Harmonic Analysis on Homogeneous Complex Bounded Domains and Noncommutative Geometry

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Abstract We define and study a noncommutative Fourier transform on every homogeneous complex bounded domain. We then give an application in noncommutative differential geometry by defining noncommutative Baumslag–Solitar tori.

Key words Strict deformation quantization • Symmetric spaces • $\star$-representation • $\star$-exponential • noncommutative manifolds

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1 Introduction

In [7] the authors developed a tracial symbolic pseudo-differential calculus on every Lie group G whose Lie algebra $\mathfrak{g}$ is a normal j-algebra in the sense of Pyatetskii-Shapiro [16]. The class of such Lie groups is in one-to-one correspondence with the class of homogeneous complex bounded domains. Each of them carries a left-invariant Kähler structure.

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As a by-product, they obtained a G-equivariant continuous linear mapping between the Schwartz space $\mathcal{S}(G)$ of such a Lie group and a subalgebra of Hilbert–Schmidt operators on a Hilbert irreducible unitary G-module. This yields a one-parameter family of noncommutative associative multiplications $\{\star_\theta\}_{\theta\in\mathbb{R}}$ on the Schwartz space, each of them endowing $\mathcal{S}(G)$ with a Fréchet nuclear algebra structure. Moreover, the resulting family of Fréchet algebras $\{(\mathcal{S}(G), \star_\theta)\}_{\theta\in\mathbb{R}}$ deforms the commutative Fréchet algebra structure on $\mathcal{S}(G)$ given by the pointwise multiplication of functions corresponding to the value $\theta = 0$ of the deformation parameter. Note that such a program was achieved in [17] for abelian Lie groups and in [8] for abelian Lie supergroups.

In this article we construct a bijective intertwiner between every noncommutative Fréchet algebra $(\mathcal{S}(G), \star_\theta)$ $(\theta \neq 0)$ and a convolution function algebra on the group G. The intertwiner's kernel consists in a complex-valued smooth function $\mathcal{E}$ on the group $G \times G$ that we call "$\star$-exponential" because of its similar nature with objects defined in [11] and studied in [2] in the context of the Weyl–Moyal quantization of coadjoint orbits of exponential Lie groups.
We then prove that the associated smooth map

$$\mathcal{E} : G \to C^\infty(G)$$

consists in a group-morphism valued in the multiplier (nuclear Fréchet) algebra $\mathcal{M}_{\star_\theta}(G)$ of $(\mathcal{S}(G), \star_\theta)$. The above group-morphism integrates the classical moment mapping

$$\lambda : \mathfrak{g} \to \mathcal{M}_{\star_\theta}(G) \ \cap \ C^\infty(G)$$

associated with the (symplectic) action of G on itself by left-translations. Next, we modify the 2-point kernel $\mathcal{E}$ by a power of the modular function of G in such a way that the corresponding Fourier-type transform consists of a unitary operator $\mathcal{F}$ on the Hilbert space of square integrable functions with respect to a left-invariant Haar measure on G.

As an application, we define a class of noncommutative tori associated to generalized Bauslag–Solitar groups in every dimension.

2 Homogeneous Complex Bounded Domains and j-Algebras

The theory of j-algebras was greatly developed by Pyatetskii-Shapiro [16] for studying in a Lie-algebraic way the structure and classification of bounded homogeneous—not necessarily symmetric—domains in $\mathbb{C}^n$. A j-algebra is roughly the Lie algebra $\mathfrak{g}$ of a transitive Lie group of analytic automorphisms of the domain, together with the data of the Lie algebra $\mathfrak{k}$ of the stabilizer of a point in the latter Lie group, an endomorphism j of $\mathfrak{g}$ coming from the complex structure on the domain, and a linear form on $\mathfrak{g}$ whose Chevalley coboundary gives the j-invariant symplectic structure coming from the Kähler structure on the domain. Pyatetskii-

Shapiro realized that among the j-algebras corresponding to a fixed bounded homogeneous domain, there always is at least one whose associated Lie group acts simply transitively on the domain, and which is realizable as upper triangular real matrices. Thoses j-algebras have the structure of *normal j-algebras* which we proceed to describe now.

Definition 2.1. A *normal j-algebra* is a triple $(\mathfrak{g}, \alpha, j)$ where

1. $\mathfrak{g}$ is a solvable Lie algebra which is split over the reals, i.e., ad_X has only real eigenvalues for all $X \in \mathfrak{g}$,
2. j is an endomorphism of $\mathfrak{g}$ such that $j^2 = -Id_{\mathfrak{g}}$ and $[X, Y] + j[jX, Y] + j[X, jY] - [jX, jY] = 0, \forall\ X, Y \in \mathfrak{g}$,
3. α is a linear form on $\mathfrak{g}$ such that $\alpha([jX, X]) > 0$ if $X \neq 0$ and $\alpha([jX, jY]) = \alpha([X, Y]), \forall\ X, Y \in \mathfrak{g}$.

If $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ which is invariant by j, then $(\mathfrak{g}', \alpha|_{\mathfrak{g}'}, j|_{\mathfrak{g}'})$ is again a normal j-algebra, said to be a **j-subalgebra** of $(\mathfrak{g}, \alpha, j)$. A j-subalgebra whose algebra is at the same time an ideal is called a **j-ideal**.

Remark 2.2. To each simple Lie algebra $\mathfrak{G}$ of Hermitian type (i.e., such that the center of the maximal compact algebra $\mathfrak{k}$ has real dimension one) we can attach a normal j-algebra $(\mathfrak{g}, \alpha, j)$ where

1. $\mathfrak{g}$ is the solvable Lie algebra underlying the Iwasawa factor $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}$ of an Iwasawa decomposition $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{G}$.
2. Denoting by $\mathbb{G}/K$ the Hermitian symmetric space associated to the pair $(\mathfrak{G}, \mathfrak{k})$ and by $\mathbb{G} = KAN$ the Iwasawa group decomposition corresponding to $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, the global diffeomorphism:

$$G := AN \longrightarrow \mathbb{G}/K : g \mapsto gK\ ,$$

endows the group G with an exact left-invariant symplectic structure as well as a compatible complex structure. The evaluations at the unit element $e \in G$ of these tensor fields define the elements $\boldsymbol{\Omega} = \mathrm{d}\alpha$ and j at the Lie algebra level.

It is important to note that not every normal j-algebra arises this way. Indeed, it is with the help of the theory of j-algebras that Pyatetskii-Shapiro discovered the first examples of nonsymmetric bounded homogeneous domains. Nevertheless, they can all be built from these "Hermitian" normal j-algebras by a semidirect product process, as we recall now.

Definition 2.3. A normal j-algebra associated with a rank one Hermitian symmetric space (i.e., $\dim \mathfrak{a} = 1$) is called *elementary*.

Lemma 2.4. *Let (V, ω_0) be a symplectic vector space of dimension $2n$, and let $\mathfrak{h}_V := V \oplus \mathbb{R}E$ be the corresponding Heisenberg algebra : $[x, y] = \omega_0(x, y)E$, $[x, E] = 0\ \forall\ x, y \in V$. Setting $\mathfrak{a} := \mathbb{R}H$, we consider the split extension of Lie algebras:*

$$0 \to \mathfrak{h}_V \to \mathfrak{s} := \mathfrak{a} \ltimes \mathfrak{h}_V \to \mathfrak{a} \to 0,$$

with extension homomorphism $\rho_{\mathfrak{h}} : \mathfrak{a} \to \mathrm{Der}(\mathfrak{h})$ *given by*

$$\rho_{\mathfrak{h}}(H)(x + \ell E) := [H, x + \ell E] := x + 2\ell E, \quad x \in V, \, \ell \in \mathbb{R}.$$

Then the Lie algebra $\mathfrak{s}$ *underlines an elementary normal* j*-algebra. Moreover, every elementary normal* j*-algebra is of that form.*

The main interest of elementary normal j-algebras is that they are the only building blocks of normal j-algebras, as shown by the following important property [16].

Proposition 2.5. *Let* $(\mathfrak{g}, \alpha, j)$ *be a normal* j*-algebra. Then,*

1. *there exists a one-dimensional ideal* $\mathfrak{z}_1$ *of* $\mathfrak{g}$*, and a vector subspace* V *of* $\mathfrak{g}$*, such that* $\mathfrak{s} = j\mathfrak{z}_1 + V + \mathfrak{z}_1$ *underlies an elementary normal* j*-ideal of* $\mathfrak{g}$*. Moreover, the associated extension sequence*

$$0 \longrightarrow \mathfrak{s} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}' \longrightarrow 0 \,,$$

   is split as a sequence of normal j*-algebras and such that*

   a. $[\mathfrak{g}', \mathfrak{a}_1 \oplus \mathfrak{z}_1] = 0,$
   b. $[\mathfrak{g}', V] \subset V.$

2. *(follows from 1.) every normal* j*-algebra admits a decomposition as a sequence of split extensions of elementary normal* j*-algebras with properties (a) and (b) above.*

2.1 Symplectic Symmetric Space Geometry of Elementary Normal *j*-Groups

In this section we briefly recall results of [6, 9].

Definition 2.6. The connected simply-connected real Lie group G whose Lie algebra $\mathfrak{g}$ underlies a normal j-algebra is called a *normal* j*-group*. The connected simply connected Lie group $\mathbb{S}$ whose Lie algebra $\mathfrak{s}$ underlies an elementary normal j-algebra is said to be an *elementary normal* j*-group*.

Elementary normal j-groups are exponential (non-nilpotent) solvable Lie groups. As an example, consider the Lie algebra $\mathfrak{s}$ of Definition 2.3 where $V = 0$. It is generated over $\mathbb{R}$ by two elements H and E satisfying $[H, E] = 2E$ and is therefore isomorphic to the Lie algebra of the group of affine transformations of the real line: in this case, $\mathbb{S}$ is the $ax + b$ group.

Now generally, the Iwasawa factor AN of the simple group $SU(1, n)$ (which corresponds to the above example in the case $n = 1$) is an elementary normal j-group.

We realize $\mathbb{S}$ on the product manifold underlying $\mathfrak{s}$:

$$\mathbb{S} = \mathbb{R} \times V \times \mathbb{R} = \{(a, x, \ell)\} .$$

The group law of $\mathbb{S}$ is given by

$$(a, x, \ell)\cdot(a', x', \ell') = \left(a + a', e^{-a'} x + x', e^{-2a'} \ell + \ell' + \frac{1}{2} e^{-a'} \omega_0(x, x')\right) \quad (1.1)$$

and the inverse by

$$(a, x, \ell)^{-1} = (-a, -e^{a} x, -e^{2a} \ell) .$$

We denote by

$$\mathrm{Ad}^* : \mathbb{S} \times \mathfrak{s}^* : (g, \xi) \mapsto \mathrm{Ad}^*_g(\xi) := \xi \circ \mathrm{Ad}_{g^{-1}}$$

the coadjoint action of $\mathbb{S}$ on the dual space $\mathfrak{s}^*$ of $\mathfrak{s} = \mathbb{R}H \oplus V \oplus \mathbb{R}E$. In the dual $\mathfrak{s}^*$, we consider the elements ${}^\flat H$ and ${}^\flat E$ as well as ${}^\flat x$ ($x \in V$) defined by

$$\begin{aligned} {}^\flat H|_{V \oplus \mathbb{R}E} &\equiv 0, \qquad \langle {}^\flat H, H \rangle = 1, \\ {}^\flat E|_{\mathbb{R}H \oplus V} &\equiv 0, \qquad \langle {}^\flat E, E \rangle = 1, \\ {}^\flat x|_{\mathbb{R}H \oplus \mathbb{R}E} &\equiv 0, \qquad \langle {}^\flat x, y \rangle = \omega_0(x, y) \quad (y \in V) . \end{aligned}$$

Proposition 2.7. *Let $\mathcal{O}_\epsilon$ denote the coadjoint orbit through the element $\epsilon\, {}^\flat E$, for $\epsilon = \pm 1$, equipped with its standard Kirillov–Kostant–Souriau symplectic structure (referred to as KKS). Then the map*

$$\mathbb{S} \to \mathcal{O}_\epsilon : (a, x, \ell) \mapsto \mathrm{Ad}^*_{(a,x,\ell)}(\epsilon\, {}^\flat E) = \epsilon(2\ell\, {}^\flat H - e^{-a}\, {}^\flat x + e^{-2a}\, {}^\flat E) \quad (1.2)$$

is a $\mathbb{S}$-equivariant global Darboux chart on $\mathcal{O}_\epsilon$ in which the KKS two-form reads

$$\omega := \omega_{\mathbb{S}} := \epsilon(2\mathrm{d}a \wedge \mathrm{d}\ell + \omega_0) .$$

Within this setting, we consider the moment map of the action of $\mathbb{S}$ on $\mathcal{O}_\epsilon \simeq \mathbb{S}$:

$$\lambda : \mathfrak{s} \to C^\infty(\mathbb{S}) : X \mapsto \lambda_X$$

defined by the relations

$$\lambda_X(g) := \langle \mathrm{Ad}^*_g \left(\epsilon\, {}^\flat E\right) , X \rangle .$$

Lemma 2.8. *Denoting for every $X \in \mathfrak{s}$ the associated fundamental vector field by*

$$X^*_g := \frac{\mathrm{d}}{\mathrm{d}t}|_0 \exp(-tX).g ,$$

one has ($y \in V$)*:*

$$H^* = -\partial_a\ , \qquad y^* = -e^{-a}\partial_y + \frac{1}{2}e^{-a}\omega_0(x,y)\partial_\ell\ , \qquad E^* = -e^{-2a}\partial_\ell\ .$$

Moreover the moment map reads

$$\lambda_H(a,x,\ell) = 2\epsilon\ell\ , \qquad \lambda_y(a,x,\ell) = e^{-a}\epsilon\omega_0(y,x)\ , \qquad \lambda_E(a,x,\ell) = \epsilon e^{-2a}\ . \tag{1.3}$$

Proposition 2.9. *The map*

$$s : \mathbb{S}\times\mathbb{S} \to \mathbb{S} : (g,g') \mapsto s_g g'$$

defined by

$$\begin{aligned} &s_{(a,x,\ell)}(a',x',\ell') \\ &= \Big(2a-a', 2\cosh(a-a')x - x', 2\cosh(2(a-a'))\ell - \ell' + \sinh(a-a')\omega_0(x,x')\Big) \end{aligned} \tag{1.4}$$

endows the Lie group $\mathbb{S}$ *with a left-invariant structure of the symmetric space in the sense of O. Loos (cf. [15]).*

Moreover the symplectic structure ω *is invariant under the symmetries: for every* $g \in \mathbb{S}$*, one has*

$$s_g^*\omega \ = \ \omega\ .$$

2.2 Normal j-Groups

The above Proposition 2.5 implies that every normal j-group G can be decomposed into a semidirect product

$$G \ = \ G_1 \ltimes_\rho \mathbb{S}_2 \tag{1.5}$$

where

$$\mathbb{S}_2 \ := \ \mathbb{R}H_2 \times V_2 \times \mathbb{R}E_2$$

is an elementary normal j-group of real dimension $2n_2 + 2$ and G_1 is a normal j-group. This means that the group law of G has the form

$$\forall\ g_1, g_1' \in G_1,\ \forall\ g_2, g_2' \in \mathbb{S}_2 \quad : \quad (g_1,g_2)\cdot(g_1',g_2') = \Big(g_1\cdot g_1',\ g_2\cdot(\rho(g_1)g_2')\Big),$$

where $\rho : G_1 \to Sp(V_2, \omega_0)$ denotes the extension homomorphism; and the inverse is given by $(g_1, g_2)^{-1} = (g_1^{-1}, \rho(g_1^{-1})g_2^{-1})$. As a consequence, every normal j-group therefore results in a sequence of semidirect products of a finite number of elementary normal j-groups.

Proposition 2.10. *Consider the decomposition (1.5). Then,*

1. *the Lie group G_1 admits an open coadjoint orbit $\mathcal{O}_1$ through an element $o_1 \in \mathfrak{g}_1^*$ which it acts on in a simply transitive way;*
2. *the coadjoint orbit $\mathcal{O}$ of G through the element $o := o_1 + \epsilon_2 {}^\flat E_2$ (same notation as in Sect. 2.1) is open in $\mathfrak{g}^*$;*
3. *denoting by $\mathcal{O}_2$ the coadjoint orbit of $\mathbb{S}_2$ through $\epsilon_2 {}^\flat E_2$, the map*

$$\phi : \mathcal{O}_1 \times \mathcal{O}_2 \to \mathcal{O} : (\mathrm{Ad}^*_{g_1} o_1, \epsilon_2 \mathrm{Ad}^*_{g_2} {}^\flat E_2) \mapsto \mathrm{Ad}^*_{(g_1,g_2)}(o) \tag{1.6}$$

is a symplectomorphism when endowing each orbit with its KKS two-form.

Proof. We proceed by induction on the dimension in proving that $\mathcal{O}$ is acted on by G in a simply transitive way. By induction hypothesis, so is $\mathcal{O}_1$ by G_1. And Proposition 2.7 implies it is the case for $\mathcal{O}_2$ as well. Now denoting $(g_1, e) =: g_1$ and $(e, g_2) =: g_2$, we observe:

$$\mathrm{Ad}^*_{(g_1,g_2)}(o) = \mathrm{Ad}^*_{g_2 g_1}(o) = \mathrm{Ad}^*_{g_2}\left(\mathrm{Ad}^*_{g_1}(o_1) + \epsilon_2 {}^\flat E_2 \circ \rho(g_1^{-1})_{*e}\right)$$

where $\rho : G_1 \to \mathrm{Aut}(\mathbb{S}_2)$ denotes the extension homomorphism.

Now for all $\xi_1 \in \mathfrak{g}_1^*$, $X_1 \in \mathfrak{g}_1$, $X_2 \in \mathfrak{s}_2$ and $g_2 \in \mathbb{S}_2$:

$$\langle \mathrm{Ad}^*_{g_2}\xi_1 \, , \, X_1 + X_2 \rangle = \langle \xi_1 \, , \, \mathrm{Ad}_{g_2^{-1}} X_1 + \mathrm{Ad}_{g_2^{-1}} X_2 \rangle = \langle \xi_1 \, , \, \mathrm{Ad}_{g_2^{-1}} X_1 \rangle \, .$$

But

$$\mathrm{Ad}_{g_2^{-1}} X_1 = \frac{\mathrm{d}}{\mathrm{d}t}|_0 (\exp(tX_1), g_2^{-1})(e, g_2) = \frac{\mathrm{d}}{\mathrm{d}t}|_0 (\exp(tX_1), g_2^{-1}\rho(\exp(tX_1))g_2) \, .$$

Hence

$$\langle \xi_1 \, , \, \mathrm{Ad}_{g_2^{-1}} X_1 \rangle = \langle \xi_1 \, , \, X_1 \oplus \left(\frac{\mathrm{d}}{\mathrm{d}t}|_0 g_2^{-1} \rho(\exp(tX_1)) g_2 \right) \rangle = \langle \xi_1 \, , \, X_1 \rangle \, .$$

Therefore $\mathrm{Ad}^*_{g_2}\xi_1 = \xi_1$ and we get

$$\mathrm{Ad}^*_{(g_1,g_2)}(o) = \mathrm{Ad}^*_{g_1}(o_1) + \epsilon_2 \, \mathrm{Ad}^*_{g_2}\left({}^\flat E_2 \circ \rho(g_1^{-1})_{*e}\right) \, .$$

The induction hypothesis thus implies that the stabilizer of element o in G is trivial, which shows in particular that the fundamental group of $\mathcal{O}$ is trivial. The map (1.6) being a surjective submersion is therefore a diffeomorphism.

It remains to prove the assertion regarding the symplectic structures. Denoting by $\omega^{\mathcal{O}}$ the KKS form on $\mathcal{O}$, we observe that with obvious notation, for all $Y_1 \in \mathfrak{g}_1$ and $Y_2 \in \mathfrak{s}_2$:

$$\phi^*\omega^{\mathcal{O}}(X_1^* \oplus X_2^*, Y_1^* \oplus Y_2^*) = \omega^{\mathcal{O}}_{\mathrm{Ad}^*_{(g_1,g_2)}(o)}(\phi_* X_1^* + \phi_* X_2^*, \phi_* Y_1^* + \phi_* Y_2^*)$$

$$= \omega^{\mathcal{O}}_{\mathrm{Ad}^*_{(g_1,g_2)}(o)}\big(\big(\mathrm{Ad}_{g_2}X_1\big)^* + X_2^*, \big(\mathrm{Ad}_{g_2}Y_1\big)^* + Y_2^*\big)$$

$$= \langle\, \mathrm{Ad}^*_{(g_1,g_2)}(o)\,,\, [\mathrm{Ad}_{g_2}X_1 + X_2\,,\, \mathrm{Ad}_{g_2}Y_1 + Y_2]\,\rangle$$

$$= \langle\, \mathrm{Ad}^*_{g_1}(o)\,,\, [X_1 + \mathrm{Ad}_{g_2^{-1}}X_2\,,\, Y_1 + \mathrm{Ad}_{g_2^{-1}}Y_2]\,\rangle$$

$$= \langle\, \mathrm{Ad}^*_{g_1}(o)\,,\, [X_1, Y_1] - \rho(Y_1)\mathrm{Ad}_{g_2^{-1}}X_2 + \rho(X_1)\mathrm{Ad}_{g_2^{-1}}Y_2 + \mathrm{Ad}_{g_2^{-1}}[X_2, Y_2]\,\rangle$$

$$= \omega^{\mathcal{O}_1}_{\mathrm{Ad}^*_{g_1}(o_1)}(X_1^*, Y_1^*) + \epsilon_2\,\omega^{\mathcal{O}_2}_{\mathrm{Ad}^*_{g_2}{}^\flat E_2}(X_2^*, Y_2^*)$$

$$+ \epsilon_2\,\langle {}^\flat E_2\,,\, \rho(g_1^{-1})_{*e}\left(-\rho(Y_1)\mathrm{Ad}_{g_2^{-1}}X_2 + \rho(X_1)\mathrm{Ad}_{g_2^{-1}}Y_2\right)\rangle\,.$$

The last term in the above expression vanishes identically. Indeed, the specific form of ρ implies that the element $v_2 := -\rho(Y_1)\mathrm{Ad}_{g_2^{-1}}X_2 + \rho(X_1)\mathrm{Ad}_{g_2^{-1}}Y_2$ lives in V_2 as well as in $\rho(g_1^{-1})_{*e}v_2$. □

Remark 2.11. Normal j-groups can be decomposed into elementary normal j-groups $\mathbb{S}_k$ as $G = \big(\ldots(\mathbb{S}_1 \ltimes_{\rho_1} \mathbb{S}_2) \ltimes_{\rho_2} \ldots\big) \ltimes_{\rho_{N-1}} \mathbb{S}_N$ and the coadjoint orbits described in Proposition 2.10 are determined by sign choices $\epsilon_k = \pm 1$ for each factor $\mathbb{S}_k$. We will denote by $\mathcal{O}_{(\epsilon)}$ the coadjoint orbit associated to the signs $(\epsilon_k)_{1\le k\le N} \in (\mathbb{Z}_2)^N$.

Example 2.12. Let us describe the following example corresponding to the six-dimensional Siegel domain $\mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(2)$. Let $G_1 = \mathbb{S}_1$ be of dimension 2 ($V_1 = 0$, G_1 is the affine group), $\mathbb{S}_2$ of dimension 4, i.e., V_2 is of dimension 2, with basis f_2, f_2' endowed with $\omega_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$), and let the action $\rho : \mathbb{S}_1 \to Sp(V_2)$ be given by

$$\rho(a_1, \ell_1) = \begin{pmatrix} e^{a_1} & 0 \\ e^{-a_1}\ell_1 & e^{-a_1}\end{pmatrix}.$$

Then the group law is

$$(a_1, \ell_1, a_2, v_2, w_2, \ell_2)\cdot(a_1', \ell_1', a_2', v_2', w_2'\ell_2') = \Big(a_1 + a_1', e^{-2a_1'}\ell_1$$

$$+ \ell_1', a_2 + a_2', e^{-a_2'}v_2 + e^{a_1}v_2', e^{-a_2'}w_2 + e^{-a_1}\ell_1 v_2' + e^{-a_1}w_2',$$

$$e^{-2a_2'}\ell_2 + \ell_2' + \frac{1}{2}e^{-a_2'}(e^{-a_1}\ell_1 v_2 v_2' + e^{-a_1}v_2 w_2' - e^{a_1}w_2 v_2')\Big),$$

where $(a_1, \ell_1) \in \mathbb{S}_1$, $(a_2, v_2, w_2, \ell_2) \in \mathbb{S}_2$ and

$$g := (a_1, \ell_1, a_2, v_2, w_2, \ell_2) = e^{a_2 H_2} e^{v_2 f_2 + w_2 f_2'} e^{\ell_2 E_2} e^{a_1 H_1} e^{\ell_1 E_1} .$$

Its Lie algebra is characterized by

$$[H_1, E_1] = 2E_1, \qquad [H_2, f_2] = f_2, \qquad [H_2, f_2'] = f_2', \qquad [f_2, f_2'] = E_2,$$
$$[H_2, E_2] = 2E_2, \qquad [H_1, f_2] = f_2, \qquad [H_1, f_2'] = -f_2', \qquad [E_1, f_2] = f_2',$$

where the other relations vanish. The coadjoint action takes the form

$$\begin{aligned}\mathrm{Ad}_g^*(\epsilon_1 {}^\flat E_1 + \epsilon_2 {}^\flat E_2) = (2\epsilon_1\ell_1 + \epsilon_2 v_2 w_2)^\flat H_1 + (\epsilon_1 e^{-2a_1} - \frac{\epsilon_2}{2} v_2^2)^\flat E_1 \\ + \epsilon_2 (2\ell_2{}^\flat H_2 - e^{-a_2} v_2{}^\flat f_2 - e^{-a_2} w_2{}^\flat f_2' + e^{-2a_2 \flat} E_2) .\end{aligned}$$

The moment map can then be extracted from this expression:

$$\lambda_{H_1} = 2\epsilon_1\ell_1 + \epsilon_2 v_2 w_2, \qquad \lambda_{E_1} = \epsilon_1 e^{-2a_1} - \frac{\epsilon_2}{2} v_2^2, \qquad \lambda_{H_2} = 2\epsilon_2\ell_2,$$
$$\lambda_{f_2} = \epsilon_2 e^{-a_2} w_2, \qquad \lambda_{f_2'} = -\epsilon_2 e^{-a_2} v_2, \qquad \lambda_{E_2} = \epsilon_2 e^{-2a_2}.$$

3 Determination of the Star-Exponential

3.1 Quantization of Elementary Groups

We follow the analysis developed in [7], where the reader can find all the proofs. In the notation of Sect. 2.1, we choose two Lagrangian subspaces in duality V_0, V_1 of the symplectic vector space (V, ω_0) of dimension $2n$ underlying the elementary group $\mathbb{S}$. We denote the corresponding coordinates $x = (v, w) \in V$ in the global chart, with $v \in V_0$ and $w \in V_1$. Let $\mathfrak{q} = \mathbb{R}H \oplus V_0$ and $Q = \exp(\mathfrak{q})$. The unitary induced representation associated to the coadjoint orbit $\mathcal{O}_\epsilon$ ($\epsilon = \pm 1$) by the method of Kirillov has the form

$$U_{\theta,\epsilon}(a, x, \ell)\varphi(a_0, v_0) = e^{\frac{i\epsilon}{\theta}\left(e^{2(a-a_0)}\ell + \omega_0(\frac{1}{2}e^{a-a_0}v - v_0, e^{a-a_0}w)\right)} \varphi(a_0 - a, v_0 - e^{a-a_0}v) \tag{1.7}$$

for $(a, x, \ell) \in \mathbb{S}$, $\varphi \in L^2(Q)$, $(a_0, v_0) \in Q$ and $\theta \in \mathbb{R}_+^*$. These representations $U_{\theta,\epsilon} : \mathbb{S} \to \mathcal{L}(\mathcal{H})$ are unitary and irreducible, and the unitary dual is described by these two representations. A multiplier $\mathbf{m}$ is a function on Q. There is a particular multiplier:

$$\mathbf{m}_0(a, v) = 2^{n+1} \sqrt{\cosh(2a)} \cosh(a)^n. \tag{1.8}$$

Let us define $\Sigma := (s_{(0,0,0)}|_Q)^*$, where s is the symmetric structure (1.4):

$$\Sigma\varphi(a,v) = \varphi(-a,-v). \tag{1.9}$$

Then, the Weyl-type quantization map is given by

$$\begin{aligned}
&\Omega_{\theta,\epsilon,\mathbf{m}_0}(a,x,\ell)\varphi(a_0,v_0) := U_{\theta,\epsilon}(a,x,\ell)\mathbf{m}_0\Sigma U_{\theta,\epsilon}(a,x,\ell)^{-1}\varphi(a_0,v_0)\\
&= 2^{n+1}\sqrt{\cosh(2(a-a_0))}\cosh(a-a_0)^n\\
&\times e^{\frac{2i\epsilon}{\theta}\left(\sinh(2(a-a_0))\ell+\omega_0(\cosh(a-a_0)v-v_0,\cosh(a-a_0)w)\right)}\\
&\times \varphi(2a-a_0, 2\cosh(a-a_0)v-v_0).
\end{aligned} \tag{1.10}$$

The operator $\Omega_{\theta,\epsilon,\mathbf{m}_0}(g)$ is a symmetric unbounded operator on $\mathcal{H}$, and $g\in\mathbb{S}\simeq\mathcal{O}_\epsilon$.

On smooth functions with compact support $f\in\mathcal{D}(\mathcal{O}_\epsilon)$, and by denoting $\kappa := \frac{1}{2^n(\pi\theta)^{n+1}}$, one has

$$\Omega_{\theta,\epsilon,\mathbf{m}_0}(f) := \kappa\int_{\mathcal{O}_\epsilon} f(g)\Omega_{\theta,\epsilon,\mathbf{m}_0}(g)\mathrm{d}\mu(g)$$

with $\mathrm{d}\mu(g) = \mathrm{d}^L g$ which corresponds to the Liouville measure of the KKS symplectic form on the coadjoint orbit $\mathcal{O}_\epsilon \simeq \mathbb{S}$. Its extension is continuous and called the quantization map $\Omega_{\theta,\epsilon,\mathbf{m}_0} : L^2(\mathcal{O}_\epsilon) \to \mathcal{L}_{HS}(\mathcal{H})$, with $\mathcal{H} := L^2(Q)$ and $\mathcal{L}_{HS}$ the Hilbert–Schmidt operators. The normalization has been chosen such that $\Omega_{\theta,\epsilon,\mathbf{m}_0}(1) = \mathbb{1}_{\mathcal{H}}$, understood in the distributional sense. Moreover, it is $\mathbb{S}$-equivariant, because of

$$\forall g,g_0\in\mathbb{S} \quad : \quad \Omega_{\theta,\epsilon,\mathbf{m}_0}(g\cdot g_0) = U_{\theta,\epsilon}(g)\Omega_{\theta,\epsilon,\mathbf{m}_0}(g_0)U_{\theta,\epsilon}(g)^{-1}.$$

The unitary representation $U_{\theta,\epsilon} : \mathbb{S}\to\mathcal{L}(\mathcal{H})$ induces a resolution of the identity.

Proposition 3.1. *By denoting the norm* $\|\varphi\|_w^2 := \int_Q |\varphi(a,v)|^2 e^{2(n+1)a}\mathrm{d}a\mathrm{d}v$ *and* $\varphi_g(q) = U_{\theta,\epsilon}(g)\varphi(q)$ *for* $g\in\mathbb{S}$, $q\in Q$ *and a nonzero* $\varphi\in\mathcal{H}$, *we have*

$$\frac{\kappa}{\|\varphi\|_w^2}\int_{\mathbb{S}} |\varphi_g\rangle\langle\varphi_g|\mathrm{d}^L g = \mathbb{1}_{\mathcal{H}}.$$

This resolution of identity shows that the trace has the form

$$\mathrm{Tr}(T) = \frac{\kappa}{\|\varphi\|_w^2}\int_{\mathbb{S}}\langle\varphi_g, T\varphi_g\rangle\mathrm{d}^L g \tag{1.11}$$

for any trace-class operator $T\in\mathcal{L}^1(\mathcal{H})$.

Theorem 3.2. *The symbol map, which is the left-inverse of the quantization map $\Omega_{\theta,\epsilon,\mathbf{m}_0}$ can be obtained via the formula*

$$\forall\ f \in L^2(\mathcal{O}_\epsilon),\ \forall\ g \in \mathcal{O}_\epsilon \quad : \quad \mathrm{Tr}(\Omega_{\theta,\epsilon,\mathbf{m}_0}(f)\Omega_{\theta,\epsilon,\mathbf{m}_0}(g)) = f(g),$$

where the trace is understood in the distributional sense in the variable $g \in \mathbb{S}$.

Then, the star-product is defined as

$$(f_1 \star_{\theta,}\ f_2)(g) := \mathrm{Tr}(\Omega_{\theta,\epsilon,\mathbf{m}_0}(f_1)\Omega_{\theta,\epsilon,\mathbf{m}_0}(f_2)\Omega_{\theta,\epsilon,\mathbf{m}_0}(g))$$

for $f_1, f_2 \in L^2(\mathcal{O}_\epsilon)$ and $g \in \mathcal{O}_\epsilon$, where we omitted the subscripts $\epsilon, \mathbf{m}_0$ for the star-product.

Proposition 3.3. *The star-product has the following expression:*

$$(f_1 \star_\theta f_2)(g) = \frac{1}{(\pi\theta)^{2n+2}} \int K_{\mathbb{S}}(g, g_1, g_2) e^{-\frac{2i}{\theta} S_{\mathbb{S}}(g,g_1,g_2)} f_1(g_1) f_2(g_2) \mathrm{d}\mu(g_1)\mathrm{d}\mu(g_2) \tag{1.12}$$

where the amplitude and the phase are

$$\begin{aligned}
K_{\mathbb{S}}(g, g_1, g_2) &= 4\sqrt{\cosh(2(a_1 - a_2))\cosh(2(a_1 - a))\cosh(2(a - a_2))}\cosh(a_2 - a)^n \\
&\quad \cosh(a_1 - a)^n \cosh(a_1 - a_2)^n, \\
\epsilon S_{\mathbb{S}}(g, g_1, g_2) &= -\sinh(2(a_1 - a_2))\ell - \sinh(2(a_2 - a))\ell_1 - \sinh(2(a - a_1))\ell_2 \\
&\quad + \cosh(a_1 - a)\cosh(a_2 - a)\omega_0(x_1, x_2) \\
&\quad + \cosh(a_1 - a)\cosh(a_1 - a_2)\omega_0(x_2, x) \\
&\quad + \cosh(a_1 - a_2)\cosh(a_2 - a)\omega_0(x, x_1),
\end{aligned}$$

with $g_i = (a_i, x_i, \ell_i) \in \mathbb{S}$. Moreover, $g \mapsto 1$ is the unit of this product, is associative, $\mathbb{S}$-invariant and satisfies the tracial identity:

$$\int f_1 \star_\theta f_2 = \int f_1 \cdot f_2. \tag{1.13}$$

Note that this product has first been found [5] by intertwining the Moyal product:

$$\begin{aligned}
(f_1 \star_\theta^0 f_2)(a, x, \ell) = \frac{4}{(\pi\theta)^{2+2n}} \int & \mathrm{d}a_i \mathrm{d}x_i \mathrm{d}\ell_i\ f_1(a_1 + a, x_1 + x, \ell_1 + \ell) \\
& f_2(a_2 + a, x_2 + x, \ell_2 + \ell) e^{-\frac{2i\epsilon}{\theta}(2a_1\ell_2 - 2a_2\ell_1 + \omega_0(x_1,x_2))}
\end{aligned}$$

for $f_1, f_2 \in L^2(\mathcal{O}_\epsilon)$ and $\mathcal{O}_\epsilon \simeq \mathbb{S} \simeq \mathbb{R}^{2n+2}$, which is $\mathfrak{s}$-covariant ($[\lambda_X, \lambda_Y]_{\star_\theta^0} = -i\theta\lambda_{[X,Y]}$) but not $\mathbb{S}$-invariant. So for smooth functions with compact support, we have $f_1 \star_\theta f_2 = T_\theta((T_\theta^{-1} f_1) \star_\theta^0 (T_\theta^{-1} f_2))$ with intertwiners:

$$T_\theta f(a,x,\ell) = \frac{1}{2\pi}\int \sqrt{\cosh(\frac{\theta t}{2})}\cosh(\frac{\theta t}{4})^n e^{\frac{2i}{\theta}\sinh(\frac{\theta t}{2})\ell - i\xi t} f(a, \cosh(\frac{\theta t}{4})x, \xi)\mathrm{d}t\mathrm{d}\xi$$

$$T_\theta^{-1} f(a,x,\ell) = \frac{1}{2\pi}\int \frac{\sqrt{\cosh(\frac{\theta t}{2})}}{\cosh(\frac{\theta t}{4})^n} e^{-\frac{2i}{\theta}\sinh(\frac{\theta t}{2})\xi + it\ell} f(a, \cosh(\frac{\theta t}{4})^{-1}x, \xi)\mathrm{d}t\mathrm{d}\xi \tag{1.14}$$

which will be useful in Sect. 3.4.

3.2 Quantization of Normal j-Groups

Let $G = G_1 \ltimes \mathbb{S}_2$ be a normal j-group, with notation as in Sect. 2.2. Taking into account its structure, the unitary representation U and the quantization map Ω of this group (dependence in $\theta \in \mathbb{R}_+^*$ will be omitted here in the subscripts) can be constructed from the ones U_1 and Ω_1 of G_1 (obtained by recurrence) and the ones U_2 and Ω_2 of $\mathbb{S}_2$, given by (1.7) and (1.10) (without $\mathbf{m}_0$ for the moment).

Let $\mathcal{H}_i$ be the Hilbert space of the representation U_i, associated to a coadjoint orbit $\mathcal{O}_i$ (in the notation of Proposition 2.10). Since U_2 is irreducible and $\rho : G_1 \to Sp(V_2)$, there exists a unique homomorphism $\mathcal{R} : G_1 \to \mathcal{L}(\mathcal{H}_2)$ such that for all $g_1 \in G_1$, for all $g_2 \in \mathbb{S}_2$,

$$U_2(\rho(g_1)g_2) = \mathcal{R}(g_1)U_2(g_2)\mathcal{R}(g_1)^{-1}.$$

$\mathcal{R}$ is actually a metaplectic-type representation associated to U_2 and ρ. The matrix $\rho(g_1)$, with smooth coefficients in g_1, is of the form

$$\rho(g_1) = \begin{pmatrix} \rho_+(g_1) & 0 \\ \rho_-(g_1) & (\rho_+(g_1)^T)^{-1} \end{pmatrix}$$

with $\rho_-(g_1)^T\rho_+(g_1) = \rho_+(g_1)^T\rho_-(g_1)$.

Proposition 3.4. *The map $\mathcal{R} : G_1 \to \mathcal{L}(\mathcal{H}_2)$ is given by for all $g_1 \in G_1$, for all $\varphi \in \mathcal{H}_2$ non-zero,*

$$\mathcal{R}(g_1)\varphi(a_0, v_0) = \frac{1}{|\det(\rho_+(g_1))|^{\frac{1}{2}}} e^{-\frac{i\epsilon_2}{2\theta} v_0\rho_-(g_1)\rho_+(g_1)^{-1}v_0} \varphi(a_0, \rho_+(g_1)^{-1}v_0)$$

and is unitary, where the sign $\epsilon_2 = \pm 1$ determines the choice of the coadjoint orbit $\mathcal{O}_2$ and the associated irreducible representation U_2.

The expression

$$U(g)\varphi := U_1(g_1)\varphi_1 \otimes U_2(g_2)\mathcal{R}(g_1)\varphi_2$$

for $g = (g_1, g_2) \in G$, $\varphi = \varphi_1 \otimes \varphi_2 \in \mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2$, defines a unitary representation $U : G \to \mathcal{L}(\mathcal{H})$. Let $\Sigma = \Sigma_1 \otimes \Sigma_2$, with Σ_2 given in (1.9). Then, the quantization map is defined as

$$\Omega(g) := U(g) \circ \Sigma \circ U(g)^{-1}.$$

Using the definition of U and $\mathcal{R}$ together with the property (see Proposition 6.55 in [7]),

$$\mathcal{R}(g_1)\Sigma_2\mathcal{R}(g_1)^{-1} = \Sigma_2,$$

it is easy to check that

$$\Omega((g_1, g_2)) = \Omega_1(g_1) \otimes \Omega_2(g_2),$$

with $(g_1, g_2) \in G$ and $\Omega_i(g_i) = U_i(g_i) \circ \Sigma_i \circ U_i(g_i)^{-1}$. Using the identification $\mathcal{O} \simeq G$ (see Proposition 2.10), we see that Ω is defined on $\mathcal{O}$ and it is again G-equivariant: for $g \in G$ and $g' \in \mathcal{O}$,

$$\Omega(g{\cdot}g') = U(g)\Omega(g')U(g)^{-1}.$$

In the same way, if $\mathbf{m}_0 := \mathbf{m}_0^1 \otimes \mathbf{m}_0^2$, where $\mathbf{m}_0^2$ is given by (1.8), we also have $\Omega_{\mathbf{m}_0}((g_1, g_2)) = \Omega_{1,\mathbf{m}_0^1}(g_1) \otimes \Omega_{2,\mathbf{m}_0^2}(g_2)$. The quantization map of functions $f \in \mathcal{D}(\mathcal{O})$ has then the form

$$\Omega_{\mathbf{m}_0}(f) := \kappa \int_{\mathcal{O}} f(g)\Omega_{\mathbf{m}_0}(g)\mathrm{d}\mu(g)$$

where $\mathrm{d}\mu(g) := \mathrm{d}\mu_1(g_1)\mathrm{d}\mu_2(g_2) = \mathrm{d}^L g_1 \mathrm{d}^L g_2$ is the Liouville measure of the KKS symplectic form on the coadjoint orbit $\mathcal{O} \simeq G$; $\kappa = \kappa_1\kappa_2$, for $G = G_1 \ltimes \mathbb{S}_2$, is defined recursively with $\kappa_2 = \frac{1}{2^{n_2}(\pi\theta)^{n_2+1}}$ and $\dim(\mathbb{S}_2) = 2n_2 + 2$ in Sect. 3.1. We then have $\Omega_{\mathbf{m}_0}(1) = \mathbb{1}$.

Note that the left-invariant measure for the group $G = G_1 \ltimes \mathbb{S}_2$ has the form

$$\mathrm{d}^L g = \mathrm{d}^L g_1 \mathrm{d}^L g_2$$

which corresponds to the Liouville measure $\mathrm{d}\mu(g)$, like the elementary case. As in Sect. 3.1, the unitary representation $U : G \to \mathcal{L}(\mathcal{H})$ induces a resolution of the identity.

Proposition 3.5. *By denoting the norm* $\|\varphi\|_w^2 := \|\varphi_1\|_w^2\|\varphi_2\|_w^2$ *for* $\varphi = \varphi_1 \otimes \varphi_2 \in \mathcal{H}$ *nonzero, we have*

$$\frac{\kappa}{\|\varphi\|_w^2}\int_G |U(g)\varphi\rangle\langle U(g)\varphi| \mathrm{d}^L g = \mathbb{1}_{\mathcal{H}}.$$

This resolution of identity shows that the trace has the form

$$\mathrm{Tr}(T) = \frac{\kappa}{\|\varphi\|_w^2}\int_G \langle U(g)\varphi, T\,U(g)\varphi\rangle \mathrm{d}^L g \tag{1.15}$$

for $T \in \mathcal{L}^1(\mathcal{H})$. In particular, for $T = T_1 \otimes T_2$ with $T_i \in \mathcal{L}^1(\mathcal{H}_i)$, one has

$$\mathrm{Tr}(T) = \frac{\kappa}{\|\varphi\|_w^2}\int_G \langle U_1(g_1)\varphi_1, T_1U_1(g_1)\varphi_1\rangle\langle U_2(g_2)\mathcal{R}(g_1)\varphi_2, T_2U_2(g_2)\mathcal{R}(g_1)\varphi_2\rangle \\ \mathrm{d}^L g_1 \mathrm{d}^L g_2 = \mathrm{Tr}(T_1)\,\mathrm{Tr}(T_2). \tag{1.16}$$

Theorem 3.6. *The symbol map, which is the left-inverse of the quantization map* $\Omega_{\mathbf{m}_0}$ *can be obtained via the formula*

$$\forall f \in L^2(\mathcal{O}),\ \forall g \in \mathcal{O} \quad : \quad \mathrm{Tr}\Big(\Omega_{\mathbf{m}_0}(f)\Omega_{\mathbf{m}_0}(g)\Big) = f(g).$$

Proof. Abstractly (in a weak sense), we have

$$\begin{aligned}
&\mathrm{Tr}(\Omega_{\mathbf{m}_0}(f)\Omega_{\mathbf{m}_0}(g)) \\
&= \kappa\int f(g_1', g_2')\,\mathrm{Tr}(\Omega_1(g_1')\Omega_1(g_1))\,\mathrm{Tr}(\Omega_2(g_2')\Omega_2(g_2))\mathrm{d}\mu_1(g_1')\mathrm{d}\mu_2(g_2') \\
&= f(g_1, g_2).
\end{aligned}$$

□

The star-product is defined as

$$(f_1 \star_\theta f_2)(g) := \mathrm{Tr}(\Omega_{\mathbf{m}_0}(f_1)\Omega_{\mathbf{m}_0}(f_2)\Omega_{\mathbf{m}_0}(g))$$

for $f_1, f_2 \in L^2(\mathcal{O})$ and $g \in \mathcal{O}$.

Proposition 3.7. *The star-product has the following expression:*

$$\begin{aligned}
&(f_1 \star_\theta f_2)(g) \\
&= \frac{1}{(\pi\theta)^{\dim(G)}}\int_{G\times G} K_G(g, g', g'')e^{-\frac{2i}{\theta}S_G(g,g',g'')} f_1(g')f_2(g'')\mathrm{d}\mu(g')\mathrm{d}\mu(g'')
\end{aligned} \tag{1.17}$$

where the amplitude and the phase are

$$K_G(g,g',g'') = K_{G_1}(g_1,g_1',g_1'')K_{\mathbb{S}_2}(g_2,g_2',g_2''),$$
$$S_G(g,g',g'') = S_{G_1}(g_1,g_1',g_1'') + S_{\mathbb{S}_2}(g_2,g_2',g_2''),$$

with $g = (g_1, g_2) \in \mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$ *due to* (1.6)*. There is also a tracial identity:*

$$\int_{\mathcal{O}} (f_1 \star_\theta f_2)(g)\mathrm{d}\mu(g) = \int_{\mathcal{O}} f_1(g) f_2(g)\mathrm{d}\mu(g).$$

3.3 Computation of the Star-Exponential

Definition 3.8. We define the **star-exponential** associated to the deformation quantization $(\star, \Omega)$ of Sect. 3.2 as

$$\forall\ g \in G,\ \forall\ g' \in \mathcal{O} \simeq G \quad : \quad \mathcal{E}_g^{\mathcal{O}}(g') = \mathrm{Tr}(U(g)\Omega_{\mathbf{m}_0}(g')),$$

where the trace has to be understood in the distributional sense in $(g, g') \in G \times \mathcal{O}$.

By using computation rules of the above sections, we can obtain recursively the number of factors of the normal j-group $G = G_1 \ltimes \mathbb{S}_2$ with the corresponding coadjoint orbit $\mathcal{O} \simeq \mathcal{O}_1 \times \mathcal{O}_2$, the expression of the star-exponential $\mathcal{E}^{\mathcal{O}} \in \mathcal{D}'(G \times \mathcal{O})$.

Theorem 3.9. *We have for all* $g, g' \in G$,

$$\begin{aligned} &\mathcal{E}_g^{\mathcal{O}}(g') \\ &= \mathcal{E}_{g_1}^{\mathcal{O}_1}(g_1') \frac{2^{n_2}|\det(\rho_+(g_1))|^{\frac{1}{2}}\sqrt{\cosh(a_2)}\cosh(\frac{a_2}{2})^{n_2}}{|\det(1+\rho_+(g_1))|} \exp\Big(\frac{i\epsilon_2}{\theta}\Big[2\sinh(a_2)\ell_2' \\ &+ e^{a_2 - 2a_2'}\ell_2 + e^{\frac{a_2}{2} - a_2'}\cosh(\frac{a_2}{2})\omega_0(x_2, x_2') + \frac{1}{2}(\tilde{x})^T M_\rho(g_1)\tilde{x}\Big]\Big), \end{aligned} \tag{1.18}$$

where

$$M_\rho(g_1) := \begin{pmatrix} -B_\rho & C_\rho^T \\ C_\rho & 0 \end{pmatrix}, \qquad \tilde{x} := e^{\frac{a_2}{2} - a_2'} x_2 - 2\cosh(\frac{a_2}{2}) x_2'$$

and with $B_\rho = (1+\rho_+^T(g_1))^{-1}\rho_+^T(g_1)\rho_-(g_1)(1+\rho_+(g_1))^{-1}$, *and* $C_\rho = \frac{1}{2}(\rho_+(g_1) - 1)(\rho_+(g_1)+1)^{-1}$, $g = (g_1, g_2)$, $g_2 = (a_2, x_2, \ell_2) \in \mathbb{S}_2$, *and* $\mathcal{E}^{\mathcal{O}_1}$ *the star-exponential of the normal* j*-group* G_1.

Proof. First, we use Proposition 3.5 and Eq. (1.16):

$$\mathcal{E}_g^{\mathcal{O}}(g') = \mathrm{Tr}(U(g)\Omega_{\mathbf{m}_0}(g')) = \mathrm{Tr}(U_1(g_1)\Omega_{1,\mathbf{m}_0^1}(g_1'))\,\mathrm{Tr}(U_2(g_2)\mathcal{R}(g_1)\Omega_{2,\mathbf{m}_0^2}(g_2')).$$

The second part of the above expression can be computed by using (1.11), and it gives

$$\mathrm{Tr}(U_2(g_2)\mathcal{R}(g_1)\Omega_{2,\mathbf{m}_0^2}(g_2')) = \frac{\kappa_2}{\|\varphi\|_w^2}\int_{\mathbb{S}_2}\langle\varphi_{g_2''}, U_2(g_2)\mathcal{R}(g_1)\Omega_{2,\mathbf{m}_0^2}(g_2')\varphi_{g_2''}\rangle \mathrm{d}^L g_2''.$$

If we replace U_2, $\Omega_{2,\mathbf{m}_0^2}$ and $\mathcal{R}$ by their expressions determined previously in (1.7), (1.10) and Proposition 3.4, we find after some integrations and simplifications that for all $g, g' \in G$,

$$\begin{aligned}\mathcal{E}_g^{\mathcal{O}}(g') = \mathcal{E}_{g_1}^{\mathcal{O}_1}(g_1')&\frac{2^{n_2}|\det(\rho_+(g_1))|^{\frac{1}{2}}\sqrt{\cosh(a_2)}\cosh(\frac{a_2}{2})^{n_2}}{|\det(1+\rho_+(g_1))|}\\ &\times\exp(\frac{i\epsilon_2}{\theta}\Big[2\sinh(a_2)\ell_2' + e^{a_2-2a_2'}\ell_2 + X^T A_\rho X\Big]),\end{aligned}\tag{1.19}$$

where

$$A_\rho = \begin{pmatrix} -B_\rho & C_\rho^T & B_\rho & (1+\rho_+^T(g_1))^{-1} \\ C_\rho & 0 & -\rho_+(g_1)(1+\rho_+(g_1))^{-1} & 0 \\ B_\rho & -\rho_+^T(g_1)(1+\rho_+^T(g_1))^{-1} & -B_\rho & C_\rho^T \\ (1+\rho_+(g_1))^{-1} & 0 & C_\rho & 0\end{pmatrix},$$

$$X = \begin{pmatrix}\frac{1}{\sqrt{2}}e^{\frac{a_2}{2}-a_2'}v_2 \\ \frac{1}{\sqrt{2}}e^{\frac{a_2}{2}-a_2'}w_2 \\ \sqrt{2}\cosh(\frac{a_2}{2})v_2' \\ \sqrt{2}\cosh(\frac{a_2}{2})w_2'\end{pmatrix}$$

and with $x_2 = (v_2, w_2)$. A straightforward computation then gives the result. □

Let us denote by $\mathcal{E}_{(g_1,g_2)}^{\mathcal{O}_2}(g_2')$ the explicit part in the RHS of (1.19) which corresponds to the star-exponential of the group $\mathbb{S}_2$ twisted by the action of $g_1 \in G_1$.

The expression (1.19) seems to be ill-defined when $\det(1+\rho_+^{-1}) = 0$. However, one can obtain in this case a degenerated expression of the star-exponential which is well defined. For example, when $\rho_+(g_1) = -\mathbb{1}_{n_2}$, we have

$$\mathcal{E}^{\mathcal{O}_2}_{(g_1,g_2)}(g'_2) = (\pi\theta)^{n_2}\frac{\sqrt{\cosh(a_2)}}{\cosh(\frac{a_2}{2})^{n_2}}$$
$$\times\exp(\frac{i\epsilon_2}{\theta}\Big[2\sinh(a_2)\ell'_2 + e^{a_2-2a'_2}\ell_2 + \frac{1}{2}e^{a_2-2a'_2}\omega_0(v_2,w_2)\Big])$$
$$\times\,\delta\Big(v'_2 - \frac{e^{\frac{a_2}{2}-a'_2}}{2\cosh(\frac{a_2}{2})}v_2\Big)\delta\Big(w'_2 - \frac{e^{\frac{a_2}{2}-a'_2}}{2\cosh(\frac{a_2}{2})}w_2\Big).$$

In the case where $\rho(g_1) = \mathbb{1}$, i.e., when the action of G_1 on $\mathbb{S}_2$ is trivial in G, we find the second part of the star-exponential

$$\mathcal{E}^{\mathcal{O}_2}_{g_2}(g'_2) = \sqrt{\cosh(a_2)}\cosh(\frac{a_2}{2})^{n_2}$$
$$\times\exp(\frac{i\epsilon_2}{\theta}\Big[2\sinh(a_2)\ell'_2 + e^{a_2-2a'_2}\ell_2 + e^{\frac{a_2}{2}-a'_2}\cosh(\frac{a_2}{2})\omega_0(x_2,x'_2)\Big]). \tag{1.20}$$

which corresponds to the star-exponential of the elementary normal j-group $\mathbb{S}_2$.

By using this characterization in terms of the quantization map, we can derive easily some properties of the star-exponential.

Proposition 3.10. *The star-exponential enjoys the following properties. For all $g, g' \in G$, for all $g_0 \in \mathcal{O}$,*

- *hermiticity:* $\overline{\mathcal{E}^{\mathcal{O}}_g(g_0)} = \mathcal{E}^{\mathcal{O}}_{g^{-1}}(g_0)$.
- *covariance:* $\mathcal{E}^{\mathcal{O}}_{g'\cdot g\cdot g'^{-1}}(g'\cdot g_0) = \mathcal{E}^{\mathcal{O}}_g(g_0)$.
- *BCH:* $\mathcal{E}^{\mathcal{O}}_g \star_\theta \mathcal{E}^{\mathcal{O}}_{g'} = \mathcal{E}^{\mathcal{O}}_{g\cdot g'}$.
- *Character formula:* $\int_G \mathcal{E}^{\mathcal{O}}_g(g_0)\mathrm{d}\mu(g_0) = \kappa^{-1}\,\mathrm{Tr}(U(g))$.

Proof. Using Theorem 3.9, we can show that $\overline{\mathcal{E}^{\mathcal{O}}_g(g_0)} = \mathrm{Tr}(U(g^{-1})\Omega_{\mathbf{m}_0}(g_0)) = \mathcal{E}^{\mathcal{O}}_{g^{-1}}(g_0)$ since $\Omega_{\mathbf{m}_0}(g_0)$ is self-adjoint. In the same way, covariance follows the G-equivariance of $\Omega_{\mathbf{m}_0}$. The BCH property is related to the fact that U is a group representation. Finally, we get

$$\int_{\mathcal{O}}\mathcal{E}^{\mathcal{O}}_g(g_0)\mathrm{d}\mu(g_0) = \mathrm{Tr}(U(g)\int_{\mathcal{O}}\Omega_{\mathbf{m}_0}(g_0)\mathrm{d}\mu(g_0)) = \kappa^{-1}\,\mathrm{Tr}(U(g))$$

using that $\Omega_{\mathbf{m}_0}(1) = \mathbb{1}$. □

Note that the BCH property makes sense in a non-formal way only in the functional space $\mathcal{M}_{\star_\theta}(G)$ determined in Sect. 4.2, where we will see that the star-exponential belongs to.

3.4 Other Determination Using PDEs

We give here another way to determine the star-exponential without using the quantization map, but directly by solving the PDE it has to satisfy. We restrict here to the case of an elementary normal j-group $G = \mathbb{S}$ for simplicity.

By using the strong-invariance of the star-product, for any $f \in \mathcal{M}_{\star_\theta}(\mathbb{S})$ (see Sect. 4.2),

$$\forall\ X \in \mathfrak{s} \quad : \quad [\lambda_X, f]_{\star_\theta} = -i\theta X^* f,$$

where λ is the moment map (1.3), and by using also the equivariance of $\Omega_{\mathbf{m}_0}$, we deduce that

$$\begin{aligned}
&[\Omega_{\mathbf{m}_0}(\lambda_X), \Omega_{\mathbf{m}_0}(f)] = \Omega_{\mathbf{m}_0}([\lambda_X, f]_{\star_\theta}) = -i\theta\Omega_{\mathbf{m}_0}(X^* f)\\
&= -i\theta\frac{d}{dt}|_0\Omega_{\mathbf{m}_0}(L^*_{e^{-tX}} f) = -i\theta\frac{d}{dt}|_0 U(e^{tX})\Omega_{\mathbf{m}_0}(f)U(e^{-tX})\\
&= -i\theta[U_*(X), \Omega_{\mathbf{m}_0}(f)]
\end{aligned}$$

Since the center of $\mathcal{M}_{\star_\theta}(\mathbb{S})$ is trivial, this means that there exists a linear map $\beta : \mathfrak{g} \to \mathbb{C}$ such that

$$\Omega_{\mathbf{m}_0}(\lambda_X) = -i\theta U_*(X) + \beta(X)\mathbb{1}.$$

The invariance of the product under Σ (see (1.9)) implies that $\beta(X) = -\beta(X)$ and finally $\beta(X) = 0$. As a consequence, we have the following proposition.

Proposition 3.11. *The star-exponential (see Definition 3.8) of an elementary normal j-group $G = \mathbb{S}$ satisfies the equation*

$$\partial_t \mathcal{E}_{e^{tX}} = \frac{i}{\theta}(\lambda_X \star_\theta \mathcal{E}_{e^{tX}}) \tag{1.21}$$

with initial condition $\lim_{t\to 0} \mathcal{E}_{e^{tX}} = 1$.

Proof. Indeed, by using $\Omega_{\mathbf{m}_0}(\lambda_X) = -i\theta U_*(X)$, we derive

$$\begin{aligned}
\partial_t \mathcal{E}_{e^{tX}}(g_0) &= \partial_t \operatorname{Tr}(U(e^{tX})\Omega_{\mathbf{m}_0}(g_0)) = \operatorname{Tr}(U_*(X)U(e^{tX})\Omega_{\mathbf{m}_0}(g_0))\\
&= \frac{i}{\theta}\operatorname{Tr}(\Omega_{\mathbf{m}_0}(\lambda_X)U(e^{tX})\Omega_{\mathbf{m}_0}(g_0)) = \frac{i}{\theta}(\lambda_X \star_\theta \mathcal{E}_{e^{tX}})(g_0)
\end{aligned}$$

□

Now we can use this equation to find directly the expression of the star-exponential. Let us do it for example for the coadjoint orbit associated to the sign $\epsilon = +1$. Since the equation (1.21) is integro-differential and complicated to solve, we will analyze the following equation:

$$\partial_t f_t = \frac{i}{\theta}(\lambda_X \star_\theta^0 f_t), \qquad \lim_{t\to 0} f_t = 1 \tag{1.22}$$

for the Moyal product $\star_\theta^0$. Indeed, we have the expression of the intertwiner T_θ from $\star_\theta^0$ to $\star_\theta$. We define the partial Fourier transformation as

$$\mathcal{F}f(a,x,\xi) := \hat{f}(a,x,\xi) := \int e^{-i\xi\ell} f(a,x,\ell)\mathrm{d}\ell. \tag{1.23}$$

Applying the partial Fourier transformation (1.23), with $X = \alpha H + y + \beta E \in \mathfrak{s}$, on the action of moment maps by the Moyal product, we find

$$\mathcal{F}(\lambda_H \star_\theta^0 f) = \left(2i\partial_\xi + \frac{i\theta}{2}\partial_a\right)\hat{f}$$

$$\mathcal{F}(\lambda_y \star_\theta^0 f) = e^{-a-\frac{\theta\xi}{4}}\left(\omega_0(y,x) + \frac{i\theta}{2}y\partial_x\right)\hat{f}$$

$$\mathcal{F}(\lambda_E \star_\theta^0 f) = e^{-2a-\frac{\theta\xi}{2}}\hat{f},$$

so that Eq. (1.22) can be reformulated as

$$\partial_t \hat{f}_t = \frac{i}{\theta}\Big[2i\alpha\partial_\xi + \frac{i\theta\alpha}{2}\partial_a + \beta e^{-2a-\frac{\theta\xi}{2}} + e^{-a-\frac{\theta\xi}{4}}(\omega_0(y,x) + \frac{i\theta}{2}y\partial_x)\Big]\hat{f}_t \tag{1.24}$$

which is a pure PDE. Then, owing to the form of the moment map (1.3), we consider the *ansatz*

$$f_t(a,x,\ell) = v(t)\exp\frac{i}{\theta}\Big[2\ell\gamma_1(t) + e^{-2a}\gamma_2(t) + e^{-a}\gamma_3(t)\omega_0(y,x)\Big] \tag{1.25}$$

whose partial Fourier transform can be expressed as

$$\hat{f}_t(a,x,\xi) = 4\pi^2\delta\Big(\xi - \frac{2\gamma_1(t)}{\theta}\Big)v(t)\exp\frac{i}{\theta}\Big[e^{-2a}\gamma_2(t) + e^{-a}\gamma_3(t)\omega_0(y,x)\Big].$$

Inserting this *ansatz* into Eq. (1.24), it gives

$$\gamma_1'(t) = \alpha,\ \gamma_2'(t) = \alpha\gamma_2(t) + \beta e^{-\gamma_1(t)},\ \gamma_3'(t) = \frac{\alpha}{2}\gamma_3(t) + e^{-\frac{\alpha t}{2}},\ v'(t) = 0.$$

We find that the solutions with initial condition $\lim_{t\to 0} f_t = 1$ are

$$\gamma_1 = \alpha t, \qquad \gamma_2 = \frac{\beta}{\alpha}\sinh(\alpha t), \qquad \gamma_3 = \frac{\sinh(\frac{\alpha t}{2})}{\alpha}, \qquad v = 1.$$

Using intertwining operators (1.14), we see that $T_\theta^{-1}\lambda_X = \lambda_X$, and $T_\theta f_t$ is then a solution of (1.21):

$$E_{\star_\theta}(t\lambda_X)(a,x,\ell) := \mathcal{E}_{e^{tX}}(a,x,\ell) = T_\theta f_t(a,x,\ell)$$
$$= \sqrt{\cosh(\alpha t)}\cosh(\frac{\alpha t}{2})^n e^{\frac{i}{\theta}\sinh(\alpha t)\left(2\ell+\frac{\beta}{\alpha}e^{-2a}+\frac{e^{-a}}{\alpha}\omega_0(y,x)\right)}.$$

To obtain the star-exponential, we need the expression of the logarithm of the group $\mathbb{S}$: $\mathcal{E}_{g_0} = E_{\star_\theta}(\lambda_{\log(g_0)})$. For $X = \alpha H + y + \beta E \in \mathfrak{s}$, the exponential of the group $\mathbb{S}$ has the expression

$$\exp(\alpha H + y + \beta E) = \Big(\alpha, \frac{2e^{-\frac{\alpha}{2}}}{\alpha}\sinh(\frac{\alpha}{2})y, \frac{\beta}{\alpha}e^{-\alpha}\sinh(\alpha)\Big),$$

and the logarithm

$$\log(a,x,\ell) = aH + \frac{a}{2}\frac{e^{\frac{a}{2}}}{\sinh(\frac{a}{2})}x + \frac{ae^a}{\sinh(a)}\ell E.$$

Therefore, we obtain

$$\mathcal{E}_{g_0}(g) = \sqrt{\cosh(a_0)}\cosh(\frac{a_0}{2})^n e^{\frac{i}{\theta}\left(2\sinh(a_0)\ell+e^{a_0-2a}\ell_0+e^{\frac{a_0}{2}-a}\cosh(\frac{a_0}{2})\omega_0(x_0,x)\right)},$$

which coincides with the expression (1.20) determined by using the quantization map $\Omega_{\mathbf{m}_0}$. Note that the BCH property (see Proposition 3.10) can also be checked directly at the level of the Lie algebra $\mathfrak{s}$. From the above expressions of the logarithm and the exponential of the group $\mathbb{S}$, we derive the BCH expression: $\mathrm{BCH}(X_1, X_2) := \log(e^{X_1}e^{X_2})$, i.e.,

$$\mathrm{BCH}(X_1,X_2) = \Big(\alpha_1+\alpha_2, \frac{(\alpha_1+\alpha_2)}{\sinh(\frac{\alpha_1+\alpha_2}{2})}\Big(\frac{e^{-\frac{\alpha_2}{2}}}{\alpha_1}\sinh(\frac{\alpha_1}{2})y_1 + \frac{e^{\frac{\alpha_1}{2}}}{\alpha_2}\sinh(\frac{\alpha_2}{2})y_2\Big),$$
$$\frac{(\alpha_1+\alpha_2)}{\sinh(\alpha_1+\alpha_2)}\Big[\frac{\beta_1}{\alpha_1}e^{-\alpha_2}\sinh(\alpha_1)$$
$$+\frac{\beta_2}{\alpha_2}e^{\alpha_1}\sinh(\alpha_2) + \frac{2}{\alpha_1\alpha_2}e^{\frac{\alpha_1-\alpha_2}{2}}\sinh(\frac{\alpha_1}{2})\sinh(\frac{\alpha_2}{2})\omega_0(y_1,y_2)\Big]\Big).$$

Then, BCH property $\mathcal{E}_g \star_\theta \mathcal{E}_{g'} = \mathcal{E}_{g\cdot g'}$ is equivalent to

$$\forall\ X_1, X_2 \in \mathfrak{s} \quad : \quad E_{\star_\theta}(\lambda_{X_1}) \star_\theta E_{\star_\theta}(\lambda_{X_2}) = E_{\star_\theta}(\lambda_{\mathrm{BCH}(X_1,X_2)}),$$

which turns out to be true for the star-product (1.12) and the star-exponential determined above.

4 Non-Formal Definition of the Star-Exponential

4.1 Schwartz Spaces

In [7], a Schwartz space adapted to the elementary normal j-group $\mathbb{S}$ has been introduced, which is different from the usual one $\mathcal{S}(\mathbb{R}^{2n+2})$ in the global chart $\{(a,x,\ell)\}$, but related to oscillatory integrals. Let us have a look at the phase (1.12) of the star-product:

$$\epsilon S_{\mathbb{S}}(0,g_1,g_2) = \sinh(2a_1)\ell_2 - \sinh(2a_2)\ell_1 + \cosh(a_1)\cosh(a_2)\omega_0(x_1,x_2)$$

with $g_i = (a_i,x_i,\ell_i) \in \mathbb{S}$. Recall that the left-invariant vector fields of $\mathbb{S}$ are given by

$$\tilde{H} = \partial_a - x\partial_x - 2\ell\partial_\ell, \quad \tilde{y} = y\partial_x + \frac{1}{2}\omega_0(x,y)\partial_\ell, \quad \tilde{E} = \partial_\ell.$$

We define the maps $\tilde{\alpha}$ by $\forall X = (X_1,X_2) \in \mathfrak{s}\oplus\mathfrak{s}$,

$$\tilde{X}\cdot e^{-\frac{2i}{\theta}S_{\mathbb{S}}(0,g_1,g_2)} =: -\frac{2i\epsilon}{\theta}\tilde{\alpha}_X(g_1,g_2)e^{-\frac{2i}{\theta}S_{\mathbb{S}}(0,g_1,g_2)}$$

since it is an oscillatory phase. For example, we have

$$\begin{aligned}\tilde{\alpha}_{(E,0)}(g_1,g_2) &= -\sinh(2a_2), \text{ and}\\ \tilde{\alpha}_{(H,0)}(g_1,g_2) &= 2\cosh(2a_1)\ell_2 + 2\sinh(2a_2)\ell_1 - e^{-a_1}\cosh(a_2)\omega_0(x_1,x_2).\end{aligned}$$

Then we set $\alpha_X(g) := \tilde{\alpha}_{(X,0)}(0,g)$ for any $X \in \mathfrak{s}$ and $g \in G$, whose expressions are

$$\alpha_H(g) = 2\ell, \qquad \alpha_y(g) = \cosh(a)\omega(y,x), \qquad \alpha_E(g) = -\sinh(2a).$$

This leads to the following definition.

Definition 4.1. The *Schwartz space* of $\mathbb{S}$ is defined as

$$\begin{aligned}&\mathcal{S}(\mathbb{S}) = \{f \in C^\infty(\mathbb{S}) \quad \forall j \in \mathbb{N}^{2n+2},\ \forall P \in \mathcal{U}(\mathfrak{s}) \text{ such that}\\ &\|f\|_{j,P} := \sup_{g\in\mathbb{S}}\left|\alpha^j(g)\tilde{P}f(g)\right| < \infty\},\end{aligned}$$

where $\alpha^j := \alpha_H^{j_1}\alpha_{e_1}^{j_2}\ldots\alpha_{e_{2n}}^{j_{2n+1}}\alpha_E^{j_{2n+2}}$.

It turns out that the space $\mathcal{S}(\mathbb{S})$ corresponds to the usual Schwartz space in the coordinates (r,x,ℓ) with $r = \sinh(2a)$. It is stable by the action of $\mathbb{S}$:

$$\forall f \in \mathcal{S}(\mathbb{S}),\ \forall g \in \mathbb{S} \quad : \quad g^* f \in \mathcal{S}(\mathbb{S}).$$

Moreover, $\mathcal{S}(\mathbb{S})$ is a Fréchet nuclear space endowed with the seminorms $(\|f\|_{j,P})$.

For $f, h \in \mathcal{S}(\mathbb{S})$, the product $f \star_\theta h$ is well defined by (1.12). However, to show that it belongs to $\mathcal{S}(\mathbb{S})$, we will use arguments close to oscillatory integral theory. Let us illustrate this concept. One can show that the following operators leave the phase $e^{-\frac{2i}{\theta}S(0,g_1,g_2)}$ invariant:

$$\mathcal{O}_{a_2} := \frac{1}{1+\tilde{\alpha}^2_{(E,0)}}(1 - \frac{\theta^2}{4}\tilde{E}^2) = \frac{1}{1+\sinh(2a_2)^2}(1 - \frac{\theta^2}{4}\partial^2_{\ell_1}),$$

$$\mathcal{O}_{a_1} := \frac{1}{1+\sinh(2a_1)^2}(1 - \frac{\theta^2}{4}\partial^2_{\ell_2}),$$

$$\mathcal{O}_{x_2} := \frac{1}{1+x_2^2}(1 - \frac{\theta^2}{4\cosh(a_1)^2\cosh(a_2)^2}\partial^2_{x_1}),$$

$$\mathcal{O}_{x_1} := \frac{1}{1+x_1^2}(1 - \frac{\theta^2}{4\cosh(a_1)^2\cosh(a_2)^2}\partial^2_{x_2}),$$

$$\mathcal{O}_{\ell_2} := \frac{1}{1+\ell_2^2}(1 - \frac{\theta^2}{4}(\frac{1}{\cosh(2a_1)}(\partial_{a_1} - \tanh(a_1)x_1\partial_{x_1}))^2),$$

$$\mathcal{O}_{\ell_1} := \frac{1}{1+\ell_1^2}(1 - \frac{\theta^2}{4}(\frac{1}{\cosh(2a_2)}(\partial_{a_2} - \tanh(a_2)x_2\partial_{x_2}))^2).$$

So we can add arbitrary powers of these operators in front of the phase without changing the expression. Then, using integrations by parts, we have for $F \in \mathcal{S}(\mathbb{S}^2)$:

$$\int e^{-\frac{2i}{\theta}S_{\mathbb{S}}(0,g_1,g_2)} F(g_1,g_2)\mathrm{d}g_1\mathrm{d}g_2$$

$$= \int e^{-\frac{2i}{\theta}S_{\mathbb{S}}(0,g_1,g_2)} (\mathcal{O}^*_{a_1})^{k_1}(\mathcal{O}^*_{a_2})^{k_2}(\mathcal{O}^*_{x_1})^{p_1}(\mathcal{O}^*_{x_2})^{p_2}(\mathcal{O}^*_{\ell_1})^{q_1}(\mathcal{O}^*_{\ell_2})^{q_2} F(g_1,g_2)\mathrm{d}g_1\mathrm{d}g_2$$

$$= \int e^{-\frac{2i}{\theta}S_{\mathbb{S}}(0,g_1,g_2)} \frac{1}{(1+\sinh^2(2a_1))^{k_1}(1+\sinh^2(2a_2))^{k_2}}$$

$$\frac{1}{(1+x_1^2)^{p_1-q_2}(1+x_2^2)^{p_2-q_1}(1+\ell_1^2)^{q_1}(1+\ell_2^2)^{q_2}} DF(g_1,g_2)\mathrm{d}g_1\mathrm{d}g_2 \quad (1.26)$$

for any $k_i, q_i, p_i \in \mathbb{N}$ such that $p_1 \geq q_2$ and $p_2 \geq q_1$, and where D is a linear combination of products of bounded functions (with every derivatives bounded) in (g_1, g_2) with powers of ∂_{ℓ_i}, ∂_{x_i} and $\frac{1}{\cosh(2a_i)}\partial_{a_i}$. The first line of (1.26) is not defined for nonintegrable functions F bounded by polynomials in $r_i := \sinh(2a_i)$, x_i and ℓ_i. However, the last two lines of (1.26) are well defined for k_i, p_i, q_i sufficiently large. Therefore it gives a sense to the first line, now understood as an oscillatory integral, i.e., as being equal to the last two lines. This definition of oscillatory integral [7, 9] is unique, in particular unambiguous in the powers

k_i, p_i, q_i. Note that this corresponds to the usual oscillatory integral [13] in the coordinates (r, x, ℓ).

The next theorem, proved in [7], can be showed by using such methods of oscillatory integrals on $\mathcal{S}(\mathbb{S})$.

Theorem 4.2. *Let $\mathcal{P} : \mathbb{R} \to C^\infty(\mathbb{R})$ be a smooth map such that $\mathcal{P}_0 \equiv 1$, and $\mathcal{P}_\theta(a)$ as well as its inverse are bounded by $C \sinh(2a)^k$, $k \in \mathbb{N}$, $C > 0$. Then, the expression (1.12) yields a $\mathbb{S}$-invariant non-formal deformation quantization.*

In particular, $(\mathcal{S}(\mathbb{S}), \star_\theta)$ is a nuclear Fréchet algebra.

In what follows we show a factorization property for this Schwartz space. First, by introducing $\gamma(a) = \sinh(2a)$ and $\mathcal{S}(A) := \gamma^*\mathcal{S}(\mathbb{R})$, we note that the group law of $\mathbb{S}$ reads in the coordinates $(r = \gamma(a), x, \ell)$:

$$(r, x, \ell)\cdot(r', x', \ell') = \Big(r\sqrt{1 + r'^2} + r'\sqrt{1 + r^2}, (c(r') - s(r'))x + x',$$

$$(\sqrt{1 + r'^2} - r')\ell + \ell' + \frac{1}{2}(c(r') - s(r'))\omega_0(x, x')\Big)$$

with the auxiliary functions:

$$c(r) = \frac{\sqrt{2}}{2}(1 + \sqrt{1 + r^2})^{\frac{1}{2}} = \cosh(\frac{1}{2}\text{arcsinh}(r)), \tag{1.27}$$

$$s(r) = \frac{\sqrt{2}}{2}\text{sgn}(r)(-1 + \sqrt{1 + r^2})^{\frac{1}{2}} = \sinh(\frac{1}{2}\text{arcsinh}(r)).$$

Proposition 4.3 (Factorization). *The map Φ defined by $\Phi(f \otimes h) = f \star_\theta h$, for $f \in \mathcal{S}(A)$ and $h \in \mathcal{S}(\mathbb{R}^{2n+1})$ realizes a continuous automorphism $\mathcal{S}(\mathbb{S}) = \mathcal{S}(A)\hat{\otimes}\mathcal{S}(\mathbb{R}^{2n+1}) \to \mathcal{S}(\mathbb{S})$.*

Proof. Due to the nuclearity of the Schwartz space, we have indeed $\mathcal{S}(\mathbb{S}) = \mathcal{S}(A)\hat{\otimes}\mathcal{S}(\mathbb{R}^{2n+1})$. For $f \in \mathcal{S}(A)$ (abuse of notation identifying $f(a)$ and $f(r) := f(\gamma^{-1}(r)))$ and $h \in \mathcal{S}(\mathbb{R}^{2n+1})$, we reexpress the star-product (1.12) in the coordinates (r, x, ℓ):

$$(f \star_\theta h)(r, x, \ell) = \frac{1}{(\pi\theta)^{2n+2}} \int \Big(1 - \frac{r_1 r_2}{\sqrt{(1 + r_1^2)(1 + r_2^2)}}\Big)$$

$$\times f(r\sqrt{1 + r_1^2} + r_1\sqrt{1 + r^2})h(\frac{1}{c(r_1)}x_2 + (c(r_2) - \frac{s(r_1)s(r_2)}{c(r_1)})x, \ell_2 + \sqrt{1 + r_2^2}\ell)$$

$$\times \frac{\sqrt{c(r_1)c(r_2)}}{\sqrt{c(r_1\sqrt{1 + r_2^2} - r_2\sqrt{1 + r_1^2})}} e^{-\frac{2i\epsilon}{\theta}(r_1\ell_2 - r_2\ell_1 + \omega_0(x_1, x_2))} \mathrm{d}r_i \mathrm{d}x_i \mathrm{d}\ell_i .$$

By using the partial Fourier transform $\hat{h}(r,\xi) = \int \mathrm{d}\ell e^{-i\ell\xi} h(x,\ell)$, and integrating over several variables, we obtain

$$(f \star_\theta h)(r,x,\ell) = \frac{1}{2\pi}\int f(r\sqrt{1+\frac{\theta^2\xi^2}{4}} + \frac{\epsilon\theta\xi}{2}\sqrt{1+r^2})\hat{h}(x,\xi)e^{i\ell\xi}\mathrm{d}\xi.$$

For $\varphi \in \mathcal{S}(\mathbb{S})$, we have now the following explicit expression for Φ:

$$\Phi(\varphi)(r,x,\ell) = \frac{1}{2\pi}\int \hat{\varphi}(r\sqrt{1+\frac{\theta^2\xi^2}{4}} + \frac{\epsilon\theta\xi}{2}\sqrt{1+r^2}, x, \xi)e^{i\ell\xi}\mathrm{d}\xi$$

which permits to deduce that Φ is valued in $\mathcal{S}(\mathbb{S})$ and continuous. Then the formula

$$\hat{\varphi}(r,x,\ell) = \int \Phi(\varphi)(r\sqrt{1+\frac{\theta^2\ell^2}{4}} - \frac{\epsilon\theta\ell}{2}\sqrt{1+r^2}, x, \xi)e^{-i\ell\xi}\mathrm{d}\xi$$

permits to obtain the inverse of Φ which is also continuous. □

For normal j-groups $G = G_1 \ltimes \mathbb{S}_2$, we define the Schwartz space recursively

$$\mathcal{S}(G) = \mathcal{S}(G_1)\hat{\otimes}\mathcal{S}(\mathbb{S}_2)$$

and obtain the same properties as before. In particular, endowed with the star-product (1.17), the Schwartz space $\mathcal{S}(G)$ is a nuclear Fréchet algebra.

4.2 Multipliers

Let us consider the topological dual $\mathcal{S}'(\mathbb{S})$ of $\mathcal{S}(\mathbb{S})$. In the coordinates ($r = \gamma(a)$, x, ℓ), it corresponds to tempered distributions. By denoting $\langle -,- \rangle$ the duality bracket between $\mathcal{S}'(\mathbb{S})$ and $\mathcal{S}(\mathbb{S})$, one can extend the product $\star_\theta$ (with tracial identity) as

$$\forall\, T \in \mathcal{S}'(\mathbb{S}),\ \forall\ f,h \in \mathcal{S}(\mathbb{S}) : \langle T\star_\theta f, h\rangle := \langle T, f\star_\theta h\rangle \text{ and } \langle f\star_\theta T, h\rangle := \langle T, h\star_\theta f\rangle,$$

which is compatible with the case $T \in \mathcal{S}(\mathbb{S})$.

Definition 4.4. The *multiplier space* associated to $(\mathcal{S}(\mathbb{S}), \star_\theta)$ is defined as

$$\mathcal{M}_{\star_\theta}(\mathbb{S}) :=$$
$$\{T \in \mathcal{S}'(\mathbb{S}),\ f \mapsto T \star_\theta f \text{ and } f \mapsto f \star_\theta T \text{ are continuous from } \mathcal{S}(\mathbb{S}) \text{ into itself}\}.$$

We can endow this space with the topology associated to the seminorms:

$$\|T\|_{B,j,P,L} = \sup_{f\in B} \|T \star f\|_{j,P} \text{ and } \|T\|_{B,j,P,R} = \sup_{f\in B} \|f \star T\|_{j,P}$$

where B is a bounded subset of $\mathcal{S}(\mathbb{S})$, $j \in \mathbb{N}^{2n+2}$, $P \in \mathcal{U}(\mathfrak{s})$ and $\|f\|_{j,P}$ is the Schwartz seminorm introduced in Definition 4.1. Note that B can be described as a set satisfying $\forall j, P$, $\sup_{f\in B} \|f\|_{j,P}$ exists.

Proposition 4.5. *The star-product can be extended to* $\mathcal{M}_{\star_\theta}(\mathbb{S})$ *by:*

$$\forall\, S, T \in \mathcal{M}_{\star_\theta}(\mathbb{S}),\ \forall\, f \in \mathcal{S}(\mathbb{S}) \quad : \quad \langle S \star_\theta T, f\rangle := \langle S, T \star_\theta f\rangle = \langle T, f \star_\theta S\rangle.$$

Then $(\mathcal{M}_{\star_\theta}(\mathbb{S}), \star_\theta)$ *is an associative Hausdorff locally convex complete and nuclear algebra, with separately continuous product called the multiplier algebra.*

Proof. For the extension of the star-product and its associativity, we can show successively for all $S, T \in \mathcal{M}_{\star_\theta}(\mathbb{S})$, for all $f, h \in \mathcal{S}(\mathbb{S})$,

$$(T \star_\theta f) \star_\theta h = T \star_\theta (f \star_\theta h) \quad , \quad (S \star_\theta T) \star_\theta f = S \star_\theta (T \star_\theta f), \text{ and}$$
$$(T_1 \star_\theta T_2) \star_\theta T_3 = T_1 \star_\theta (T_2 \star_\theta T_3),$$

each time by evaluating the distribution on a Schwartz function $\varphi \in \mathcal{S}(\mathbb{S})$ and by using the factorization property (Proposition 4.3).

$\mathcal{M}_{\star_\theta}(\mathbb{S})$ is the intersection of $\mathcal{M}_L$, the left multipliers, and $\mathcal{M}_R$, the right multipliers. By definition, each space $\mathcal{M}_L$ and $\mathcal{M}_R$ is topologically isomorphic to $\mathcal{L}(\mathcal{S}(\mathbb{S}))$ endowed with the strong topology. Since $\mathcal{S}(\mathbb{S})$ is Fréchet and nuclear, so is $\mathcal{L}(\mathcal{S}(\mathbb{S}))$, as well as $\mathcal{M}_L$, $\mathcal{M}_R$ and finally $\mathcal{M}_{\star_\theta}(\mathbb{S})$ (see [18] Propositions 50.1, 50.5 and 50.6). □

Due to the definition of $\mathcal{S}(G)$ for a normal j-group $G = G_1 \ltimes \mathbb{S}_2$ and to the expression of the star-product (1.17), the multiplier space associated to $(\mathcal{S}(G), \star_\theta)$ takes the form

$$\mathcal{M}_{\star_\theta}(G) = \mathcal{M}_{\star_\theta}(G_1)\hat{\otimes}\mathcal{M}_{\star_\theta}(\mathbb{S}_2), \tag{1.28}$$

and is also an associative Hausdorff locally convex complete and nuclear algebra, with separately continuous product. Remember that we have identified coadjoint orbits $\mathcal{O}$ described in Proposition 2.10 with the group G itself, so that we can speak also about the multiplier algebra $\mathcal{M}_{\star_\theta}(\mathcal{O})$.

4.3 Non-Formal Star-Exponential

Theorem 4.6. *Let G be a normal j-group and $\star_\theta$ the star-product* (1.17)*. Then for any $g \in G$, the star-exponential* (1.19) $\mathcal{E}_g^{\mathcal{O}}$ *lies in the multiplier algebra* $\mathcal{M}_{\star_\theta}(\mathcal{O})$.

Proof. Let us focus for the moment on the case of the elementary group $\mathcal{S}$. The general case can then be obtained recursively due to the structure of the star-exponential (1.19) and of the multiplier algebra (1.28). We use the same notations as before. For $f, h \in \mathcal{S}(A)$, $f \star_\theta h = f \cdot h$. If T belongs to the multiplier space $\mathcal{M}(\mathcal{S}(A))$ of $\mathcal{S}(A)$ for the usual commutative product, we have in particular $T \in \mathcal{S}'(\mathbb{S})$ and by duality $T \star_\theta f = T \cdot f$. Then,

$$\forall\, f \in \mathcal{S}(A),\ \forall\, h \in \mathcal{S}(\mathbb{R}^{2n+1}) \quad : \quad T \star_\theta (f \star_\theta h) = (T \cdot f) \star_\theta h.$$

By the factorization property (Proposition 4.3), it means that $T \in \mathcal{M}_{\star_\theta}(\mathbb{S})$, and we have an embedding $\mathcal{M}(\mathcal{S}(A)) \hookrightarrow \mathcal{M}_{\star_\theta}(\mathbb{S})$. If we note as before $\mathbb{R}^{2n+1} = V \oplus \mathbb{R}E$, we can show in the same way that there is another embedding $\mathcal{M}(\mathcal{S}(\mathbb{R}E)) \hookrightarrow \mathcal{M}_{\star_\theta}(\mathbb{S})$. Since $x' \in V \mapsto \mathcal{E}^{\mathcal{O}_2}_{(g_1,g_2)}(0, x', 0)$ is an imaginary exponential of a polynomial of degree less or equal than 2 in x' and since the product $\star_\theta$ coincides with the Moyal product on V, it turns out that $x' \in V \mapsto \mathcal{E}^{\mathcal{O}_2}_{(g_1,g_2)}(0, x', 0)$ is in $\mathcal{M}_{\star_\theta}(\mathcal{S}(V))$. Then, the star-exponential $\mathcal{E}^{\mathcal{O}_2}$ in (1.19) lies in $\mathcal{M}(\mathcal{S}(A))\hat{\otimes}\mathcal{M}_{\star_\theta}(\mathcal{S}(V))\hat{\otimes}\mathcal{M}(\mathcal{S}(\mathbb{R}E))$, and it belongs also to $\mathcal{M}_{\star_\theta}(\mathbb{S})$. $\square$

5 Adapted Fourier Transformation

5.1 *Definition*

As in the case of the Moyal–Weyl quantization treated in [1,2], we can introduce the notion of adapted Fourier transformation. For normal j-groups $G = G_1 \ltimes \mathbb{S}_2$, which are not unimodular, it is relevant for that to introduce a **modified star-exponential**

$$\tilde{\mathcal{E}}^{\mathcal{O}}_g(g') := \mathrm{Tr}(U(g) d^{\frac{1}{2}} \Omega(g')),$$

where d is the formal dimension operator associated to U (see [10, 12]) and $\mathcal{O}$ is the coadjoint orbit determining the irreducible representation U. Such an operator d is used to regularize the expressions since $\int f(g) U(g) d^{\frac{1}{2}}$ is a Hilbert–Schmidt operator whenever f is in $L^2(G)$. So the trace in the definition of $\tilde{\mathcal{E}}^{\mathcal{O}}$ is understood as a distribution only in the variable $g' \in \mathcal{O}$.

By denoting Δ the modular function, defined by $\mathrm{d}^L(g \cdot g') = \Delta(g') \mathrm{d}^L g$, whose computation gives

$$\Delta(g) = \Delta_1(g_1)\Delta_2(g_2), \text{ with } \Delta_2(a_2, x_2, \ell_2) = e^{-2(n_2+1)a_2},$$

the operator d is defined (up to a positive constant) by the relation

$$\forall\, g \in G \quad : \quad U(g) d U(g)^{-1} = \Delta(g)^{-1} d.$$

Since $\mathcal{R}(g_1)d_2\mathcal{R}(g_1)^{-1} = d_2$, it can therefore be expressed as $d = d_1 \otimes d_2$, for d_i the dimension operator associated to U_i, and with for all $\varphi_2 \in \mathcal{H}_2$, for all $(a_0, v_0) \in Q_2$,

$$(d_2\varphi_2)(a_0, v_0) = \kappa_2^2 e^{-2(n_2+1)a_0}\varphi_2(a_0, v_0)$$

where we recall that $\dim(\mathbb{S}_2) = 2(n_2 + 1)$. Note that d_2 is independent here of the choice of the irreducible representation U_2 ($\epsilon_2 = \pm 1$).

Proposition 5.1. *The expression of the modified star-exponential can then be computed the same notation as for Theorem 3.9:*

$$\tilde{\mathcal{E}}_g^{\mathcal{O}}(g') = \tilde{\mathcal{E}}_{g_1}^{\mathcal{O}_1}(g_1')\frac{e^{(n_2+1)(\frac{a_2}{2}-a_2')}}{(\pi\theta)^{n_2+1}}\frac{\sqrt{\cosh(a_2)}\cosh(\frac{a_2}{2})^{n_2}|\det(\rho_+(g_1))|^{\frac{1}{2}}}{|\det(1+\rho_+(g_1))|}$$
$$\exp\left(\frac{i\epsilon_2}{\theta}\Big[2\sinh(a_2)\ell_2' + e^{a_2-2a_2'}\ell_2 + X^T A_\rho X\Big]\right).$$

Definition 5.2. We can now define the *adapted Fourier transformation*: for $f \in \mathcal{S}(G)$ and $g' \in \mathcal{O}$,

$$\mathcal{F}_{\mathcal{O}}(f)(g') := \int_G f(g)\tilde{\mathcal{E}}_g^{\mathcal{O}}(g')\mathrm{d}^L g.$$

We see that this definition is a generalization of the usual (symplectic) Fourier transformation. For example in the case of the group $\mathbb{R}^2$, the star-exponential associated to the Moyal product is indeed given by $\exp(\frac{2i}{\theta}(a\ell' - a'\ell))$.

5.2 Fourier Analysis

Proposition 5.3. *The modified star-exponential satisfies an orthogonality relation: for $g', g'' \in G$,*

$$\int_G \overline{\tilde{\mathcal{E}}_g^{\mathcal{O}}(g')}\tilde{\mathcal{E}}_g^{\mathcal{O}}(g'')\mathrm{d}^L g = \frac{1}{\Delta(g'')}\delta(g''\cdot(g')^{-1}).$$

Note that $\Delta(g_2'')^{-1}\delta(g_2''\cdot(g_2')^{-1}) = \delta(a_2''-a_2')\delta(x_2''-x_2')\delta(\ell_2''-\ell_2')$. This orthogonality relation does not hold for the unmodified star-exponential.

Proof. We use the expression of Proposition 5.1:

$$\int_G \overline{\tilde{\mathcal{E}}_g^{\mathcal{O}}(g')}\tilde{\mathcal{E}}_g^{\mathcal{O}}(g'')\mathrm{d}^L g = \int_{G_1} \overline{\tilde{\mathcal{E}}_{g_1}^{\mathcal{O}_1}(g_1')}\tilde{\mathcal{E}}_{g_1}^{\mathcal{O}_1}(g_1'')\int_{\mathbb{S}_2}\frac{e^{(n_2+1)(a_2-a_2'-a_2'')}}{(\pi\theta)^{2(n_2+1)}}$$

$$\frac{|\det(\rho_+(g_1))|\cosh(a_2)\cosh(\frac{a_2}{2})^{2n_2}}{|\det(1+\rho_+(g_1))|^2} e^{\frac{i\epsilon_2}{\theta}(2\sinh(a_2)(\ell_2''-\ell_2')+e^{a_2}(e^{-2a_2''}-e^{-2a_2'})\ell_2)}$$
$$e^{\frac{i\epsilon_2}{\theta}((X'')^T A_\rho X''-(X')^T A_\rho X')}\mathrm{d}^L g_2 \mathrm{d}^L g_1$$

with

$$X' = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{\frac{a_2}{2}-a_2'} v_2 \\ \frac{1}{\sqrt{2}} e^{\frac{a_2}{2}-a_2'} w_2 \\ \sqrt{2}\cosh(\frac{a_2}{2}) v_2' \\ \sqrt{2}\cosh(\frac{a_2}{2}) w_2' \end{pmatrix} \quad \text{and} \quad X'' = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{\frac{a_2}{2}-a_2''} v_2 \\ \frac{1}{\sqrt{2}} e^{\frac{a_2}{2}-a_2''} w_2 \\ \sqrt{2}\cosh(\frac{a_2}{2}) v_2'' \\ \sqrt{2}\cosh(\frac{a_2}{2}) w_2'' \end{pmatrix}.$$

Integration over ℓ_2 leads to the contribution $\delta(a_2' - a_2'')$. Since A_ρ depends only on g_1, and $a_2' = a_2''$, we see that the gaussian part in (v_2, w_2) disappears and integration over these variables brings $\frac{|\det(1+\rho_+(g_1))|^2}{|\det(\rho_+(g_1))|}\delta(v_2' - v_2'')\delta(w_2' - w_2'')$. Eventually, integration on a_2 can be performed and we find

$$\int_G \overline{\tilde{\mathcal{E}}_g^{\mathcal{O}}(g')}\tilde{\mathcal{E}}_g^{\mathcal{O}}(g'')\mathrm{d}^L g = \Big(\int_{G_1} \overline{\tilde{\mathcal{E}}_{g_1}^{\mathcal{O}_1}(g_1')}\tilde{\mathcal{E}}_{g_1}^{\mathcal{O}_1}(g_1'')\mathrm{d}^L g_1\Big)\, \Delta(g_2'')^{-1}\delta(g_2''\cdot(g_2')^{-1})$$

which leads to the result recursively. □

Proposition 5.4. *The adapted Fourier transformation satisfies the following property:* $\forall f_1, f_2 \in \mathcal{S}(G)$,

$$\mathcal{F}_{\mathcal{O}}(f_1 \times f_2) = \frac{\Delta^{\frac{1}{2}}}{\kappa}\big(\Delta^{-\frac{1}{2}}\mathcal{F}_{\mathcal{O}}(f_1)\big) \star_\theta \big(\Delta^{-\frac{1}{2}}\mathcal{F}_{\mathcal{O}}(f_2)\big),$$

with $(f_1 \times f_2)(g) = \int_G f_1(g') f_2((g')^{-1}g)\mathrm{d}^L g'$ *the usual convolution.*

Proof. Due to the BCH property (see Proposition 3.10) and to the computation of the modified star-exponential $\tilde{\mathcal{E}}_g^{\mathcal{O}}(g') = \tilde{\mathcal{E}}_{g_1}^{\mathcal{O}_1}(g_1')\frac{\kappa_2}{\Delta_2(g_2(g_2')^{-2})^{\frac{1}{2}}}\mathcal{E}_g^{\mathcal{O}_2}(g_2')$, we have the modified the BCH property

$$\tilde{\mathcal{E}}_{g\cdot g'}^{\mathcal{O}}(g'') = \frac{\Delta(g'')^{\frac{1}{2}}}{\kappa}\Big(\Delta^{-\frac{1}{2}}\tilde{\mathcal{E}}_g^{\mathcal{O}}\Big) \star_\theta \Big(\Delta^{-\frac{1}{2}}\tilde{\mathcal{E}}_{g'}^{\mathcal{O}}\Big)(g'')$$

which leads directly to the result by using the expression of the adapted Fourier transform and the convolution. □

As in Remark 2.11, we consider the coadjoint orbit $\mathcal{O}_{(\epsilon)} = \mathcal{O}_{1,(\epsilon_1)} \times \mathcal{O}_{2,\epsilon_2}$ of the normal j-group $G = G_1 \ltimes \mathbb{S}_2$ determined by the sign choices $(\epsilon) = ((\epsilon_1), \epsilon_2) \in (\mathbb{Z}_2)^N$, with $(\epsilon_1) \in (\mathbb{Z}_2)^{N-1}$ and $\epsilon_2 \in \mathbb{Z}_2$. Due to Proposition 5.1, we can write the modified star-exponential as

$$\tilde{\mathcal{E}}_g^{\mathcal{O}_{(\epsilon)}}(g') = \tilde{\mathcal{E}}_{g_1}^{\mathcal{O}_{1,(\epsilon_1)}}(g_1')\,\tilde{\mathcal{E}}_{(g_1,g_2)}^{\mathcal{O}_{2,\epsilon_2}}(g_2'),$$

with $g = (g_1, g_2) \in G$ and $g' = (g_1', g_2') \in \mathcal{O}_{(\epsilon)}$.

Theorem 5.5. *We have the following inversion formula for the adapted Fourier transformation: for $f \in \mathcal{S}(G)$ and $g \in G$,*

$$f(g) = \sum_{(\epsilon)\in(\mathbb{Z}_2)^N} \int_{\mathcal{O}_{(\epsilon)}} \overline{\tilde{\mathcal{E}}_g^{\mathcal{O}_{(\epsilon)}}(g')}\mathcal{F}_{\mathcal{O}_{(\epsilon)}}(f)(g')\mathrm{d}\mu(g').$$

Moreover, the Parseval–Plancherel theorem is true:

$$\int_G |f(g)|^2 \mathrm{d}^L g = \sum_{(\epsilon)\in(\mathbb{Z}_2)^N} \int_{\mathcal{O}_{(\epsilon)}} |\mathcal{F}_{\mathcal{O}_{(\epsilon)}}(f)(g')|^2 \mathrm{d}\mu(g').$$

Proof. Let us show the dual property to Proposition 5.3, i.e.,

$$\sum_{(\epsilon)\in(\mathbb{Z}_2)^N} \int_{\mathcal{O}_{(\epsilon)}} \overline{\tilde{\mathcal{E}}_{g'}^{\mathcal{O}_{(\epsilon)}}(g)}\tilde{\mathcal{E}}_{g''}^{\mathcal{O}_{(\epsilon)}}(g)\mathrm{d}\mu(g) = \frac{1}{\Delta(g'')}\delta(g''\cdot(g')^{-1}). \tag{1.29}$$

First, we have

$$\begin{aligned}
&\int_{\mathcal{O}_{(\epsilon)}} \overline{\tilde{\mathcal{E}}_{g'}^{\mathcal{O}_{(\epsilon)}}(g)}\tilde{\mathcal{E}}_{g''}^{\mathcal{O}_{(\epsilon)}}(g)\mathrm{d}\mu(g) \\
&= \int_{\mathcal{O}_{1,(\epsilon_1)}} \overline{\tilde{\mathcal{E}}_{g_1'}^{\mathcal{O}_{1,(\epsilon_1)}}(g_1)}\tilde{\mathcal{E}}_{g_1''}^{\mathcal{O}_{1,(\epsilon_1)}}(g_1) \int_{\mathcal{O}_{2,\epsilon_2}} \frac{e^{(n_2+1)(\frac{a_2'+a_2''}{2}-2a_2)}}{(\pi\theta)^{2(n_2+1)}} \\
&\times \frac{|\det(\rho_+(g_1'))\det(\rho_+(g_1''))|^{\frac{1}{2}}\sqrt{\cosh(a_2')\cosh(a_2'')}\cosh(\frac{a_2'}{2})^{n_2}\cosh(\frac{a_2''}{2})^{n_2}}{|\det(1+\rho_+(g_1'))\det(1+\rho_+(g_1''))|} \\
&\times \exp\Big[\frac{i\epsilon_2}{\theta}(-2\sinh(a_2')\ell_2 + 2\sinh(a_2'')\ell_2 - e^{a_2'-2a_2}\ell_2' + e^{a_2''-2a_2}\ell_2'' \\
&+ (X'')^T A_\rho(g_1'')X'' - (X')^T A_\rho(g_1')X'\Big]\mathrm{d}\mu_2(g_2)\mathrm{d}\mu_1(g_1)
\end{aligned}$$

with

$$X' = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\frac{a_2'}{2}-a_2}v_2' \\ \frac{1}{\sqrt{2}}e^{\frac{a_2'}{2}-a_2}w_2' \\ \sqrt{2}\cosh(\frac{a_2'}{2})v_2 \\ \sqrt{2}\cosh(\frac{a_2'}{2})w_2 \end{pmatrix} \quad \text{and} \quad X'' = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\frac{a_2''}{2}-a_2}v_2'' \\ \frac{1}{\sqrt{2}}e^{\frac{a_2''}{2}-a_2}w_2'' \\ \sqrt{2}\cosh(\frac{a_2''}{2})v_2 \\ \sqrt{2}\cosh(\frac{a_2''}{2})w_2 \end{pmatrix}.$$

We want to compute the sum over $(\epsilon) \in (\mathbb{Z}_2)^N$ of such terms. By recurrence, we can suppose that

$$\sum_{(\epsilon_1)\in(\mathbb{Z}_2)^{N-1}} \int_{\mathcal{O}_{1,(\epsilon_1)}} \overline{\tilde{\mathcal{E}}_{g_1'}^{\mathcal{O}_{1,(\epsilon_1)}}(g_1)} \tilde{\mathcal{E}}_{g_1''}^{\mathcal{O}_{1,(\epsilon_1)}}(g_1) \mathrm{d}\mu_1(g_1) = \frac{1}{\Delta(g_1'')} \delta(g_1''\cdot(g_1')^{-1}),$$

which means that $g_1'' = g_1'$ in the following. The integration over ℓ_2 brings a contribution in $\delta(a_2' - a_2'')$. Since $g_1'' = g_1'$ and $a_2' = a_2''$, the gaussian part in (v_2, w_2) disappears and integration over these variables brings $\frac{|\det(1+\rho_+(g_1'))|^2}{|\det(\rho_+(g_1'))|}\delta(v_2' - v_2'')\delta(w_2' - w_2'')$. The remaining term is proportional to

$$\sum_{\epsilon_2=\pm 1} \int_{\mathbb{R}} e^{a_2'-2a_2} e^{\frac{i\epsilon_2}{\theta} e^{a_2'-2a_2}(\ell_2''-\ell_2')} \mathrm{d}a_2 = \pi\theta\delta(\ell_2'' - \ell_2').$$

The property (1.29) permits showing the inversion formula

$$\begin{aligned}
&\sum_{(\epsilon)\in(\mathbb{Z}_2)^N} \int_{\mathcal{O}_{(\epsilon)}} \overline{\tilde{\mathcal{E}}_g^{\mathcal{O}_{(\epsilon)}}(g')} \mathcal{F}_{\mathcal{O}_{(\epsilon)}}(f)(g') \mathrm{d}\mu(g') \\
&= \sum_{(\epsilon)\in(\mathbb{Z}_2)^N} \int \overline{\tilde{\mathcal{E}}_g^{\mathcal{O}_{(\epsilon)}}(g')} f(g'') \tilde{\mathcal{E}}_{g''}^{\mathcal{O}_{(\epsilon)}}(g') \mathrm{d}^L g'' \mathrm{d}\mu(g') = f(g),
\end{aligned}$$

as well as the Parseval–Plancherel theorem

$$\begin{aligned}
&\sum_{(\epsilon)\in(\mathbb{Z}_2)^N} \int_{\mathcal{O}_{(\epsilon)}} |\mathcal{F}_{\mathcal{O}_{(\epsilon)}}(f)(g')|^2 \mathrm{d}\mu(g') \\
&= \sum_{(\epsilon)\in(\mathbb{Z}_2)^N} \int \overline{f(g)\tilde{\mathcal{E}}_g^{\mathcal{O}_{(\epsilon)}}(g')} f(g'') \tilde{\mathcal{E}}_{g''}^{\mathcal{O}_{(\epsilon)}}(g') \mathrm{d}^L g'' \mathrm{d}^L g \mathrm{d}\mu(g') = \int_G |f(g)|^2 \mathrm{d}^L g.
\end{aligned}$$

$\square$

Corollary 5.6. *The map*

$$\mathcal{F} := \bigoplus_{(\epsilon)\in(\mathbb{Z}_2)^N} \mathcal{F}_{\mathcal{O}_{(\epsilon)}} : L^2(G, \mathrm{d}^L g) \to \bigoplus_{(\epsilon)} L^2(\mathcal{O}_{(\epsilon)}, \mu),$$

defined by $\mathcal{F}(f) := \bigoplus_{(\epsilon)}(\mathcal{F}_{\mathcal{O}_{(\epsilon)}} f)$ *realizes an isometric isomorphism.*

Proof. From Proposition 5.3, we deduce that $\forall(\epsilon) \in (\mathbb{Z}_2)^N$, $\mathcal{F}_{\mathcal{O}_{(\epsilon)}}\mathcal{F}^*_{\mathcal{O}_{(\epsilon)}} = \mathbb{1}$. And the Parseval–Plancherel means that $\sum_{(\epsilon)} \mathcal{F}^*_{\mathcal{O}_{(\epsilon)}}\mathcal{F}_{\mathcal{O}_{(\epsilon)}} = \mathbb{1}$. Moreover, we can show that for all $(\epsilon), (\epsilon') \in (\mathbb{Z}_2)^N$, with $(\epsilon) \neq (\epsilon')$, $\mathcal{F}_{\mathcal{O}_{(\epsilon)}}\mathcal{F}^*_{\mathcal{O}_{(\epsilon')}} = 0$. Indeed, if $k \leq N$ is such that $\epsilon_k \neq \epsilon'_k$, then the computation of $\int_G \overline{\tilde{\mathcal{E}}_g^{\mathcal{O}_{(\epsilon')}}(g')}\tilde{\mathcal{E}}_g^{\mathcal{O}_{(\epsilon)}}(g'')\mathrm{d}^L g$ corresponds to having a factor $e^{\frac{i\epsilon_k}{\theta}e^{a_2}(e^{-2a''_2}+e^{-2a'_2})\ell_2}$ in the proof of Proposition 5.3. Integration over ℓ_2 makes this expression vanish.

For each $(\epsilon) \in (\mathbb{Z}_2)^N$ (i.e., for each $(\epsilon) = (\epsilon_1, \cdots \epsilon_N)$ with $\epsilon_j = \pm 1$), we will consider a function $f_{(\epsilon)} \in L^2(\mathcal{O}_{(\epsilon)}, \mu)$. We denote by $\bigoplus_{(\epsilon)} f_{(\epsilon)}$ the 2^N-uplet of these functions on the different orbits. By using the three properties above and the fact that

$$\mathcal{F}^*(\bigoplus_{(\epsilon)} f_{(\epsilon)}) = \sum_{(\epsilon)} \mathcal{F}^*_{\mathcal{O}_{(\epsilon)}}(f_{(\epsilon)}),$$

we obtain that

$$\mathcal{F}^*\mathcal{F}(f) = \sum_{(\epsilon)} \mathcal{F}^*_{\mathcal{O}_{(\epsilon)}}\mathcal{F}_{\mathcal{O}_{(\epsilon)}}(f) = f, \text{ and}$$

$$\mathcal{F}\mathcal{F}^*(\bigoplus_{(\epsilon)} f_{(\epsilon)}) = \bigoplus_{(\epsilon)}(\mathcal{F}_{\mathcal{O}_{(\epsilon)}}\mathcal{F}^*_{\mathcal{O}_{(\epsilon)}} f_{(\epsilon)}) = \bigoplus_{(\epsilon)} f_{(\epsilon)}.$$

□

5.3 *Fourier Transformation and Schwartz Spaces*

Given such an adapted Fourier transformation, we can wonder wether the Schwartz space $\mathcal{S}(G)$ defined in [7] (see Sect. 4.1) is stable by this transformation, as it is true in the flat case: the usual transformation stabilizes the usual Schwartz space on $\mathbb{R}^n$. However, the answer appears to be wrong here. Let us focus on the case of the elementary normal j-group $\mathbb{S}$. The Schwartz space $\mathcal{S}(\mathbb{S})$ of Definition 4.1 corresponds to the usual Schwartz space in the coordinates $(r = \sinh(2a), x, \ell)$. These coordinates are adapted to the phase of the kernel of the star-product (1.12). For the star-exponential of $\mathbb{S}$ given in (1.20), we need also to consider the coordinates corresponding to the moment maps (1.3):

$$\mu : \mathbb{S} \to \mathbb{R}^*_+ \times \mathbb{R}^{2n+1}, \quad (a, x, \ell) \mapsto (e^{-2a}, e^{-a}x, \ell).$$

We will denote the new variables $(s, z, \ell) = \mu(a, x, \ell)$.

Definition 5.7. We define the *moment-Schwartz space* of $\mathbb{S}$ to be

$$\begin{aligned}\mathcal{S}_\lambda(\mathbb{S}) =& \{f \in C^\infty(\mathbb{S}) \quad (\mu^{-1})^* f \in \mathcal{S}(\mathbb{R}^*_+ \times \mathbb{R}^{2n+1}) \\ & \text{and } s^{-\frac{n+1}{2}}(\mu^{-1})^* f(s, z, \ell) \text{ is smooth in } s = 0\}.\end{aligned}$$

The space $\mathcal{S}_\lambda(\mathbb{S})$ corresponds to the usual Schwartz space in the coordinates (s, z, ℓ) (for $s > 0$) with some boundary regularity condition in $s = 0$. As before, we identify the group $\mathbb{S}$ with the coadjoint orbit $\mathcal{O}_\epsilon$ ($\epsilon = \pm 1$).

Theorem 5.8. *The adapted Fourier transformation restricted to the Schwartz space induces an isomorphism*

$$\mathcal{F} : \mathcal{S}(\mathbb{S}) \to \mathcal{S}_\lambda(\mathcal{O}_+) \oplus \mathcal{S}_\lambda(\mathcal{O}_-).$$

Proof. Let $f \in \mathcal{S}(\mathbb{S})$. The Fourier transform reads as

$$\mathcal{F}_{\mathcal{O}_\epsilon}(f)(s,z,\ell)=\frac{1}{(\pi\theta)^{n+1}}\int \mathrm{d}r'\mathrm{d}x'\mathrm{d}\ell'\frac{f(r',x',\ell')}{(1+r'^2)^{\frac14}}(\sqrt{1+r'^2}+r')^{\frac{n+1}{2}}s^{\frac{n+1}{2}}c(r')^n$$
$$e^{\frac{i\epsilon}{\theta}\left(2r'\ell+(\sqrt{1+r'^2}+r')s\ell'+\frac12(\sqrt{1+r'^2}+r'+1)\omega_0(x',z)\right)}.$$

Here we use the function $c(r')$ defined in (1.27), the coordinates $s = e^{-2a}$, $z = e^{-a}x$, $r' = \sinh(a')$ and the fact that f is Schwartz in the variable $\sinh(a)$ if and only if it is in the variable $\sinh(2a)$. We denote again by f the function in the new coordinates by a slight abuse of language. We have to check that $h(s, z, \ell) = s^{-\frac{n+1}{2}}\mathcal{F}_{\mathcal{O}_\epsilon}(f)(s, z, \ell)$ is Schwartz in (s, z, ℓ), i.e., we want to estimate expressions of the type

$$\int \mathrm{d}s\mathrm{d}z\mathrm{d}\ell \; |(1+s^2)^{k_1}(1+z^2)^{p_1}(1+\ell^2)^{q_1}\partial_s^{k_2}\partial_z^{p_2}\partial_\ell^{q_2}\; h(s,z,\ell)|.$$

Let us provide an analysis in terms of oscillatory integrals.

- Polynomial in ℓ: controlled by an adapted power of the following operator (invariant acting on the phase) $\frac{1}{1+\ell^2}\Big(1 - \frac{\theta^2}{4}\Big(\partial_{r'} - \frac{\ell'}{\sqrt{1+r'^2}}\partial_{\ell'} + \frac{(\sqrt{1+r'^2}+r')}{\sqrt{1+r'^2}(\sqrt{1+r'^2}+r'+1)}x'\partial_{x'}\Big)^2\Big)$ (see Sect. 4.1). Indeed, powers and derivatives in the variables r', x', ℓ' are controlled by the Schwartz function f inside the integral.
- Polynomial in z: controlled by an adapted power of the (invariant) operator $\frac{1}{1+z^2}(1 - \frac{4\theta^2}{(\sqrt{1+r'^2}+r'+1)^2}\partial_{x'}^2)$.
- Polynomial in s: controlled by an adapted power of the (invariant) operator $\frac{1}{1+s^2}(1 - \frac{\theta^2}{(\sqrt{1+r'^2}+r')^2}\partial_{\ell'}^2)$. Note that the function $\frac{1}{(\sqrt{1+r'^2}+r')^2}$ is estimated by a polynomial in r' for $r' \to \pm\infty$, as its derivatives.
- Derivations in s: produce terms like powers of $(\sqrt{1+r'^2}+r')\ell'$ which are controlled.
- Derivations in z: produce terms like powers of $(\sqrt{1+r'^2}+r'+1)x'$ which are controlled.
- Derivations in ℓ: produce terms like powers of r'.

This shows that h is Schwartz in (s, z, ℓ), so $\mathcal{F}(f) \in \mathcal{S}_\lambda(\mathbb{S})$.

Conversely, let $f_\epsilon \in \mathcal{S}_\lambda(\mathcal{O}_\epsilon)$. Due to Theorem 5.5, we can write the inverse of the Fourier transform as:

$$\begin{aligned}
&\mathcal{F}^{-1}(f_+, f_-)(r, x, \ell) \\
&= \sum_{\epsilon=\pm 1} \frac{1}{2(\pi\theta)^{n+1}} \int \mathrm{d}s'\mathrm{d}z'\mathrm{d}\ell' \frac{f_\epsilon(s', z', \ell')}{s'^{\frac{n+1}{2}}} (\sqrt{1+r^2}+r)^{\frac{n+1}{2}} \sqrt{1+r^2} c(r)^n \\
&\times e^{-\frac{i\epsilon}{\theta}\left(2r\ell' + (\sqrt{1+r^2}+r)s'\ell + \frac{1}{2}(\sqrt{1+r^2}+r+1)\omega_0(x,z')\right)}.
\end{aligned}$$

Here we use now the coordinates $s' = e^{-2a'}$, $z' = e^{-a'}x'$, $r = \sinh(a)$. We want to estimate expressions of the type

$$\int \mathrm{d}r\mathrm{d}x\mathrm{d}\ell \, |(1+r^2)^{k_1}(1+x^2)^{p_1}(1+\ell^2)^{q_1} \partial_r^{k_2} \partial_x^{p_2} \partial_\ell^{q_2} \, \mathcal{F}^{-1}(f_+, f_-)(r, x, \ell)|.$$

Let us provide also an analysis in terms of oscillatory integrals.

- Polynomial in ℓ: controlled by an adapted power of the following operator (invariant acting on the phase) $\frac{1}{1+\ell^2}(1 - \frac{\theta^2}{(\sqrt{1+r^2}+r)^2}\partial^2_{s'})$. As before, powers and derivatives in the variables s', z', ℓ' are controlled by the Schwartz function f inside the integral. Note that $\frac{f_\epsilon(s',z',\ell')}{s'^{\frac{n+1}{2}}}$ is smooth in $s = 0$ so that the integral is well-defined for $s \in \mathbb{R}_+$.
- Polynomial in x: controlled by an adapted power of the (invariant) operator:

$$\frac{1}{1+x^2}(1 - \frac{4\theta^2}{(\sqrt{1+r^2}+r+1)^2}\partial^2_{z'}) \ .$$

- Polynomial in r: controlled by an adapted power of the (invariant) operator $\frac{1}{1+r^2}(1 - \frac{\theta^2}{4}\partial^2_{\ell'})$.
- Derivations in r: produce terms like powers of $(\sqrt{1+r^2} + r)$, $\frac{1}{\sqrt{1+r^2}}$, $r, c'(r), \ell', \frac{(\sqrt{1+r^2}+r')}{\sqrt{1+r^2}} s'\ell, \omega_0(x, z'), \ldots$ which are controlled (see just above).
- Derivations in x: produce terms like powers of $(\sqrt{1+r^2} + r + 1)z'$ which are controlled.
- Derivations in ℓ: produce terms like powers of $(\sqrt{1+r^2} + r)s'$ which are also controlled.

This shows that $\mathcal{F}^{-1}(f_+, f_-) \in \mathcal{S}(\mathbb{S})$. $\square$

5.4 Application to Noncommutative Baumslag–Solitar Tori

We consider the decomposition of G into elementary normal j-groups of Sect. 2.2

$$G = \big(\dots(\mathbb{S}_1 \ltimes_{\rho_1} \mathbb{S}_2) \ltimes_{\rho_2} \dots\big) \ltimes_{\rho_{N-1}} \mathbb{S}_N$$

and the associated basis

$$\mathfrak{B} := \Big(H_1, (f_1^{(i)})_{1\le i\le 2n_1}, E_1, \dots, H_N, (f_N^{(i)})_{1\le i\le 2n_N}, E_N\Big)$$

of its Lie algebra $\mathfrak{g}$, where $(f_j^{(i)})_{1\le i\le 2n_j}$ is a canonical basis of the symplectic space V_j contained in $\mathbb{S}_j$. We note G_{BS} the subgroup of G generated by $\{e^{\theta X},\ X \in \mathfrak{B}\}$ and call it the *Baumslag–Solitar subgroup* of G. Indeed, in the case of the "$ax+b$" group (two-dimensional elementary normal j-group), and if $e^{2\theta} \in \mathbb{N}$, this subgroup corresponds to the Baumslag–Solitar group [3]:

$$\mathrm{BS}(1,m) := \langle\ e_1, e_2 \quad | \quad e_1 e_2 (e_1)^{-1} = (e_2)^m\ \rangle.$$

We have seen before that the star-exponential associated to a coadjoint orbit $\mathcal{O}$ is a group morphism $\mathcal{E} : G \to \mathcal{M}_{\star_\theta}(\mathcal{O}) \simeq \mathcal{M}_{\star_\theta}(G)$. Composed with the quantization map Ω, it coincides with the unitary representation $U = \Omega \circ \mathcal{E}$. So, if we now take the subalgebra of $\mathcal{M}_{\star_\theta}(G)$ generated by the star-exponential of G_{BS}, i.e., by elements $\{\mathcal{E}_{e^{\theta X}},\ X \in \mathfrak{B}\}$, then it is closed for the complex conjugation and it can be completed into a C*-algebra $\mathbf{A}_G$ with norm $\|\Omega(\cdot)\|_{\mathcal{L}(\mathcal{H})}$. This C*-algebra is canonically associated to the group G. Moreover, if $\theta \to 0$, this C*-algebra is commutative and corresponds thus to a certain torus.

Definition 5.9. Let G be a normal j-group. We define the *noncommutative Baumslag–Solitar torus* of G to be the C*-algebra $\mathbf{A}_G$ constructed above.

It turns out that the relation between the generators $\mathcal{E}_{e^{\theta X}}$ ($X \in \mathfrak{B}$) of $\mathbf{A}_G$ can be computed explicitly by using the BCH formula of Proposition 3.10. Let us see some examples.

Example 5.10. In the elementary group case $G = \mathbb{S}$, let

$$U(a,x,\ell) := \mathcal{E}_{(\theta,0,0)}(a,x,\ell) = \sqrt{\cosh(\theta)}\cosh(\frac{\theta}{2})^n e^{\frac{2i}{\theta}\sinh(\theta)\ell},$$

$$V(a,x,\ell) := \mathcal{E}_{(0,0,\theta)}(a,x,\ell) = e^{ie^{-2a}},$$

$$W_i(a,x,\ell) := \mathcal{E}_{(0,\theta e_i,0)}(a,x,\ell) = e^{ie^{-a}\omega_0(e_i,x)},$$

where (e_i) is a canonical basis of the symplectic space (V,ω_0) of dimension $2n$ (i.e. $\omega_0(e_i,e_{i+n}) = 1$ if $i \le n$). Then, we can compute relations like

$$U \star_\theta V = V^{e^{2\theta}} \star_\theta U.$$

by using the BCH property of the star-exponential (see Proposition 3.10). We obtain (by omitting the notation $\star$):

$$UV = V^{e^{2\theta}}U \qquad (\text{ and } UV^{\beta} = V^{\beta e^{2\theta}}U),$$
$$UW_i = W_i^{e^{\theta}}U, \qquad W_i W_{i+n} = V^{\theta} W_{i+n} W_i$$

where the other commutation relations are trivial. Note that these relations become trivial at the commutative limit $\theta \to 0$. In the two-dimensional case, where $\mathbb{S}$ is the "ax+b group", the relation $UV = V^{e^{2\theta}}U$ has already been obtained in another way in [14].

Example 5.11. Let us consider the Siegel domain of dimension 6 (see Example 2.12 for definitions and notations). As before, we can define the following generators:

$$U(g) := \mathcal{E}_{(0,0,\theta,0,0,0)}(g) = \sqrt{\cosh(\theta)}\cosh(\frac{\theta}{2})e^{\frac{2i}{\theta}\sinh(\theta)\ell_2},$$
$$V(g) := \mathcal{E}_{(0,0,0,0,0,\theta)}(g) = e^{ie^{-2a_2}},$$
$$W_1(g) := \mathcal{E}_{(0,0,0,\theta,0,0)}(g) = e^{ie^{-a_2}w_2},$$
$$W_2(g) := \mathcal{E}_{(0,0,0,0,\theta,0)}(g) = e^{-ie^{-a_2}v_2},$$
$$R(g) := \mathcal{E}_{(\theta,0,0,0,0,0)}(g) = \frac{e^{\frac{\theta}{2}}\sqrt{\cosh(\theta)}}{\cosh(\frac{\theta}{2})}e^{\frac{2i}{\theta}(\sinh(\theta)\ell_1+\tanh(\frac{\theta}{2})v_2w_2)},$$
$$S(g) := \mathcal{E}_{(0,\theta,0,0,0,0)}(g) = e^{i(e^{-2a_1}+\frac{1}{2}v_2^2)}.$$

We obtain the relationship:

$$UV = V^{e^{2\theta}}U, \quad UW_1 = (W_1)^{e^{\theta}}U, \quad UW_2 = (W_2)^{e^{\theta}}U, \quad W_1W_2 = V^{\theta}W_2W_1,$$
$$RS = S^{e^{2\theta}}R, \quad RW_1 = (W_1)^{e^{\theta}}R, \quad RW_2 = (W_1)^{e^{-\theta}}R, \quad SW_1 = V^{\frac{\theta^2}{2}}(W_2)^{\theta}W_1S,$$

where the other commutation relations are trivial.

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