A Risk-Averse Differential Game Approach to Multi-agent Tracking and Synchronization with Stochastic Objects and Command Generators

Khanh Pham and Meir Pachter

Abstract This chapter presents the formulation of a class of distributed stochastic multi-agent systems where local interconnections among cautious and defensive decision makers and/or trackers are supported by connectivity graphs. Associated with autonomous decision makers and/or trackers are finite-horizon performance measures for conflict-free coordination and cohesive object and/or command tracking. The current analysis is limited to the class of distributed linear stochastic systems and measurement subsystems. It is shown that optimal rules for ordering uncertain prospects are feasible for all self-directed decision makers and/or trackers with output-feedback Nash decision making and risk-averse utility functions.

Keywords Distributed Control • Performance-Measure Statistics • Downside Performance Risk Measure • Connectivity Graphs • Person-by-Person Decision and Control

1 Introduction

One of the best ways to understand the growing interest in multi-agent tracking and distributed control systems is to review the history of these old and new engineering problems. Examples include multi-agent architectures for tracking and estimation [7, 10, 15] and control design with pre-specified information structures and control under communication constraints [8]. Yet relatively little work has focused on understanding quantitatively the downside risk measures in multi-agent systems for various tasks in terms of performance robustness and risk aversion.

In noncooperative stochastic games and distributed controls, there are more than two capable decision makers who optimize different goals and utilities. Each

K. Pham (✉)
Air Force Research Laboratory, Kirtland A.F.B., NM 87117, Rome, NewYork
e-mail: AFRL.RVSV@kirtland.af.mil

M. Pachter
Air Force Institute of Technology, Wright-Patterson A.F.B., OH 45433, USA

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decision maker wishes to influence to his/her advantage a shared interaction process by exerting his/her control decisions. To the best knowledge of the authors, most studies, e.g., [2] and [4], have mainly concentrated on the selection of open- and/or closed-loop Nash strategy equilibria in accordance of expected utilities under the structural constraints of linear system dynamics, quadratic cost functionals, and additive independent white Gaussian noises corrupting the system dynamics and measurements. Very little work, if any, has been published on the subject of higher-order assessment of performance uncertainty and risks beyond expected performance.

For this reason attention in the research investigation that follows is directed primarily toward a linear-quadratic class of noncooperative stochastic games and/or distributed controls, which in turn has linear system dynamics, quadratic rewards and/or costs, and independent white zero-mean Gaussian noises additively corrupting the system dynamics and output measurements. Notice that, under these conditions, the quadratic rewards or costs are random variables with the generalized chi-squared probability distributions. If a measure of uncertainty such as the variance of the possible rewards or costs was used in addition to the expected reward or costs, the decision makers should be able to correctly order preferences for alternatives. This claim seems plausible, but it is not always correct. Various investigations have indicated that any evaluation scheme based on just the expected reward or cost and reward/cost variance would necessarily imply indifference between some courses of action; therefore, no criterion based solely on the two attributes of means and variances can correctly represent their preferences. See [14] and [9] for further details.

The present research contributions include significant extensions of the existing results [11] toward some completely unexplored areas as such: i) the design of distributed filtering via private observations for self-directed decision makers and/or autonomous controllers with distributed noisy information structures about the uncertain interaction process; ii) an efficiently computational procedure for all the mathematical statistics associated with the generalized chi-squared rewards/costs when respective mean-risk aware utilities are formed; and iii) the synthesis of distributed risk-sensitive decision policies with output feedback for distributed noncooperative solutions of Nash type that now guarantee performance robustness with certainty much stronger than ensemble averaging measures of performance.

The remainder of the chapter is organized as follows: In Sect. 2, the setting which involved necessary background and terminologies associated with a class of distributed multi-agent tracking and synchronization is provided. The purpose of Sect. 3 is to continue the discussion of the research development in using preferences of risk, dynamic game decision optimization, and distributed decision making with local output-feedback measurements tailored toward the worst-case scenarios. The feasibility of person-by-person risk-averse strategies supported by distributed Kalman-like estimators is subsequently put forward in Sect. 4. Some final remarks are given in Sect. 5.

2 The Setting

In this section, some preliminaries are in order. For instance, a fixed probability space with filtration is denoted by $(\Omega, \mathbb{F}, \{\mathbb{F}_{t_0,t} : t \in [t_0, t_f], \mathbb{P}\})$ where all filtrations are right continuous and complete. In addition, $L^2_{\mathbb{F}_{t_f}}([t_0, t_f]; \mathbb{R}^n)$ denotes the space of $\mathbb{F}_{t_f}$-adapted random processes $\{z(t) : t \in [t_0, t_f]\}$ such that $E\{\int_{[t_0,t_f]} ||z(t)||^2_{\mathbb{R}^n} dt\} < \infty$ and $\mathbb{F}_{t_f} \triangleq \{\mathbb{F}_{t_0,t} : t \in [t_0, t_f]\}$.

2.1 Distributed Multi-agent Tracking and Synchronization

As for a model specification, there is a stochastic object or command generator that evolved in the fixed probability space $(\Omega, \mathbb{F}, \{\mathbb{F}_{t_0,t} : t \in [t_0, t_f], \mathbb{P}\})$ and is subject to the following stochastic dynamical decision system

$$dx_o(t) = Ax_o(t)dt + G_o dw_o(t)\,, \quad x_o(t_0) \tag{1}$$

where the initial state $x_o(t_0) = x_o^0$, the state space is in $\mathbb{R}^{n_0}$, and the exogenous state noise $\{w_o(t) : t \in [t_0, t_f]\}$ is an $\mathbb{R}^{n_o}$-valued stationary Wiener process adapted to $\mathbb{F}_{t_f}$, independent of $x_o(t_0)$ and having the correlation of independent increments

$$E\left\{[w_o(\tau_1) - w_o(\tau_2)][w_o(\tau_1) - w_o(\tau_2)]^T\right\} = W_o|\tau_1 - \tau_2|\,, \quad \forall\ \tau_1, \tau_2 \in [t_0, t_f].$$

Moreover, there are N identical decision makers and/or trackers which are also described by

$$dx_{ti}(t) = (Ax_{ti}(t) + Bu_{ti}(t))dt + Gdw_{ti}(t)\,, \quad x_{ti}(t_0)\,. \tag{2}$$

Of note, each stochastic dynamical decision system i and $i \in \overline{N} \triangleq \{1, \ldots, N\}$ has an initial state $x_{ti}(t_0) = x_{ti}^0$, state space $\mathbb{R}^{n_{ti}}$, an action space $\mathbb{A}^i \subset \mathbb{R}^{m_i}$, and an exogenous state noise space $\{w_{ti}(t) : t \in [t_0, t_f]\}$ defined by an $\mathbb{R}^{p_{ti}}$-valued stationary Wiener process adapted to $\mathbb{F}_{t_f}$, independent of $x_{ti}(t_0)$ and having the correlation of independent increments

$$E\left\{[w_{ti}(\tau_1) - w_{ti}(\tau_2)][w_{ti}(\tau_1) - w_{ti}(\tau_2)]^T\right\} = W_{ti}|\tau_1 - \tau_2|\,, \quad \forall\ \tau_1, \tau_2 \in [t_0, t_f].$$

The decentralized partial information structure available to decision maker i or u_{ti} is generated by noisy relative observation

$$dy_{ti}(t) = C(x_{ti}(t) - x_o(t))dt + Hdv_{ti}(t)\,, \quad i \in \overline{N} \tag{3}$$

where the exogenous measurement noise is an $\mathbb{R}^{q_{ti}}$-valued stationary Wiener process adapted to $\mathbb{F}_{t_f}$ and independent of $\{w_{ti}(t) : t \in [t_0, t_f]\}$ with the correlation of independent increments

$$E\left\{[v_{ti}(\tau_1) - v_{ti}(\tau_2)][v_{ti}(\tau_1) - v_{ti}(\tau_2)]^T\right\} = V_{ti}|\tau_1 - \tau_2|\,, \quad \forall\ \tau_1, \tau_2 \in [t_0, t_f].$$

A concern has grown in the relative states, e.g., $x_i \triangleq x_{ti} - x_o$ that the evolutions are then described by

$$dx_i(t) = (Ax_i(t) + Bu_{ti}(t))dt + Gdw_{ti}(t) - G_o dw_o(t)\,, \quad x_i(t_0) = x_i^0 \tag{4}$$

with the local noisy observations

$$dy_{ti}(t) = Cx_i(t)dt + Hdv_{ti}(t)\,, \quad i \in \overline{N}\,. \tag{5}$$

With the advent of connectivity graphs widely used in cooperative control and formation among unmanned systems [5] and [16], the role of formation graphs supporting networks of local decision makers or trackers herein has also been significant. To this end, a vertex of the graph corresponds to a decision maker or tracker and the edges of the graph convey the dependence of the interconnections. For instance, a directed graph $\mathcal{G}^i = (\mathcal{V}^i, \mathcal{E}^i)$ associated with decision maker or tracker i consists of a set of vertices $\mathcal{V}^i \triangleq \{v_{i_1}, \ldots, v_{i_{N_i}}\}$, indexed by local decision makers or trackers in the N_i-neighborhood and a set of edges $\mathcal{E}^i \triangleq \{(v_{i_1}, v_{i_2}) \in \mathcal{V}^i \times \mathcal{V}^i\}$, containing ordered pairs of distinct vertices.

As part of the effort to approach distributed multi-agent tracking and synchronization, a local neighborhood of N_i immediate decision makers (or trackers) associated with decision maker (or tracker) i and supported by an appropriate directed graph includes two key elements. First, the augmented vectors are sought to be

$$z_i \triangleq \begin{bmatrix} x_{i_1} \\ \vdots \\ x_{i_{N_i}} \end{bmatrix}, \quad w_i \triangleq \begin{bmatrix} w_{ti_1} \\ \vdots \\ w_{ti_{N_i}} \\ w_o^T \end{bmatrix}, \quad v_i \triangleq \begin{bmatrix} v_{ti_1} \\ \vdots \\ v_{ti_{N_i}} \end{bmatrix}, \quad i \in \overline{N}.$$

Second, the Kronecker product of matrices are defined as follows:

$$A_{N_i} \triangleq I_{N_i \times N_i} \otimes A\,, \qquad G_{N_i} \triangleq \left[\, I_{N_i \times N_i} \otimes G \; -1_{N_i} \otimes G_o \,\right]$$
$$B_i \triangleq \left[0 \ldots 0\ 1\ 0 \ldots 0\right]^T \otimes B\,, \qquad C_{N_i} \triangleq I_{N_i \times N_i} \otimes C$$
$$H_{N_i} \triangleq I_{N_i \times N_i} \otimes H$$

where $I_{N_i \times N_i}$ is the $N_i \times N_i$ identity matrix and $1_{N_i} \triangleq \left[1 \ldots 1 \right]^T$ is the column vector of one of size N_i and will result in the distributed stochastic system dynamics with controls and observations from decision maker or tracker i and all of its immediate neighbors $\overline{N}_i$ and $\overline{N}_i \triangleq \{i_1, \ldots, i_{N_i}\}$

$$dz_i(t) = (A_{N_i} z_i(t) + B_i u_{ti}(t) + \sum_{j \in \overline{N}_i} e_{i_j} B_{i_j}(t) u_{tj}(t))dt + G_{N_i} dw_i(t) \tag{6}$$

$$dy_i(t) = C_{N_i} z_i(t)dt + H_{N_i} dv_i(t) \tag{7}$$

where e_{i_j} is the edge weights and $z_i(t_0)$ is the initial system state with the value of $z_i^0 \triangleq \left[x_{ti_1}^T(t_0) \ldots x_{ti_{N_i}}^T(t_0) \right]^T$.

Continuing the practice of private observations, each decision maker or tracker i can presumably observe a noise corrupted version of all best responses $\sum_{j \in \overline{N}_i} e_{i_j} B_{i_j}(t) u_{tj}(t)$ from the immediate neighbors

$$du_{-ti}(t) \triangleq u_{-ti}(t)dt = \sum_{j \in \overline{N}_i} e_{i_j} B_{i_j}(t) u_{tj}(t)dt + d\eta_i(t) \,. \tag{8}$$

Notice that decision makers or trackers i operate within their own noisy environments modeled by the uncorrelated p_i-, n_i-, and q_i-dimensional stationary Wiener processes adapted for $[t_0, t_f]$

$$E\left\{[w_i(\tau_1) - w_i(\tau_2)][w_i(\tau_1) - w_i(\tau_2)]^T\right\} = W_i|\tau_1 - \tau_2|$$
$$E\left\{[\eta_i(\tau_1) - \eta_i(\tau_2)][\eta_i(\tau_1) - \eta_i(\tau_2)]^T\right\} = M_i|\tau_1 - \tau_2|$$
$$E\left\{[v_i(\tau_1) - v_i(\tau_2)][v_i(\tau_1) - v_i(\tau_2)]^T\right\} = V_i|\tau_1 - \tau_2|$$

whose a priori second-order statistics $W_i \triangleq diag(W_{ti_1}, \ldots, W_{ti_{N_i}}, W_o) > 0$, $M_i > 0$, and $V_i \triangleq diag(V_{ti_1}, \ldots, V_{ti_{N_i}}) > 0$ for $i \in \overline{N}$ are also assumed known.

For decentralized filtering, each decision maker or tracker i has σ-algebras

$$\mathbb{F}_{t_0,t}^i \triangleq \sigma\{z_i(t_0), w_i(s), v_i(s) : t_0 \le s \le t\} \tag{9}$$

$$\mathcal{G}_{t_0,t}^{y^{i,u}} \triangleq \sigma\{y_i(s) : t_0 \le s \le t\}\,, \quad t \in [t_0, t_f], \quad i \in \overline{N} \tag{10}$$

and the minimum σ-algebras generated by (9)–(10) are therefore given by

$$\mathbb{F}_{t_0,t} \triangleq \vee_{i=1}^N \mathbb{F}_{t_0,t}^i \tag{11}$$

$$\mathcal{G}_{t_0,t}^{y^u} \triangleq \vee_{i=1}^N \mathcal{G}_{t_0,t}^{y^{i,u}}\,. \tag{12}$$

2.2 Distributed Decision/Control and Filtering

As a critical element of the effort to move toward the distributed decision strategies, $\mathcal{G}_{t_f}^{y^{i,u}} \triangleq \{\mathcal{G}_{t_0,t}^{y^{i,u}} : t \in [t_0, t_f]\} \subset \{\mathbb{F}_{t_0,t} : t \in [t_0, t_f]\}$ is denoted for the information available to decision maker and/or tracker i and $i \in \overline{N}$. The admissible set of distributed feedback strategies for decision maker and/or tracker i is defined by

$$\mathbb{U}^{y^{i,u}}[t_0, t_f] \triangleq \{u_{ti} \in L^2_{\mathcal{G}_{t_f}^{y^{i,u}}}([t_0, t_f], \mathbb{R}^{m_i}) : u_{ti}^t \in \mathbb{A}^i \subset \mathbb{R}^{m_i}, \text{almost everywhere}$$
$$t \in [t_0, t_f], \mathbb{P} - \text{almost surely}\}, \quad i \in \overline{N} \tag{13}$$

where $\mathbb{U}^{y^{i,u}}[t_0, t_f]$ is a closed convex subset of $L^2_{\mathbb{F}_{t_f}}([t_0, t_f]; \mathbb{R}^{m_i})$ for $i \in \overline{N}$.

At this point, each of the N distributed filters whose outputs are the state estimates $\hat{z}_i(t) \triangleq E\{z_i(t)|\mathcal{G}_{t_0,t}^{y^u}\}$ of (6) has the form

$$\begin{aligned} d\hat{z}_i(t) &= (A_{N_i}(t)\hat{z}_i(t) + B_i(t)u_{ti}(t) + u_{-ti}(t))dt \\ &\quad + L_i(t)(dy_i(t) - C_{N_i}\hat{z}_i(t)dt), \quad \hat{z}_i(t_0) = z_i^0 \end{aligned} \tag{14}$$

where the local filter gain $L_i(t)$ is given by

$$L_i(t) = \Sigma_i(t)C_{N_i}^T(H_{N_i}V_iH_{N_i})^{-1} \tag{15}$$

and the estimate error covariance $\Sigma_i(t) \triangleq E\{[z_i(t) - \hat{z}_i(t)][z_i(t) - \hat{z}_i(t)]^T|\mathcal{G}_{t_0,t}^{y^u}\}$

$$\begin{aligned} \frac{d}{dt}\Sigma_i(t) &= A_{N_i}\Sigma_i(t) + \Sigma_i(t)A_{N_i}^T + G_{N_i}W_iG_{N_i}^T + M_i \\ &\quad - \Sigma_i(t)C_{N_i}^T(H_{N_i}V_iH_{N_i})^{-1}C_{N_i}\Sigma_i(t), \quad \Sigma_i(t_0) = 0. \end{aligned} \tag{16}$$

In the background is the substitution of (6), (8), and (14) in a setting shaped by the estimate errors, e.g., $\tilde{z}_i(t) \triangleq z_i(t) - \hat{z}_i(t)$. Thus, it can be shown that

$$\begin{aligned} d\tilde{z}_i(t) &= (A_{N_i} - L_i(t)C_{N_i})\tilde{z}_i(t)dt \\ &\quad + G_{N_i}dw_i(t) - L_i(t)H_{N_i}dv_i(t) - d\eta_i(t), \quad \tilde{z}_i(t_0) = 0. \end{aligned} \tag{17}$$

2.3 Person-by-Person Performance Measure

Recall that decision maker or tracker i is assumed to act purely on the basis of his own information, e.g.,

$$\mathcal{G}_{t_f}^{y^{i,u}} \triangleq \{\mathcal{G}_{t_0,t}^{y^{i,u}} : t \in [t_0, t_f]\} \subset \{\mathbb{F}_{t_0,t} : t \in [t_0, t_f]\}.$$

And the set of admissible decentralized feedback policies $\mathbb{U}^{y^{i,u}}[t_0, t_f]$ is a closed convex subset of $L^2_{\mathbb{F}_{t_f}}([t_0, t_f]; \mathbb{R}^{m_i})$ for $i \in \overline{N}$. The objective of distributed multi-agent tracking and synchronization is then to regulate the dynamical states of all the decision makers or trackers to those of stochastic command generators or objects while being subject to transient trade-offs between the state regulatory and effectiveness of decision policies and/or control inputs.

Associated with each admissible 2-tuple $(u_{ti}(\cdot), u_{-ti}(\cdot))$ is the person-by-person performance measure with the generalized chi-squared type for each decision maker and/or tracker i defined as

$$J_i(u_{ti}, u_{-ti}) = g_i(t_f, z_i(t_f)) + \int_{t_0}^{t_f} C_i(\tau, z_i(\tau), u_{ti}(\tau), u_{-ti}(\tau)) d\tau\,, \quad i \in \overline{N} \quad (18)$$

where the cohesive tracking and regulation criteria are given by

$$g_i(t_f, z_i(t_f)) = \sum_{(v_{i_r}, v_{i_s}) \in \mathcal{E}^i} w_{i_r i_s} ||x_{ti_r}(t_f) - x_{ti_s}(t_f)||^2 + ||x_{ti}(t_f)||^2_{S_{if}} = ||z_i(t_f)||^2_{Q_{if}}$$

and

$$\begin{aligned} C_i(\tau, z_i(\tau), u_{ti}(\tau), u_{-ti}(\tau)) = & \sum_{(v_{i_r}, v_{i_s}) \in \mathcal{E}^i} \nu_{i_r i_s} ||x_{ti_r}(\tau) - x_{ti_s}(\tau)||^2 \\ & + ||x_{ti}(\tau)||^2_{S_i} + ||u_{ti}(\tau)||^2_{R_i} = ||z_i(\tau)||^2_{Q_i} + ||u_{ti}(\tau)||^2_{R_i} \end{aligned}$$

provided that the design parameters S_{if}, S_i, and R_i are positive semidefinite with R_i invertible and

$$\begin{aligned} Q_{if} &= \hat{D}_i \hat{W}_{if} \hat{D}_i^T + diag(0_{n_{ti} \times n_{ti}}, \ldots, 0_{n_{ti} \times n_{ti}}, S_{if}, 0_{n_{ti} \times n_{ti}}, \ldots, 0_{n_{ti} \times n_{ti}}) \\ Q_i &= \hat{D}_i \hat{W}_i \hat{D}_i^T + diag(0_{n_{ti} \times n_{ti}}, \ldots, 0_{n_{ti} \times n_{ti}}, S_i, 0_{n_{ti} \times n_{ti}}, \ldots, 0_{n_{ti} \times n_{ti}}) \\ \hat{D}_i &= D_i \otimes I_{n_{ti} \times n_{ti}}; \quad \hat{W}_i = W_i \otimes I_{n_{ti} \times n_{ti}}; \quad \hat{W}_{if} = W_{if} \otimes I_{n_{ti} \times n_{ti}} \\ W_{if} &= diag(w_{i_r i_s}) \text{ of dimension } |\mathcal{E}^i|; \quad W_i = diag(\nu_{i_r i_s}) \text{ of dimension } |\mathcal{E}^i| \\ D_i &= \text{incidence matrix of the directed graph } \mathcal{G}^i(\mathcal{V}^i, \mathcal{E}^i) \text{ withsize } N_i \times |\mathcal{E}^i|\,. \end{aligned}$$

2.4 Person-by-Person Decision and/or Control Policies

The realization of admissible feedback policies is discussed next. In the case of incomplete information, an admissible feedback policy u_{ti} for a local best response to relevant immediate decision makers or trackers u_{-ti} must be of the form, for some $\eth^i(\cdot,\cdot)$,

$$u_{ti}(t) = \eth^i(t, y_i(\tau)), \quad \tau \in [t_0, t], \quad i \in \overline{N}. \tag{19}$$

In general, the conditional density $p^i(z_i(t)|\mathcal{G}^{y^u}_{t_0,t})$, which is the density of $z_i(t)$ conditioned on $\mathcal{G}^{y^u}_{t_0,t}$ (i.e., induced by the observation $\{y_i(\tau) : \tau \in [t_0, t]\}$), represents the sufficient statistics for describing the conditional stochastic effects of future feedback policy u_{ti}. Under the linear-Gaussian assumption the conditional density $p^i(z_i(t)|\mathcal{G}^{y^u}_{t_0,t})$ is parameterized by the locally available state estimate $\hat{z}_i(t)$ and estimate error covariance $\Sigma_i(t)$. In addition, $\Sigma_i(t)$ is independent of feedback policy $u_{ti}(t)$ and observations $\{y_i(\tau) : \tau \in [t_0, t]\}$. Henceforth, to look for an optimal control and/or decision policy $u_{ti}(t)$ of the form (19), it is only required that

$$u_{ti}(t) = \gamma^i(t, \hat{z}_i(t)), \quad t \in [t_0, t_f], \quad i \in \overline{N}.$$

Given the linear-quadratic properties of the distributed multi-agent tracking and synchronization problem governed by (6), (7), and (18), the search for an optimal feedback solution is productively restricted to a linear time-varying feedback policy generated from the locally accessible state $\hat{z}_i(t)$ by

$$u_{ti}(t) = K^i(t)\hat{z}_i(t), \quad t \in [t_0, t_f], \quad i \in \overline{N} \tag{20}$$

with $K^i \in C([t_0, t_f]; \mathbb{R}^{m_i \times n_i})$ an admissible feedback form whose further defining properties will be stated shortly.

For the admissible pair (t_0, z_i^0), the a priori knowledge about neighboring disturbances $u_{-ti}(\cdot)$ and the admissible feedback policy (20), the aggregation of the dynamics (14) and (17) associated with decision maker or tracker i, is described by the controlled stochastic differential equation

$$dz^i(t) = (F^i(t)z^i(t) + E^i(t)u_{-ti}(t))dt + G^i(t)dw^i(t), \quad z^i(t_0) = z_0^i \tag{21}$$

and the performance measure (18) is rewritten as follows:

$$J_i(u_{ti}, u_{-ti}) = (z^i)^T(t_f)N_f^i z^i(t_f) + \int_{t_0}^{t_f} (z^i)^T(\tau)N^i(\tau)z^i(\tau)d\tau \tag{22}$$

where the aggregate dynamical states and system coefficients are given by

$$z^i(t) \triangleq \begin{bmatrix} \hat{z}_i(t) \\ \tilde{z}_i(t) \end{bmatrix}, \quad F^i(t) \triangleq \begin{bmatrix} A_{N_i} + B_i K^i(t) & L_i(t)C_{N_i} \\ 0 & A_{N_i} - L_i(t)C_{N_i} \end{bmatrix}$$

$$G^i(t) \triangleq \begin{bmatrix} 0 & 0 & L_i(t)H_{N_i} \\ G_{N_i} & -I_{n_i\times n_i} & -L_i(t)H_{N_i} \end{bmatrix}, \; E^i(t) \triangleq \begin{bmatrix} I_{n_i\times n_i} \\ 0 \end{bmatrix}, \; W^i \triangleq \begin{bmatrix} W_i & 0 & 0 \\ 0 & M_i & 0 \\ 0 & 0 & V_i \end{bmatrix}$$

$$N^i(t) \triangleq \begin{bmatrix} Q_i + (K^i)^T(t)R_i(t)K^i(t) & Q_i \\ Q_i & Q_i \end{bmatrix}, \qquad N^i_f \triangleq \begin{bmatrix} Q_{if} & Q_{if} \\ Q_{if} & Q_{if} \end{bmatrix},$$

whereas the aggregate Wiener process noise $w^i \triangleq \begin{bmatrix} w_i^T & \eta_i^T & v_i^T \end{bmatrix}^T$ has the correlation of independent increments $E\left\{[w^i(\tau_1) - w^i(\tau_2)][w^i(\tau_1) - w^i(\tau_2)]^T\right\} = W^i|\tau_1 - \tau_2|$ for all $\tau_1, \tau_2 \in [t_0, t_f]$.

2.5 *Person-by-Person Downside Risk Measures*

In the sequel, moving from the background of the generalized chi-squared random performance (22) and its complex behavior, one productive step involved in the discussion of the use of downside risk measures in person-by-person decision and/or control analysis is modeling and management of all the mathematical statistics (also known as semi-invariants) associated with (22). The major target in the downside risk measure debate is the measure of all the higher-order statistics associated with (22) as used in mean-risk optimization. To this end, the results that follow highlight the rather crucial role played by the endeavor of extracting higher-order statistics pertaining to random distributions of (22).

Theorem 1 (Person-by-Person Cumulant-Generating Function). *Let the states $z^i(\cdot)$ of the distributed stochastic dynamics (21) subject to the performance measure (22) be associated with risk-averse decision maker or tracker i. Further, let initial states $z^i(\tau) \equiv z^i_\tau$ and $\tau \in [t_0, t_f]$ and moment-generating functions with risk-sensitive parameter θ^i be defined by*

$$\varphi^i\left(\tau, z^i_\tau, \theta^i\right) \triangleq \varrho^i\left(\tau, \theta^i\right) \exp\left\{(z^i_\tau)^T \Upsilon^i(\tau, \theta^i) z^i_\tau + 2(z^i_\tau)^T \ell^i(\tau, \theta^i)\right\} \tag{23}$$

$$\upsilon^i\left(\tau, \theta^i\right) \triangleq \ln\{\varrho^i\left(\tau, \theta^i\right)\}, \qquad \theta^i \in \mathbb{R}^+ . \tag{24}$$

Then, the cumulant-generating function is quadratic affine

$$\psi^i\left(\tau, z^i_\tau, \theta^i\right) = (z^i_\tau)^T \Upsilon^i(\tau, \theta^i) z^i_\tau + 2(z^i_\tau)^T \ell^i(\tau, \theta^i) + \upsilon^i\left(\tau, \theta^i\right) \tag{25}$$

where the backward-in-time scalar-valued $\upsilon^i\left(\tau, \theta^i\right)$ satisfies

$$\frac{d}{d\tau}\upsilon^i\left(\tau, \theta^i\right) = -\mathrm{Tr}\left\{\Upsilon^i(\tau, \theta^i) G^i(\tau) W^i (G^i)^T(\tau)\right\}, \quad \upsilon^i\left(t_f, \theta^i\right) = 0, \tag{26}$$

whereas the backward-in-time matrix $\Upsilon^i(\tau,\theta^i)$ and vector $\ell^i(\tau,\theta^i)$ solutions are satisfying

$$\frac{d}{d\tau}\Upsilon^i(\tau,\theta^i) = -(F^i)^T(\tau)\Upsilon^i(\tau,\theta^i) - \Upsilon^i(\tau,\theta^i)F^i(\tau) - \theta^i N^i(\tau) \tag{27}$$
$$- 2\Upsilon^i(\tau,\theta^i)G^i(\tau)W^i(G^i)^T(\tau)\Upsilon^i(\tau,\theta^i)\,, \quad \Upsilon^i(t_f,\theta^i) = \theta^i N^i_f$$
$$\frac{d}{d\tau}\ell^i(\tau,\theta^i) = -\Upsilon^i(\tau,\theta^i)E^i(t)u_{-ti}(\tau)\,, \quad \ell^i(t_f,\theta^i) = 0\,. \tag{28}$$

Proof. For notional simplicity, it is convenient to define

$$\varpi^i\left(\tau, z^i_\tau, \theta^i\right) \triangleq \exp\left\{\theta^i J_i\left(\tau, z^i_\tau\right)\right\}\,, \quad i \in \overline{N}$$

in which the person-by-person performance measure (22) is rewritten as the cost-to-go function from an arbitrary state z^i_τ at a running time $\tau \in [t_0, t_f]$

$$J_i(\tau, z^i_\tau) = (z^i)^T(t_f)N^i_f z^i(t_f) + \int_\tau^{t_f}(z^i)^T(t)N^i(t)z^i(t)dt \tag{29}$$

subject to

$$dz^i(t) = (F^i(t)z^i(t) + E^i(t)u_{-ti}(t))dt + G^i(t)dw^i(t)\,, \quad z^i(\tau) = z^i_\tau\,. \tag{30}$$

By definition, the moment-generating function is

$$\varphi^i(\tau, z^i_\tau, \theta^i) \triangleq E\{\varpi^i(\tau, z^i_\tau, \theta^i)\}\,.$$

Thus, the total time derivative of $\varphi^i(\tau, z^i_\tau, \theta^i)$ is obtained as

$$\frac{d}{d\tau}\varphi^i\left(\tau, z^i_\tau, \theta^i\right) = -\theta^i(z^i_\tau)^T N^i(\tau)z^i_\tau\varphi^i\left(\tau, z^i_\tau, \theta^i\right)\,.$$

Using the standard Ito's formula, it follows

$$d\varphi^i\left(\tau, z^i_\tau, \theta^i\right) = E\left\{d\varpi^i\left(\tau, z^i_\tau, \theta^i\right)\right\} = E\Big\{\varpi^i_\tau\left(\tau, z^i_\tau, \theta^i\right)d\tau + \varpi^i_{z^i_\tau}\left(\tau, z^i_\tau, \theta^i\right)dz^i_\tau$$
$$+ \frac{1}{2}\mathrm{Tr}\left\{\varpi^i_{z^i_\tau z^i_\tau}(\tau, z^i_\tau, \theta^i)G^i(\tau)W^i(G^i)^T(\tau)\right\}d\tau\Big\}$$
$$= \varphi^i_\tau(\tau, z^i_\tau, \theta^i)d\tau + \varphi^i_{z^i_\tau}(\tau, z^i_\tau, \theta^i)(F^i(\tau)z^i_\tau + E^i(\tau)u_{-ti}(\tau))d\tau$$
$$+ \frac{1}{2}\mathrm{Tr}\left\{\varphi^i_{z^i_\tau z^i_\tau}(\tau, z^i_\tau, \theta^i)G^i(\tau)W^i(G^i)^T(\tau)\right\}d\tau$$

which under the definition of the moment-generating function or the first characteristic function

$$\varphi^i\left(\tau, z^i_\tau, \theta^i\right) = \varrho^i\left(\tau, \theta^i\right) \exp\left\{(z^i_\tau)^T \Upsilon^i(\tau, \theta^i) z^i_\tau + 2(z^i_\tau)^T \ell^i(\tau, \theta^i)\right\}$$

and its partial derivatives leads to the result

$$\begin{aligned}
-\theta^i (z^i_\tau)^T N^i(\tau) z^i_\tau \varphi^i\left(\tau, z_\tau, \theta^i\right) = \Bigg\{ & \frac{\frac{d}{d\tau}\varrho^i\left(\tau, \theta^i\right)}{\varrho^i\left(\tau, \theta^i\right)} + (z^i_\tau)^T \frac{d}{d\tau}\Upsilon^i(\tau, \theta^i) z^i_\tau \\
& + 2(z^i_\tau)^T \frac{d}{d\tau}\ell^i(\tau, \theta^i) \\
& + (z^i_\tau)^T \left[(F^i)^T(\tau)\, \Upsilon^i(\tau, \theta^i) + \Upsilon^i(\tau, \theta^i) F^i(\tau)\right] z^i_\tau \\
& + 2(z^i_\tau)^T \Upsilon^i(\tau, \theta^i) E^i(\tau) u_{-ti}(\tau) \\
& + 2(z^i_\tau)^T \Upsilon^i(\tau, \theta^i) G^i(\tau)\, W^i (G^i)^T(\tau)\, \Upsilon^i(\tau, \theta^i) z^i_\tau \\
& + \mathrm{Tr}\left\{\Upsilon^i(\tau, \theta^i) G^i(\tau) W^i (G^i)^T(\tau)\right\}\Bigg\} \varphi^i\left(\tau, z^i_\tau, \theta^i\right).
\end{aligned}$$

To have constant, linear, and quadratic terms independent of arbitrary z^i_τ, it requires that the following expressions hold true:

$$\begin{aligned}
\frac{d}{d\tau}\Upsilon^i(\tau, \theta^i) &= -(F^i)^T(\tau)\Upsilon^i(\tau, \theta^i) - \Upsilon^i(\tau, \theta^i) F^i(\tau) - \theta^i N^i(\tau) \\
&\quad - 2\Upsilon^i(\tau, \theta^i) G^i(\tau) W^i (G^i)^T(\tau) \Upsilon^i(\tau, \theta^i) \\
\frac{d}{d\tau}\ell^i(\tau, \theta^i) &= -\Upsilon^i(\tau, \theta^i) E^i(\tau) u_{-ti}(\tau) \\
\frac{d}{d\tau}\varrho^i(\tau, \theta^i) &= -\varrho^i\left(\tau, \theta^i\right) \mathrm{Tr}\left\{\Upsilon^i(\tau, \theta^i) G^i(\tau) W^i (G^i)^T(\tau)\right\}
\end{aligned}$$

where the terminal-value conditions $\Upsilon^i(t_f, \theta^i) = \theta^i N^i_f$, $\ell^i(t_f, \theta^i) = 0$, and $\varrho^i(t_f, \theta^i) = 1$. Finally, the backward-in-time differential equation satisfied by $\upsilon^i(\tau, \theta^i)$ becomes

$$\frac{d}{d\tau}\upsilon^i(\tau, \theta^i) = -\mathrm{Tr}\left\{\Upsilon^i(\tau, \theta^i) G^i(\tau) W^i (G^i)^T(\tau)\right\}, \quad \upsilon^i(t_f, \theta^i) = 0,$$

which completes the proof.

Specifically, a MacLaurin series expansion of the cumulant-generating function (25) is employed to infer behaviors regarding probabilistic distributions of (22) through the knowledge representation of the mathematical statistics

$$\psi^i\left(\tau, z^i_\tau, \theta^i\right) = \sum_{r=1}^{\infty} \frac{\partial^{(r)}}{\partial\theta^{(r)}} \psi^i(\tau, z^i_\tau, \theta^i)\bigg|_{\theta^i=0} \frac{(\theta^i)^r}{r!} \tag{31}$$

where all $\kappa^i_r \triangleq \frac{\partial^{(r)}}{\partial(\theta^i)^{(r)}} \psi^i(\tau, z^i_\tau, \theta^i)\Big|_{\theta^i=0}$ are performance-measure statistics available at risk-averse decision maker or tracker i

$$\kappa^i_r = (z^i_\tau)^T \frac{\partial^{(r)}}{\partial(\theta^i)^{(r)}} \Upsilon^i(\tau, \theta^i)\bigg|_{\theta^i=0} z^i_\tau + 2(z^i_\tau)^T \frac{\partial^{(r)}}{\partial(\theta^i)^{(r)}} \ell^i(\tau, \theta^i)\bigg|_{\theta^i=0} + \frac{\partial^{(r)}}{\partial(\theta^i)^{(r)}} \upsilon^i(\tau, \theta^i)\bigg|_{\theta^i=0}. \tag{32}$$

For notational convenience, the change of variables

$$H^i_r(\tau) \triangleq \frac{\partial^{(r)}\Upsilon^i(\tau, \theta^i)}{\partial(\theta^i)^{(r)}}\bigg|_{\theta^i=0}, \quad \breve{D}^i_r(\tau) \triangleq \frac{\partial^{(r)}\ell^i(\tau, \theta^i)}{\partial(\theta^i)^{(r)}}\bigg|_{\theta^i=0}$$

$$D^i_r(\tau) \triangleq \frac{\partial^{(r)}\upsilon^i(\tau, \theta^i)}{\partial(\theta^i)^{(r)}}\bigg|_{\theta^i=0}, \quad \tau \in [t_0, t_f]; \quad r \in \mathbb{N} \tag{33}$$

is introduced so that the next result will provide an effective and accurate capability for forecasting all the higher-order characteristics associated with performance uncertainty (22).

Theorem 2 (Person-by-Person Performance-Measure Statistics). *Let (A_{N_i}, B_i) and (C_{N_i}, A_{N_i}) associated with the coupling constraint (21) and the goal function (22) be stabilizable and detectable. For $k^i \in \mathbb{N}$, the k^ith performance-measure statistic of (22) concerned by risk-averse decision maker or tracker i and $i \in \overline{N}$ is given by*

$$\kappa^i_k = (z^i_0)^T H^i_{k^i}(t_0) z^i_0 + 2(z^i_0)^T \breve{D}^i_{k^i}(t_0) + D^i_{k^i}(t_0) \tag{34}$$

where the supporting variables $\{H^i_r(\tau)\}_{r=1}^{k^i}$, $\{\breve{D}^i_r(\tau)\}_{r=1}^{k^i}$, and $\{D^i_r(\tau)\}_{r=1}^{k^i}$ evaluated at $\tau = t_0$ satisfy the differential equations (with the dependence of $H^i_r(\tau)$, $\breve{D}^i_r(\tau)$, and $D_r(\tau)$ upon the admissible feedback policy gain $K^i(\tau)$ and other observable policies $u_{-ti}(\tau)$ suppressed)

$$\frac{d}{d\tau} H^i_1(\tau) = -(F^i)^T(\tau) H^i_1(\tau) - H^i_1(\tau) F^i(\tau) - N^i(\tau) \tag{35}$$

$$\frac{d}{d\tau}H_r^i(\tau) = -(F^i)^T(\tau)H_r^i(\tau) - H_r^i(\tau)F^i(\tau)$$
$$-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}H_s^i(\tau)G^i(\tau)W^i(G^i)^T(\tau)H_{r-s}^i(\tau)\,, \quad 2 \le r \le k^i \tag{36}$$

and

$$\frac{d}{d\tau}\breve{D}_r^i(\tau) = -H_r^i(\tau)E^i(\tau)u^{-i}(\tau)\,, \qquad 1 \le r \le k^i \tag{37}$$

and, finally,

$$\frac{d}{d\tau}D_r^i(\tau) = -\mathrm{Tr}\left\{H_r^i(\tau)G^i(\tau)W^i(G^i)^T(\tau)\right\}\,, \quad 1 \le r \le k^i \tag{38}$$

provided that the terminal-value conditions $H_1^i(t_f) = N_f^i$, $H_r^i(t_f) = 0$ *for* $2 \le r \le k^i$, $\breve{D}_r^i(t_f) = 0$ *for* $1 \le r \le k^i$, *and* $D_r^i(t_f) = 0$ *for* $1 \le r \le k^i$.

Proof. The expression of performance-measure statistics described in (34) is readily justified by using the result (32) and the definition (33). What remains is to show that the solutions $H_r^i(\tau)$, $\breve{D}_r^i(\tau)$, and $D_r^i(\tau)$ for $1 \le r \le k^i$ indeed satisfy the dynamical equations (35)–(38). Notice that these backward-in-time equations (35)–(38) satisfied by the matrix-valued $H_r^i(\tau)$, vector-valued $\breve{D}_r^i(\tau)$, and scalar-valued $D_r^i(\tau)$ solutions are then obtained by successively taking derivatives with respect to θ of the supporting equations (26)–(28) and subject to the assumptions of (A_{N_i}, B_i) and (C_{N_i}, A_{N_i}) being uniformly stabilizable and detectable on $[t_0, t_f]$.

3 Problem Statements

In the context of risk-averse decision making, cautious decision makers or trackers who realize performance risk akin to a costly preference for certainty will have to leverage higher-order statistics of the probability distribution (22) for downside risk measures and optimizing risk-averse decisions. For such a problem it is important to have a compact statement of the risk-averse decision and control optimization so as to aid mathematical manipulations. Precisely, one may think of the k^i-tuple state variables

$$\mathcal{H}^i(\cdot) \triangleq (\mathcal{H}_1^i(\cdot), \ldots, \mathcal{H}_{k^i}^i(\cdot))\,,$$
$$\breve{\mathcal{D}}^i(\cdot) \triangleq (\breve{\mathcal{D}}_1^i(\cdot), \ldots, \breve{\mathcal{D}}_{k^i}^i(\cdot))\,,$$
$$\mathcal{D}^i(\cdot) \triangleq (\mathcal{D}_1^i(\cdot), \ldots, \mathcal{D}_{k^i}^i(\cdot))\,,$$

whose continuously differentiable state variables $\mathcal{H}_r^i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{2n_i \times 2n_i})$, $\breve{\mathcal{D}}_r^i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{2n_i \times 1})$, and $\mathcal{D}_r^i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ having the representations

$$\mathcal{H}_r^i(\cdot) \triangleq H_r^i(\cdot)\,, \qquad \breve{\mathcal{D}}_r^i(\cdot) \triangleq \breve{D}_r^i(\cdot)\,, \qquad \mathcal{D}_r^i(\cdot) \triangleq D_r^i(\cdot)$$

with the right members satisfying the dynamics (35)–(38) are defined on $[t_0, t_f]$.

In the remainder of the development, the convenient mappings are introduced as

$$\begin{aligned}
\mathcal{F}_r^i : [t_0, t_f] \times (\mathbb{R}^{2n_i \times 2n_i})^{k^i} \times \mathbb{R}^{m_i \times n_i} &\mapsto \mathbb{R}^{2n_i \times 2n_i} \\
\breve{\mathcal{G}}_r^i : [t_0, t_f] \times (\mathbb{R}^{2n_i \times 1})^{k^i} &\mapsto \mathbb{R}^{2n_i \times 1} \\
\mathcal{G}_r^i : [t_0, t_f] \times (\mathbb{R}^{2n_i \times 2n_i})^{k^i} &\mapsto \mathbb{R}
\end{aligned}$$

where the rules of action are given by

$$\begin{aligned}
\mathcal{F}_1^i(\tau, \mathcal{H}^i, K^i) &\triangleq -(F^i)^T(\tau)\mathcal{H}_1^i(\tau) - \mathcal{H}_1^i(\tau)F^i(\tau) - N^i(\tau) \\
\mathcal{F}_r^i(\tau, \mathcal{H}^i, K^i) &\triangleq -(F^i)^T(\tau)\mathcal{H}_r^i(\tau) - \mathcal{H}_r^i(\tau)F^i(\tau) \\
&\quad - \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{H}_s^i(\tau)G^i(\tau)W^i(G^i)^T(\tau)\mathcal{H}_{r-s}^i(\tau)\,, \quad 2 \le r \le k^i \\
\breve{\mathcal{G}}_r^i(\tau, \mathcal{H}^i) &\triangleq -\mathcal{H}_r^i(\tau)E^i(\tau)u^{-i}(\tau)\,, \quad 1 \le r \le k^i \\
\mathcal{G}_r^i(\tau, \mathcal{H}^i) &\triangleq -\mathrm{Tr}\left\{\mathcal{H}_r^1(\tau)G^i(\tau)W^i(G^i)^T(\tau)\right\}\,, \quad 1 \le r \le k^i\,.
\end{aligned}$$

The product mappings that follow are necessary for a compact formulation; for example,

$$\begin{aligned}
\mathcal{F}_1^i \times \cdots \times \mathcal{F}_{k^i}^i : [t_0, t_f] \times (\mathbb{R}^{2n_i \times 2n_i})^{k^i} \times \mathbb{R}^{m_i \times n_i} &\mapsto (\mathbb{R}^{2n_i \times 2n_i})^{k^i} \\
\breve{\mathcal{G}}_1^i \times \cdots \times \breve{\mathcal{G}}_{k^i}^i : [t_0, t_f] \times (\mathbb{R}^{2n_i \times 1})^{k^i} &\mapsto (\mathbb{R}^{2n_i \times 1})^{k^i} \\
\mathcal{G}_1^i \times \cdots \times \mathcal{G}_{k^i}^i : [t_0, t_f] \times (\mathbb{R}^{2n_i \times 2n_i})^{k^i} &\mapsto \mathbb{R}^{k^i}
\end{aligned}$$

where the corresponding notations

$$\mathcal{F}^i \triangleq \mathcal{F}_1^i \times \cdots \times \mathcal{F}_{k^i}^i\,, \quad \breve{\mathcal{G}}^i \triangleq \breve{\mathcal{G}}_1^i \times \cdots \times \breve{\mathcal{G}}_{k^i}^i\,, \quad \mathcal{G}^i \triangleq \mathcal{G}_1^i \times \cdots \times \mathcal{G}_{k^i}^i$$

are used. Thus, the dynamical equations (35)–(38) can be rewritten as

$$\frac{d}{d\tau}\mathcal{H}^i(\tau) = \mathcal{F}^i(\tau, \mathcal{H}^i(\tau), K^i(\tau)), \qquad \mathcal{H}^i(t_f) \equiv \mathcal{H}_f^i \tag{39}$$

$$\frac{d}{d\tau}\breve{\mathcal{D}}^i(\tau) = \breve{\mathcal{G}}^i\left(\tau, \mathcal{H}^i(\tau)\right), \qquad \breve{\mathcal{D}}^i(t_f) \equiv \breve{\mathcal{D}}^i_f \tag{40}$$

$$\frac{d}{d\tau}\mathcal{D}^i(\tau) = \mathcal{G}^i\left(\tau, \mathcal{H}^i(\tau)\right), \qquad \mathcal{D}^i(t_f) \equiv \mathcal{D}^i_f \tag{41}$$

where the k^i-tuple terminal conditions $\mathcal{H}^i_f \triangleq (N^i_f, 0, \ldots, 0)$, $\breve{\mathcal{D}}^i_f \triangleq (0, \ldots, 0)$, and $\mathcal{D}^i_f \triangleq (0, \ldots, 0)$.

Notice that the product system (39)–(41) uniquely determines the state matrices $\mathcal{H}^i$, $\breve{\mathcal{D}}^i$, and $\mathcal{D}^i$ once the admissible feedback policy gain K^i and observable policies u_{-ti} by decision maker and/or tracker i are specified. Henceforth, these state variables are considered as

$$\mathcal{H}^i \equiv \mathcal{H}^i(\cdot, K^i, u_{-ti}), \quad \breve{\mathcal{D}}^i \equiv \breve{\mathcal{D}}^i(\cdot, K^i, u_{-ti}), \quad \mathcal{D}^i \equiv \mathcal{D}^i(\cdot, K^i, u_{-ti}).$$

Given terminal data $(t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f)$, the class of admissible person-by-person feedback decision/control gains employed by risk-averse decision maker and/or tracker i is next defined.

Definition 1 (Person-by-Person Feedback Decision/Control Gains). Let compact subset $\overline{K}^i \subset \mathbb{R}^{m_i \times n}$ be the set of allowable feedback form values. For the given $k^i \in \mathbb{N}$ and sequence $\mu^i = \{\mu^i_r \geq 0\}_{r=1}^{k^i}$ with $\mu^i_1 > 0$, the set of feedback gains $\mathcal{K}^i_{t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f; \mu^i}$ is assumed to be the class of $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i \times n_i})$ with values $K^i(\cdot) \in \overline{K}^i$, for which the solutions to the dynamic equations (39)–(41) with the terminal-value conditions $\mathcal{H}^i(t_f) = \mathcal{H}^i_f$, $\breve{\mathcal{D}}^i(t_f) = \breve{\mathcal{D}}^i_f$, and $\mathcal{D}^i(t_f) = \mathcal{D}^i_f$ exist on the interval of optimization $[t_0, t_f]$.

An obvious fact about the private set of design freedom $\mu^i = \{\mu^i_r \geq 0\}_{r=1}^{k^i}$ with $\mu^i_1 > 0$ is that risk sensitivity entails the lack of certainty equivalence, in the sense that any performance index formed only by the first statistic of (22) does not lead to optimal decisions. In addition, it is important to recognize that this finite set of custom weights is quite different from those of infinite sets of series expansion coefficients as in [1, 6, 17], just to name a few.

On $\mathcal{K}^i_{t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f; \mu^i}$ the performance index with mean-risk considerations is subsequently defined as follows.

Definition 2 (Mean-Risk Aware Performance Index). Let cautious decision maker and/or tracker i select $k^i \in \mathbb{N}$ and the set of custom weights $\mu^i = \{\mu^i_r \geq 0\}_{r=1}^{k^i}$ with $\mu^i_1 > 0$. Then, for the given z^i_0, the mean-risk aware performance index

$$\phi^i_0 : \{t_0\} \times (\mathbb{R}^{2n_i \times 2n_i})^{k^i} \times (\mathbb{R}^{2n_i})^{k^i} \times \mathbb{R}^{k^i} \mapsto \mathbb{R}^+$$

pertaining to person-by-person risk-averse decision making over $[t_0, t_f]$ is

$$\phi_0^i(t_0, \mathcal{H}^i, \breve{\mathcal{D}}^i, \mathcal{D}^i) \triangleq \underbrace{\mu_1^i \kappa_1^i}_{\text{Mean}} + \underbrace{\mu_2^i \kappa_2^i + \cdots + \mu_{k^i}^i \kappa_{k^i}^i}_{\text{Risk}}$$

$$= \sum_{r=1}^{k^i} \mu_r^i [(z_0^i)^T \mathcal{H}_r^i(t_0) z_0^i + 2(z_0^i)^T \breve{\mathcal{D}}_r^i(t_0) + \mathcal{D}_r^i(t_0)] \qquad (42)$$

where additional design freedom μ_r^i's utilized by cautious and defensive decision maker and/or tracker i are tailored to meet different levels of performance-based reliability requirements, e.g., mean, variance, anti-symmetry, heavy tails of the reward/cost density (22), etc., pertaining to closed-loop performance uncertainties and whereas the supporting solutions $\{\mathcal{H}_r^i(\tau)\}_{r=1}^{k^i}$, $\{\breve{\mathcal{D}}_r^i(\tau)\}_{r=1}^{k^i}$ and $\{\mathcal{D}_r^i(\tau)\}_{r=1}^{k^i}$ evaluated at $\tau = t_0$ satisfy the dynamical equations (39)–(41).

The technical challenge faced by a cautious decision maker and/or tracker i and $i \in \overline{N}$ is that the correspondent mean-risk aware performance index (42) depends on the observable decision and/or control policies from neighboring decision makers or trackers u_{-ti}. The basic question the decision maker and/or tracker i faces is whether or not a sort of noncooperative equilibrium or Nash solution is possible at all. Defensive decision maker and/or tracker i's rationale for choosing K_*^i is to force the immediate neighbors to hold u_{-ti}^*, so as to secure the Nash payoff. Thus, it is precisely in this sense that a Nash solution is risk-averse by nature.

Definition 3 (Feedback Nash Equilibrium). Let K_*^i constitute a feedback Nash strategy such that

$$\phi_0^i(K_*^i, u_{-ti}^*) \leq \phi_0^i(K^i, u_{-ti}^*), \qquad i \in \overline{N}_i \qquad (43)$$

for all admissible $K^i \in \mathcal{K}^i_{t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f; \mu^i}$, upon which the solutions to the dynamical systems (39)–(41) exist on $[t_0, t_f]$.

Then, $\left(K_*^1, \ldots, K_*^{N_i}\right)$ when restricted to $[t_0, \tau]$ is still a N_i-tuple feedback Nash equilibrium solution for the multiperson Nash decision problem with the appropriate terminal-value condition $(\tau, \mathcal{H}_*^i(\tau), \breve{\mathcal{D}}_*^i(\tau), \mathcal{D}_*^i(\tau))$ for all $\tau \in [t_0, t_f]$.

Of note, an N_i-tuple of decision and/or control policies $\left(K_*^1, \ldots, K_*^{N_i}\right)$ is said to constitute an interactive feedback Nash equilibrium solution for an N_i-agent differential graphical game if, for all $i \in \overline{N}_i$, the following Nash condition holds

$$\phi_0^i(K_*^i, u_{-ti}^*) \leq \phi_0^i(K^i, u_{-ti}^*) .$$

In addition, there exist decision and/or control policies $\underline{K}^j$ and $\overline{K}^j$ such that

$$\phi_0^i(\underline{K}^j, u_{-tj}^*) \neq \phi_0^i(\overline{K}^j, u_{-tj}^*)\,, \quad \forall\, i, j \in \overline{N}_i\,.$$

The interpretation is that the variation of person-by-person performance index pertaining to decision maker and/or tracker i is resulted while the rest of the immediate decision makers and/or trackers in the local neighborhood of decision maker and/or tracker j supported by the corresponding connectivity graph assume their optimal strategies.

Now, the objective of cautious decision maker and/or tracker i is to minimize (42) over $K^i = K^i(\cdot)$ in $\mathcal{K}^i_{t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f; \mu^i}$ and subject to the neighboring feedback Nash policies u_{-ti}^*.

Definition 4 (Person-by-Person Optimization). Given the profile of risk-averse attitudes $\mu^i = \{\mu_r^i \geq 0\}_{r=1}^{k^i}$ with $\mu_1^i > 0$, the decision optimization problem defined by

$$\min_{K^i(\cdot) \in \mathcal{K}_{t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f; \mu^i}} \phi_0^i(K^i, u_{-ti}^*) \tag{44}$$

is subject to the dynamical equations (39)–(41) on $[t_0, t_f]$.

In conformity with the dynamic programming approach, the terminal time and states $(t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f)$ are parameterized as $(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ whereby $\mathcal{Y}^i \triangleq \mathcal{H}^i(\varepsilon)$, $\breve{\mathcal{Z}}^i \triangleq \breve{\mathcal{D}}^i(\varepsilon)$, and $\mathcal{Z}^i \triangleq \mathcal{D}^i(\varepsilon)$. Thus, the value function of (44) now depends on the parameterization of the terminal-value conditions.

Definition 5 (Value Function). Let $(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \in [t_0, t_f] \times (\mathbb{R}^{2n_i \times 2n_i})^{k^i} \times (\mathbb{R}^{2n_i \times 1})^{k^i} \times \mathbb{R}^{k^i}$. Then, the value function $\mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ is defined by

$$\mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \triangleq \inf_{K^i(\cdot) \in \mathcal{K}_{\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i; \mu^i}} \phi_0^i(K^i, u_{-ti}^*)\,.$$

For convention, $\mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \triangleq \infty$ when $\mathcal{K}_{\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i; \mu^i}$ is empty. Some candidates for the value function are also constructed with the help of the concept of reachable set.

Definition 6 (Reachable Sets). Let a reachable set of decision maker i be defined by $\mathcal{Q}^i \triangleq \{(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) \in [t_0, t_f] \times (\mathbb{R}^{2n \times 2n})^{k^i} \times (\mathbb{R}^{2n \times 1})^{k^i} \times \mathbb{R}^{k^i} : \mathcal{K}^i_{\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i; \mu^i} \neq \emptyset\}$.

Formally, it can be shown that the value function associated with decision maker i is satisfying partial differential equation (e.g., Hamilton-Jacobi-Bellman (HJB) equation) at interior points of $\mathcal{Q}^i$, at which it is differentiable.

Theorem 3 (HJB Equation-Mayer Problem). *Let* $(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ *be any interior point of* $\mathcal{Q}^i$*, at which the value function* $\mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ *is differentiable. If there exists a feedback Nash strategy* $K^i_* \in \mathcal{K}^i_{t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f; \mu^i}$*, then the partial differential equation is satisfied:*

$$
\begin{aligned}
0 = \min_{K^i \in \overline{K}^i} \Big\{ & \frac{\partial}{\partial \varepsilon} \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) + \frac{\partial}{\partial\, vec(\mathcal{Y}^i)} \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) vec(\mathcal{F}^i(\varepsilon, \mathcal{Y}^i, K^i)) \\
& + \frac{\partial}{\partial\, vec(\breve{\mathcal{Z}}^i)} \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) vec(\breve{\mathcal{G}}^i(\varepsilon, \mathcal{Y}^i) \\
& + \frac{\partial}{\partial\, vec(\mathcal{Z}^i)} \mathcal{V}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i) vec(\mathcal{G}^i(\varepsilon, \mathcal{Y}^i) \Big\}
\end{aligned}
\tag{45}
$$

and $\mathcal{V}^i(t_0, \mathcal{Y}^i(t_0), \breve{\mathcal{Z}}^i(t_0), \mathcal{Z}^i(t_0)) = \phi^i_0(\mathcal{H}^i(t_0), \breve{\mathcal{D}}^i(t_0), \mathcal{D}^i(t_0))$.

Proof. Similar to that of [12] and hence is omitted.

Finally, the sufficient condition for verification of a feedback Nash strategy by cautious decision maker and/or tracker i and $i \in \overline{N}$ is given below.

Theorem 4 (Verification Theorem). *Let* $\mathcal{W}^i(\varepsilon, \mathcal{Y}^i, \breve{\mathcal{Z}}^i, \mathcal{Z}^i)$ *be continuously differentiable solution of the HJB equation (45) with the boundary condition*

$$
\mathcal{W}^i(t_0, \mathcal{H}^i(t_0), \breve{\mathcal{D}}^i(t_0), \mathcal{D}^i(t_0)) = \phi^i_0(t_0, \mathcal{H}^i(t_0), \breve{\mathcal{D}}^i(t_0), \mathcal{D}^i(t_0)) .
$$

Let $(t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f) \in \mathcal{Q}^i$*,* $K^i \in \mathcal{K}^i_{t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f; \mu^i}$*,* $(\mathcal{H}^i(\cdot), \breve{\mathcal{D}}^i(\cdot), \mathcal{D}^i(\cdot))$ *be the trajectory solutions of (39)–(41). Then,* $\mathcal{W}^i(\tau, \mathcal{H}^i(\tau), \breve{\mathcal{D}}^i(\tau), \mathcal{D}^i(\tau))$ *is a time-backward increasing function of* $\tau \in [t_0, t_f]$.

If K^i_* *is in* $\mathcal{K}^i_{t_f, \mathcal{H}^i_f, \breve{\mathcal{D}}^i_f, \mathcal{D}^i_f; \mu^i}$ *with the associative solutions* $(\mathcal{H}^i_*(\cdot), \breve{\mathcal{D}}^i_*(\cdot), \mathcal{D}^i_*(\cdot))$ *of the equations (39)–(41) such that*

$$
\begin{aligned}
0 = & \frac{\partial}{\partial \varepsilon} \mathcal{W}^i(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau)) \\
& + \frac{\partial}{\partial\, vec(\mathcal{Y}^i)} \mathcal{W}^i(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau)) vec(\mathcal{F}^i(\tau, \mathcal{H}^i_*(\tau), K^i_*(\tau))) \\
& + \frac{\partial}{\partial\, vec(\breve{\mathcal{Z}}^i)} \mathcal{W}^i(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau)) vec(\breve{\mathcal{G}}^i(\tau, \mathcal{H}^i_*(\tau))) \\
& + \frac{\partial}{\partial\, vec(\mathcal{Z}^i)} \mathcal{W}_{is}(\tau, \mathcal{H}^i_*(\tau), \breve{\mathcal{D}}^i_*(\tau), \mathcal{D}^i_*(\tau)) vec(\mathcal{G}^i(\tau, \mathcal{H}^i_*(\tau))),
\end{aligned}
\tag{46}
$$

then K_*^i *is a feedback Nash strategy in* $\mathcal{K}^i_{t_f,\mathcal{H}^i_f,\breve{\mathcal{D}}^i_f,\mathcal{D}^i_f;\mu^i}$,

$$\mathcal{W}^i(\varepsilon,\mathcal{Y}^i,\breve{\mathcal{Z}}^i,\mathcal{Z}^i)=\mathcal{V}^i(\varepsilon,\mathcal{Y}^i,\breve{\mathcal{Z}}^i,\mathcal{Z}^i)\,. \tag{47}$$

Proof. The proof follows the same manner as [12].

4 Person-By-Person Risk-Averse Strategies

To this end, the initial state z_0^i is recognized to contribute linearly and quadratically to the mean-risk performance index (42). Henceforth, it is beneficial to infer that a candidate for the value function is expected to take the form

$$\begin{aligned}\mathcal{W}^i(\varepsilon,\mathcal{Y}^i,\breve{\mathcal{Z}}^i,\mathcal{Z}^i)&=(z_0^i)^T\sum_{r=1}^{k^i}\mu_r^i(\mathcal{Y}_r^i+\mathcal{E}_r^i(\varepsilon))\,z_0^i\\&\quad+2(z_0^i)^T\sum_{r=1}^{k^i}\mu_r^i(\breve{\mathcal{Z}}_r^i+\breve{\mathcal{T}}_r^i(\varepsilon))+\sum_{r=1}^{k^i}\mu_r^i(\mathcal{Z}_r^i+\mathcal{T}_r^i(\varepsilon))\end{aligned} \tag{48}$$

where the functions $\mathcal{E}_r^i\in\mathcal{C}^1([t_0,t_f];\mathbb{R}^{2n_i\times 2n_i})$, $\breve{\mathcal{T}}_r^i\in\mathcal{C}^1([t_0,t_f];\mathbb{R}^{2n_i\times 1})$, and $\mathcal{T}_r^i\in\mathcal{C}^1([t_0,t_f];\mathbb{R})$ are time parameterized and yet to be determined.

As reported in [13], the time derivative of $\mathcal{W}^i(\varepsilon,\mathcal{Y}^i,\breve{\mathcal{Z}}^i,\mathcal{Z}^i)$ can be shown as follows:

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{W}^i(\varepsilon,\mathcal{Y}^i,\breve{\mathcal{Z}}^i,\mathcal{Z}^i)&=(z_0^i)^T\sum_{r=1}^{k^i}\mu_r^i[\mathcal{F}_r^i(\varepsilon,\mathcal{Y}^i,K^i)+\frac{d}{d\varepsilon}\mathcal{E}_r^i(\varepsilon)]z_0^i\\&\quad+2(z_0^i)^T\sum_{r=1}^{k^i}\mu_r^i[\breve{\mathcal{G}}_r^i(\varepsilon,\mathcal{Y}^i)+\frac{d}{d\varepsilon}\breve{\mathcal{T}}_r^i(\varepsilon)]\\&\quad+\sum_{r=1}^{k^i}\mu_r^i[\mathcal{G}_r^i(\varepsilon,\mathcal{Y}^i)+\frac{d}{d\varepsilon}\mathcal{T}_r^i(\varepsilon)]\,.\end{aligned} \tag{49}$$

The substitution of this candidate (48) for the value function into the HJB equation (45) and making use of (49) yield

$$\begin{aligned}0=\min_{K^i\in\overline{K}^i}\Big\{&(z_0^i)^T\sum_{r=1}^{k^i}\mu_r^i[\mathcal{F}_r^i(\varepsilon,\mathcal{Y}^i,K^i)+\frac{d}{d\varepsilon}\mathcal{E}_r^i(\varepsilon)]z_0^i\\&+2(z_0^i)^T\sum_{r=1}^{k^i}\mu_r^i[\breve{\mathcal{G}}_r^i(\varepsilon,\mathcal{Y}^i)+\frac{d}{d\varepsilon}\breve{\mathcal{T}}_r^i(\varepsilon)]+\sum_{r=1}^{k^i}\mu_r^i[\mathcal{G}_r^i(\varepsilon,\mathcal{Y}^i)+\frac{d}{d\varepsilon}\mathcal{T}_r^i(\varepsilon)]\Big\}.\end{aligned} \tag{50}$$

Taking the gradient with respect to K^i of the expression within the bracket of (50) yields the necessary conditions for an extremum of (42) on $[t_0, \varepsilon]$ where $I_0^T \triangleq \left[I_{n_i \times n_i} \; 0 \right]$

$$K^i = -R_i^{-1}(B_i)^T I_0^T \sum_{r=1}^{k^i} \hat{\mu}_r^i \mathcal{Y}_r^i I_0((I_0^T I_0)^{-1})^T \tag{51}$$

in which $\hat{\mu}_r^i \triangleq \mu_r^i / \mu_1^i$ for $\mu_1^i > 0$. With the feedback Nash strategy (51) replaced in the expression of the bracket (50) and having $\{\mathcal{Y}_r^i\}_{r=1}^{k^i}$ evaluated on the optimal solution trajectories (39)–(41), the time parametric functions $\mathcal{E}_r^i(\varepsilon)$, $\breve{\mathcal{T}}_r^i(\varepsilon)$, and $\mathcal{T}_r^i(\varepsilon)$ are thus chosen so that the sufficient condition (46) in the verification theorem is satisfied in spite of the arbitrary values z_0^i; for example,

$$\dot{\mathcal{E}}_1^i(\varepsilon) = (F_*^i)^T(\varepsilon)\mathcal{H}_{1*}^i(\varepsilon) + \mathcal{H}_{1*}^i(\varepsilon)F_*^i(\varepsilon) + N_*^i(\varepsilon)\,, \quad \mathcal{E}_1^i(t_0) = 0$$

$$\begin{aligned}\dot{\mathcal{E}}_r^i(\varepsilon) &= (F_*^i)^T(\varepsilon)\mathcal{H}_{r*}^i(\varepsilon) + \mathcal{H}_{r*}^i(\varepsilon)F_*^i(\varepsilon) \\ &\quad + \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{H}_{s*}^i(\varepsilon)G^i(\varepsilon)W^i(G^i)^T(\varepsilon)\mathcal{H}_{r-s*}^i(\varepsilon), \quad \mathcal{E}_r^i(t_0) = 0\,, \quad r \geq 2\end{aligned}$$

$$\dot{\breve{\mathcal{T}}}{}_r^i(\varepsilon) = \mathcal{H}_{r*}^i(\varepsilon)E^i(\varepsilon)u_{-ti}^*(\varepsilon), \quad \breve{\mathcal{T}}_r^i(t_0) = 0, \quad 1 \leq r \leq k^i$$

$$\dot{\mathcal{T}}_r^i(\varepsilon) = \mathrm{Tr}\left\{\mathcal{H}_{1*}^i(\varepsilon)G^i(\varepsilon)W^i G^{iT}(\varepsilon)\right\}\,, \quad \mathcal{T}_r^i(t_0) = 0\,, \quad 1 \leq r \leq k^i\,.$$

Before closing the section, it is important to note that the sufficient condition (46) of the verification theorem is satisfied. Hence, the extremizing feedback strategy (51) associated with cautious decision maker i becomes optimal.

Theorem 5 (Person-by-Person Risk-Averse Decision/Control Strategies). *Consider the multi-agent tracking and synchronization supported by connectivity graphs wherein cautious and defensive decision maker and/or tracker i and $i \in \overline{N}$ have complete knowledge of the coupling constraints (6), (7), (18), and mean-risk aware performance index (42). When all decision makers i have the similar risk attitudes, an imitative or Nash equilibrium exists and is enabled by the risk-averse class of feedback strategies*

$$u_{ti}^*(t) = K_*^i(t)\hat{z}_i^*(t), \quad t \triangleq t_0 + t_f - \tau \tag{52}$$

$$K_*^i(\tau) = -R_i^{-1}(B_i)^T I_0^T \sum_{r=1}^{k^i} \hat{\mu}_r^i \mathcal{H}_{r*}^i(\tau) I_0((I_0^T I_0)^{-1})^T$$

where all the parametric design freedom through $\hat{\mu}_r^i$ represent the risk-averse preferences toward performance distributions; the optimal trajectory solutions $\mathcal{H}_{r*}^i(\cdot)$ are satisfying (39).

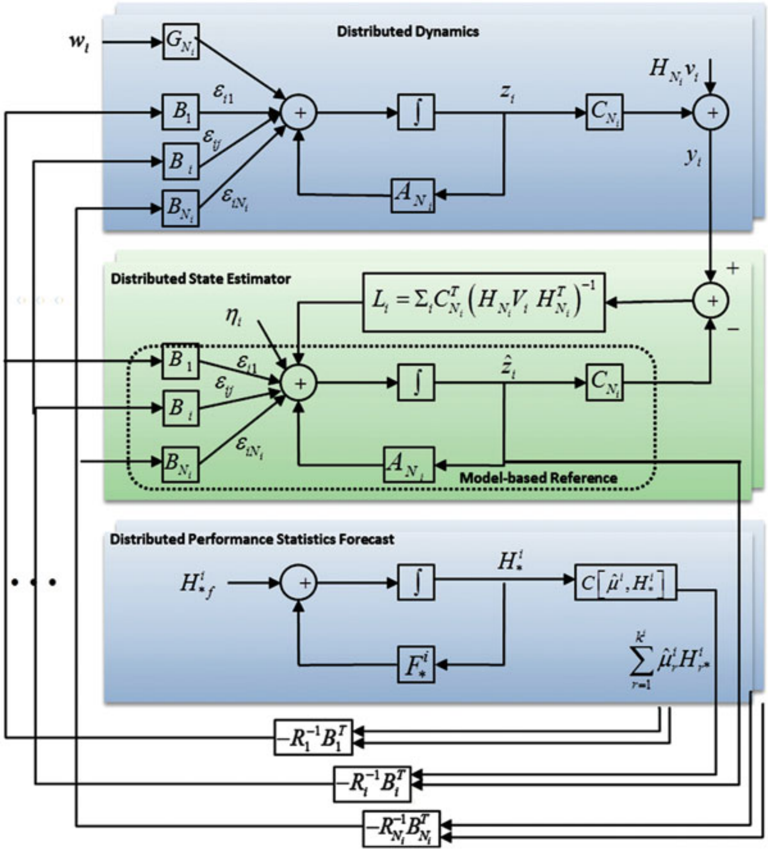

Fig. 1 Distributed decision architecture for performance risk aversion

Notice that, to have the distributed feedback Nash policy (52) of decision maker i be defined and continuous for all $\tau \in [t_0, t_f]$, the solutions $\mathcal{H}^i_{r*}(\tau)$ to the equations (39) when evaluated at $\tau = t_0$ must also exist. Therefore, it is necessary that $\mathcal{H}^i_{r*}(\tau)$ are finite for all $\tau \in [t_0, t_f)$. Moreover, the solutions of (39) exist and are continuously differentiable in a neighborhood of t_f. Under the assumption of (A_{N_i}, B_i) and (C_{N_i}, A_{N_i}) being stabilizable and detectable, the result from [3] concludes that these solutions can further be extended to the left of t_f as long as $\mathcal{H}^i_{r*}(\tau)$ remain finite. Hence, the existence of unique and continuously differentiable solutions to the equations (39) is certain if $\mathcal{H}^i_{r*}(\tau)$ are bounded for all $\tau \in [t_0, t_f)$. As the result, the candidate value functions $\mathcal{W}^i(\tau, \mathcal{H}^i, \breve{\mathcal{D}}^i, \mathcal{D}^i)$ are continuously differentiable as well.

As illustrated by Fig. 1, the person-by-person decisions and controls, generated by risk-averse policies $u^*_{ti}(t)$ for $r = 1, \ldots, N$, not only depend on the basis of information $\hat{z}^*_i(t)$ about the conditional probability distribution for coupling interactions from the local neighborhood supported by connectivity graphs, but also rely on the robust prediction of higher-order characteristics for person-by-person performance uncertainty, e.g., mean, variance, skewness, etc. The need for the control decision laws to take into account accurate estimations of performance uncertainty is one form of interaction between two interdependent functions of a decision and/or control strategy: i) anticipation of performance uncertainty and ii) proactive decisions for mitigating downside performance risk measures. This form of interaction between these two decision and/or control strategy functions gives rise to what are now termed as *performance probing* and *performance cautioning* and thus are of particular importance in the newly developed theory of statistical optimal control.

5 Conclusions

In this chapter, the research emphasis and contributions have been the generalization of the results known for linear-quadratic classes of noncooperative stochastic games and distributed controls. Specifically, under risk attitudes toward performance uncertainties, the risk-averse feedback decision laws are not only the functions of higher-order statistics of the chi-squared rewards or costs but also dependent of a priori knowledge of common process noises as well as subjective observation noises. Thus, both certainty equivalence and separation principle do not hold. Also important is that the existence of the Nash equilibrium as proposed herein is conditional upon the custom sets of selective weights, which in turn relate to risk parameters residing at cautious decision makers or controllers. An extension of the results obtained in this exposition may be worthy of future investigation, when there are presence of mistrust and excessive risk aversion; such results could constitute fundamentals and principles in adversarial systems sciences and flexibly survivable decision making.

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