

Preface

In 1842 the Belgian mathematician Eugène Charles Catalan asked whether 8 and 9 are the only consecutive pure powers of nonzero integers. One hundred and sixty years after, the question was answered affirmatively by the Swiss mathematician of Romanian origin Preda Mihăilescu. In other words, $3^2 - 2^3 = 1$ is the only solution of the equation $x^p - y^q = 1$ in integers x, y, p, q with $xy \neq 0$ and $p, q \geq 2$.

Since 2002, the different steps of the proof have been presented by various authors; see, for instance, the expository articles [9, 10, 82]. Complete proofs appeared in monographs by Schoof [124] and Cohen [25, 26].

In this book we give a complete and (almost) self-contained exposition of Mihăilescu's work, which must be understandable by a curious university student not necessarily specializing in number theory. We assume very modest background: a standard university course of algebra, including basic Galois theory, and working knowledge of basic algebraic number theory such as ideal decomposition, units (including the Dirichlet unit theorem), ideal classes, and finiteness of the class group. From the ramification theory we use only one basic fact: a prime number is ramified in a number field if and only if it divides the discriminant. All necessary facts from algebraic number theory are gathered (without proofs) in Appendix A.

We do not assume any knowledge about cyclotomic fields; everything needed is defined or proved in the book.

With our minimalistic approach, some omissions were inevitable. For instance, an experienced reader can notice that many arguments in this book have an obvious non-Archimedean flavor. Nevertheless, we resisted the temptation of broader use of the language of (non-Archimedean) valuations. Our main motivation was that a matured reader will easily reveal the non-Archimedean context wherever it is hidden, but abusing the non-Archimedean language may create problems for a less knowledgeable reader.

Another example is restricting to commutative groups and rings in Appendices C and D. Of course, certain results from these appendices (like the theorem of Maschke) extend to noncommutative case as well. We, however, assume commutativity, because this makes the arguments technically simpler and is sufficient for our purposes.

One more notable omission is Runge's method. We are aware, of course, that certain proofs, especially in Chaps. 3, 8, and 9, use Runge-style arguments. We feel, however, that Runge's method can be correctly explained only by using the language of algebraic curves, which is foreign in this book.

Chapters 4, 5, 7, 10, and 12 are dedicated to the general theory of cyclotomic fields. Catalan's problem is treated in Chaps. 2, 3, 6, 8, 9, and 11. Chapter 13 is quite isolated and independent of the others. In it we give a concise introduction to Baker's method and prove the Theorem of Tijdeman, which was the top achievement in Catalan's problem before Mihăilescu's work.

The book has six appendices. As we already mentioned, Appendix A is a very brief account of basic algebraic number theory. Other appendices treat miscellaneous topics in algebra and number theory used in the book. While Appendix A contains almost no proofs, in the other appendices, we prove everything we need.

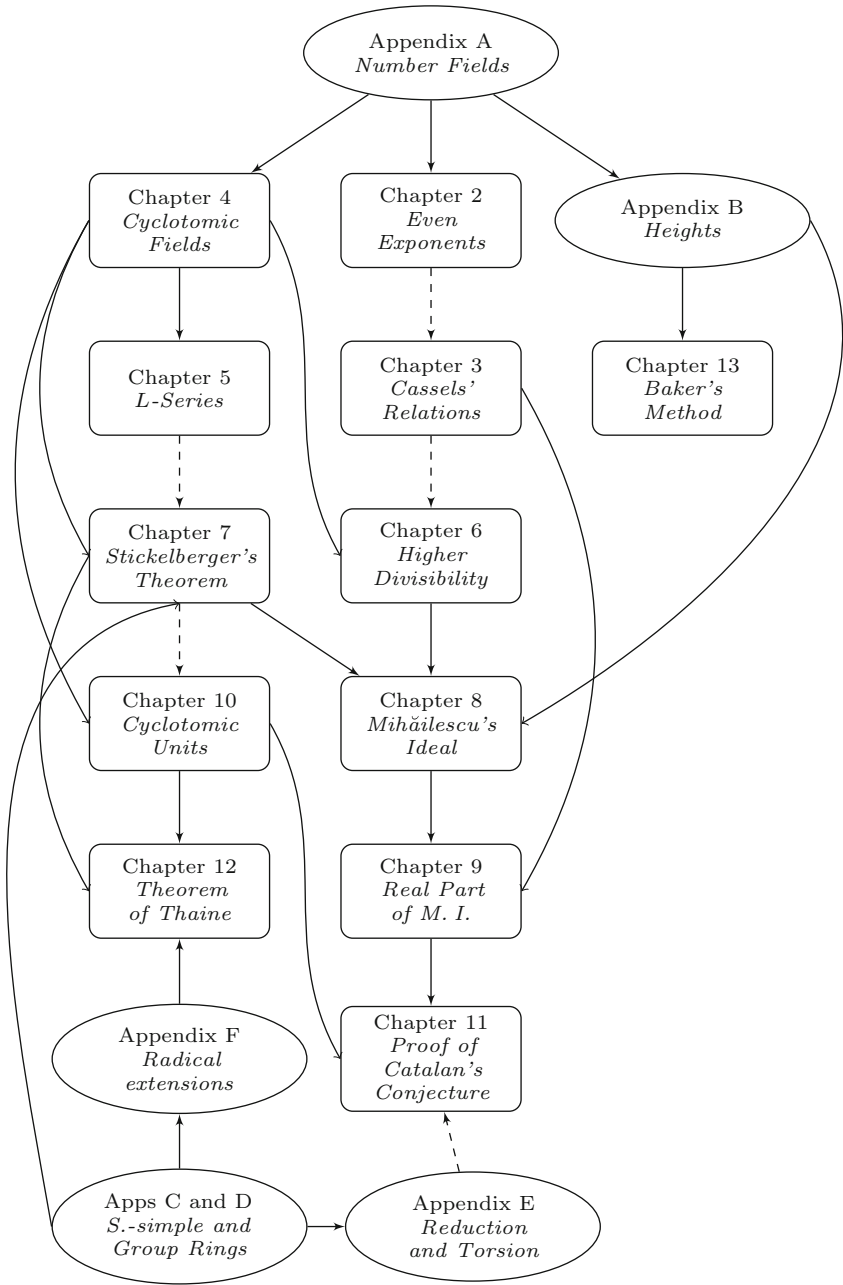
Our notation is mainly standard. We denote by $|S|$ the cardinality of a finite set S . We use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the lower and the upper integral part of $x \in \mathbb{R}$, respectively:

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}, \quad \lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}.$$

Unless the contrary is stated explicitly, letters p and q denote distinct odd prime numbers; in particular, q never denotes a power of p .

On page vii we display the logical dependence of chapters and appendices. Dashed lines indicate weak dependence.

Logical Dependence of Chapters and Appendices



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