3, 0)$. Using differential notation we have $dt = -0.1 - 0 = -0.1$ and

$$\begin{aligned} d\,\mathbf{s} &= \mathbf{s}'(0)\,dt \\ &\approx (-1, 3, 0)\,(-0.1) \\ &\approx (0.1, -0.3, 0). \end{aligned}$$

Since $d\mathbf{s} = \mathbf{s}(-0.1) - \mathbf{s}(0)$, we have

$$\begin{aligned} \mathbf{s}(-0.1) &\approx \mathbf{s}(0) + d\mathbf{s} \\ &\approx (0, 5, 1) + (0.1, -0.3, 0) \\ &\approx (0.1, 4.7, 1). \end{aligned}$$

We can also plot both sides of Eq. (2.2) to get a geometric view of the derivative.

Example 90. Consider the function $f(x) = e^x$. Then $f'(x) = e^x$ and $f(0) = f'(0) = 1$. This gives us

$$e^x \approx 1 + 1 \cdot x.$$

The plot shows that the line $y = 1 + x$ is a good approximation to $y = e^x$ around the point $(0,1)$, see Fig. 2.21 on page 65.

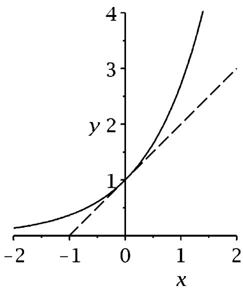


Fig. 2.21 Tangent line approximation to $y = e^x$ at $x = 0$

The same basic picture works for vector valued functions.

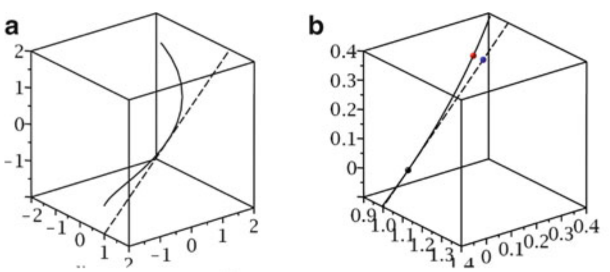
Example 91. Consider the function $\mathbf{f}(t) = (\cos(t), \sin(t), 2t)$. The derivative of this function is $\mathbf{f}'(t) = (-\sin(t), \cos(t), 2)$. At $t = 0$ we get that

$$\begin{aligned} \mathbf{f}(t) &\approx \mathbf{f}(0) + \mathbf{f}'(0)\,t \\ &\approx (1,0,0) + t\,(0,1,2) \\ &\approx (1,t,2t). \end{aligned}$$

Plotting the left and right sides of the last equation gives Fig. 2.22a on page 65. When we try to approximate $\mathbf{f}(0.2)$ we get

$$\begin{aligned} \mathbf{f}(0.2) &\approx \mathbf{f}(0) + \mathbf{f}'(0)\,(0.2) \\ &(1,0.2,0.4). \end{aligned}$$

This is plotted in Fig. 2.22b on page 65.



The tangent line to $\mathbf{f}(t) = (\cos(t), \sin(t), 2t)$

The tangent line approximation to $\mathbf{f}(t) = (\cos(t), \sin(t), 2t)$

Fig. 2.22

Exercises

1. In each part of this problem a function value and derivative of a function are given at a point a. Find an approximation for the function value at the given b.

(a) $f(1)=2$, $f'(1)=3$, $b=1.3$
(b) $f(0)=0$, $f'(0)=-2$, $b=-0.5$
(c) $f(-1)=2$, $f'(-1)=1/4$, $b=-1.3$
(d) $f(3)=2$, $f'(3)=-2$, $b=2.9$
(e) $\mathbf{r}(1)=(1,-2)$, $\mathbf{r}'(1)=(-1,1)$, $b=1.2$
(f) $\mathbf{r}(0)=(-1,0)$, $\mathbf{r}'(0)=(2,-1)$, $b=-0.2$
(g) $\mathbf{r}(-10)=(3,2,-1)$, $\mathbf{r}'(-10)=(4,-2,1)$, $b=-9.7$
(h) $\mathbf{r}(3)=(0,-1,1)$, $\mathbf{r}'(3)=(-1,2,-2)$, $b=2.7$

2. Given are a function value at a, a derivative value at a, and a b. Use this information to approximate $f(b)$.

(a) $f(1)=1$, $f'(1)=-2$, $b=1.5$
(b) $f(2)=-1$, $f'(2)=-1$, $b=2.5$
(c) $f(3)=1.3$, $f'(3)=0.25$, $b=1.35$
(d) $\mathbf{r}(1)=(1,0)$, $\mathbf{r}'(1)=(3,1/3)$, $b=0.95$
(e) $\mathbf{f}(0)=(2,-1)$, $\mathbf{f}'(0)=(0.3,-0.3)$, $b=0.33$
(f) $\mathbf{r}(5)=(-1,2,1)$, $\mathbf{r}'(5)=(-0.2,0.2,0)$, $b=5.15$

3. Given are a function, an a, and a b. Use this information to approximate $f(b)$ using the value and the derivative of the function at a.

(a) $f(x)=3x+3$, $a=2$, $b=2.5$
(b) $f(x)=\sin(x)+\cos(x)$, $a=\pi/4$, $b=0.9$
(c) $g(w)=w^3-4w^2+4$, $a=3$, $b=2.9$
(d) $\mathbf{r}(t)=(t^2,t^3-3t)$, $a=-2$, $b=-1.95$
(e) $\mathbf{f}(t)=(\sin(t),\cos(t),t)$, $a=3\pi/2$, $b=4.8$
(f) $\mathbf{r}(s)=(s^2,2-s^3,s^2-4s+2)$, $a=-1$, $b=-1.2$

4. Use differential notation to write df for the following functions.

(a) $f(x)=3x+3$
(b) $f(x)=\sin(x)$
(c) $f(x)=e^x+2x$
(d) $f(x)=\cos(x)+\ln(x)$
(e) $\mathbf{f}(x)=(x^3-3x+4,\ln(x)+3)$
(f) $\mathbf{f}(w)=\left(6w^2,5w^{10},6e^w\right)$

5. The surface area of a sphere with radius r is $A(r)=4\pi r^2$. Use a linear approximation for $A(r)$ at $r=4$ to approximate the area of a sphere with radius $r=4.08$. What is the error in this approximation?

6. The volume of a sphere with radius r is $V(r)=\frac{4}{3}\pi r^3$. Use a linear approximation for $V(r)$ at $r=2$ to approximate the volume of a sphere with radius $r=1.94$. What is the error in this approximation?

7. The surface are of a right circular cylinder with height 5 cm and radius r is $A(r) = 2\pi r^2 + 10\pi r\,\text{cm}$. Use a linear approximation to $A(r)$ at $r = 3\,\text{cm}$ to approximate the volume of a cylinder with radius 2.95 cm.

8. The volume of a right circular cone with height sm and base diameter sm is $V(s) = \frac{1}{12}\pi\,\mathrm{s}^3\,\mathrm{m}^3$. Using a linear approximation to $V(s)$ at $s = 5\,\text{m}$, approximate the volume of a cone with $s = 5.11\,\text{m}$.

9. The position of a projectile is given by $\mathbf{r}(t) = (10t, -4.9t^2 + 60t)\,\text{m}$ where the first coordinate is the distance down range from the firing point and the second coordinate is the height above the ground of the projectile. (Distances are in meters and time is measured in seconds.) Using a linear approximation to the position function at $t = 8\,\text{s}$, estimate when and where the projectile lands.

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