

# The Calderón Brothers, a Happy Mathematical Relation

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It is not often that one has the opportunity to observe at close range two remarkable brothers, mathematicians. I had the privilege of being Alberto Calderón's wife—second wife—which made me Calixto Calderón's sister-in-law and gave me an unusual vantage point. Allow me then to say a few words about the remarkable Calderón brothers.

Alberto and Calixto were born some 20 years apart in Argentina, in the city of Mendoza, that golden city at the foot of the Andes, the eternally snow-capped Andes. With its luxuriant vineyards and olive groves where children roamed freely, Mendoza helped shape Alberto and later Calixto, as they were growing up. For them Mendoza never lost its magic spell.

Alberto and Calixto Calderón were in fact half-brothers: same father, different mothers. Alberto's mother, Haydée, a spirited woman—reputed to be the first woman in Mendoza to drive a car—died unexpectedly, prematurely. Sometime later, the father, Dr. Pedro Calderón—a renowned surgeon in Mendoza—remarried. His second wife was Matilde, a charming, much younger woman, Calixto's mother. I met Matilde a number of years ago. Dr. Pedro Calderón had a natural affinity for arithmetic and music. He would have undoubtedly subscribed to Leibniz's famous saying that: "Music is the secret arithmetic of the soul, unaware of its act of counting." The fact is that he tried to instill in his sons, at an early age, a keen interest in mathematics and music. "At the dinner table he would challenge Alberto, a boy of six or seven, to make rapid mental calculations; or he would play classical music for Alberto and his older sister Nenacha." This scenario repeated itself, at the dinner table, some 20 years later, with Calixto and his older sister, Matilde.

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Considering the large gap in age between Alberto and Calixto, it is probably fair to say that Alberto was more of a father-figure than a brother-figure for Calixto. Given Alberto's meteoric rise in the world of mathematics, his air of quiet authority but unassuming manner, his impact on his much younger, impressionable brother was unavoidable. Indeed, as a teenager, Calixto felt closer to Alberto than to his own parents and knew all life long that he could trust Alberto, count on his kindness, infinite patience, affection. As a matter of fact, generosity, loyalty and unmistakable chivalry were traits of character that the Calderón brothers shared.

Dr. Pedro Calderón was a man of authority, a stern father. It was not surprising, therefore, that Alberto and Calixto's education should follow more or less similar patterns. In elementary school both boys passed through the Colegio San José of the Maristas Brothers. Alberto developed a passion for Mathematics early on. A number of years later, Calixto became strongly attracted to Theoretical Physics. Dr. Pedro Calderón, however, was firmly convinced that you could not earn a living as a mathematician, or for that matter, a theoretical physicist. I do not know how much soul-searching went on, but the fact is that Alberto went to the School of Engineering of the University of Buenos Aires and graduated as a civil engineer, while sometime later Calixto attended the Engineering School of the University of Cuyo where he received basic training in Mathematics, Physics, Technical Drawing, and Chemistry. But the final destination, the preordained destiny was the same for both, namely, Mathematics! The Institute of Mathematics at the University of Buenos Aires was a powerful magnet that had attracted several brilliant Spanish refugees, and that was brimming over with mathematical activity. At the center of the Institute, as we had already learned this morning, was the legendary personality of Dr. Alberto González Domínguez, a man of vast humanistic culture and human wisdom, who had left behind Greek, Latin and Philology, for the sake of Mathematics. Dr. González Domínguez became Alberto's mentor, protector, devoted friend, and unsurpassed "fan." Years later he became Calixto Calderón's doctoral thesis advisor. The story of how Alberto got his doctorate at the University of Chicago is familiar to older mathematicians, so I shall not repeat it.

One of the things that make this country great is the vast influx of talent from abroad. Alberto and Calixto's academic careers zigzagged through various mathematical centers in the USA and Argentina, including important stints at MIT for Alberto and the University of Minnesota for Calixto. In the end both Alberto, and years later Calixto, settled in Chicago: Alberto at the University of Chicago, Calixto at the University of Illinois. Alberto took early retirement from the University of Chicago and returned to Buenos Aires when his wife—his first wife Mabel—became seriously ill. He returned to the University of Chicago on a post-retirement appointment in 1989. Calixto retired from the University of Illinois in 2003, and taught afterwards at Oakton College and DePaul University for a number of years. Alberto and Calixto had parallel professional lives. Their many collaborators and doctoral students made these professional lives most lively and interesting. As a matter of fact Calixto's first doctoral student, the one and only Luis Caffarelli, and one of Alberto's late doctoral students, the incomparable Carlos Kenig, both performed at this meeting, earlier today.

As far as professional interests go, Alberto roamed widely but stayed within the confines of Mathematics, with a side interest in mathematical education. Calixto ventured outside and tried related fields, such as Biomathematics, History of Calculus and Biographies of Scientists. Noteworthy are the “*Biography of Copernicus*,” the article on “*16th century Iberian Calculatores*” (these were the precursors of Galileo’s Modern Mechanics and of Newton’s Calculus), as well as the brief but highly interesting essay “*Dr. Pedro Calderón and Urology in Mendoza*” (in his heyday, Dr. Pedro Calderón was the preeminent urologist in Mendoza.) The “*Biography of Copernicus*,” in particular, took Calixto several years of research. There is an unforgettable quote in this biography of the great astronomer. A close friend and adviser of Martin Luther, by the name of Melancthon, refers to Copernicus, sarcastically in Latin: “*Il Sarmaticus Astronomus qui movet Terram et figit Solem*” (“*The Sarmatian Astronomer who moves the Earth but fixes, immobilizes the Sun.*”) “Sarmatian” here is used disparagingly to mean “outsider,” a “barbarian” for Calixto goes on to explain that Sarmatia in ancient times was a vast geographic area that stretched from the basin of the Vistula to the Caspian Sea.

Alberto admired the facility and grace with which Calixto was able to write. He also admired Calixto’s unusual erudition and exceptional memory: “Where does he store all this information?” Alberto wondered. For Alberto writing a letter, a review, an essay was not exactly a pleasant task. He nevertheless had a real feeling for poetic beauty: he did a stunning translation into English of Gustavo Adolfo Becquer’s classic poem “*Volverán las oscuras golondrinas*” just before we were married.

Over the years I often heard mathematicians comment on the Bernoulli brothers of Basel, Switzerland, Jacob Bernoulli, the older brother, the founder of the Bernoulli “dynasty,” and Johann Bernoulli, his much younger brother. The Bernoulli brothers lived more than 3 and a half centuries ago, and were endowed with magnificent creative gifts, but their relationship was fraught with difficulties, animosity, bitter rivalry. Not so in the case of the Calderón brothers. Nothing illustrates this better, in my opinion, than “the posthumous” paper by Alberto Calderón and Calixto Calderón.

Let me backtrack a little. Alberto Calderón died in April 1998. The volume “*Selected papers of Alberto Calderón with Commentary*” or for short *Calderón Selecta* took 10 years to see the light of day, but the AMS did a fine job. The Editors were: Paul Malliavin—the great French probabilist and Alberto’s oldest mathematical friend outside Argentina, Carlos Kenig—Alberto’s former doctoral student, and myself—Alberto’s wife (second wife). This project was a tribute to Alberto’s mathematical legacy and a labor of love. I would like to single out three of the papers in this volume:

- (1) Paul Malliavin, “*On the analytical side of the proof of the Index Theorem, some personal recollections.*”

This essay (Commentary) contains Malliavin’s account of the history of the Index Theorem and the role played by Alberto Calderón.

- (2) Yves Meyer, “Complex analysis and operator theory in Alberto Calderón’s work.”

This is one of the most substantial, beautiful and at times poetic essays (Commentary) in the volume.

And last but perhaps most important and relevant to this talk, the “posthumous” paper, the joint paper that Calixto Calderón published in the Indiana University Mathematics Journal in 2000 and that we included in the *Calderón Selecta*,

- (3) A. P. Calderón and C. P. Calderón, “A representation Formula and its Applications to Singular Integrals.”

Let me now go back to the year 1977. Alberto Calderón had just published the landmark paper on the “Cauchy Integral on Lipschitz curves,” the Cauchy integral here being the singular integral on the curve, i.e. the analog of the Hilbert transform on the real line. This triggered a frenzy of activity. First, mathematicians worked hard to remove the bound on the norm of the Lipschitz curve. For the next 20 years or so, first-rate analysts were busy finding new proofs of the boundedness in  $L^2$  of this Cauchy operator. In the process, they discovered connections with such diverse areas of Mathematics as: the Traveling Salesman problem, Ahlfors Regular Curves, Menger curvature, to mention only a few. This last, shall we say “geometric” proof, given by Melnikov and Verdera in 1995, and using as a tool the Menger curvature, is considered by many to be the simplest and most beautiful proof of the boundedness of the Cauchy Integral on Lipschitz curves.

If we are in the complex plane  $\mathbb{C}$ , have an open disc  $D$ , a function  $F(z)$  that is analytic on the closure of  $D$ , we know we can use the classical Cauchy kernel, integrate on the boundary of  $D$ ,  $\partial D$ , to recapture the function inside  $D$

$$\frac{1}{w-z} = \frac{1}{1-\frac{z}{w}} \cdot \frac{1}{w} \quad (1)$$

This Cauchy formula representation can be extended to the  $n$ -dimensional complex space,  $\mathbb{C}^n$  and functions  $F(z_1, z_2, \dots, z_n)$  analytic in a polydisc, by using the standard “product” Cauchy kernel,

$$\prod_{i=1}^n \frac{1}{1-\frac{z_i}{w_i}} \cdot \frac{1}{w_1 w_2 \cdots w_n} \quad (2)$$

This is indeed the Cauchy kernel that Hormander introduced in 1966. But there is another side of the story, and this is where the Calderón brothers come in.

There is another Cauchy type kernel in  $n$  dimensions, more elusive, mysterious, but gloriously beautiful, that had been known to Alberto and Calixto since the mid-seventies and that had an underground kind of existence all these years. Alberto Calderón died in April, 1998. A few months before Alberto died, I remember Calixto coming to visit us: I remember Calixto and his older brother deeply engrossed in mathematical conversation, trying to recapture their “elusive” Cauchy

type kernel. But they were unable to; memory did not cooperate. This was all the more frustrating since apparently Alberto and Calixto had made use of this “kernel,” without explicitly exhibiting it, in their 1978 paper, the only other paper that Alberto and Calixto wrote together: this was a paper with five authors, called “*Applications of the Cauchy integral on Lipschitz curves.*” After Alberto died, Calixto was determined to complete the job. He was finally able to recover their “elusive” kernel and to write up the paper. As I said before, the paper, the “posthumous” paper, appeared in the *Indiana Journal of Mathematics* in 2000. We included it also in the *Calderón Selecta*. It is the supreme tribute and gesture of love that Calixto paid to his older brother.

A few words now about this “elusive” Cauchy type kernel which I shall refer to as “*the Calderóns*” kernel. Unlike “*the Hörmander*” kernel which is based on “multiplication,” the Calderóns kernel is based on “addition.”

$$\frac{1}{1 - \sum_{i=1}^n \frac{z_i}{w_i}} \cdot \frac{1}{w_1 w_2 \cdots w_n} \quad (3)$$

This kernel too yields a Cauchy type formula, i.e. the representation of a function analytic in a polydisc, by integration on the boundary of a transformed function.<sup>1</sup> Furthermore, the Calderóns kernel permits to answer the important Calderón Conjecture which Yves Meyer calls the *magic key* opening new chapters in complex analysis, linear PDE and nonlinear PDE, namely, let:

$$K(x, y) = F \left( \frac{A(x) - A(y)}{x - y} \right) \frac{1}{x - y} \quad (4)$$

where:  $A : \mathbb{R} \rightarrow \mathbb{R}^n$  is Lipschitz and  $F$  is analytic. Then the singular integral operator defined by the kernel  $K(x, y)$  is continuous in  $L^2$ .

What a pity that the Calderóns kernel was not publicized several decades ago. It is to be hoped, however, that in the future it will be put to further good use!

Talk given at “Special Functions, Partial Differential Equations and Harmonic Analysis,” A Conference in honor of Calixto P. Calderón, November 16–18, 2012, Department of Mathematics and Actuarial Sciences, Roosevelt University.

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<sup>1</sup>If  $a_{k_1, \dots, k_n}$  are the coefficients of the function, those of the “transformed” one are  $C \frac{k_1! \cdots k_n!}{(k_1 + \cdots + k_n)!} a_{k_1, \dots, k_n}$

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