Chapter 2

Basic Inequalities

In so far as the theorems of mathematics relate to reality, they are not certain, and in so far as they are certain they do not relate to reality.

Every thing should be made as simple as possible but not simpler.

Albert Einstein (1879–1955).

This chapter deals with the basic inequalities used in the rest of the book. The chapter is divided into seven sections and is organized as follows. In Sect. 2.1 we consider Young type inequalities which will be used in the proof of the Hölder and Minkowski inequalities. Section 2.2 discusses Jensen's inequality on time scales and Sect. 2.3 considers Hölder type inequalities. In Sect. 2.4 we consider the Minkowski inequality and Sect. 2.5 is devoted to Steffensen type inequalities on time scales. Section 2.6 considers Hermite–Hadamard type inequalities and finally Sect. 2.7 discusses Čebyšev type inequalities on time scales.

2.1 Young Inequalities

In 1912, Young [157] presented the following highly intuitive integral inequality

$$ab \leq \int_0^a f(t)dt + \int_0^b (f^{-1})(s)ds, \tag{2.1.1}$$

for any real-valued continuous function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ with f strictly increasing on $[0, \infty)$ and a, $b \in [0, \infty)$. The equality holds if

R. Agarwal et al., *Dynamic Inequalities On Time Scales*,
DOI 10.1007/978-3-319-11002-8_2

and only if $b = f(a)$. A useful consequence of this inequality, by taking $f(t) = t^{p-1}$ and $q = \frac{p}{p-1}$, is the classical Young inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{2.1.2}$$

Hardy, Littlewood, and Pólya included (2.1.1) in their classical book [72]. The purpose of this section is to establish this inequality and its extensions on time scales. These will be used in the next sections to prove Hölder and Minkowski inequalities on time scales. The results are adapted from [25, 29, 151].

Theorem 2.1.1 *Let $g \in C_{rd}([0,c]_{\mathbb{T}}, \mathbb{R})$ be a strictly increasing function with $c > 0$. If $g(0) = 0$, $a \in [0, c]_{\mathbb{T}}$ and $b \in [0, g(c)]_{g(\mathbb{T})}$, then*

$$ab \leq \int_0^a g^\sigma(x)\Delta x + \int_0^b (g^{-1})^\sigma(y)\Delta y.$$

Proof. Since $g^{-1}(x)$ is strictly increasing and $\sigma(s) \geq s$, we see that

$$\int_0^b (g^{-1})^\sigma(x)\Delta x = \int_0^b (g^{-1})(\sigma(x))\Delta x \geq \int_0^b (g^{-1}(x))\Delta x. \tag{2.1.3}$$

Letting $v(x) = g(x)$ and $f(x) = x$ in Lemma 1.1.2, we see that

$$\int_0^{g^{-1}(b)} g^\Delta(x)x\Delta x = \int_{g(0)}^{g(g^{-1}(b))} g^{-1}(y)\Delta y = \int_0^b g^{-1}(y)\Delta y. \tag{2.1.4}$$

Integration by parts yields

$$\begin{aligned}
\int_0^{g^{-1}(b)} g^\Delta(x)x\Delta x &= g(x)x|_0^{g^{-1}(b)} - \int_0^{g^{-1}(b)} g^\sigma(x)\Delta x \\
&= bg^{-1}(b) - \int_0^{g^{-1}(b)} g^\sigma(x)\Delta x.
\end{aligned}$$

Thus, (2.1.3) and (2.1.4) imply that

$$\int_0^a g^\sigma(x)\Delta x + \int_0^b (g^{-1})^\sigma(y)\Delta y \geq bg^{-1}(b) + \int_{g^{-1}(b)}^0 g^\sigma(x)\Delta x. \tag{2.1.5}$$

Case (a)**.** $a > g^{-1}(b)$.

It follows from the strictly increasing property of g that

$$\begin{aligned}
\int_{g^{-1}(b)}^a g^\sigma(x)\Delta x &\geq \int_{g^{-1}(b)}^a g(\sigma(g^{-1}(b)))\Delta x \geq \int_{g^{-1}(b)}^a g(g^{-1}(b))\Delta x \\
&= b(a - g^{-1}(b)) = ab - bg^{-1}(b).
\end{aligned}$$

This and (2.1.5) imply

$$\int_0^a g^\sigma(x)\Delta x + \int_0^b (g^{-1})^\sigma(y)\Delta y \geq ab.$$

Case (b)**.** $a < g^{-1}(b)$.

Let $h = g^{-1}$. Then $a < h(b)$. Applying case (a) yields

$$ab \leq \int_0^b h^\sigma(x)\Delta x + \int_0^a (h^{-1})^\sigma(y)\Delta y = \int_0^b \left(g^{-1}\right)^\sigma(x)\Delta x + \int_0^a (g)^\sigma(y)\Delta y.$$

Combining Case (a) and Case (b), we get the desired inequality. The proof is complete. ■

As an application of Theorem 2.1.1 by taking $g(x) = x^{p-1}$ on $[0,\infty)_{\mathbb{T}}$ and $g^{-1}(y) = y^{q-1}$ on $[0,\infty)_{\mathbb{T}}$, we get the following result.

Corollary 2.1.1 *Let $p > 1$ and $q > 1$ with $1/p + 1/q = 1$. If $a \geq 0$ and $b \geq 0$, then*

$$ab \leq \int_0^a (\sigma(x))^{p-1}\Delta x + \int_0^b (\sigma(y))^{q-1}\Delta y.$$

Example 2.1.1 *Let $\mathbb{T} = \mathbb{R}$, then Corollary 2.1.1 says, note that in $\mathbb{R}$ $\sigma(x) = x$, that*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{2.1.6}$$

which is the classical Young inequality.

Example 2.1.2 *Let $\mathbb{T} = \mathbb{Z}$ and $g(t) = t$, then Theorem 2.1.1 says that*

$$ab \leq \sum_{t=0}^{a-1}(t+1) + \sum_{y=0}^{b-1}(y+1) = \frac{1}{2}a(a+1) + \frac{1}{2}b(b+1). \tag{2.1.7}$$

Theorem 2.1.2 *Let $\mathbb{T}$ be any time scale (unbounded above) with $0 \in \mathbb{T}$. Further suppose that $f : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ is a real-valued function satisfying*

(1). *$f(0) = 0$;*

(2). *f is continuous on $[0,\infty)_{\mathbb{T}}$, right-dense continuous at 0;*

(3). *f is strictly increasing on $[0,\infty)_{\mathbb{T}}$ such that $\tilde{\mathbb{T}} = f(\mathbb{T})$ is also a time scale.*

Then for any $a \in [0,\infty)_{\mathbb{T}}$ and $b \in [0,\infty)_{\tilde{\mathbb{T}}}$, we have

$$\int_0^a f(t)\Delta t + \int_0^a f(t)\nabla t + \int_0^b f^{-1}(y)\Delta y + \int_0^b f^{-1}(y)\nabla y \geq 2ab, \tag{2.1.8}$$

with equality if and only if $b = f(a)$.

Proof. From the continuity assumption (2), we see that f is both delta and nabla integrable. For simplicity, define

$$F(a,b) := \int_0^a f(t)\Delta t + \int_0^a f(t)\nabla t + \int_0^b f^{-1}(y)\Delta y + \int_0^b f^{-1}(y)\nabla y - 2ab.$$

Then it is enough to prove that $F(a,b) \geq 0$.

(I). We will first show that

$$F(a,b) \geq F(a, f(a)), \quad a \in [0,\infty)_{\mathbb{T}} \text{ and } b \in [0,\infty)_{\tilde{\mathbb{T}}},$$

with equality if and only if $b = f(a)$. For any such a and b, we have

$$\begin{aligned} F(a,b) - F(a,f(a)) &= \int_{f(a)}^b [f^{-1}(y) - a]\Delta y + \int_{f(a)}^b [f^{-1}(y) - a]\nabla y \\ &= \int_b^{f(a)} [a - f^{-1}(y)]\Delta y + \int_b^{f(a)} [a - f^{-1}(y)]\nabla y. \end{aligned}$$

There are two cases to consider. The first case is $b > f(a)$. Here, whenever $y \in [f(a), b]_{\tilde{\mathbb{T}}}$, we have $f^{-1}(b) \geq f^{-1}(y) \geq f^{-1}(f(a))=a$. Consequently,

$$F(a,b) - F(a,f(a)) = \int_b^{f(a)} [a - f^{-1}(y)]\Delta y + \int_b^{f(a)} [a - f^{-1}(y)]\nabla y \geq 0.$$

Since $f^{-1}(y) - a$ is continuous and strictly increasing for $y \in [f(a), b]_{\tilde{\mathbb{T}}}$, equality will hold if and only if $b = f(a)$. The second case is $b \leq f(a)$. Here whenever $y \in [f(a), b] \cap f(\mathbb{T})$, we have $f^{-1}(b) \leq f^{-1}(y) \leq f^{-1}(f(a)) = a$. Consequently,

$$F(a,b) - F(a,f(a)) = \int_b^{f(a)} [a - f^{-1}(y)]\Delta y + \int_b^{f(a)} [a - f^{-1}(y)]\nabla y \geq 0.$$

Since $a - f^{-1}(y)$ is continuous and strictly decreasing for $y \in [b, f(a)]_{\tilde{\mathbb{T}}}$, equality will hold if and only if $b = f(a)$.

(II). We will next show that $F(a, f(a)) = 0$.

Now, for brevity, we put $\delta(a) = F(a, f(a))$, that is

$$\delta(a) = \int_0^a f(t)\Delta t + \int_0^a f(t)\nabla t + \int_0^{f(a)} f^{-1}(y)\Delta y + \int_0^{f(a)} f^{-1}(y)\nabla y - 2af(a).$$

First, assume a is right scattered point. Then

$$\begin{aligned} \delta^\sigma(a) - \delta(a) &= [\sigma(a) - a]f(a) + [\sigma(a) - a]f^\sigma(a) \\ &\quad + [f^\sigma(a) - f(a)]f^{-1}(f(a)) + [f^\sigma(a) - f(a)]f^{-1}(f^\sigma(a)) \\ &\quad - 2[\sigma(a)f^\sigma(a) - af(a)] \\ &= [\sigma(a) - a][f(a) + f^\sigma(a)] + [f^\sigma(a) - f(a)][\sigma(a) + a] \\ &\quad - 2[\sigma(a)f^\sigma(a) - af(a)] = 0. \end{aligned}$$

Therefore if a is right-scattered point, then $\delta^{\Delta}(a) = 0$. Next, assume a is a right-dense point. Let $\{a_n\}_{n\in\mathbb{N}} \subset [a,\infty)_{\mathbb{T}}$ be a decreasing sequence converging to a. Then

$$\begin{aligned}
& \delta(a_n) - \delta(a) \\
= & \int_a^{a_n} f(t)\Delta t + \int_a^{a_n} f(t)\nabla t + \int_{f(a)}^{f(a_n)} f^{-1}(y)\Delta y + \int_{f(a)}^{f(a_n)} f^{-1}(y)\nabla y \\
& -2a_n f(a_n) + 2af(a). \\
= & \int_a^{a_n} [f(t) - f(a_n)]\Delta t + \int_a^{a_n} [f(t) - f(a_n)]\nabla t + \int_{f(a)}^{f(a_n)} [f^{-1}(y) - a]\Delta y \\
& + \int_{f(a)}^{f(a_n)} [f^{-1}(y) - a]\nabla y.
\end{aligned}$$

Since the functions f and f^{-1} are strictly increasing, we get that

$$\begin{aligned}
\delta(a_n) - \delta(a) \geq & \int_a^{a_n} [f(a) - f(a_n)]\Delta t + \int_a^{a_n} [f(a) - f(a_n)]\nabla t \\
& + \int_{f(a)}^{f(a_n)} [f^{-1}(f(a)) - a]\Delta y + \int_{f(a)}^{f(a_n)} [f^{-1}(f(a)) - a]\nabla y \\
= & \ 2(a_n - a)[f(a) - f(a_n)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\delta(a_n) - \delta(a) \leq & \int_a^{a_n} [f(a_n) - f(a_n)]\Delta t + \int_a^{a_n} [f(a_n) - f(a_n)]\nabla t \\
& + \int_{f(a)}^{f(a_n)} [f^{-1}(f(a_n)) - a_n]\Delta y + \int_{f(a)}^{f(a_n)} [f^{-1}(f(a)) - a]\nabla y \\
= & \ 2(a_n - a)[f(a_n) - f(a)].
\end{aligned}$$

Therefore

$$\begin{aligned}
0 = & \lim_{n\to\infty} 2[f(a_n) - f(a)] \leq \lim_{n\to\infty} \frac{\delta(a_n) - \delta(a)}{(a_n - a)} \\
\leq & \lim_{n\to\infty} 2[f(a_n) - f(a)] = 0.
\end{aligned}$$

It follows that $\delta^{\Delta}(a)$ exists, and $\delta^{\Delta}(a) = 0$ for right-dense a as well. As $\delta(0) = 0$, by a uniqueness theorem for initial value problems, we have that $\delta(a) = 0$ for all $a \in [0,\infty)_{\mathbb{T}}$. This implies that $F(a,b) \geq F(a, f(a)) = 0$, with equality if and only if $b = f(a)$. The proof is complete. ■

As an application of Theorem 2.1.2 when $f(t) = t^{p-1}$ and $f^{-1}(y) = y^{q-1}$, we have the following result.

Corollary 2.1.2 *Let $\mathbb{T}$ be any time scale (unbounded above) with $0 \in \mathbb{T}$. Let $p, q > 1$ be real numbers with $1/p + 1/q = 1$. Then for any $a \in [0, \infty)_{\mathbb{T}}$ and $b \in [0, \infty)_{\mathbb{T}^*}$ where $\mathbb{T}^* = \{t^{p-1} : t \in \mathbb{T}\}$, we have*

$$\int_0^a t^{p-1}\Delta t + \int_0^a t^{p-1}\nabla t + \int_0^b y^{q-1}\Delta y + \int_0^b y^{q-1}\nabla y \geq 2ab,$$

with equality if and only if $b = a^{p-1}$.

Example 2.1.3 *If $\mathbb{T} = \mathbb{R}$, we see that $\sigma(t) = t$ and then Theorem 2.1.2 yields the classical Young inequality (2.1.1).*

Example 2.1.4 *If $\mathbb{T} = \mathbb{Z}$, we see that $\sigma(t) = t + 1$ and then Theorem 2.1.2 yields Young's discrete inequality*

$$2ab \leq \sum_{t=0}^{a-1}[f(t) + f(t+1)] + \sum_{y \in [0,b) \cap f(\mathbb{Z})}^{b-1} \mu(y)[2f^{-1}(y) + 1],$$

since here $f^{-1}(\sigma(y)) = \sigma(f^{-1}(y)) = f^{-1}(y) + 1$.

Theorem 2.1.3 *Let $\mathbb{T}$ be any time scale (unbounded above) with $0 \in \mathbb{T}$. Further suppose that $f : [0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is a real-valued function satisfying:*

(1). *$f(0) = 0$;*

(2). *f is continuous on $[0, \infty)_{\mathbb{T}}$, right-dense continuous at 0;*

(3). *f is strictly increasing on $[0, \infty)_{\mathbb{T}}$ such that $\tilde{\mathbb{T}} = f(\mathbb{T})$ is also a time scale.*

Then for any $a \in [0, \infty)_{\mathbb{T}}$ and $b \in [0, \infty)_{\tilde{\mathbb{T}}}$, we have

$$\int_0^a [f(t) + f^{\sigma}(t)]\,\Delta t + \int_0^b \left[f^{-1}(y) + f^{-1}(\sigma(y))\right]\Delta y \geq 2ab, \tag{2.1.9}$$

with equality if and only if $b = f(a)$.

Proof. For a continuous function g and $a \in [0, \infty)_{\mathbb{T}}$, define the function

$$G(a) = \int_0^a g(t)\Delta t + \int_0^a g(t)\nabla t - \int_0^a [g(t) + g^{\sigma}(t)]\,\Delta t.$$

Then $G(0) = 0$, and

$$G^{\Delta}(a) = g(a) + g^{\sigma}(a) - [g(a) + g^{\sigma}(a)] = 0.$$

Therefore $G \equiv 0$, and Theorem 2.1.3 follows from Theorem 2.1.2. The proof is complete. ■

Next we establish Young integral inequalities with upper and lower bounds for the remainder.

Theorem 2.1.4 *Let $\mathbb{T}$ be any time scale (unbounded above) with $\alpha_1 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Further suppose that $f : [\alpha_1, \infty)_{\mathbb{T}} \to \mathbb{R}$ is a real-valued function satisfying*

(i). $f(\alpha_1) = \beta_1$;

(ii). f is continuous on $[\alpha_1, \infty)_{\mathbb{T}}$, right-dense continuous at α_1;

(iii). f is strictly increasing on $[\alpha_1, \infty)_{\mathbb{T}}$ such that $\tilde{\mathbb{T}} = f(\mathbb{T})$ is also a time scale.

Then for any $a \in [\alpha_1, \infty)_{\mathbb{T}}$ and $b \in [\beta_1, \infty)_{\tilde{\mathbb{T}}}$, we have

$$ab \leq \int_{\alpha_1}^{a} f(t)\Delta t + \int_{\beta_1}^{b} f^{-1}(y)\tilde{\nabla} y + \alpha_1\beta_1, \tag{2.1.10}$$

with equality if and only if $b \in \{f^{\rho}(a), f(a)\}$ for fixed a or with equality if and only if $a \in \{f^{-1}(b), \sigma(f^{-1}(b))\}$ for fixed b. The inequality (2.1.10) is reversed if f is strictly decreasing.

Proof. By the continuity assumption (ii), we see that the function f is delta integrable and the function f^{-1} is nabla integrable. For simplicity, we define

$$F(a,b) = \int_{\alpha_1}^{a} f(t)\Delta t + \int_{\beta_1}^{b} f^{-1}(y)\tilde{\nabla} y + \alpha_1\beta_1 - ab. \tag{2.1.11}$$

To prove (2.1.10), we need to show that $F(a,b) \geq 0$.

(I). We will first show that

$$F(a,b) \geq F(a, f(a)), \text{ for } a \in [\alpha_1, \infty)_{\mathbb{T}} \text{ and } b \in [\beta_1, \infty)_{\tilde{\mathbb{T}}},$$

with equality if and only if $b \in \{f^{\rho}(a), f(a)\}$. For any such a and b, we have

$$F(a,b) - F(a, f(a)) = \int_{f(a)}^{b} [f^{-1}(y) - a]\tilde{\nabla} y. \tag{2.1.12}$$

Clearly if $b = f(a)$, then the integral equals to zero and if $b = f^{\rho}(a)$, then

$$\begin{aligned} F(a, f^{\rho}(a)) - F(a, f(a)) &= \int_{f^{\rho}(a)}^{f(a)} [a - f^{-1}(y)]\tilde{\nabla} y \\ &= [f(a) - f^{\rho}(a)][a - f^{-1}(f(a))] = 0. \end{aligned}$$

Otherwise, since $f^{-1}(y)$ is continuous and strictly increasing for $y \in \tilde{\mathbb{T}}$, the integrals in (2.1.12) are strictly positive for $b < f^{\rho}(a)$ and $b > f(a)$.

(II). We will next show that $F(a, f(a)) = F(a, f^{\rho}(a)) = 0$.

Now, for brevity, we put $\varphi(a) = F(a, f(a))$, that is

$$\varphi(a) = \int_{\alpha_1}^{a} f(t)\Delta t + \int_{\beta_1}^{f(a)} f^{-1}(y)\tilde{\nabla} y - af(a) + \alpha_1\beta_1.$$

First, assume that a is right scattered point. Then

$$\begin{aligned}
& \varphi^\sigma(a)-\varphi(a) \\
= \;& \int_{\alpha_1}^{\sigma(a)} f(t)\Delta t+\int_{f(a)}^{f^\sigma(a)} f^{-1}(y)\tilde{\nabla} y-\sigma(a)f^\sigma(a)+af(a) \\
= \;& [\sigma(a)-a]f(a)+[f^\sigma(a)-f(a)]f^{-1}(f^\sigma(a))-\sigma(a)f^\sigma(a)+af(a) \\
= \;& 0.
\end{aligned}$$

Therefore if a is right-scattered point, then $\varphi^\Delta(a)=0$. Next, assume a is a right-dense point. Let $\{a_n\}_{n\in\mathbb{N}}\subset[a,\infty)_{\mathbb{T}}$ be a decreasing sequence converging to a. Then

$$\begin{aligned}
& \varphi(a_n)-\varphi(a) \\
= \;& \int_a^{a_n} f(t)\Delta t+\int_{f(a)}^{f(a_n)} f^{-1}(y)\tilde{\nabla} y-a_nf(a_n)+af(a) \\
\geq \;& (a_n-a)f)a)+[f(a)-f(a_n)]a-a_nf(a_n)+af(a) \\
= \;& (a_n-a)[f(a)-f(a_n)],
\end{aligned}$$

since the functions f and f^{-1} are strictly increasing. Similarly,

$$\varphi(a_n)-\varphi(a)\leq(a_n-a)[f(a_n)-f(a)].$$

Therefore

$$0=\lim_{n\to\infty}[f(a_n)-f(a)]\leq\lim_{n\to\infty}\frac{\varphi(a_n)-\varphi(a)}{(a_n-a)}\leq\lim_{n\to\infty}[f(a_n)-f(a)]=0.$$

It follows that $\varphi^\Delta(a)$ exists, and $\varphi^\Delta(a)=0$ for right-dense a as well. In other words, in either case $\varphi^\Delta(a)=0$ for $a\in[\alpha_1,\infty)_{\mathbb{T}}$. As $\varphi(\alpha_1)=0$, by a uniqueness theorem for initial value problems, we have that $\varphi(a)=0$ for all $a\in[\alpha_1,\infty)_{\mathbb{T}}$. As $F(a,f(a))=F(a,f^\rho(a))=0$, we have that $F(a,b)\geq F(a,f(a))=0$, with equality if and only if $b=f(a)$ or $b=f^\rho(a)$. The case with $a\in\{f^{-1}(b),\sigma(f^{-1}(b))\}$ for fixed b is similar and thus omitted. If f is strictly decreasing, it is straightforward to see that the inequality (2.1.10) is reversed. The proof is complete. ■

Now to establish upper bounds for Young's integral inequality we need the following result.

Lemma 2.1.1 *Let f satisfy the hypotheses of Theorem 2.1.4, and let $F(a,b)$ be given as in (2.1.11). Then for any a, $\alpha\in\mathbb{T}$ and $b,\beta\in\tilde{\mathbb{T}}$, we have*

$$F(a,b)+F(\alpha,\beta)\geq-(\alpha-a)(\beta-b), \tag{2.1.13}$$

with equality if and only if $\alpha\in\{f^{-1}(b),\sigma(f^{-1}(b))\}$ and $\beta\in\{f^\rho(a),\ f(a)\}$.

Proof. Fix $a \in \mathbb{T}$ and $b \in \tilde{\mathbb{T}}$. By Young's integral inequality (2.1.10), we see that

$$\int_{\alpha_1}^{a} f(t)\Delta t + \int_{\beta_1}^{\beta} f^{-1}(y)\tilde{\nabla} y + \alpha_1\beta_1 \geq a\beta, \tag{2.1.14}$$

and

$$\int_{\alpha_1}^{\alpha} f(t)\Delta t + \int_{\beta_1}^{\beta} f^{-1}(y)\tilde{\nabla} y + \alpha_1\beta_1 \geq \alpha b, \tag{2.1.15}$$

with equality if and only if $\beta \in \{f^\rho(a),\ f(a)\}$ and $\alpha \in \{f^{-1}(b),\ \sigma(f^{-1}(b))\}$, respectively. By rearranging it follows that

$$\begin{aligned}
& \int_{\alpha_1}^{a} f(t)\Delta t + \int_{\beta_1}^{b} f^{-1}(y)\tilde{\nabla} y + \alpha_1\beta_1 - ab \\
& + \int_{\alpha_1}^{\alpha} f(t)\Delta t + \int_{\beta_1}^{\beta} f^{-1}(y)\tilde{\nabla} y + \alpha_1\beta_1 - \alpha b \\
= \quad & \int_{\alpha_1}^{a} f(t)\Delta t + \int_{\beta_1}^{\beta} f^{-1}(y)\tilde{\nabla} y + \alpha_1\beta_1 \\
& + \int_{\alpha_1}^{\alpha} f(t)\Delta t + \int_{\beta_1}^{b} f^{-1}(y)\tilde{\nabla} y + \alpha_1\beta_1 - ab - \alpha\beta \\
\geq \quad & a\beta + \alpha b - ab - \alpha\beta = -(\alpha - a)(\beta - b).
\end{aligned}$$

Note that equality holds here if and only if it holds in (2.1.14) and (2.1.15), and this happens if and only if $\beta \in \{f^\rho(a),\ f(a)\}$ and $\alpha \in \{f^{-1}(b), \sigma(f^{-1}(b))\}$. The proof is complete. ■

Theorem 2.1.5 *Let $\mathbb{T}$ be any time scale and $f : [\alpha_1, \alpha_2]_{\mathbb{T}} \to [\beta_1, \beta_2]_{\tilde{\mathbb{T}}}$ be a continuous strictly increasing function such that $\tilde{\mathbb{T}} = f(\mathbb{T})$ is also a time scale. Then for every $a,\ A \in [\alpha_1, \alpha_2]_{\mathbb{T}}$ and $b,\ B \in [\beta_1, \beta_2]_{\tilde{\mathbb{T}}}$, we have*

$$\begin{aligned}
(f^{-1}(B) - A)(f^\rho(A) - B)ab \quad & \leq \quad \int_{A}^{a} f(t)\Delta t + \int_{B}^{b} f^{-1}(y)\tilde{\nabla} y - ab + AB \\
& \leq \quad -(f^{-1}(b) - a)(f^\rho(a) - b),
\end{aligned} \tag{2.1.16}$$

with equality if and only if $B \in \{f^\rho(A), f(A)\}$ and $b \in \{f^\rho(a), f(a)\}$. The inequalities are reversed if f is strictly decreasing.

Proof. Considering F as in (2.1.11) and (2.1.13) with $\alpha = f^{-1}(b)$ and $\beta = f(a)$, we have the equality

$$F(a, b) + F(f^{-1}(b), f^\rho(a)) = -(f^{-1}(b) - a)(f^\rho(a) - b).$$

As $f^{-1} \in [\alpha_1, \alpha_2]_{\mathbb{T}}$ and $f^{\rho} \in [\beta_1, \beta_2]_{\widetilde{\mathbb{T}}}$, via Young's inequality (2.1.10), we see that $F(f^{-1}(b), f^{\rho}(a)) \geq 0$. Consequently, we have that

$$0 \leq F(a,b) \leq -(f^{-1}(b) - a)(f^{\rho}(a) - b), \tag{2.1.17}$$

and inequality holds if and only if $b \in \{f^{\rho}(a), f(a)\}$. Thus for any $A \in [\alpha_1, \alpha_2]_{\mathbb{T}}$ and $B \in [\beta_1, \beta_2]_{\widetilde{\mathbb{T}}}$, we have from (2.1.17) that

$$0 \leq -(f^{-1}(B) - A)(f^{\rho}(A) - B) - F(A,B), \tag{2.1.18}$$

with equality if and only if $B \in \{f^{\rho}(A), f(A)\}$. Combining (2.1.17) and (2.1.18), we get

$$\begin{aligned} 0 &\leq F(a,b) - (f^{-1}(B) - A)(f^{\rho}(A) - B) - F(A,B) \\ &\leq -(f^{-1}(b) - a)(f^{\rho}(a) - b) - (f^{-1}(B) - A)(f^{\rho}(A) - B) - F(A,B), \end{aligned}$$

which can be rewritten to obtain (2.1.16). If f strictly decreasing the proof is similar and omitted. The proof is complete. ■

In the following, we establish a theorem which can be considered as a modification of Theorem 2.1.5 above. This theorem allows us to get a Young type integral inequality without having to find f^{-1}.

Theorem 2.1.6 *Let the hypotheses of Theorem 2.1.5 hold. Then for any $a, \alpha, A, \Lambda \in [\alpha_1, \alpha_2]_{\mathbb{T}}$, we have*

$$\begin{aligned} (\Lambda - A)(f^{\rho}(A) - f(\Lambda)) &\leq \int_A^a f(t)\Delta t - \int_{\Lambda}^{\alpha} f(t)\Delta t \\ &\quad +(\alpha - a)f(\alpha) + (A - \Lambda)f(\Lambda) \\ &\leq -(\alpha - a)(f^{\rho}(a) - f(\alpha)), \end{aligned} \tag{2.1.19}$$

where equalities hold if and only if $\Lambda \in \{\rho(A), A\}$ and $\alpha \in \{\rho(a), a\}$.

Proof. By Theorem 2.1.5 with $A = \Lambda$, $B = f(\Lambda)$, $a = \alpha$ and $b = f(\alpha)$, we have

$$\int_{f(\Lambda)}^{f(\alpha)} f^{-1}(y)\widetilde{\nabla} y = \alpha f(\alpha) - \Lambda f(\Lambda) - \int_{\Lambda}^{\alpha} f(t)\Delta t, \tag{2.1.20}$$

for any $\alpha, \Lambda \in [\alpha_1, \alpha_2]_{\mathbb{T}}$. Since $\alpha, \Lambda \in [\alpha_1, \alpha_2]_{\mathbb{T}}$ are arbitrary, we substitute (2.1.20) into (2.1.16) to obtain (2.1.19). The proof is complete. ■

In the following, we apply the results when $\mathbb{T} = \mathbb{Z}$ and derive some discrete inequalities. Recall that $[\alpha_1, \alpha_2]_{\mathbb{Z}} = \{\alpha_1, \alpha_1 + 1, \ldots, \alpha_2 - 1, \alpha_2\}$. The first two theorems are direct translations to $\mathbb{T} = \mathbb{Z}$ of Theorem 2.1.5 and Theorem 2.1.6, respectively.

Theorem 2.1.7 *Let $f : [\alpha_1, \alpha_2]_{\mathbb{Z}} \to [\beta_1, \beta_2]_{\tilde{\mathbb{Z}}}$ be strictly increasing, where $\tilde{\mathbb{Z}} = f(\mathbb{Z})$. Then for every a, $A \in [\alpha_1, \alpha_2]_{\mathbb{Z}}$ and b, $B \in [\beta_1, \beta_2]_{\tilde{\mathbb{Z}}}$, we have*

$$\begin{aligned} & \left[f^{-1}(B) - A)\right] (f(A-1) - B) \\ \leq & \sum_{n=A}^{a-1} f(n) + \sum_{m \in (B,b) \cap \tilde{\mathbb{Z}}}^{a-1} f^{-1}(m) \widetilde{\nu}(m) - ab + AB \\ \leq & -(f^{-1}(b) - a)(f(a-1) - b), \end{aligned}$$

where equalities hold if and only if $B \in \{f(A-1),\ f(A)\}$ and $b \in \{f(a-1),\ f(a)\}$.

Theorem 2.1.8 *Let $f : \mathbb{Z} \to \mathbb{R}$ be strictly increasing. Then for every a, A, α, Λ, we have*

$$\begin{aligned} & [\Lambda - A)] (f(A-1) - f(\Lambda)) \\ \leq & \sum_{n=A}^{a-1} f(n) - \sum_{m=\Lambda}^{\alpha-1} f(m) + (\alpha - a) f(\alpha) + (A - \Lambda) \\ \leq & -(\alpha - a)(f(a-1) - f(\alpha)), \end{aligned}$$

where equalities holds if and only if $\Lambda \in \{(A-1),\ (A)\}$ and $\alpha \in \{(a-1),\ a\}$.

Example 2.1.5 *Consider the factorial function*

$$f_k(t) = t^{(k)} = t(t-1)\dots(t-k+1), \ \text{for} \ \ t,\ k \in \mathbb{Z}.$$

It is clear that f_k is increasing on the interval $[k-1, \infty)_{\mathbb{Z}}$. By Theorem 2.1.8, we have

$$(a - \alpha) f_k(\alpha) \leq \frac{1}{k+1} [f_{k+1}(a) - f_{k+1}(\alpha)] \leq (a - \alpha) f_k(a-1),$$

for a, $\alpha \in \{k-1, k, k+1, \dots\}$, where equalities hold if and only if $\alpha \in \{a-1, a\}$.

Example 2.1.6 *Let $f(t) = \sin[\pi t / 2k]$ for $k \in \mathbb{N}$. Then f is increasing on $[-k, k]$, so that for any $a \geq \alpha \in [-k, k]_{\mathbb{Z}}$, we have by Theorem 2.1.8 that*

$$\begin{aligned} \sin \frac{\alpha\pi}{2k} & \leq \frac{1}{2(a-\alpha)} \left(\cos \left[\frac{(2\alpha - 1)\,\pi}{4k} \right] - \cos \left[\frac{(2a - 1)\,\pi}{4k} \right] \right) \csc \frac{\pi}{4k} \\ & \leq \sin \frac{(a-1)\pi}{2k}, \end{aligned}$$

with equalities if and only if $\alpha \in \{a-1, a\}$.

2.2 Jensen Inequalities

The original Jensen inequality proved by Jensen states that if $g \in C([a, b], (c, d))$ and $F \in C([a, b], \mathbb{R})$ is convex, then

$$F\left(\frac{\int_a^b g(s)ds}{b-a}\right) \leq \frac{1}{b-a}\int_a^b F(g(s))ds. \tag{2.2.1}$$

In this section we give extensions of this inequality on time scales. The inequalities will be proved for delta derivative, nabla derivative as well as for diamond-α derivative. The results are adapted from [11, 23, 30, 39, 115, 150].

We begin with a lemma adapted from [67].

Lemma 2.2.1 *Let $f \in C((c, d), \mathbb{R})$ be convex. Then for each $t \in (c, d)$, there exits $\beta_t \in \mathbb{R}$ such that*

$$f(x) - f(t) \geq \beta_t(x-t), \quad \text{for all } x \in (c, d). \tag{2.2.2}$$

If f is strictly convex, then the inequality sign $\geq$ in (2.2.2) should be replaced by $>$.

Theorem 2.2.1 *Let $a, b \in \mathbb{T}$ and c, $d \in \mathbb{R}$. Let $g \in C_{rd}([a, b], (c, d))$ and $F \in C((c, d), \mathbb{R})$ is convex. Then*

$$F\left(\frac{\int_a^b g(s)\Delta s}{b-a}\right) \leq \frac{1}{b-a}\int_a^b F(g(s))\Delta s. \tag{2.2.3}$$

If F is strictly convex, then the inequality $\leq$ can be replaced by $<$.

Proof. Since F is convex, it follows from Lemma 2.2.1 that for each $t \in (c, d)$, there exists $\beta_t \in \mathbb{R}$ such that (2.2.2) holds. Let

$$t = \frac{1}{b-a}\int_a^b g(s)\Delta s.$$

Now

$$\begin{aligned}
&\int_a^b F(g(s))\Delta s - (b-a)\, F\left(\frac{\int_a^b g(s)\Delta s}{b-a}\right) \\
= &\int_a^b F(g(s))\Delta s - (b-a)\, F\,(t) \\
\geq &\ \beta_t \int_a^b [g(s) - t]\, \Delta s = \beta\left[\int_a^b g(s)\Delta s - t(b-a)\right] = 0.
\end{aligned}$$

The proof is complete. ■

Example 2.2.1 *As a special case let* $\mathbb{T} = \mathbb{R}$ *and* $F = -\log$. *Note* F *is convex and continuous on* $(0, \infty)$. *Apply Theorem 2.2.1 with* $a = 0$ *and* $b = 1$ *to obtain* $\log \int_0^1 g(t)dt \geq \int_0^1 \log(g(t))dt$, *and hence* $\int_0^1 g(t)dt \geq \exp\left(\int_0^1 \log(g(t))dt\right)$, *whenever* $g \in C([0,1),(0,\infty))$ *is continuous.*

Example 2.2.2 *Let* $\mathbb{T} = \mathbb{N}$ *and* $N \in \mathbb{N}$. *Apply Jensen's inequality (Theorem 2.2.1) with* $a = 1$ *and* $b = N+1$ *and* $g : [1, N+1]_{\mathbb{N}} \to (0, \infty)$ *to find*

$$\begin{aligned}
\log\left[\frac{1}{N}\sum_{n=1}^{N} g(n)\right] &\geq \log\left[\frac{1}{N}\int_1^{N+1} g(t)\Delta t\right] \\
&\geq \frac{1}{N}\int_1^{N+1} \log(g(t))\Delta t \\
&= \frac{1}{N}\sum_{n=1}^{N} \log(g(n)) = \log\left(\prod_{n=1}^{N} g(n)\right)^{1/N},
\end{aligned}$$

and hence

$$\frac{1}{N}\sum_{n=1}^{N} g(n) \geq \left(\prod_{n=1}^{N} g(n)\right)^{1/N}.$$

This is the well-known arithmetic-mean geometric-mean inequality.

Example 2.2.3 *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$ *and* $N \in \mathbb{N}$. *Apply Jensen's inequality (Theorem 2.2.1) with* $a = 1$ *and* $b = 2^N$ *and* $g : [1, 2^N]_{2^{\mathbb{N}_0}} \to (0, \infty)$ *to find*

$$\begin{aligned}
&\log\left[\frac{1}{2^N - 1}\sum_{n=0}^{N-1} 2^n g(2^n)\right] \\
\geq\ &\log\left[\frac{1}{2^N - 1}\int_1^{2^N} g(t)\Delta t\right] \\
\geq\ &\frac{1}{2^N - 1}\int_1^{2^N} \log(g(t))\Delta t = \frac{1}{2^N - 1}\sum_{n=0}^{N-1} 2^n \log(g(2^n)) \\
=\ &\frac{1}{2^N - 1}\sum_{n=0}^{N-1} \log((g(2^n))^{2^n} = \log\left(\prod_{n=1}^{N} ((g(2^n))^{2^n}\right)^{1/(2^N-1)},
\end{aligned}$$

and hence

$$\frac{1}{2^N - 1}\sum_{n=0}^{N-1} 2^n g(2^n) \geq \left(\prod_{n=1}^{N} ((g(2^n))^{2^n}\right)^{1/(2^N-1)}.$$

Theorem 2.2.2 *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C_{rd}([a,b],(c,d))$ and $h \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with*

$$\int_a^b |h(s)| \Delta s > 0.$$

If $F \in C((c,d), \mathbb{R})$ is convex, then

$$F\left(\frac{\int_a^b |h(s)| g(s)\Delta s}{\int_a^b |h(s)| \Delta s}\right) \leq \int_a^b \frac{|h(s)| F(g(s))\Delta s}{\int_a^b |h(s)| \Delta s}. \tag{2.2.4}$$

If F is strictly convex, then the inequality $\leq$ can be replaced by $<$.

Proof. Since F is convex it follows from Lemma 2.2.1 that for each $t \in (c,d)$, there exists $\beta_t \in \mathbb{R}$ such that (2.2.2) holds. Let

$$t = \frac{\int_a^b |h(s)| g(s)\Delta s}{\int_a^b |h(s)| \Delta s}.$$

Thus

$$\begin{aligned}
& \int_a^b |h(s)| F(g(s))\Delta s - \left(\int_a^b |h(s)| \Delta s\right) F\left(\frac{\int_a^b |h(s)| g(s)\Delta s}{\int_a^b |h(s)| \Delta s}\right) \\
= \;& \int_a^b |h(s)| F(g(s))\Delta s - \left(\int_a^b |h(s)| \Delta s\right) F(t) \\
= \;& \int_a^b |h(s)| \left[F(g(s)) - F(t)\right] \Delta s \geq \beta_t \int_a^b |h(s)| \left[g(s) - t\right] \Delta s \\
= \;& \beta_t \left[\int_a^b |h(s)| g(s)\Delta s - t \int_a^b |h(s)| \Delta s\right] \\
= \;& \beta_t \left[\int_a^b |h(s)| g(s)\Delta s - \frac{\int_a^b |h(s)| g(s)\Delta s}{\int_a^b |h(s)| \Delta s} \int_a^b |h(s)| \Delta s\right] = 0.
\end{aligned}$$

The proof is complete. ■

Remark 2.2.1 *If the condition of convexity of the function F is changed to concavity, then the inequality sign of the inequality (2.2.4) is reversed.*

As a special case of Theorem 2.2.2, when $g(t) \geq 0$ on $[a,b]$ and $F(t) = t^\gamma$ on $[0,\infty)$, we see that F is convex on $[0,\infty)$ for $\alpha < 0$ or $\alpha > 1$ and F is concave on $[0,\infty)$ for $\alpha \in (0,1)$.

Corollary 2.2.1 *Let $g \in C_{rd}([a,b],(c,d))$ such that $g(t) \geq 0$ on $[a,b]$ and $h \in C_{rd}([a,b], \mathbb{R})$ with*

$$\int_a^b |h(s)| \Delta s > 0,$$

where $a, b \in \mathbb{T}$ and $(c, d) \subset \mathbb{R}$. Then

$$\left(\frac{\int_a^b |h(s)|\, g(s)\Delta s}{\int_a^b |h(s)|\, \Delta s}\right)^{\alpha} \leq \frac{\int_a^b |h(s)|\, g^{\alpha}(s)\Delta s}{\int_a^b |h(s)|\, \Delta s}, \quad \text{for } \alpha < 0 \text{ or } \alpha > 1,$$

and

$$\left(\frac{\int_a^b |h(s)|\, g(s)\Delta s}{\int_a^b |h(s)|\, \Delta s}\right)^{\alpha} \geq \frac{\int_a^b |h(s)|\, g^{\alpha}(s)\Delta s}{\int_a^b |h(s)|\, \Delta s}, \quad \text{for } \alpha \in (0, 1).$$

We now present nabla Jensen inequalities.

Theorem 2.2.3 *Let $a, b \in \mathbb{T}$ and c, $d \in \mathbb{R}$, and $h \in C_{ld}([a, b]_{\mathbb{T}}, \mathbb{R})$ and $g \in C_{ld}([a, b], (c, d))$ with $\int_a^b |h(\tau)|\, \nabla\tau > 0$, and $\phi \in C((c, d), \mathbb{R})$ is convex, then*

$$\phi\left(\frac{\int_a^b |h(\tau)|\, g(\tau)\nabla\tau}{\int_a^b |h(\tau)|\, \nabla\tau}\right) \leq \frac{\int_a^b |h(\tau)|\, \phi(g(\tau))\nabla\tau}{\int_a^b |h(\tau)|\, \nabla\tau}. \tag{2.2.5}$$

If ϕ is strictly convex, then the inequality $\leq$ can be replaced by $<$.

Proof. Since ϕ is convex, it follows from Lemma 2.2.1 that for each $t \in (c, d)$, there exists $\beta_t \in \mathbb{R}$ such that (2.2.2) holds. Let

$$t = \frac{\int_a^b |h(s)|\, g(s)\nabla s}{\int_a^b |h(s)|\, \nabla s}.$$

Thus

$$\begin{aligned}
& \int_a^b |h(s)|\, \phi(g(s))\nabla s - \left(\int_a^b |h(s)|\, \nabla s\right) \phi\left(\frac{\int_a^b |h(s)|\, g(s)\nabla s}{\int_a^b |h(s)|\, \nabla s}\right) \\
= \;& \int_a^b |h(s)|\, \phi(g(s))\Delta s - \left(\int_a^b |h(s)|\, \nabla s\right) \phi\,(t) \\
= \;& \int_a^b |h(s)|\, [\phi(g(s)) - \phi\,(t)]\, \Delta s \geq \beta_t \int_a^b |h(s)|\, [g(s) - t]\, \nabla s \\
= \;& \beta_t \left[\int_a^b |h(s)|\, g(s)\nabla s - t \int_a^b |h(s)|\, \nabla s\right] \\
= \;& \beta_t \left[\int_a^b |h(s)|\, g(s)\nabla s - \frac{\int_a^b |h(s)|\, g(s)\nabla s}{\int_a^b |h(s)|\, \Delta s} \int_a^b |h(s)|\, \nabla s\right] = 0.
\end{aligned}$$

The proof is complete. ■

As a consequence of Theorem 2.2.3, we have the following result.

Theorem 2.2.4 *Let $a, b \in \mathbb{T}$ and c, $d \in \mathbb{R}$. If $h \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$ and $g \in C_{ld}([a,b],(c,d))$ are nonnegative, with $\int_a^b h(t)\nabla t > 0$, and $\phi : (c,d) \to \mathbb{R}$ is continuous and convex, then*

$$\phi\left(\frac{\int_a^b h(t)g(t)\nabla t}{\int_a^b h(t)\nabla t}\right) \le \frac{\int_a^b h(t)\phi(g(t))\nabla t}{\int_a^b h(t)\nabla t}.$$

If ϕ is strictly convex, then the inequality $\le$ can be replaced by $<$.

Now, we give some generalized versions of Jensen's inequality on time scales via the diamond-α integral.

Theorem 2.2.5 *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ and c, $d \in \mathbb{R}$. Suppose that $g \in C([a,b]_{\mathbb{T}},(c,d))$ and $F \in C((c,d), \mathbb{R})$ is convex. Then*

$$F\left(\frac{\int_a^b g(s)\Diamond_\alpha s}{b-a}\right) \le \frac{1}{b-a}\int_a^b F(g(s))\Diamond_\alpha s. \tag{2.2.6}$$

If F is strictly convex, then the inequality $\le$ can be replaced by $<$.

Proof. Since F is convex, we have

$$F\left(\frac{\int_a^b g(s)\Diamond_\alpha \Delta s}{b-a}\right) = F\left(\frac{\alpha}{b-a}\int_a^b g(s)\Delta s + \frac{(1-\alpha)}{b-a}\int_a^b g(s)\nabla s\right)$$

$$\le \quad \alpha F\left(\frac{1}{b-a}\int_a^b g(s)\Delta s\right) + (1-\alpha)F\left(\frac{1}{b-a}\int_a^b g(s)\nabla s\right).$$

Now, using delta and nabla Jensen inequalities, we get that

$$\begin{aligned} F\left(\frac{\int_a^b g(s)\Diamond_\alpha \Delta s}{b-a}\right) &\le \frac{\alpha}{b-a}\left(\int_a^b F(g(s))\Delta s\right) + \frac{(1-\alpha)}{b-a}\left(\int_a^b F(g(s))\nabla s\right) \\ &= \frac{1}{b-a}\left[\left(\int_a^b F(g(s))\Delta s\right) + \left(\int_a^b F(g(s))\nabla s\right)\right] \\ &= \frac{1}{b-a}\left(\int_a^b F(g(s))\Diamond_\alpha \Delta s\right). \end{aligned}$$

The proof is complete. ■

In the following, we give a generalization of (2.2.6) on time scales.

Theorem 2.2.6 *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a,b],(c,d))$ and $h \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |h(s)| \Diamond_\alpha s > 0$. If $F \in C((c,d), \mathbb{R})$ is convex, then*

$$F\left(\frac{\int_a^b |h(s)|\, g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}\right) \le \frac{\int_a^b |h(s)|\, F(g(s))\Diamond_\alpha s}{\int_a^b |h(s)|\Diamond_\alpha s}. \tag{2.2.7}$$

If F is strictly convex, then the inequality $\le$ can be replaced by $<$.

Proof. Since F is convex, it follows from Lemma 2.2.1 that for each $t \in (c,d)$, there exists $\beta_t \in \mathbb{R}$ such that (2.2.2) holds. Setting

$$t = \frac{\int_a^b |h(s)|\, g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\, \Diamond_\alpha s},$$

we get that

$$\begin{aligned}
& \int_a^b |h(s)|\, F(g(s))\Diamond_\alpha s - \left(\int_a^b |h(s)|\, \Diamond_\alpha s\right) F\left(\frac{\int_a^b |h(s)|\, g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\, \Diamond_\alpha s}\right) \\
= & \int_a^b |h(s)|\, F(g(s))\Diamond_\alpha s - \left(\int_a^b |h(s)|\, \Delta s\right) F(t) \\
= & \int_a^b |h(s)|\, [F(g(s)) - F(t)]\, \Diamond_\alpha s \geq \beta_t \int_a^b |h(s)|\, [g(s) - t]\, \Diamond_\alpha s \\
= & \beta_t \left[\int_a^b |h(s)|\, g(s)\Diamond_\alpha s - t\int_a^b |h(s)|\, \Diamond_\alpha s\right] \\
= & \beta_t \left[\int_a^b |h(s)|\, g(s)\Diamond_\alpha s - \frac{\int_a^b |h(s)|\, g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\, \Diamond_\alpha s}\int_a^b |h(s)|\, \Diamond_\alpha s\right] = 0.
\end{aligned}$$

The proof is complete. ■

Remark 2.2.2 *If the convexity condition of the function F is changed to concavity, then the inequality sign of the inequality (2.2.7) is reversed.*

As a special case of Theorem 2.2.6, when $F(t) = t^\gamma$ on $[0,\infty)$, we see that F is convex on $[0,\infty)$ for $\gamma < 0$ or $\gamma > 1$ and F is concave on $[0,\infty)$ for $\gamma \in (0,1)$. This gives us the following result.

Corollary 2.2.2 *Let $g \in C([a,b],(c,d))$ such that $g(t) > 0$ on $[a,b]_{\mathbb{T}}$ and $h \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with*

$$\int_a^b |h(s)|\, \Diamond_\alpha s > 0,$$

where $a, b \in \mathbb{T}$ and $(c,d) \subset \mathbb{R}$. Then

$$\left(\frac{\int_a^b |h(s)|\, g(s)\Diamond_\alpha s}{\int_a^b |h(s)|\, \Diamond_\alpha s}\right)^\gamma \leq \frac{\int_a^b |h(s)|\, g^\gamma(s)\Diamond_\alpha s}{\int_a^b |h(s)|\, \Diamond_\alpha s}, \text{ for } \gamma < 0 \text{ or } \gamma > 1,$$

and

$$\left(\frac{\int_a^b |h(s)|\, g(s)\Delta s}{\int_a^b |h(s)|\, \Diamond_\alpha s}\right)^\gamma \geq \frac{\int_a^b |h(s)|\, g^\gamma(s)\Diamond_\alpha s}{\int_a^b |h(s)|\, \Diamond_\alpha s}, \quad \text{for } \gamma \in (0,1).$$

Example 2.2.4 *Let $g(t) > 0$ on $[a,b]_{\mathbb{T}}$ and $F(t) = \ln(t)$ on $(0,\infty)$. Now, since F is concave on $(0,\infty)$, it follows from Theorem 2.2.6 that*

$$\ln\left(\frac{\int_a^b |h(s)|\, g(s)\Delta s}{\int_a^b |h(s)|\, \Diamond_\alpha s}\right) \geq \frac{\int_a^b |h(s)| \ln\left(g(s)\right) \Diamond_\alpha s}{\int_a^b |h(s)|\, \Diamond_\alpha s}.$$

Example 2.2.5 *Let $\mathbb{T} = \mathbb{Z}$ and $n \in \mathbb{N}$. Fix $a = 1$ and $b = N+1$ and consider $g : [1, N+1]_{\mathbb{N}} \to (0,\infty)$ and let $F(t) = -\ln t$. Now F is convex and continuous on $(0,\infty)$. Apply the Jensen inequality (2.2.7) to obtain*

$$\begin{aligned}
&\ln\left[\frac{\alpha}{N}\sum_{n=1}^{N} g(n) + \frac{1-\alpha}{N}\sum_{n=2}^{N+1} g(n)\right] \\
= \;& \ln\left(\int_1^{N+1} \frac{1}{N} g(t) \Diamond_\alpha t\right) \\
\geq \;& \frac{1}{N}\int_1^{N+1} \ln(g(t))\Diamond_\alpha t = \frac{\alpha}{N}\sum_{n=1}^{N} \ln g(n) + \frac{1-\alpha}{N}\sum_{n=2}^{N+1} \ln g(n) \\
= \;& \ln\left(\prod_{n=1}^{N} g(n)\right)^{\frac{\alpha}{N}} + \ln\left(\prod_{n=2}^{N+1} g(n)\right)^{\frac{1-\alpha}{N}},
\end{aligned}$$

and hence

$$\frac{1}{N}\left[\alpha\sum_{n=1}^{N} g(n) + (1-\alpha)\sum_{n=2}^{N+1} g(n)\right] \geq \left(\prod_{n=1}^{N} g(n)\right)^{\frac{\alpha}{N}} \left(\prod_{n=2}^{N+1} g(n)\right)^{\frac{1-\alpha}{N}}.$$

When $\alpha = 1$, we obtain the well-known arithmetic-mean geometric-mean inequality

$$\frac{1}{N}\sum_{n=1}^{N} g(n) \geq \left(\prod_{n=1}^{N} g(n)\right)^{\frac{1}{N}},$$

and when $\alpha = 0$, we obtain

$$\frac{1}{N}\sum_{n=2}^{N+1} g(n) \geq \left(\prod_{n=2}^{N+1} g(n)\right)^{\frac{1}{N}}.$$

Example 2.2.6 *Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $F(t) = -\ln t$. Apply the Jensen inequality (Theorem 2.2.6) with $a = 1$ and $b = 2^N$ and $g : [1, 2^N]_{2^{\mathbb{N}_0}} \to (0,\infty)$, we find that*

$$\begin{aligned}
& \ln\left[\frac{1}{2^N-1}\int_1^{2^N} g(t)\Diamond_\alpha t\right] \\
= & \ln\left[\frac{\alpha}{2^N-1}\int_1^{2^N} g(t)\Delta t + \frac{1-\alpha}{2^N-1}\int_1^{2^N} g(t)\nabla t\right] \\
= & \ln\left[\frac{\alpha}{2^N-1}\sum_{n=0}^{N-1} 2^n \log(g(2^n)) + \frac{1-\alpha}{2^N-1}\sum_{n=1}^{N} 2^n \log(g(2^n))\right] \\
\geq & \left[\frac{1}{2^N-1}\int_1^{2^N} \ln g(t)\Diamond_\alpha t\right] \\
= & \frac{\alpha}{2^N-1}\sum_{n=0}^{N-1} 2^n \log((g(2^n)) + \frac{1-\alpha}{2^N-1}\sum_{n=1}^{N} 2^n \log((g(2^n)) \\
= & \frac{\alpha}{2^N-1}\sum_{n=0}^{N-1} \log((g(2^n))^{2^n} + \frac{1-\alpha}{2^N-1}\sum_{n=1}^{N} \log((g(2^n))^{2^n} \\
= & \frac{1}{2^N-1}\ln\prod_{n=0}^{N-1}((g(2^n))^{\alpha 2^n} + \frac{1}{2^N-1}\ln\prod_{n=1}^{N}((g(2^n))^{(1-\alpha)2^n} \\
= & \ln\left(\prod_{n=1}^{N}((g(2^n))^{2^n}\right)^{\frac{1}{2^N-1}} + \ln\left(\prod_{n=1}^{N}((g(2^n))^{(1-\alpha)2^n}\right)^{\frac{1}{2^N-1}}.
\end{aligned}$$

From this we conclude that

$$\begin{aligned}
& \ln\left[\frac{\alpha}{2^N-1}\sum_{n=0}^{N-1} 2^n \log(g(2^n)) + \frac{1-\alpha}{2^N-1}\sum_{n=1}^{N} 2^n \log(g(2^n))\right] \\
\geq & \ln\left[\left(\prod_{n=1}^{N}((g(2^n))^{2^n}\right)^{\frac{1}{2^N-1}}\left(\prod_{n=1}^{N}((g(2^n))^{(1-\alpha)2^n}\right)^{\frac{1}{2^N-1}}\right],
\end{aligned}$$

and hence

$$\begin{aligned}
& \frac{\alpha}{2^N-1}\sum_{n=0}^{N-1} 2^n \log(g(2^n)) + \frac{1-\alpha}{2^N-1}\sum_{n=1}^{N} 2^n \log(g(2^n)) \\
\geq & \left(\prod_{n=1}^{N}((g(2^n))^{2^n}\right)^{\frac{1}{2^N-1}}\left(\prod_{n=1}^{N}((g(2^n))^{(1-\alpha)2^n}\right)^{\frac{1}{2^N-1}}.
\end{aligned}$$

Since

$$\begin{aligned}
& \frac{1}{2^N-1}\left[\alpha\sum_{n=0}^{N-1} 2^n \log(g(2^n)) + (1-\alpha)\sum_{n=1}^{N} 2^n \log(g(2^n))\right] \\
= & \frac{1}{2^N-1}\sum_{n=1}^{N-1} 2^n \log(g(2^n)) + \alpha g(1) + (1-\alpha)2^N g(2^N),
\end{aligned}$$

we get that

$$\begin{aligned} &\frac{1}{2^N-1}\sum_{n=1}^{N-1} 2^n \log(g(2^n)) + \alpha g(1) + (1-\alpha)2^N g(2^N) \\ \geq\ & \left(\prod_{n=1}^{N} ((g(2^n))^{2^n}\right)^{\frac{1}{2^N-1}} \left(\prod_{n=1}^{N} ((g(2^n))^{(1-\alpha)2^n}\right)^{\frac{1}{2^N-1}}. \end{aligned}$$

As an application of Theorem 2.2.6, we have the following result.

Theorem 2.2.7 *Let $\mathbb{T}$ be a time scale, a, $b \in \mathbb{T}$ with $a < b$ and f, g, $h \in C([a,b]_{\mathbb{T}}, (0,\infty))$.*

(i) If $p > 1$, then

$$\begin{aligned} &\left[\left(\int_a^b h(s)f(s)\Diamond_\alpha s\right)^p + \left(\int_a^b h(s)g(s)\Diamond_\alpha s\right)^p\right]^{1/p} \\ \leq\ & \int_a^b h(s)\left[f^p(s) + g^p(s)\right]^{1/p} \Diamond_\alpha s. \end{aligned} \tag{2.2.8}$$

(ii) If $0 < p < 1$, then

$$\begin{aligned} &\left[\left(\int_a^b h(s)f(s)\Diamond_\alpha s\right)^p + \left(\int_a^b h(s)g(s)\Diamond_\alpha s\right)^p\right]^{1/p} \\ \geq\ & \int_a^b h(s)\left[f^p(s) + g^p(s)\right]^{1/p} \Diamond_\alpha s. \end{aligned} \tag{2.2.9}$$

Proof. We prove only (i), since the proof of (ii) is similar. Inequality (2.2.8) is trivially true when f is zero. Otherwise, applying Theorem 2.2.6 with $F(x) = (1 + x^p)^{1/p}$, which is clearly convex on $(0,\infty)$, we obtain

$$\left(1 + \frac{\int_a^b h(s)f(s)\Diamond_\alpha s}{\int_a^b h(s)\Diamond_\alpha s}\right)^{1/p} \leq \frac{\int_a^b h(s)(1 + f^p(s))^{1/p}\Diamond_\alpha s}{\int_a^b h(s)\Diamond_\alpha s}.$$

In other words

$$\left(\int_a^b h(s)\Diamond_\alpha s + \int_a^b h(s)f(s)\Diamond_\alpha s\right)^{1/p} \leq \int_a^b h(s)(1 + f^p(s))^{1/p}\Diamond_\alpha s.$$

Changing h and f with $hf/\int_a^b h(s)f(s)\Diamond_\alpha s$ and g/f in the last inequality we obtain (2.2.8). The proof is complete. ■

Using the fact that the time scale integral is an isotonic linear functional, we prove some Jensen type inequalities on time scales.

Definition 2.2.1 *Let E be a nonempty set and L be a linear class of real-valued functions $f : E \to \mathbb{R}$, having the following properties:*

(L_1). *If f, $g \in L$ and a, $b \in \mathbb{R}$, then $(af + bg) \in L$.*

(L_2). *If $f(t) = 1$ for all $t \in E$, then $f \in L$.*

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:

(A_1). *If f, $g \in L$ and a, $b \in \mathbb{R}$, then $A(af + bg) = aA(f) + bA(g)$.*

(A_2). *If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.*
Furthermore, if the functional A has a property

(A_3). *$A(\mathbf{1}) = 1$, where $\mathbf{1}(t) = 1$ for all $t \in E$, then we will say that A is normalized.*

Our next theorem proves that the Cauchy integral on time scales is an isotonic functional. The proof is straightforward from its definition and properties presented in [51, Defintion 1.58 and Theorem 1.77].

Theorem 2.2.8 *Let $\mathbb{T}$ be a time scale, a, $b \in \mathbb{T}$ with $a < b$ and let*

$$E = [a, b) \cap \mathbb{T}, \ L = C_{rd}([a, b), \mathbb{R}). \tag{2.2.10}$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t)\Delta t, \tag{2.2.11}$$

where the integral is the Cauchy delta time-scale integral. Then (A_1) and (A_2) are satisfied.

Example 2.2.7 *If $\mathbb{T} = \mathbb{R}$ in Theorem 2.2.8, then $L = C([a, b], \mathbb{R})$ and $A(f) = \int_a^b f(t)dt$. If $\mathbb{T} = \mathbb{Z}$ in Theorem 2.2.8, then L consists of real-valued functions on $[a, b-1] \cap \mathbb{Z}$ and $A(f) = \sum\limits_{n=a}^{b-1} f(n)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$, in Theorem 2.2.8, then L consists of real-valued functions on $[a, b/q] \cap q^{\mathbb{N}_0}$ and $A(f) = (q-1) \sum\limits_{n=\log_q(a)}^{\log_q(b)-1} q^n f(q^n)$.*

Theorem 2.2.8 also has corresponding versions for the nabla and the α-diamond integral.

Theorem 2.2.9 *Let $\mathbb{T}$ be a time scale, a, $b \in \mathbb{T}$ with $a < b$ and let*

$$E = (a, b] \cap \mathbb{T}, \ L = C_{ld}((a, b], \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t)\nabla t,$$

where the integral is the Cauchy nabla time-scale integral. Then (A_1) and (A_2) are satisfied.

Theorem 2.2.10 *Let $\mathbb{T}$ a time scale, a, $b \in \mathbb{T}$ with $a < b$ and let*

$$E = [a, b] \cap \mathbb{T}, \quad L = C([a, b], \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t)\Diamond_\alpha t,$$

where the integral is the Cauchy α-diamond time-scale integral. Then (A_1) and (A_2) are satisfied.

The Riemann multiple integral is also an isotonic linear functional.

Theorem 2.2.11 *Let $\mathbb{T}_1, \ldots, \mathbb{T}_n$ a time scales. For a_i, $b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \leq i \leq n$, let*

$$E \subset ([a_1, b_1) \cap \mathbb{T}_1 \times \ldots \times [a_n, b_n) \cap \mathbb{T}_n,$$

be Jordan Δ-measurable and let L be the set of all bounded Δ-integrable functions from E to $\mathbb{R}$. Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_E f(t)\Delta t,$$

where the integral is the multiple Riemann delta-time scale integral. Then (A_1) and (A_2) are satisfied.

Theorem 2.2.12 *Let $\mathbb{T}_1, \ldots, \mathbb{T}_n$ be time scales. For a_i, $b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \leq i \leq n$, let*

$$E \subset ([a_1, b_1) \cap \mathbb{T}_1 \times \ldots \times [a_n, b_n) \cap \mathbb{T}_n,$$

be Lebesgue Δ-measurable and let L be the set of all bounded Δ-integrable functions from E to $\mathbb{R}$. Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_E f(t)\Delta t,$$

where the integral is the multiple Lebesgue delta-time scale integral. Then (A_1) and (A_2) are satisfied.

Theorem 2.2.13 *Let the assumptions of Theorem 2.2.12 be satisfied. Let $A(f)$ be replaced by*

$$A(f) = \frac{\int_E |h(t)|\, f(t)\Delta t}{\int_E |h(t)|\, \Delta t},$$

where $h : \mathbb{E} \to \mathbb{R}$ is Δ-integrable such that $\int_E |h(t)|\, \Delta t > 0$. Then A is an isotonic linear functional satisfying $A(1) = 1$.

We next note the following theorem that has been proved by Jessen [87] (see also [117]).

Theorem 2.2.14 *Let L satisfy properties (L_1) and (L_2). Assume $\Phi \in C(\mathbb{I},\mathbb{R})$ is convex where $\mathbb{I} \subset \mathbb{R}$ is an interval. If A satisfies (A_1) and (A_2) such that $A(1) = 1$, then for all $f \in L$ such that $\Phi(f) \in L$, one has $A(f) \in \mathbb{I}$ and*

$$\Phi(A(f)) \leq A(\Phi(f)).$$

Now, the application of Theorems 2.2.13 and 2.2.14 gives the following result.

Theorem 2.2.15 *Assume that $\Phi \in C(\mathbb{I},\mathbb{R})$ is convex where $\mathbb{I} \subset \mathbb{R}$ is an interval. Let $E \subset \mathbb{R}^n$ be as in Theorem 2.2.12 and suppose that f is Δ-integrable on E such that $f(E) = \mathbb{I}$. Moreover, let $h : E \to \mathbb{R}$ be Δ-integrable such that $\int_E |h(t)|\, \Delta t > 0$. Then*

$$\Phi\left(\frac{\int_E |h(t)|\, f(t)\Delta t}{\int_E |h(t)|\, \Delta t}\right) \leq \frac{\int_E |h(t)|\, \Phi(f(t))\Delta t}{\int_E |h(t)|\, \Delta t}.$$

The concept of superquadratic functions in one variable, as a generalization of the class of convex functions was introduced by S. Abramovich, G. Jameson, and G. Sinnamon in [1, 2].

Definition 2.2.2 *A function $\varphi : [0,\infty) \to \mathbb{R}$ is called superquadratic if there exists a function $C : [0,\infty) \to \mathbb{R}$ such that*

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C(x)(y-x), \quad \textit{for all } x, y > 0.$$

We say that φ is subquadratic if $-\varphi$ is superquadratic.

For example, the function $\varphi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $p \in (0,2]$.

Lemma 2.2.2 *Let φ be a superquadratic function with C as in Definition 2.2.2. Then* *(i)* $\varphi(0) \leq 0$,

(ii) *if* $\varphi(0) = \varphi^{'}(0)$, *then* $C(x) = \varphi^{'}(x)$ *whenever* φ *is differentiable at* $x > 0$,

(iii) *if* $\varphi \geq 0$, *then* φ *is convex and* $\varphi(0) = \varphi^{'}(0) = 0$.

In the following, we prove a Jensen type inequality on time scales for superquadratic functions.

Theorem 2.2.16 *Let* a, $b \in \mathbb{T}$. *Suppose* $f \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$ *and* $\varphi : [0,\infty) \to \mathbb{R}$ *is continuous and superquadratic. Then*

$$\varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{1}{b-a}\int_a^b \left[\varphi(f(s)) - \varphi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\right]\Delta s. \tag{2.2.12}$$

Proof. Since $\varphi : [0,\infty) \to \mathbb{R}$ is a superquadratic function, then there exists a function $C : [0,\infty) \to \mathbb{R}$ such that

$$\varphi(y) \geq \varphi(x_0) + \varphi(|y - x_0|) + C(x_0)(y - x_0), \text{ for all } x_0, y > 0. \tag{2.2.13}$$

Let

$$x_0 = \frac{1}{(b-a)}\int_a^b f(t)\Delta t.$$

Applying (2.2.13) with $y = f(s)$, we see that

$$\begin{aligned} \varphi(f(s)) \quad &\geq \quad \varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) + \varphi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right) \\ &\quad + C(x_0)(f(s) - x_0). \end{aligned}$$

Integrating from a to b, we see that

$$\begin{aligned} &\int_a^b \left[\varphi(f(s)) - \varphi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right) - \varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)\right]\Delta s \\ \geq \quad & C(x_0)\int_a^b (f(s) - x_0)\Delta s = C(x_0)\left[\int_a^b f(s)\Delta s - (b-a)x_0\right] = 0. \end{aligned}$$

This implies that

$$\varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{1}{b-a}\int_a^b \left[\varphi(f(s)) - \varphi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\right]\Delta s,$$

which is the desired inequality (2.2.12). The proof is complete. ■

2.3 Hölder Inequalities

In 1889 Hölder [84] proved that

$$\sum_{k=1}^{n} x_k y_k \leq \left(\sum_{k=1}^{n} x_k^p\right)^{1/p} \left(\sum_{k=1}^{n} y_k^q\right)^{1/q}, \tag{2.3.1}$$

where x_n and y_n are positive sequences and p and q are two positive numbers such that $1/p + 1/q = 1$. The inequality reverses if either p or q is negative. The integral form of this inequality is

$$\int_a^b |f(t)g(t)|\, dt \leq \left[\int_a^b |f(t)|^p\, dt\right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q\, dt\right]^{\frac{1}{q}}, \tag{2.3.2}$$

where $a,\ b \in \mathbb{R}$ and $f,\ g \in C([a,b],\ \mathbb{R})$. In this section, we discuss various versions of the Hölder inequality on time scales which not only give a unification of (2.3.1) and (2.3.2) but can be applied on different types of time scales. The results in this section are adapted from [11, 24, 30, 39, 145, 155]. We begin with the proof of the classical Hölder inequality on time scales.

Theorem 2.3.1 *Let $a,\ b \in \mathbb{T}$. For $f,\ g \in C_{rd}(\mathbb{I},\ \mathbb{R})$, we have*

$$\int_a^b |f(t)g(t)|\, \Delta t \leq \left[\int_a^b |f(t)|^p\, \Delta t\right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q\, \Delta t\right]^{\frac{1}{q}}, \tag{2.3.3}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For nonnegative real numbers α and β, the classical Young inequality

$$\alpha^{1/p}\beta^{1/q} \leq \frac{\alpha}{p} + \frac{\beta}{q}, \tag{2.3.4}$$

holds. Now suppose without loss of generality that

$$\left(\int_a^b |f(t)|^p\, \Delta t\right)\left(\int_a^b |g(t)|^q\, \Delta t\right) \neq 0.$$

Apply (2.3.4) with

$$\alpha = \frac{|f(t)|^p}{\left(\int_a^b |f(s)|^p\, \Delta s\right)}, \text{ and } \beta = \frac{|g(t)|^q}{\int_a^b |g(s)|^q\, \Delta s},$$

and integrate the obtained inequality between a and b (this is possible since all functions are rd-continuous), we find that

$$\int_a^b \frac{|f(t)|}{\left(\int_a^b |f(s)|^p \Delta s\right)^{1/p}} \frac{|g(t)|}{\left(\int_a^b |g(s)|^q \Delta s\right)^{1/q}} \Delta t = \int_a^b \alpha^{1/p}(t)\beta^{1/q}(t)\Delta t$$

$$\leq \int_a^b \left(\frac{\alpha(t)}{p} + \frac{\beta(t)}{q}\right)\Delta t = \int_a^b \left[\frac{|f(t)|^p}{p\left(\int_a^b |f(s)|^p \Delta s\right)} + \frac{|g(t)|^q}{q\int_a^b |g(s)|^q \Delta s}\right]\Delta t$$

$$= \frac{\int_a^b |f(t)|^p \Delta t}{p\left(\int_a^b |f(s)|^p \Delta s\right)} + \frac{\int_a^b |g(t)|^q \Delta t}{q\int_a^b |g(s)|^q \Delta s} = \frac{1}{p} + \frac{1}{q} = 1,$$

which is the desired inequality (2.3.3). The proof is complete. ■

As a special case when $p = q = 2$, we have the following Schwarz's inequality.

Theorem 2.3.2 *Let $a, b \in \mathbb{T}$. For $f, g \in C_{rd}(\mathbb{I}, \mathbb{R})$, we have*

$$\int_a^b |f(t)g(t)| \Delta t \leq \left[\int_a^b |f(t)|^2 \Delta t\right]^{\frac{1}{2}} \left[\int_a^b |g(t)|^2 \Delta t\right]^{\frac{1}{2}}. \tag{2.3.5}$$

Setting

$$\alpha = \frac{|h(t)|^{1/p}|f(t)|}{\left(\int_a^b |h(s)||f(s)|^p \Delta s\right)^{1/p}}, \text{ and } \beta = \frac{|h(t)|^{1/q}|f(t)|}{\left(\int_a^b |h(s)||g(s)|^q \Delta s\right)^{1/q}},$$

in the proof of Theorem 2.3.1 and applying the Young inequality, we have the following inequality.

Theorem 2.3.3 *Let $h, f, g \in C_r([a,b]_{\mathbb{T}}, [0,\infty))$. If $1/p + 1/q = 1$, with $p > 1$, then*

$$\int_a^b h(t)f(t)g(t)\Delta t \leq \left(\int_a^b h(t)f^p(t)\Delta t\right)^{1/p} \left(\int_a^b h(t)g^q(t)\Delta t\right)^{1/q}. \tag{2.3.6}$$

Now we give the nabla Hölder type inequality on time scales.

Theorem 2.3.4 *Let $a, b \in \mathbb{T}$. For $f, g, h \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R})$, we have*

$$\int_a^b |h(t)||f(t)g(t)| \nabla t \leq \left[\int_a^b |h(t)||f(t)|^p \nabla t\right]^{\frac{1}{p}} \left[\int_a^b |h(t)||g(t)|^q \nabla t\right]^{\frac{1}{q}}, \tag{2.3.7}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Setting

$$A=\frac{|h(t)|^{1/p}\,|f(t)|}{\left(\int_a^b |h(s)|\,|f(s)|^p\,\nabla s\right)^{1/p}},\ \text{and}\ B=\frac{|h(t)|^{1/q}\,|f(t)|}{\left(\int_a^b |h(s)|\,|g(s)|^q\,\nabla s\right)^{1/q}},$$

and applying the Young inequality $AB\le\frac{A^p}{p}+\frac{B^q}{q}$, where A, B are nonnegative, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we see that

$$\begin{aligned}
\int_a^b A(t)B(t)\nabla t &\le \int_a^b\left(\frac{A^p}{p}+\frac{B^q}{q}\right)\nabla t\\
&= \int_a^b\left[\frac{|h(t)|\,|f(t)|^p}{p\left(\int_a^b |h(s)|\,|f(s)|^p\,\nabla s\right)}+\frac{|h(t)|\,|g(t)^q|}{q\int_a^b |h(s)|\,|g(s)|^q\,\Delta s}\right]\nabla t\\
&= \frac{\int_a^b |h(t)|\,|f(t)|^p\,\nabla t}{p\left(\left(\int_a^b |h(s)|\,|f(s)|^p\,\nabla s\right)\right)}+\frac{\int_a^b |h(t)|\,|g(t)^q|\,\nabla t}{q\int_a^b |h(s)|\,|g(s)|^q\,\nabla s}\\
&= \frac{1}{p}+\frac{1}{q}=1,
\end{aligned}$$

which is the desired inequality (2.3.7). The proof is complete. ■

As a special case of Theorem 2.3.4 when $p=q=2$, we have the following result.

Theorem 2.3.5 *Let $a,\ b\in\mathbb{T}$. For $f,\ g,\ h\in C_{ld}([a,b]_{\mathbb{T}},\ \mathbb{R})$, we have*

$$\int_a^b |h(t)|\,|f(t)g(t)|\,\nabla t\le\left[\int_a^b |h(t)|\,|f(t)|^2\,\nabla t\right]^{\frac{1}{2}}\left[\int_a^b |h(t)|\,|g(t)|^2\,\nabla t\right]^{\frac{1}{2}}. \tag{2.3.8}$$

Theorem 2.3.6 *Let $a,\ b\in\mathbb{T}$. For $f,\ g,\ h\in C_{ld}([a,b]_{\mathbb{T}},\ \mathbb{R})$, we have*

$$\int_a^b |h(t)|\,|f(t)g(t)|\,\nabla t\ge\left[\int_a^b |h(t)|\,|f(t)|^p\,\nabla t\right]^{\frac{1}{p}}\left[\int_a^b |h(t)|\,|g(t)|^q\,\nabla t\right]^{\frac{1}{q}}, \tag{2.3.9}$$

where $p<0$ or $q<0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Without loss of generality, we assume that $p<0$. Set $P=-p/q$ and $Q=1/q$. Then $1/P+1/Q=1$ with $P>1$ and $Q>1$. From (2.3.7) we have

$$\int_a^b |h(t)|\,|F(t)G(t)|\,\nabla t\le\left[\int_a^b |h(t)|\,|F(t)|^P\,\nabla t\right]^{\frac{1}{P}}\left[\int_a^b |h(t)|\,|G(t)|^Q\,\nabla t\right]^{\frac{1}{Q}}.$$

Letting $F(t) = f^{-q}(t)$ and $G(t) = f^q(t)g^q(t)$ in the last inequality, we get the desired inequality (2.3.9). The proof is complete. ■

As an application of Hölder inequality (2.3.3), we have the following theorem.

Theorem 2.3.7 *Let $a,\ b \in \mathbb{T}$ with $a < b$ and f and g be two positive functions defined on the interval $[a,b]_{\mathbb{T}}$ such that $0 < m \leq f/g \leq M < \infty$. Then for $p > 1$ and $q > 1$ with $1/p + 1/q = 1$, we have*

$$\int_a^b f^{1/p}(t)g^{1/q}(t)\Delta t \leq \frac{M^{1/p^2}}{m^{1/q^2}} \int_a^b f^{1/q}(t)g^{1/p}(t)\Delta t, \tag{2.3.10}$$

and then

$$\int_a^b f^{1/p}(t)g^{1/q}(t)\Delta t \leq \frac{M^{1/p^2}}{m^{1/q^2}} \left(\int_a^b f(t)\Delta t\right)^{1/q} \left(\int_a^b g(t)\Delta t\right)^{1/p}.$$

Proof. From inequality (2.3.3), we obtain

$$\int_a^b f^{1/p}(t)g^{1/q}(t)\Delta t \leq \left(\int_a^b f(t)\Delta t\right)^{1/p} \left(\int_a^b g(t)\Delta t\right)^{1/q},$$

that is

$$\int_a^b f^{1/p}(t)g^{1/q}(t)\Delta t \leq \left(\int_a^b f^{1/p}(t)f^{1/q}(t)\Delta t\right)^{1/p} \left(\int_a^b g^{1/q}(t)g^{1/p}(t)\Delta t\right)^{1/q}.$$

Since $f^{1/p}(t) \leq M^{1/p}g^{1/p}(t)$ and $g^{1/q}(t) \leq m^{-1/q}f^{1/q}(t)$, then from the above inequality it follows that

$$\begin{aligned}\int_a^b f^{1/p}(t)g^{1/q}(t)\Delta t \quad &\leq \quad M^{1/p^2}m^{-1/q^2}\left(\int_a^b f^{1/q}(t)g^{1/p}(t)\Delta t\right)^{1/p} \\ &\quad \times \left(\int_a^b f^{1/q}(t)g^{1/p}(t)\Delta t\right)^{1/q},\end{aligned}$$

that is

$$\int_a^b f^{1/p}(t)g^{1/q}(t)\Delta t \leq M^{1/p^2}m^{-1/q^2} \int_a^b f^{1/q}(t)g^{1/p}(t)\Delta t. \tag{2.3.11}$$

Hence, the inequality (2.3.10) is proved. The proof is complete. ■

The following theorems give the reverse Hölder type inequality on time scales.

Theorem 2.3.8 *Let a, $b \in \mathbb{T}$ with $a < b$ and f and g be two positive functions defined on the interval $[a, b]_{\mathbb{T}}$ such that $0 < m \le f^p/g^q \le M < \infty$. Then for $p > 1$ and $q > 1$ with $1/p + 1/q = 1$, we have*

$$\left(\int_a^b f^p(t)\Delta t\right)^{1/p}\left(\int_a^b g^q(t)\Delta t\right)^{1/q} \le \left(\frac{M}{m}\right)^{\frac{1}{pq}}\int_a^b f(t)g(t)\Delta t. \tag{2.3.12}$$

Proof. Since $f^p/g^q \le M$, then we have $g \ge M^{-1/q}f^{p/q}$. Therefore

$$fg \ge M^{\frac{-1}{q}}f^{\frac{p}{q}+1} = M^{\frac{-1}{q}}f^{\frac{p+q}{q}} = M^{\frac{-1}{q}}f^p,$$

and so

$$\left(\int_a^b f^p(t)\Delta t\right)^{\frac{1}{p}} \le M^{\frac{1}{pq}}\left(\int_a^b f(t)g(t)\Delta t\right)^{\frac{1}{p}}. \tag{2.3.13}$$

Also since $m \le f^p/g^q$, then we have $f \ge m^{1/p}g^{q/p}$. Then

$$\int_a^b f(t)g(t)\Delta t \ge m^{1/p}\int_a^b g^{1+q/p}(t)\Delta t = m^{1/p}\int_a^b g^q(t)\Delta t,$$

and so

$$\left(\int_a^b f(t)g(t)\Delta t\right)^{1/q} \ge m^{\frac{1}{pq}}\left(\int_a^b g^q(t)\Delta t\right)^{\frac{1}{q}}. \tag{2.3.14}$$

Combining (2.3.13) and (2.3.14), we have the desired inequality (2.3.12). The proof is complete. ■

In Theorem 2.3.8, if we replace f^p and g^q by f and g, we obtain the reverse Hölder type inequality

$$\left(\int_a^b f(t)\Delta t\right)^{1/p}\left(\int_a^b g(t)\Delta t\right)^{1/q} \le \left(\frac{M}{m}\right)^{\frac{1}{pq}}\int_a^b f^{1/p}(t)g^{1/q}(t)\Delta t. \tag{2.3.15}$$

Theorem 2.3.9 *Let a, $b \in \mathbb{T}$ with $a < b$ and f and g be two positive functions defined on the interval $[a, b]_{\mathbb{T}}$ such that $0 < m \le f^p \le M < \infty$. Then for $p > 1$ and $q > 1$ with $1/p + 1/q = 1$, we have*

$$\left(\int_a^b f^{1/p}(t)\Delta t\right)^{p} \ge (b-a)^{\frac{p+1}{q}}\left(\frac{m}{M}\right)^{\frac{p+1}{pq}}\left(\int_a^b f^p(t)\Delta t\right)^{1/p}. \tag{2.3.16}$$

Proof. Putting $g = 1$ in Theorem 2.3.8, we obtain

$$\left(\int_a^b f^p(t)\Delta t\right)^{1/p}(b-a)^{1/q} \le \left(\frac{m}{M}\right)^{\frac{-1}{pq}}\int_a^b f(t)\Delta t.$$

Therefore, we get

$$\left(\int_a^b f^p(t)\Delta t\right)^{1/p} \leq \left(\frac{m}{M}\right)^{\frac{-1}{pq}} (b-a)^{-1/q} \int_a^b f(t)\Delta t. \tag{2.3.17}$$

Substituting g in (2.3.15) leads to

$$\left(\int_a^b f(t)\Delta t\right)^{1/p} \leq \left(\frac{m}{M}\right)^{\frac{-1}{pq}} (b-a)^{-1/q} \int_a^b f^{1/p}(t)\Delta t,$$

and so

$$\int_a^b f(t)\Delta t \leq \left(\frac{m}{M}\right)^{\frac{-1}{q}} (b-a)^{-p/q} \left(\int_a^b f^{1/p}(t)\Delta t\right)^p. \tag{2.3.18}$$

Combining (2.3.17) with (2.3.18), we obtain

$$\left(\int_a^b f^{1/p}(t)\Delta t\right)^p \geq \left(\frac{m}{M}\right)^{\frac{p+1}{pq}} (b-a)^{(p+1)/q} \left(\int_a^b f^p(t)\Delta t\right)^{1/p},$$

which is the desired inequality (2.3.16). The proof is complete. ■

Next we prove a Hölder type inequality in two dimensionals on time scales.

Theorem 2.3.10 *Let $a,\ b \in \mathbb{T}$ with $a < b$ and f and g be two rd-continuous functions defined on the interval $[a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$. Then*

$$\begin{aligned} &\int_a^b \int_a^b |f(x,y)g(x,y)|\, \Delta x \Delta y \qquad (2.3.19)\\ \leq\ &\left(\int_a^b \int_a^b |f(x,y)|^p\, \Delta x \Delta y\right)^{1/p} \left(\int_a^b \int_a^b |g(x,y)|^q\, \Delta x \Delta y\right)^{1/q}, \end{aligned}$$

where $p > 1$ and $q = p/((p-1)$.

Proof. Suppose without loss of generality that

$$\left(\int_a^b \int_a^b |f(x,y)|^p\, \Delta x \Delta y\right) \int_a^b \int_a^b |g(x,y)|^q\, \Delta x \Delta y \neq 0.$$

Apply the Young inequality $\alpha^{1/p}\beta^{1/q} \leq \frac{\alpha}{p} + \frac{\beta}{q}$(2.3.4) with

$$\begin{aligned} \alpha(x,y) &= \frac{|f(x,y)|^p}{\int_a^b \int_a^b |f(\tau_1,\tau_2)|^p\, \Delta \tau_1 \Delta \tau_2}, \\ \beta(x,y) &= \frac{|g(x,y)|^q}{\int_a^b \int_a^b |g(\tau_1,\tau_2)|^q\, \Delta \tau_1 \Delta \tau_2}, \end{aligned}$$

and integrate the obtained inequality between a and b to get

$$\begin{aligned}
&\int_a^b \int_a^b \alpha^{1/p}(x,y)\beta^{1/q}(x,y)\Delta x\Delta y \\
\leq\ & \int_a^b \int_a^b \left(\frac{\alpha(x,y)}{p} + \frac{\beta(x,y)}{q}\right)\Delta x\Delta y \\
=\ & \frac{\int_a^b \int_a^b |f(x,y)|^p \,\Delta x\Delta y}{p\int_a^b \int_a^b |f(\tau_1,\tau_2)|^p \,\Delta\tau_1\Delta\tau_2} + \frac{\int_a^b \int_a^b |g(x,y)|^q \,\Delta x\Delta y}{q\int_a^b \int_a^b |g(\tau_1,\tau_2)|^q \,\Delta\tau_1\Delta\tau_2} \\
=\ & \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned}$$

The proof is complete. ∎

Now, we give the diamond α-Hölder inequalities on time scales by applying the diamond α-Jensen inequalities on time scales. As an application of the diamond α-Jensen inequality proved in Theorem 2.2.6 by taking $F(t) = t^p$ for $p > 1$ and g and $|h|$ be replaced by $ug^{-p/q}$ and hg^q, we have the following Hölder inequality.

Theorem 2.3.11 *Let h, u, $g \in C([a,b]_{\mathbb{T}},\mathbb{R})$ with $\int_a^b h(t)g^q(t)\diamond_\alpha t > 0$. If $1/p + 1/q = 1$, with $p > 1$, then*

$$\int_a^b |h(t)|\,|u(t)g(t)|\,\diamond_\alpha t \leq \left(\int_a^b |h(t)|\,|u(t)|^p \,\diamond_\alpha t\right)^{1/p} \left(\int_a^b |h(t)|\,|g(t)|^q \,\diamond_\alpha t\right)^{1/q}. \tag{2.3.20}$$

In the particular case $h = 1$, Theorem 2.3.11 gives the diamond-α version of the classical Hölder inequality:

$$\int_a^b |u(t)g(t)|\,\diamond_\alpha t \leq \left(\int_a^b |u(t)|^p \,\diamond_\alpha t\right)^{1/p} \left(\int_a^b |g(t)|^q \,\diamond_\alpha t\right)^{1/q}, \tag{2.3.21}$$

where $p > 1$ and $q = p/(p-1)$. In the special case $p = q = 2$, the inequality (2.3.21) reduces to the following diamond-α Cauchy–Schwarz integral inequality on time scales

$$\int_a^b |u(t)g(t)|\,\diamond_\alpha t \leq \sqrt{\left(\int_a^b |u(t)|^2 \,\diamond_\alpha t\right)\left(\int_a^b |g(t)|^2 \,\diamond_\alpha t\right)}. \tag{2.3.22}$$

Theorem 2.3.12 *Let h, u, $g \in C([a,b]_{\mathbb{T}},\mathbb{R})$ with $\int_a^b h(t)g^q(t)\Diamond_\alpha t > 0$. If $1/p + 1/q = 1$, with $p < 0$ or $q < 0$, then*

$$\begin{aligned}\int_a^b |h(t)|\,|u(t)g(t)|\,\Diamond_\alpha t &\geq \left(\int_a^b |h(t)|\,|u(t)|^p\,\Diamond_\alpha t\right)^{1/p} \\ &\quad\times\left(\int_a^b |h(t)|\,|g(t)|^q\,\Diamond_\alpha t\right)^{1/q}.\end{aligned}$$

Theorem 2.3.13 *Let a, $b \in \mathbb{T}$ with $a < b$ and f and g be two positive functions defined on the interval $[a,b]_{\mathbb{T}}$ such that $0 < m \leq f^p/g^q \leq M < \infty$. Then for $p > 1$ with $1/p + 1/q = 1$, we have*

$$\left(\int_a^b f^p(t)\Diamond_\alpha t\right)^{1/p}\left(\int_a^b g^q(t)\Diamond_\alpha t\right)^{1/q} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}}\int_a^b f(t)g(t)\Diamond_\alpha t. \quad (2.3.23)$$

Proof. As in the proof of Theorem 2.3.8, we get that

$$\left(\int_a^b f^p(t)\Diamond_\alpha t\right)^{\frac{1}{p}} \leq M^{\frac{1}{pq}}\left(\int_a^b f(t)g(t)\Diamond_\alpha t\right)^{\frac{1}{p}},$$

and

$$\left(\int_a^b f(t)g(t)\Diamond_\alpha t\right)^{1/q} \geq (m)^{\frac{1}{pq}}\left(\int_a^b g^q(t)\Diamond_\alpha t\right)^{\frac{1}{q}}.$$

Combining these two inequalities, we have the desired inequality (2.3.23). The proof is complete. ■

Now, we give the diamond α-Hölder type inequality in two dimensions on time scales. In this case, we assume that the double integral is defined as an iterated integral. Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$, $a < b$, and f be a real-valued function on $\mathbb{T} \times \mathbb{T}$. Because we need notation for partial derivatives with respect to time scale variables x and y we denote the time scale partial derivative of $f(x,y)$ with respect to x by $f^{\Diamond_\alpha^1}(x,y)$ and let $f^{\Diamond_\alpha^2}(x,y)$ denote the time scale partial derivative with respect to y. Fix an arbitrary $y \in \mathbb{T}$. Then the diamond-α derivative of the function

$$\mathbb{T} \to \mathbb{R}, \qquad x \to f(x,y)$$

is denoted by $f^{\Diamond_\alpha^1}$. Let now $x \in \mathbb{T}$. The diamond-α derivative of the function

$$\mathbb{T} \to \mathbb{R}, \qquad y \to f(x,y)$$

is denoted by $f^{\Diamond_\alpha^{21}}$. If the function f has a $\Diamond_\alpha^1$ antiderivative A, i.e., $A^{\Diamond_\alpha^1} = f$, and A has a $\Diamond_\alpha^2$ antiderivative B, i.e., $B^{\Diamond_\alpha^2} = A$, then

$$\begin{aligned}\int_a^b\int_a^b f(x,y)\Diamond_\alpha x\Diamond_\alpha y &= \int_a^b (A(b,y) - A(a,y))\Diamond_\alpha y \\ &= B(b,b) - B(b,a) - B(a,b) + B(a,a).\end{aligned}$$

Note that $\left(B^{\Diamond_\alpha^2}\right)^{\Diamond_\alpha^1} = (A)^{\Diamond_\alpha^1} = f$.

Now we are ready to state and prove the diamond α-Hölder inequality in two dimensions on time scales.

Theorem 2.3.14 *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, with $a < b$, f, g, $h : [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}} \to \mathbb{R}$, be $\diamondsuit_\alpha$ integrable functions, and $1/p + 1/q = 1$ with $p > 1$. Then,*

$$\begin{aligned}
&\int_a^b \int_a^b |h(x,y) f(x,y) g(x,y)| \diamondsuit_\alpha x \diamondsuit_\alpha y \qquad (2.3.24)\\
\leq \; & \left(\int_a^b \int_a^b |h(x,y) f(x,y)|^p \diamondsuit_\alpha x \diamondsuit_\alpha y \right)^{1/p} \\
& \left(\int_a^b \int_a^b |h(x,y) g(x,y)|^q \diamondsuit_\alpha x \diamondsuit_\alpha y \right)^{1/q} .
\end{aligned}$$

Proof. Inequality (2.3.24) is trivially true in the case when f, or g, or h is identically zero. Suppose that

$$\left(\int_a^b \int_a^b |h(x,y) f(x,y)|^{1/p} \diamondsuit_\alpha x \diamondsuit_\alpha y \right) \left(\int_a^b \int_a^b |h(x,y) g(x,y)|^{1/q} \diamondsuit_\alpha x \diamondsuit_\alpha y \right) \neq 0,$$

and let

$$\begin{aligned}
A(x,y) &= \frac{|h(x,y)|^{1/p} |f(x,y)|}{\left(\int_a^b \int_a^b |h(x,y)| \, |f(x,y)|^p \diamondsuit_\alpha x \diamondsuit_\alpha y \right)^{1/p}}, \\
B(x,y) &= \frac{|h(x,y)|^{\frac{1}{q}} |g(x,y)|}{\left(\int_a^b \int_a^b |h(x,y)| \, |g(x,y)| \diamondsuit_\alpha x \diamondsuit_\alpha y \right)^{1/q}}.
\end{aligned}$$

Applying the Young inequality $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$, we have that

$$\begin{aligned}
&\int_a^b \int_a^b A(x,y) B(x,y) \diamondsuit_\alpha x \diamondsuit_\alpha y \leq \frac{1}{p} \frac{\int_a^b \int_a^b |h(x,y)| \, |f(x,y)|^p \diamondsuit_\alpha x \diamondsuit_\alpha y}{\left(\int_a^b \int_a^b |h(x,y)| \, |f(x,y)|^p \diamondsuit_\alpha x \diamondsuit_\alpha y \right)} \\
&+ \frac{1}{q} \frac{\int_a^b \int_a^b |h(x,y)| \, |g(x,y)|^q \diamondsuit_\alpha x \diamondsuit_\alpha y}{\left(\int_a^b \int_a^b |h(x,y)| \, |g(x,y)|^q \diamondsuit_\alpha x \diamondsuit_\alpha y \right)} \\
= \; & \frac{1}{p} + \frac{1}{q} = 1,
\end{aligned}$$

and the desired inequality follows. The proof is complete. ■

As a special case of Theorem 2.3.14, when $p = q = 2$, we get the two dimensional diamond-α Cauchy Schwartz's inequality.

Corollary 2.3.1 *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, with $a < b$, f, g, h : $[a, b]_{\mathbb{T}} \times [a, b]_{\mathbb{T}} \to \mathbb{R}$, be $\Diamond_\alpha$ integrable functions, and $1/p + 1/q = 1$ with $p > 1$. Then,*

$$\begin{aligned} &\int_a^b \int_a^b |h(x,y)f(x,y)g(x,y)| \Diamond_\alpha x \Diamond_\alpha y \\ \leq\ &\left(\int_a^b \int_a^b |h(x,y)f(x,y)|^2 \Diamond_\alpha x \Diamond_\alpha y\right)^{1/2} \left(\int_a^b \int_a^b |h(x,y)g(x,y)|^2 \Diamond_\alpha x \Diamond_\alpha y\right)^{1/2}. \end{aligned}$$

Now, we apply the theory of isotonic linear functional which was presented in Sect. 2.2 to derive a Hölder type inequality on time scales. The results are adapted from [30]. We need the following theorem to prove the main results [117].

Theorem 2.3.15 *Let E, L, and A be such that (L_1), (L_2), (A_1) and (A_2) in Definition 2.2.1 are satisfied. For $p \neq 1$, define $q = p/(p-1)$. Assume $|\omega| |f|^p$, $|\omega| |g|^q$, $|\omega f g| \in L$. If $p > 1$, then*

$$A(|\omega f g|) \leq A^{1/p}(|\omega| |f|^p) A^{1/q}(|\omega| |g|^q).$$

This inequality is reversed if $0 < p < 1$ and $A(|\omega| |g|^q) > 0$ and also it is reversed if $p < 0$ and $A(|\omega| |f|^p) > 0$.

Now, the application of Theorems 2.2.12 and 2.3.15 gives us the following Hölder's inequality.

Theorem 2.3.16 *For $p > 1$, define $q = p/(p-1)$. Let $E \subset \mathbb{R}^n$ be as in Theorem 2.2.12. Assume that $|\omega| |f|^p$, $|\omega| |g|^q$, $|\omega f g|$ are Δ-integrable on E. If $p > 1$, then*

$$\int_E |\omega(t) f(t) g(t)| \Delta t \leq \left(\int_E |\omega(t)| |f(t)|^p \Delta t\right)^{1/p} \left(\int_E |\omega(t)| |g(t)|^q \Delta t\right)^{1/q}.$$

This inequality is reversed if $0 < p < 1$ and $\int_E |\omega(t)| |g(t)|^q \Delta t > 0$ and also it is reversed if $p < 0$ and $\int_E |\omega(t)| |f(t)|^p \Delta t > 0$.

2.4 Minkowski Inequalities

The well-known Minkowski integral inequality is given in [3, 72, 110]. Let f and g be real-valued functions defined on $[a, b]$ such that the functions $|f(x)|^p$ and $|g(x)|^p$ for $p > 1$ are integrable on $[a, b]$. Then

$$\left(\int_a^b |f(x) + g(x)|^p dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}.$$

Equality holds if and only if $f(x) = 0$ almost everywhere or $g(x) = \lambda f(x)$ almost everywhere with a constant $\lambda \geq 0$. The discrete version of Minkowski inequality is given by

$$\left(\sum_{i=1}^{n}|f(i)+g(i)|^{p}\right)^{1/p} \leq \left(\sum_{i=1}^{n}|f(i)|^{p}\right)^{1/p} + \left(\sum_{i=1}^{n}|g(i)|^{p}\right)^{1/p},$$

where $f(n)$ and $g(n)$ are two positive-tuples and $p > 1$. Equality holds if and only f and g are proportional.

In this section we establish the Minkowski integral inequality and its extensions on time scales. The results in this section are adapted from [23, 30, 39, 45, 115, 150, 155].

Theorem 2.4.1 *Let f, g, $h \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ and $p > 1$. Then*

$$\left(\int_a^b |h(x)|\,|f(x)+g(x)|^p\,\Delta x\right)^{1/p} \leq \left(\int_a^b |h(x)|\,|f(x)|^p\,\Delta x\right)^{1/p} + \left(\int_a^b |h(x)|\,|g(x)|^p\,\Delta x\right)^{1/p}. \quad (2.4.1)$$

Proof. Note

$$\begin{aligned}
\int_a^b |h(x)|\,|f(x)+g(x)|^p\,\Delta x &= \int_a^b |h(x)|\,|f(x)+g(x)|^{p-1}\,|f(x)+g(x)|\,\Delta x \\
&\leq \int_a^b |h(x)|\,|f(x)+g(x)|^{p-1}\,|f(x)|\,\Delta x \\
&\quad + \int_a^b |h(x)|\,|f(x)+g(x)|^{p-1}\,|g(x)|\,\Delta x.
\end{aligned}$$

Applying the Hölder inequality (2.3.6), we get that

$$\begin{aligned}
&\int_a^b |h(x)|\,|f(x)+g(x)|^p\,\Delta x \\
\leq\;& \left(\int_a^b |h(x)|\left(|f(x)+g(x)|^{p-1}\right)^q\Delta x\right)^{1/q}\left(\int_a^b |h(x)|\,|f(x)|^p\,\Delta x\right)^{1/p} \\
&+\left(\int_a^b |h(x)|\left(|f(x)+g(x)|^{p-1}\right)^q\Delta x\right)^{1/q}\left(\int_a^b |h(x)|\,|g(x)|^p\,\Delta x\right)^{1/p} \\
=\;& \left(\int_a^b |h(x)|\,|f(x)+g(x)|^p\,\Delta x\right)^{1/q} \\
&\times\left[\left(\int_a^b |h(x)|\,|f(x)|^p\,\Delta x\right)^{1/p}+\left(\int_a^b |h(x)|\,|g(x)|^p\,\Delta x\right)^{1/p}\right].
\end{aligned}$$

Therefore

$$\begin{aligned} & \left(\int_a^b |h(x)|\,|f(x)+g(x)|^p\,\Delta x\right)^{1/p} \\ = & \left(\int_a^b |h(x)|\,|f(x)+g(x)|^p\,\Delta x\right)^{1-1/q} \\ = & \left[\left(\int_a^b |h(x)|\,|f(x)|^p\,\Delta x\right)^{1/p} + \left(\int_a^b |h(x)|\,|g(x)|^p\,\Delta x\right)^{1/p}\right], \end{aligned}$$

which is the desired inequality (2.4.1). The proof is complete. ■

As a special case when $h(x) = 1$, we obtain the time scale classical Minkowski inequality

$$\left(\int_a^b |f(x)+g(x)|^p\,\Delta x\right)^{1/p} \leq \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}.$$

As in the proof of Theorem 2.4.1 (using (2.3.7)) we obtain the following nabla Minkowski inequality.

Theorem 2.4.2 *Let $f,\ g,\ h \in C_{ld}([a,b]_{\mathbb{T}},\mathbb{R})$ and $p > 1$. Then*

$$\begin{aligned} & \left(\int_a^b |h(x)|\,|f(x)+g(x)|^p\,\nabla x\right)^{1/p} \\ \leq & \left(\int_a^b |h(x)|\,|f(x)|^p\,\nabla x\right)^{1/p} + \left(\int_a^b |h(x)|\,|g(x)|^p\,\nabla x\right)^{1/p}. \end{aligned}$$

Applying the diamond-α Hölder inequality (2.3.20) we have the following diamond-α Minkowski's inequality.

Theorem 2.4.3 *Let $f,\ g,\ h \in C([a,b]_{\mathbb{T}},\ \mathbb{R})$ and $p > 1$. Then*

$$\begin{aligned} & \left(\int_a^b |h(x)|\,|f(x)+g(x)|^p\,\diamondsuit_\alpha x\right)^{1/p} \\ \leq & \left(\int_a^b |h(x)|\,|f(x)|^p\,\diamondsuit_\alpha x\right)^{1/p} + \left(\int_a^b |h(x)|\,|g(x)|^p\,\diamondsuit_\alpha x\right)^{1/p}. \end{aligned}$$

Theorem 2.4.4 *Let $f,\ g : [a,b]_{\mathbb{T}} \to \mathbb{R}$, are positive rd-continuous functions and satisfying $0 < m \leq f/g \leq M < \infty$ on $[a,b]_{\mathbb{T}}$ and for $p > 1$ define $q = p/(p-1)$. Then*

$$\left(\int_a^b f^p(x)\Delta x\right)^{1/p} + \left(\int_a^b g^p(x)\Delta x\right)^{1/p} \leq c\left(\int_a^b (f(x)+g(x))^p\,\Delta x\right)^{\frac{1}{p}}, \tag{2.4.2}$$

where $c = \left(\frac{m}{M}\right)^{\frac{1}{pq}}$.

Proof. To prove the inequality (2.4.2), we apply Theorem 2.3.8. The inner term in the right-hand side can be rewritten as

$$
\begin{aligned}
& \int_a^b (f(x)+g(x))^p \,\Delta x \\
= & \int_a^b (f(x)+g(x))^{p-1} f(x)\Delta x \\
& + \int_a^b (f(x)+g(x))^{p-1} g(x)\Delta x \\
\geq & \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}} \left(\int_a^b (f(x)+g(x))^{q(p-1)} \,\Delta x\right)^{\frac{1}{q}} \\
& + \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\int_a^b g^p(x)\Delta x\right)^{\frac{1}{p}} \left(\int_a^b (f(x)+g(x))^{q(p-1)} \,\Delta x\right)^{\frac{1}{q}} \\
= & \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\int_a^b (f(x)+g(x))^p \,\Delta x\right)^{\frac{1}{q}} \\
& \times \left[\left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)\Delta x\right)^{\frac{1}{p}}\right].
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)\Delta x\right)^{\frac{1}{p}} & \leq \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_a^b (f(x)+g(x))^p \,\Delta x\right)^{1-\frac{1}{q}} \\
& = \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_a^b (f(x)+g(x))^p \,\Delta x\right)^{\frac{1}{p}},
\end{aligned}
$$

which is the desired inequality (2.4.2). The proof is complete. ■

Now, we apply the theory of isotonic linear functional that was presented in Sect. 2.2 to derive a Minkowski inequality on time scales. To do this we need the following theorem as given in [117].

Theorem 2.4.5 *Let E, L, and A be such that (L_1), (L_2), (A_1) and (A_2), as in Definition 2.2.1, are satisfied. For $p \in \mathbb{R}$, assume $|\omega|\,|f|^p$, $|\omega|\,|g|^p$, $|\omega|\,|f+g|^p \in L$. If $p > 1$, then*

$$A^{1/p}(|\omega|\,|f+g|^p) \leq A^{1/p}(|\omega|\,|f|^p) + A^{1/p}(|\omega|\,|g|^p).$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ provided that $A(|\omega|\,|g|^p) > 0$ and $A(|\omega|\,|f|^p) > 0$ hold.

Now, the application of Theorems 2.2.12 and 2.4.5 gives us the following Minkowski inequality.

Theorem 2.4.6 *Let $E \subset \mathbb{R}^n$ be as in Theorem 2.2.12. For $p \in \mathbb{R}$, assume $|\omega| |f|^p$, $|\omega| |g|^p$, $|\omega| |f+g|^p$ are Δ-integrable on E. If $p > 1$, then*

$$\left(\int_E |\omega(t)| |f(t)+g(t)|^p \Delta t\right)^{1/p} \leq \left(\int_E |\omega(t)| |f(t)|^p \Delta t\right)^{1/p} + \left(\int_E |\omega(t)| |g(t)|^p \Delta t\right)^{1/p}. \quad (2.4.3)$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ provided that $\int_E |\omega(t)| |g(t)|^q \Delta t > 0$ and $\int_E |\omega(t)| |f(t)|^p \Delta t > 0$.

In the following we obtain generalizations of Minkowski inequalities on time scales. The inequalities will be proved for several variables and based on the definitions of the multiple Riemann and Lebesgue Δ-integration on time scales given in [53].

Let $n \in \mathbb{N}$ be fixed. For $i \in \{1, 2, \ldots, n\}$, let $\mathbb{T}_i$ denote a time scale and

$$\Lambda^n = \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_n = \{t = (t_1, t_2, \ldots, t_n) : t_i \in \mathbb{T}_i, \quad 1 \leq i \leq n\},$$

as the n-dimensional time scale. Let μ_Δ be the σ-additive Lebesuge Δ-measure on Λ^n and $\mathcal{F}$ be the family of Δ-measurable subsets of Λ^n. Let $E \subset \mathcal{F}$ and $(E, \mathcal{F}, \mu_\Delta)$ be a time scale measure space. Then for a Δ-measurable function $f : E \to \mathbb{R}$, the corresponding Δ-integral of f over E will be denoted by

$$\int_E f(t_1, t_2, \ldots, t_n)\Delta_1 t_1 \Delta_2 t_2 \ldots \Delta_n t_n, \text{ or } \int_E f(t)\Delta t,$$
$$\text{or } \int_E f d\mu_\Delta, \text{ or } \int_E f(t) d\mu_\Delta(t).$$

Here, we state the Fubini theorem for integrals. It is used in the proofs of our main results.

Theorem 2.4.7 *Let (X, M, μ_Δ) and $(Y, \mathcal{L}, \nu_\Delta)$ be two finite-dimensional time scale measure space. If $f : X \times Y \to \mathbb{R}$ is a Δ-integrable function. Setting*

$$\varphi(y) = \int_X f(x,y) d\mu_\Delta(x), \text{ for } y \in Y,$$

and

$$\psi(x) = \int_Y f(x,y) dv_\Delta(y), \text{ for } x \in X,$$

then φ is Δ-integrable on Y and ψ is Δ-integrable on X and

$$\int_X d\mu_\Delta(x) \int_Y f(x,y) dv_\Delta(y) = \int_Y dv_\Delta(y) \int_X f(x,y) d\mu_\Delta(x). \quad (2.4.4)$$

We mention here that all theorems in Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue Δ-integral on Λ^n. This means that all the classical inequalities including Jensen's inequalities, Hölder inequalities, Minkowski inequalities, and their converses for multiple integration on time scales hold for both Riemann and Lebesuge integrals on time scales.

Theorem 2.4.8 *Let* $(E, \mathcal{F}, \mu_\Delta)$ *be a time scale measure space. For* $p \in \mathbb{R}$, *assume* w, f, g *are nonnegative functions such that* ωf^p, ωg^p, $\omega (f+g)^p$ *are* Δ*-integrable on* E. *If* $p > 1$, *then*

$$\left(\int_E \omega(t)\left(f(t)+g(t)\right)^p d\mu_\Delta t\right)^{1/p} \leq \left(\int_E \omega(t) f^p(t) d\mu_\Delta t\right)^{1/p} + \left(\int_E \omega(t) g^p(t) d\mu_\Delta t\right)^{1/p}.$$

Note that Theorem 2.4.8 also holds if we have a finite number of functions. The next theorem gives an inequality of Minkowski type for infinitely many functions. We assume that all integrals are finite.

Theorem 2.4.9 *Let* $(X, \L, \mu_\Delta)$ *and* (Y, λ, ν_Δ) *be two finite-dimensional time scale measure space and let* u, v f *be* Δ*-integrable functions on* X, Y *and* $X \times Y$, *respectively. If* $p > 1$, *then*

$$\left[\int_X \left(\int_Y f(x,y) v(y) d\nu_\Delta y\right)^p u(x) d\mu_\Delta x\right]^{1/p} \leq \int_Y \left(\int_X f^p(x,y) u(x) d\mu_\Delta x\right)^{1/p} v(y) d\nu_\Delta y, \tag{2.4.5}$$

holds provided all integrals in (2.4.5) exists. If $0 < p < 1$ *and*

$$\int_X \left(\int_Y f v d\nu_\Delta\right)^p u d\mu_\Delta > 0 \text{ and } \int_Y f v d\nu_\Delta > 0, \tag{2.4.6}$$

holds, then (2.4.5) is reversed. If $p < 0$ *and (2.4.6) and*

$$\int_X f^p(x,y) u(x) d\mu_\Delta x > 0, \tag{2.4.7}$$

hold, then (2.4.5) is reversed as well.

Proof. Let $p > 1$. Put

$$H(x) = \int_Y f(x,y) v(y) d\nu_\Delta y.$$

Now, by using Fubini's Theorem 2.4.7 and Hölder inequality in Theorem 2.3.16 on time scales, we have

$$
\begin{aligned}
\int_X H^p(x) u d\mu_\Delta &= \int_X H^{p-1}(x) H(x) u(x) d\mu_\Delta x \\
&= \int_X \left(\int_Y f(x,y) v(y) d\nu_\Delta y \right) H^{p-1}(x) u(x) d\mu_\Delta x \\
&= \int_Y \left(\int_X f(x,y) H^{p-1}(x) u(x) d\mu_\Delta x \right) v(y) d\nu_\Delta y \\
&\leq \int_Y \left(\int_X f^p(x,y) u(x) d\mu_\Delta x \right)^{1/p} \\
&\quad \times \left(\int_X H^p(x) u(x) d\mu_\Delta x \right)^{\frac{p-1}{p}} v(y) d\nu_\Delta y \\
&= \int_Y \left(\int_X f^p(x,y) u(x) d\mu_\Delta x \right)^{1/p} v(y) d\nu_\Delta y \\
&\quad \times \left(\int_X H^p(x) u(x) d\mu_\Delta x \right)^{\frac{p-1}{p}},
\end{aligned}
$$

and hence

$$
\left(\int_X H^p(x) u(x) d\mu_\Delta x \right)^{1/p} \leq \int_Y \left(\int_X f^p(x,y) u(x) d\mu_\Delta x \right)^{1/p} v(y) d\nu_\Delta y,
$$

which is the desired inequality (2.4.5). For $p < 0$ and $0 < p < 1$, the corresponding result can be obtained similarly. The proof is complete. ■

2.5 Steffensen Inequalities

In 1918 Steffensen [142] proved the following inequality. Let a and b be real numbers such that $a < b$, f, and g are integrable functions from $[a, b]$ into $\mathbb{R}$ such that f is decreasing and for every $t \in [a, b]$, $0 \leq g(t) \leq 1$. Then

$$
\int_a^{a-\lambda} f(t) dt \leq \int_a^b f(t) g(t) dt \leq \int_a^{a+\lambda} f(t) dt, \tag{2.5.1}
$$

where $\lambda = \int_a^b g(t) dt$. The discrete analogue of Steffensen's inequality is given by

$$
\sum_{i=n-k_2+1}^{n} x_i \leq \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{k_1} x_i,
$$

where $(x_i)_{i=1}^n$ is a nonincreasing finite sequence of nonnegative real numbers and $(y_i)_{i=1}^n$ is a finite sequence of real numbers such that for every i, $0 \leq y_i \leq 1$ and $k_2 \leq \sum_{i=1}^n y_i \leq k_1$ for $k_1, k_2 \in \{1, 2, \ldots, n\}$.

In this section, we prove some Steffensen inequalities on time scales The results in this section are adapted from [26, 114].

Theorem 2.5.1 *Let* a, $b \in \mathbb{T}_k^k$ *with* $a < b$ *and* f, $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *be* Δ*-integrable functions such that* f *of one sign and decreasing and* $0 \le g(t) \le 1$ *for every* $t \in [a,b]_{\mathbb{T}}$. *Suppose that also* l, $\gamma \in [a,b]_{\mathbb{T}}$ *such that*

$$\begin{aligned} b - l &\le \int_a^b g(t)\Delta t \le \gamma - a, \quad \text{if } f > 0 \text{ for all } t \in [a,b]_{\mathbb{T}}, \\ \gamma - a &\le \int_a^b g(t)\Delta t \le b - l, \quad \text{if } f < 0 \text{ for all } t \in [a,b]_{\mathbb{T}}, \end{aligned}$$

then

$$\int_l^b f(t)\Delta t \le \int_a^b f(t)g(t)\Delta t \le \int_a^\gamma f(t)\Delta t. \tag{2.5.2}$$

Proof. We consider the case when $f > 0$ and prove the left inequality. Now

$$\begin{aligned} &\int_a^b f(t)g(t)\Delta t - \int_l^b f(t)\Delta t \\ = &\int_a^l f(t)g(t)\Delta t + \int_l^b f(t)g(t)\Delta t - \int_l^b f(t)\Delta t \\ = &\int_a^l f(t)g(t)\Delta t - \int_l^b f(t)[1 - g(t)]\Delta t \\ \ge &\int_a^l f(t)g(t)\Delta t - f(l)\int_l^b [1 - g(t)]\Delta t \\ = &\int_a^l f(t)g(t)\Delta t - f(l)(b - l) + f(l)\int_l^b g(t)\Delta t \\ \ge &\int_a^l f(t)g(t)\Delta t - f(l)\int_a^b g(t)\Delta t + f(l)\int_l^b g(t)\Delta t \\ = &\int_a^l f(t)g(t)\Delta t - f(l)[\int_a^b g(t)\Delta t - \int_l^b g(t)\Delta t] \\ = &\int_a^l f(t)g(t)\Delta t - f(l)\int_a^l g(t)\Delta t = \int_a^l [f(t) - f(l)]\int_a^l g(t)\Delta t \ge 0, \end{aligned}$$

since f is decreasing and g is nonnegative. The proof of the right inequality is similar. The proof is complete. ■

Note that in Theorem 2.5.1 above we could easily replace the delta integral with the nabla integral under the same hypotheses.

Theorem 2.5.2 *Let a, $b \in \mathbb{T}_k^k$ with $a < b$ and f, $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be ∇-integrable functions such that f is of one sign and decreasing and $0 \le g(t) \le 1$ on $[a,b]_{\mathbb{T}}$. Suppose that also l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\begin{aligned} b-l &\le \int_a^b g(t)\nabla t \le \gamma - a, \quad \text{if } f>0 \text{ for all } t\in[a,b]_{\mathbb{T}},\\ \gamma-a &\le \int_a^b g(t)\nabla t \le b-l, \quad \text{if } f<0 \text{ for all } t\in[a,b]_{\mathbb{T}}. \end{aligned}$$

Then

$$\int_l^b f(t)\nabla t \le \int_a^b f(t)g(t)\nabla t \le \int_a^\gamma f(t)\nabla t. \tag{2.5.3}$$

The following theorems more closely resemble the theorem in the continuous case (the proofs are identical to that above and omitted).

Theorem 2.5.3 *Let a, $b \in \mathbb{T}_k^k$ with $a < b$ and f, $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be Δ-integrable functions such that f is of one sign and decreasing and $0 \le g \le 1$ for every $t \in [a,b]_{\mathbb{T}}$. Assume that $\lambda = \int_a^b g(t)\Delta t$ such that $b-\lambda$, $a+\lambda \in \mathbb{T}$. Then*

$$\int_{b-\lambda}^b f(t)\Delta t \le \int_a^b f(t)g(t)\Delta t \le \int_a^{a+\lambda} f(t)\Delta t.$$

Theorem 2.5.4 *Let a, $b \in \mathbb{T}_k^k$ with $a < b$ and f, $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be ∇-integrable functions such that f is of one sign and decreasing and $0 \le g \le 1$ for every $t \in [a,b]_{\mathbb{T}}$. Assume that $\lambda = \int_a^b g(t)\nabla t$ such that $b-\lambda$, $a+\lambda \in \mathbb{T}$. Then*

$$\int_{b-\lambda}^b f(t)\nabla t \le \int_a^b f(t)g(t)\nabla t \le \int_a^{a+\lambda} f(t)\nabla t.$$

In the following, we prove the diamond-α Steffensen inequality using the diamond-α derivative on time scales. We begin with the following lemma that will be needed later.

Lemma 2.5.1 *Let a, $b \in \mathbb{T}_k^k$ with $a < b$ and f, g, $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\diamondsuit_\alpha$-integrable functions. Suppose that also l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\int_a^\gamma h(t)\diamondsuit_\alpha t = \int_a^b g(t)\diamondsuit_\alpha t = \int_l^b h(t)\diamondsuit_\alpha t. \tag{2.5.4}$$

Then

$$\begin{aligned} \int_a^b f(t)g(t)\diamondsuit_\alpha t &= \int_\gamma^b [f(t)-f(\gamma)]g(t)\diamondsuit_\alpha t \qquad (2.5.5)\\ &\quad + \int_a^\gamma \{f(t)h(t) - [f(t)-f(\gamma)][h(t)-g(t)]\}\diamondsuit_\alpha t, \end{aligned}$$

$$\int_a^b f(t)g(t)\Diamond_\alpha t = \int_a^l [f(t)-f(l)]g(t)\Diamond_\alpha t + \int_l^b \{f(t)h(t)-[f(t)-f(l)][h(t)-g(t)]\}\Diamond_\alpha t. \tag{2.5.6}$$

Proof. We prove (2.5.5). By direct computation, we have

$$\begin{aligned} &\int_a^\gamma \{f(t)h(t)-[f(t)-f(\gamma)][h(t)-g(t)]\}\Diamond_\alpha t - \int_a^b f(t)g(t)\Diamond_\alpha t \\ =\ &\int_a^\gamma \{f(t)h(t)-f(t)g(t)-[f(t)-f(\gamma)][h(t)-g(t)]\}\Diamond_\alpha t \\ &+\int_a^\gamma f(t)g(t)\Diamond_\alpha t - \int_a^b f(t)g(t)\Diamond_\alpha t \\ =\ &\int_a^\gamma f(\gamma)[h(t)-g(t)]\Diamond_\alpha t - \int_\gamma^b f(t)g(t)\Diamond_\alpha t \\ =\ &f(\gamma)\int_a^\gamma h(t)\Diamond_\alpha t - f(\gamma)\int_a^\gamma g(t)\Diamond_\alpha t - \int_\gamma^b f(t)g(t)\Diamond_\alpha t. \end{aligned}$$

Applying the assumption $\int_a^\gamma h(t)\Diamond_\alpha t = \int_a^b g(t)\Diamond_\alpha t$, we see that

$$\begin{aligned} &\int_a^\gamma \{f(t)h(t)-[f(t)-f(\gamma)][h(t)-g(t)]\}\Diamond_\alpha t - \int_a^b f(t)g(t)\Diamond_\alpha t \\ =\ &f(\gamma)\int_a^b g(t)\Diamond_\alpha t - f(\gamma)\int_a^\gamma g(t)\Diamond_\alpha t - \int_\gamma^b f(t)g(t)\Diamond_\alpha t \\ =\ &f(\gamma)\left(\int_a^b g(t)\Diamond_\alpha t - \int_a^\gamma g(t)\Diamond_\alpha t\right) - \int_\gamma^b f(t)g(t)\Diamond_\alpha t \\ =\ &f(\gamma)\int_\gamma^b g(t)\Diamond_\alpha t - \int_\gamma^b f(t)g(t)\Diamond_\alpha t = \int_\gamma^b [f(\gamma)-f(t)]g(t)\Diamond_\alpha t, \end{aligned}$$

which is the desired inequality (2.5.5). The proof of (2.5.6) is similar and thus is omitted. The proof is complete. ■

Theorem 2.5.5 *Let a, $b \in \mathbb{T}_k^k$ with $a < b$ and f, g, $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions such that f is of one sign and decreasing and $0 \le g(t) \le h(t)$ for every $t \in [a,b]_{\mathbb{T}}$. Assume l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\begin{cases} \int_l^\gamma h(t)\Diamond_\alpha t \le \int_a^b g(t)\Diamond_\alpha t \le \int_a^\gamma h(t)\Diamond_\alpha t, \ \text{if } f \ge 0,\ t \in [a,b]_{\mathbb{T}}, \\ \int_a^\gamma h(t)\Diamond_\alpha t \le \int_a^b g(t)\Diamond_\alpha t \le \int_l^\gamma h(t)\Diamond_\alpha t, \ \ \text{if } f \le 0,\ t \in [a,b]_{\mathbb{T}}. \end{cases} \tag{2.5.7}$$

Then

$$\int_l^b f(t)h(t)\Diamond_\alpha t \le \int_a^b f(t)g(t)\Diamond_\alpha t \le \int_a^\gamma f(t)h(t)\Diamond_\alpha t. \tag{2.5.8}$$

Proof. We prove the left inequality in (2.5.8), in the case $f \geq 0$. The proofs of the other cases are similar. Since f is decreasing and g is nonnegative, we see that

$$\begin{aligned}
& \int_a^b f(t)g(t)\Diamond_\alpha t - \int_l^b f(t)h(t)\Diamond_\alpha t \\
= \; & \int_a^l f(t)g(t)\Diamond_\alpha t + \int_l^b f(t)g(t)\Diamond_\alpha t - \int_l^b f(t)h(t)\Diamond_\alpha t \\
= \; & \int_a^l f(t)g(t)\Diamond_\alpha t - \int_l^b f(t)\left[h(t) - g(t)\right]\Diamond_\alpha t \\
\geq \; & \int_a^l f(t)g(t)\Diamond_\alpha t - f(l)\int_l^b \left[h(t) - g(t)\right]\Diamond_\alpha t \\
= \; & \int_a^l f(t)g(t)\Diamond_\alpha t - f(l)\int_l^b h(t)\Diamond_\alpha t + f(l)\int_l^b g(t)\Diamond_\alpha t \\
\geq \; & \int_a^l f(t)g(t)\Diamond_\alpha t - f(l)\int_a^b g(t)\Diamond_\alpha t + f(l)\int_l^b g(t)\Diamond_\alpha t \\
= \; & \int_a^l f(t)g(t)\Diamond_\alpha t - f(l)\left[\int_a^b g(t)\Diamond_\alpha t - \int_l^b g(t)\Diamond_\alpha t\right] \\
= \; & \int_a^l f(t)g(t)\Diamond_\alpha t - f(l)\int_a^l g(t)\Diamond_\alpha t \\
= \; & \int_a^l \left[f(t) - f(l)\right] g(t)\Diamond_\alpha t \geq 0.
\end{aligned}$$

■

As a special case of Theorem 2.5.5 when $\alpha = 1$ and $\alpha = 0$, we have the following results.

Corollary 2.5.1 *Let a, $b \in \mathbb{T}^k$ with $a < b$ and f, g, $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be Δ-integrable functions such that f is of one sign and decreasing and $0 \leq g(t) \leq h(t)$ for every $t \in [a,b]_{\mathbb{T}}$. Assume l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\begin{cases} \int_l^\gamma h(t)\Delta t \leq \int_a^b g(t)\Delta t \leq \int_a^\gamma h(t)\Delta t, & \text{if } f \geq 0,\ t \in [a,b]_{\mathbb{T}}, \\ \int_a^\gamma h(t)\Delta t \leq \int_a^b g(t)\Delta t \leq \int_l^\gamma h(t)\Delta t, & \text{if } f \leq 0,\ t \in [a,b]_{\mathbb{T}}. \end{cases} \tag{2.5.9}$$

Then

$$\int_l^b f(t)h(t)\Delta t \leq \int_a^b f(t)g(t)\Delta t \leq \int_a^\gamma f(t)h(t)\Delta t. \tag{2.5.10}$$

Corollary 2.5.2 *Let a, $b \in \mathbb{T}^k$ with $a < b$ and f, g, $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be ∇-integrable functions such that f is of one sign and decreasing and $0 \leq g(t) \leq h(t)$ for every $t \in [a,b]_{\mathbb{T}}$. Assume l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\begin{cases} \int_l^\gamma h(t)\nabla t \le \int_a^b g(t)\nabla t \le \int_a^\gamma h(t)\nabla t, \ \textit{if } f \ge 0,\ t \in [a,b]_{\mathbb{T}}, \\ \int_a^\gamma h(t)\nabla t \le \int_a^b g(t)\nabla t \le \int_l^\gamma h(t)\nabla t, \ \ \textit{if } f \le 0,\ t \in [a,b]_{\mathbb{T}}. \end{cases} \tag{2.5.11}$$

Then

$$\int_l^b f(t)h(t)\nabla t \le \int_a^b f(t)g(t)\nabla t \le \int_a^\gamma f(t)h(t)\nabla t. \tag{2.5.12}$$

Theorem 2.5.6 *Let $a,\ b \in \mathbb{T}_k^k$ with $a < b$ and $f,\ g,\ h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions such that f is of one sign and decreasing and $0 \le g(t) \le h(t)$ for every $t \in [a,b]_{\mathbb{T}}$. Assume $l,\ \gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\int_a^\gamma h(t)\Diamond_\alpha t = \int_a^b g(t)\Diamond_\alpha t = \int_l^b h(t)\Diamond_\alpha t. \tag{2.5.13}$$

Then

$$\begin{aligned} \int_l^b f(t)h(t)\Diamond_\alpha t &\le \int_l^b (f(t)h(t) - [f(t) - f(l)][h(t) - g(t)])\Diamond_\alpha t \\ &\le \int_a^b f(t)g(t)\Diamond_\alpha t \qquad (2.5.14) \\ &\le \int_a^\gamma (f(t)h(t) - [f(t) - f(\gamma)][h(t) - g(t)])\Diamond_\alpha t \\ &\le \int_a^\gamma f(t)h(t)\Diamond_\alpha t. \end{aligned}$$

Proof. In view of the assumption that the function f is decreasing and that $0 \le g(t) \le h(t)$ on $[a,b]_{\mathbb{T}}$, we see that

$$\int_a^l [f(t) - f(l)]g(t)\Diamond_\alpha t \ge 0, \quad \int_l^b [f(l) - f(t)][h(t) - g(t)]\Diamond_\alpha t \ge 0. \tag{2.5.15}$$

Using the integral identity (2.5.6) together with the integrals in (2.5.15), we have

$$\begin{aligned} \int_l^b f(t)h(t)\Diamond_\alpha t &\le \int_l^b (f(t)h(t) - [f(t) - f(l)][h(t) - g(t)])\Diamond_\alpha t \qquad (2.5.16) \\ &\le \int_a^b f(t)g(t)\Diamond_\alpha t. \end{aligned}$$

In the same way as above, we obtain that

$$\begin{aligned} \int_a^b f(t)g(t)\Diamond_\alpha t &\le \int_a^\gamma (f(t)h(t) - [f(t) - f(\gamma)][h(t) - g(t)])\Diamond_\alpha t \\ &\qquad (2.5.17) \\ &\le \int_a^\gamma f(t)h(t)\Diamond_\alpha t. \end{aligned}$$

The proof of (2.5.14) is completed by combining (2.5.16) and (2.5.17). The proof is complete. ■

As a special case of Theorem 2.5.6, when $\alpha = 1$ and $\alpha = 0$, we have the following results.

Corollary 2.5.3 *Let a, $b \in \mathbb{T}^k$ with $a < b$ and f, g, $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be Δ-integrable functions such that f is of one sign and decreasing and $0 \le g(t) \le h(t)$ for every $t \in [a,b]_{\mathbb{T}}$. Assume l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\int_a^{\gamma} h(t)\Delta t = \int_a^b g(t)\Delta t = \int_l^b h(t)\Delta t. \tag{2.5.18}$$

Then

$$\begin{aligned} &\int_l^b f(t)h(t)\Delta t \\ \le\ &\int_l^b (f(t)h(t) - [f(t) - f(l)][h(t) - g(t)])\Delta t \le \int_a^b f(t)g(t)\Delta t \\ \le\ &\int_a^{\gamma} (f(t)h(t) - [f(t) - f(\gamma)][h(t) - g(t)])\Delta t \le \int_a^{\gamma} f(t)h(t)\Delta t. \end{aligned}$$

Corollary 2.5.4 *Let a, $b \in \mathbb{T}_k$ with $a < b$ and f, g, $h : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be ∇-integrable functions such that f is of one sign and decreasing and $0 \le g(t) \le h(t)$ for every $t \in [a,b]_{\mathbb{T}}$. Assume l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\int_a^{\gamma} h(t)\nabla t = \int_a^b g(t)\nabla t = \int_l^b h(t)\nabla t. \tag{2.5.19}$$

Then

$$\begin{aligned} &\int_l^b f(t)h(t)\nabla t \\ \le\ &\int_l^b (f(t)h(t) - [f(t) - f(l)][h(t) - g(t)])\nabla t \le \int_a^b f(t)g(t)\nabla t \\ \le\ &\int_a^{\gamma} (f(t)h(t) - [f(t) - f(\gamma)][h(t) - g(t)])\nabla t \le \int_a^{\gamma} f(t)h(t)\nabla t. \end{aligned}$$

Theorem 2.5.7 *Let a, $b \in \mathbb{T}_k^k$ with $a < b$ and f, g, h and $\varphi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha$-integrable functions such that f is of one sign and decreasing and $0 \le \varphi(t) \le g(t) \le h(t) - \varphi(t)$ for every $t \in [a,b]_{\mathbb{T}}$. Assume l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\int_a^{\gamma} h(t)\Diamond_\alpha t = \int_a^b g(t)\Diamond_\alpha t = \int_l^b h(t)\Diamond_\alpha t. \tag{2.5.20}$$

Then

$$\begin{aligned} &\int_l^b f(t)h(t)\Diamond_\alpha t + \int_a^b |[f(t) - f(l)]\varphi(t)|\,\Diamond_\alpha t \\ \le\ &\int_a^b f(t)g(t)\Diamond_\alpha t \le \int_a^{\gamma} f(t)h(t) - \int_a^{\gamma} |[f(t) - f(\gamma)]\varphi(t)|\,\Diamond_\alpha t. \end{aligned} \tag{2.5.21}$$

Proof. From the assumption that the function f is decreasing and that

$$0 \leq \varphi(t) \leq g(t) \leq h(t) - \varphi(t) \ on \ \ [a,b]_{\mathbb{T}},$$

it follows that

$$\begin{aligned}
& \int_a^{\gamma} [f(t) - f(\gamma)][h(t) - g(t)]\Diamond_\alpha t + \int_{\gamma}^{b} [f(\gamma) - f(t)]g(t)\Diamond_\alpha t \\
= & \int_a^{\gamma} |f(t) - f(\gamma)|\,[h(t) - g(t)]\Diamond_\alpha t + \int_{\gamma}^{b} |f(\gamma) - f(t)|\,g(t)\Diamond_\alpha t \\
\geq & \int_a^{\gamma} |f(t) - f(\gamma)|\,\varphi(t)\Diamond_\alpha t + \int_{\gamma}^{b} |f(\gamma) - f(t)|\,\varphi(t)\Diamond_\alpha t \\
= & \int_a^{b} |f(t) - f(\gamma)|\,\varphi(t)\Diamond_\alpha t.
\end{aligned} \tag{2.5.22}$$

Similarly, we find that

$$\begin{aligned}
& \int_a^{l} [f(t) - f(l)]g(t)\Diamond_\alpha t + \int_{l}^{b} [f(l) - f(t)][h(t) - g(t)]\Diamond_\alpha t \\
\geq & \int_a^{b} |f(t) - f(l)|\,\varphi(t)\Diamond_\alpha t.
\end{aligned} \tag{2.5.23}$$

By combining the integrals in (2.5.5) and (2.5.6) and the inequalities (2.5.22) and (2.5.23), we have the inequality (2.5.21). The proof is complete. ■

2.6 Hermite–Hadamard Inequalities

The Hermite–Hadamard inequality was published in [70]. For the convex function $f : [a,b] \to \mathbb{R}$, the integral of f can be estimated by the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

We note that the left-hand side of the Hermite–Hadamard inequality is a special case of the Jensen inequality.

The results in this section are adapted from [26, 63, 64]. First, we begin with an inequality containing the delta derivative on time scales.

Theorem 2.6.1 *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be delta differentiable function such that $m \leq f^{\Delta}(t) \leq M$ for every $t \in [a,b]_{\mathbb{T}}$ for some numbers $m < M$. If there exist l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\gamma - a \leq \frac{[f(b) - f(a) - m(b-a)]}{M - m} \leq b - l,$$

then

$$\begin{aligned} mh_2(a,b)+(M-m)h_2(a,\gamma) &\leq (b-a)f(b)-\int_a^b f(t)\Delta t \\ &\leq Mh_2(a,b)+(m-M)h_2(a,l), \end{aligned} \tag{2.6.1}$$

where $h(t,s)$ is defined as in (1.4.5).

Proof. Let

$$k(t):=\frac{[f(t)-m(t-b)]}{M-m}, \quad F(t):=h_1(a,\sigma(t)),$$

and

$$G(t):=k^{\Delta}(t)=\frac{[f^{\Delta}(t)-m]}{M-m}\in[0,1].$$

Clearly F is decreasing and nonpositive, and

$$\int_a^b G(t)\Delta t=\frac{[f(b)-f(a)-m(b-a)]}{M-m}\in[\gamma-a,b-l].$$

Note

$$\int_l^b F(t)\Delta t=\int_l^b h_1(a,\sigma(t))\Delta t=-\left.h_2(a,t)\right|_l^b=-h_2(a,b)+h_2(a,l),$$

and

$$\int_a^{\gamma} F(t)\Delta t=-\left.h_2(a,t)\right|_a^{\gamma}=-h_2(a,\gamma).$$

Moreover, using the formula for integration by parts for delta integrals, we see that

$$\begin{aligned} \int_a^b F(t)G(t)\Delta t &= \int_a^b F(t)k^{\Delta}(t)\Delta t=\left.h_1(a,t)k(t)\right|_a^b-\int_a^b h_1^{\Delta}(a,t)k(t)\Delta t \\ &= \frac{1}{M-m}\left[-(b-a)f(b)+\int_a^b f(t)\Delta t+mh_2(a,b)\right]. \end{aligned}$$

Using Steffensen's inequality for delta integrals, we obtain that

$$\begin{aligned} -h_2(a,b)+h_2(a,l) &\leq \frac{1}{M-m}\left[-(b-a)f(b)+\int_a^b f(t)\Delta t+mh_2(a,b)\right] \\ &\leq -h_2(a,\gamma), \end{aligned}$$

which yields the desired inequality (2.6.1). The proof is complete. ■

Suppose that f is $(n+1)$ times nabla differentiable on $\mathbb{T}_{\kappa^{n+1}}$. Using Taylor's Theorem 1.4.4, we define the remainder function by

$$\check{R}_{-1,f}(.,s)=f(s),$$

and for $n > -1$,

$$\check{R}_{n,f}(t,s) = f(s) - \sum_{k=0}^{n} \hat{h}_k(t,s) f^{\nabla^k}(s) = \int_t^s \hat{h}_n(s,\rho(\xi)) f^{\nabla^{n+1}}(\xi)\nabla\xi.$$

The proof of the next result is by induction (and we omit the proof).

Lemma 2.6.1 *Suppose f is $(n+1)$ times nabla differentiable on $\mathbb{T}_{\kappa^{n+1}}$. Then*

$$\int_a^b \hat{h}_{n+1}(t,\rho(s)) f^{\nabla^{n+1}}(s)\nabla s = \int_a^t \check{R}_{n,f}(a,s)\nabla s + \int_t^b \check{R}_{n,f}(b,s)\nabla s.$$

Corollary 2.6.1 *Suppose f is $(n+1)$ times nabla differentiable on $\mathbb{T}_{\kappa^{n+1}}$. Then*

$$\begin{aligned} \int_a^b \hat{h}_{n+1}(a,\rho(s)) f^{\nabla^{n+1}}(s)\nabla s &= \int_a^b \check{R}_{n,f}(b,s)\nabla s, \\ \int_a^b \hat{h}_{n+1}(b,\rho(s)) f^{\nabla^{n+1}}(s)\nabla s &= \int_a^b \check{R}_{n,f}(a,s)\nabla s. \end{aligned}$$

Our next result follows by induction (we leave the details to the reader).

Lemma 2.6.2 *Suppose f is $(n+1)$ times delta differentiable on $\mathbb{T}^{\kappa^{n+1}}$. Then*

$$\int_a^b h_{n+1}(t,\sigma(s)) f^{\Delta^{n+1}}(s)\Delta s = \int_a^t R_{n,f}(a,s)\Delta s + \int_t^b R_{n,f}(b,s)\Delta s,$$

where

$$R_{n,f}(t,s) = f(s) - \sum_{j=0}^{n} h_j(s,t) f^{\Delta^j}(t).$$

Theorem 2.6.2 *Let f be an $(n+1)$ times nabla differentiable function such that $f^{\nabla^{n+1}}(s)$ is increasing and f^{∇^n} is monotonic (either increasing or decreasing) on $[a,b]_{\mathbb{T}}$. Assume l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\begin{aligned} b-l &\le \frac{\hat{h}_{n+2}(b,a))}{\hat{h}_{n+1}(b,\rho(a))} \le \gamma - a, \text{ if } f^{\nabla^n} \text{ is decreasing,} \\ \gamma - a &\le \frac{\hat{h}_{n+2}(b,a))}{\hat{h}_{n+1}(b,\rho(a))} \le b - l, \text{ if } f^{\nabla^n} \text{ is increasing.} \end{aligned}$$

Then

$$f^{\nabla^n}(\gamma) - f^{\nabla^n}(a) \le \frac{\int_a^b \check{R}_{n,f}(a,s)\nabla s}{\hat{h}_{n+1}(b,\rho(a))} \le f^{\nabla^n}(b) - f^{\nabla^n}(l). \tag{2.6.2}$$

Proof. Assume that f^{∇^n} is decreasing (the case where f^{∇^n} is increasing is similar and is omitted). Let $F = -f^{\nabla^{n+1}}$. Now, since f^{∇^n} is decreasing, we have $F \geq 0$ and decreasing on $[a,b]_{\mathbb{T}}$. Define

$$g(t) = \frac{\hat{h}_{n+1}(b,\rho(t))}{\hat{h}_{n+1}(b,\rho(a))} \in [0,1], \text{ for } t \in [a,b]_{\mathbb{T}} \text{ and } n \geq -1.$$

We will apply Steffensen's inequality (see Theorem 2.5.2). Using the fact that

$$\hat{h}^{\nabla}_{k+1}(t,s) = -\hat{h}_k(t,\rho(s)), \tag{2.6.3}$$

we see that

$$\int_a^b g(t)\nabla t = \frac{1}{\hat{h}_{n+1}(b,\rho(a))}\int_a^b \hat{h}_{n+1}(b,\rho(t))\nabla t = \frac{\hat{h}_{n+2}(b,a))}{\hat{h}_{n+1}(b,\rho(a))}.$$

That is

$$b - l \leq \frac{\hat{h}_{n+2}(b,a))}{\hat{h}_{n+1}(b,\rho(a))} \leq \gamma - a,$$

then

$$\int_l^b F(t)\nabla t \leq \int_a^b g(t)F(t)\nabla t \leq \int_a^\gamma F(t)\nabla t.$$

By Corollary 2.6.1 this simplifies to

$$f^{\nabla^n}(t)\Big|_{t=a}^{\gamma} \leq \frac{1}{\hat{h}_{n+1}(b,\rho(a))}\int_a^b \check{R}_{n,f}(a,s)\nabla s \leq f^{\nabla^n}(t)\Big|_{t=l}^{\gamma},$$

which gives the desired inequality (2.6.2). The proof is complete. ■

It is evident that an analogous result can be found for the delta integral case using the delta results in Corollary 2.5.1 by putting $h(t) = 1$. As usual a twice nabla differentiable function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is convex on $[a,b]_{\mathbb{T}}$ if and only if $f^{\nabla^2} \geq 0$ on $[a,b]_{\mathbb{T}}$.

Corollary 2.6.2 *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be convex and monotonic. Assume l, $\gamma \in [a,b]_{\mathbb{T}}$ such that*

$$\begin{aligned} l &\geq b - \frac{\hat{h}_2(b,a)}{b-\rho(a)}, \quad \gamma \geq \frac{\hat{h}_2(b,a)}{b-\rho(a)} + a, \text{ if } f \text{ is decreasing,} \\ l &\leq b - \frac{\hat{h}_2(b,a)}{b-\rho(a)}, \quad \gamma \leq \frac{\hat{h}_2(b,a)}{b-\rho(a)} + a, \text{ if } f \text{ is increasing.} \end{aligned}$$

Then

$$f(\gamma) + \frac{\rho(a)-a}{b-\rho(a)}f(a) \leq \frac{1}{b-\rho(a)}\int_a^b f(t)\nabla t \leq f(b) + \frac{b-a}{b-\rho(a)}f(a) - f(l).$$

Another slightly different form of the Hermite–Hadamard inequality is the following inequality which is given by applying the Steffensen inequality proved in Theorem 2.5.2.

Theorem 2.6.3 *Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be convex and monotonic. Assume l, $\gamma \in [a, b]_{\mathbb{T}}$ such that*

$$\begin{aligned} l &\geq a + \frac{\hat{h}_2(b,a)}{b-a}, \quad \gamma \geq b - \frac{\hat{h}_2(b,a)}{b-a}, \text{ if } f \text{ is decreasing,} \\ l &\leq a + \frac{\hat{h}_2(b,a)}{b-a}, \quad \gamma \leq b - \frac{\hat{h}_2(b,a)}{b-a}, \text{ if } f \text{ is increasing.} \end{aligned}$$

Then

$$f(\gamma) \leq \frac{1}{b-a} \int_a^b f^\rho(t) \nabla t \leq f(b) + f(a) - f(l). \tag{2.6.4}$$

Proof. Assume that f is decreasing and convex. Then $f^{\nabla^2} \geq 0$ and $f^\nabla \leq 0$. Then $F = -f^\nabla$ is decreasing and satisfies $F \geq 0$. For $G(t) = \frac{b-t}{b-a}$, we see for every $t \in [a, b]$ that $0 \leq G(t) \leq 1$ and F and G satisfy the hypotheses in Theorem 2.5.2. Now, the inequality

$$b - l \leq \int_a^b G(t) \nabla t \leq \gamma - a,$$

can be rewritten in the form

$$b - l \leq \frac{1}{b-a} \int_a^b (b-t) \nabla t \leq \gamma - a.$$

We consider the left hand inequality which takes the form

$$l \geq b - \frac{1}{b-a} \int_a^b (b-t) \nabla t = b - \frac{1}{b-a} \int_a^b (b - a + t - a) \nabla t,$$

which simplifies to

$$l \geq a + \frac{\hat{h}_2(b,a)}{b-a}.$$

Similarly

$$\gamma \geq b - \frac{\hat{h}_2(b,a)}{b-a}.$$

Furthermore, note that $\int_r^s F(t) \nabla t = f(r) - f(s)$, and integrating by parts yields that

$$\int_a^b F(t) G(t) \nabla t = \int_a^b \frac{(t-b)}{b-a} f^\nabla(t) \nabla t = f(a) - \frac{1}{b-a} \int_a^b f^\rho(t) \nabla t.$$

It follows that Steffensen's inequality takes the form

$$f(l) - f(b) \leq f(a) - \frac{1}{b-a} \int_a^b f^{\rho}(t) \nabla t \leq f(a) - f(\gamma),$$

which can be arranged to match the desired inequality (2.6.4). The case where f is increasing is similar and is omitted. The proof is complete. ■

Theorem 2.6.4 *Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be an $n+1$ times nabla differentiable function such that $m \leq f^{\nabla^{n+1}}(t) \leq M$ for every $t \in [a, b]_{\mathbb{T}}$ for some numbers $m < M$. If there exist l, $\gamma \in [a, b]_{\mathbb{T}}$ such that*

$$b - l \leq \frac{\left[f^{\nabla^n}(b) - f^{\nabla^n}(a) - m(b-a)\right]}{M - m} \leq \gamma - a,$$

then

$$\begin{aligned} & m\hat{h}_{n+2}(b, a) + (M - m)\hat{h}_{n+2}(b, l) \leq \int_a^b \check{R}_{n,f}(a, t) \nabla t \\ \leq \quad & M\hat{h}_{n+2}(b, a) + (m - M)\hat{h}_{n+2}(b, \gamma). \end{aligned} \tag{2.6.5}$$

where $\hat{h}_n(t, s)$ is defined as in (1.4.7).

Proof. Let

$$k(t) = \frac{1}{M-m} \left[f(t) - m\hat{h}_{n+1}(t, a)\right], \; F(t) = \hat{h}_{n+1}(b, \rho(t)),$$

and

$$G(t) = k^{\nabla^{n+1}}(t) = \frac{1}{M-m} \left[f^{\nabla^{n+1}}(t) - m\right] \in [0, 1].$$

Observe that F is nonnegative and decreasing, and

$$\int_a^b G(t) \nabla t = \frac{1}{M-m} \left[f^{\nabla^n}(b) - f^{\nabla^n}(a) - m(b-a)\right].$$

Now by (2.6.3), we get that

$$\int_l^b F(t) \nabla t = \int_l^b \hat{h}_{n+1}(b, \rho(t)) \nabla t = \hat{h}_{n+2}(b, l),$$

and

$$\int_a^\gamma F(t) \nabla t = \hat{h}_{n+2}(b, a) - \hat{h}_{n+2}(b, \gamma).$$

Moreover, using Corollary 2.6.1, we have

$$\begin{aligned}\int_a^b G(t)F(t)\nabla t &\leq \frac{1}{M-m}\int_a^b \hat{h}_{n+1}(b,\rho(t))\left(f^{\nabla^n}(t)-m\right)\nabla t\\ &= \frac{1}{M-m}\int_a^b \check{R}_{n,f}(a,t)\nabla t+\frac{m}{M-m}\,\hat{h}_{n+2}(b,t)\Big|_a^b\\ &= \frac{1}{M-m}\int_a^b \check{R}_{n,f}(a,t)\nabla t-\frac{m}{M-m}\hat{h}_{n+2}(b,a).\end{aligned}$$

Using Steffensen's inequality (2.5.3), we have

$$\begin{aligned}\hat{h}_{n+2}(b,l) &\leq \frac{1}{M-m}\left[\int_a^b \check{R}_{n,f}(a,t)\nabla t-m\hat{h}_{n+2}(b,a)\right]\\ &\leq \hat{h}_{n+2}(b,a)-\hat{h}_{n+2}(b,\gamma),\end{aligned}$$

which yields the desired inequality (2.6.5). The proof is complete. ■

The following inequality is an inequality of Hermite–Hadamard type for nabla derivative and is derived from Theorem 2.6.4 with $n=0$.

Theorem 2.6.5 *Let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be nabla differentiable function such that $m\leq f^{\nabla}\leq M$ for every $t\in[a,b]_{\mathbb{T}}$ for some numbers $m<M$. If there exist $l,\gamma\in[a,b]_{\mathbb{T}}$ such that*

$$b-l\leq\frac{[f(b)-f(a)-m(b-a)]}{M-m}\leq\gamma-a,$$

then

$$\begin{aligned}m\hat{h}_2(b,a)+(M-m)\hat{h}_2(b,l) &\leq \int_a^b f(t)\nabla t-(b-a)f(a)\\ &\leq M\hat{h}_2(b,a)+(m-M)\hat{h}_2(b,\gamma),\end{aligned}$$

where $\hat{h}_n(t,s)$ is defined as in (1.4.7).

Next we present some inequalities of Hermite–Hadamard type for diamond-α derivative on time scales. We start with a few technical lemmas. The first lemma gives the relation between the integrals of delta, nabla, and classical integrals on $\mathbb{R}$ and we present it without proof.

Lemma 2.6.3 *Let $f:\mathbb{T}\to\mathbb{R}$ be a continuous function and $a,b\in\mathbb{T}$.*

(i) If f is nondecreasing on $\mathbb{T}$, then

$$(b-a)f(a)\leq\int_a^b f(t)\Delta t\leq\int_a^b \tilde{f}(t)dt\leq\int_a^b f(t)\nabla t\leq(b-a)f(b),$$

where $\tilde{f}:\mathbb{R}\to\mathbb{R}$ is a continuous nondecreasing function such that $f(t)=\tilde{f}(t)$ for all $t\in\mathbb{T}$.

(ii) If f is nonincreasing on $\mathbb{T}$, then

$$(b-a)f(a) \geq \int_a^b f(t)\Delta t \geq \int_a^b \tilde{f}(t)dt \geq \int_a^b f(t)\nabla t \geq (b-a)f(b),$$

where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a continuous nonincreasing function such that $f(t) = \tilde{f}(t)$ for all $t \in \mathbb{T}$.

In both cases, there exists an

$$\alpha_T = \frac{\int_a^b \tilde{f}(t)dt - \int_a^b f(t)\nabla t}{\int_a^b f(t)\Delta t - \int_a^b f(t)\nabla t} \in [0,1],$$

such that

$$\int_a^b f(t)\Diamond_{\alpha_T} t = \int_a^b \tilde{f}(t)dt.$$

Remark 2.6.1 *(i). If f is nondecreasing on $\mathbb{T}$, then for $\alpha \leq \alpha_T$, we have*

$$\int_a^b f(t)\Diamond_\alpha t \geq \int_a^b \tilde{f}(t)dt,$$

while if $\alpha \geq \alpha_T$, we have

$$\int_a^b f(t)\Diamond_\alpha t \leq \int_a^b \tilde{f}(t)dt.$$

(ii). If f is nonincreasing on $\mathbb{T}$, then for $\alpha \leq \alpha_T$, we have

$$\int_a^b f(t)\Diamond_\alpha t \leq \int_a^b \tilde{f}(t)dt,$$

while if $\alpha \geq \alpha_T$, we have

$$\int_a^b f(t)\Diamond_\alpha t \geq \int_a^b \tilde{f}(t)dt.$$

(iii) If $\mathbb{T} = [a,b]$ or f is a constant, then α_T can be any real number from $[0,1]$. Otherwise $\alpha_T \in (0,1)$.

Next we present a lemma which gives a relation between the existence of the delta integral of a linear function and its corresponding nabla integral.

Lemma 2.6.4 *Let $f : \mathbb{T} \to \mathbb{R}$ be linear function and let $\tilde{f} : [a,b] \to \mathbb{R}$ be the corresponding linear function. If $\int_a^b f(t)\Delta t = \int_a^b \tilde{f}(t)dt - C$, with $C \in \mathbb{R}$, then $\int_a^b f(t)\nabla t = \int_a^b \tilde{f}(t)dt + C$.*

Let

$$x_\alpha = \frac{1}{b-a}\int_a^b t\Diamond_\alpha t,$$

and call it the α-center of the time scale interval $[a,b]_{\mathbb{T}}$. Now, we are in a position to state and prove diamond-α Hermite–Hadamard type inequalities on time scales.

Theorem 2.6.6 *Let $\mathbb{T}$ be a time scale and a, $b \in \mathbb{T}$. Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a continuous convex function. Then*

$$f(x_\alpha) \le \frac{1}{b-a}\int_a^b f(t)\Diamond_\alpha t \le \frac{b-x_\alpha}{b-a}f(a) + \frac{x_\alpha - a}{b-a}f(b). \tag{2.6.6}$$

Proof. For every convex function, we have

$$f(t) \le f(a) + \frac{f(b)-f(a)}{b-a}(t-a). \tag{2.6.7}$$

By taking the diamond-α integral we get

$$\begin{aligned}
\int_a^b f(t)\Diamond_\alpha t &\le \int_a^b f(a)\Diamond_\alpha t + \int_a^b \frac{f(b)-f(a)}{b-a}(t-a)\Diamond_\alpha t \\
&= (b-a)f(a) + \frac{f(b)-f(a)}{b-a}\left(\int_a^b t\Diamond_\alpha t - a(b-a)\right),
\end{aligned}$$

that is

$$\frac{1}{b-a}\int_a^b f(t)\Diamond_\alpha t \le \frac{b-x_\alpha}{b-a}f(a) + \frac{x_\alpha - a}{b-a}f(b),$$

which is the right-hand side of (2.6.6). For the left-hand side, we use Theorem 2.2.5, by taking $g(s) = s$ and $F = f$ to get that

$$f\left(\frac{\int_a^b s\Diamond_\alpha s}{b-a}\right) \le \frac{\int_a^b f(s)\Diamond_\alpha s}{b-a}.$$

Hence, we have

$$f(x_\alpha) \le \frac{1}{b-a}\int_a^b f(s)\Diamond_\alpha s,$$

which is the right-hand side of (2.6.6). The proof is complete. ■

Remark 2.6.2 *The right-hand side of the Hermite–Hadamard inequality (2.6.6) remains true for all $0 \le \alpha \le \lambda$, including the nabla integral, if $f(b) \le f(a)$ and for all $\lambda \le \alpha \le 1$, including the delta derivative, if $f(b) \ge f(a)$, where x_λ is the λ-center of the time scale interval $[a,b]_{\mathbb{T}}$.*

Let us suppose that $f(b) \geq f(a)$. Then by taking the diamond-α integral of the inequality (2.6.7), we get that

$$\begin{aligned}\int_a^b f(t)\diamondsuit_\alpha t &\leq (b-a)f(a) + \frac{f(b)-f(a)}{b-a}\left(\int_a^b t\diamondsuit_\alpha t - a(b-a)\right)\\ &\leq (b-a)f(a) + (f(b)-f(a))(x_\lambda - a)\\ &\leq (b-x_\lambda)f(a) + f(b)(x_\lambda - a).\end{aligned}$$

According to Lemma 2.6.3, the last inequality is true for $\int_a^b t\diamondsuit_\alpha t \leq \int_a^b t\diamondsuit_\lambda t$, that is for $\alpha \geq \lambda$. The same arguments work for $\lambda \geq \alpha$.

Remark 2.6.3 *The left-hand side of the Hermite–Hadamard inequality (2.6.6) remains true for all $0 \leq \alpha \leq \lambda$, including the nabla integral, if f is nonincreasing for all $\lambda \leq \alpha \leq 1$, including the delta derivative, if f is nondecreasing*

Let us suppose that f is nonincreasing. Then using Theorem 2.2.5, by taking $g(s) = s$ and $F = f$, we have

$$f\left(\frac{\int_a^b s\diamondsuit_\alpha s}{b-a}\right) \leq \frac{\int_a^b f(s)\diamondsuit_\alpha s}{b-a}.$$

For $\alpha \geq \lambda$, we have $\int_a^b t\diamondsuit_\alpha t \leq \int_a^b t\diamondsuit_\lambda t$ and so

$$f\left(\frac{\int_a^b s\diamondsuit_\lambda s}{b-a}\right) \leq f\left(\frac{\int_a^b s\diamondsuit_\alpha s}{b-a}\right) \leq \frac{\int_a^b f(s)\diamondsuit_\alpha s}{b-a},$$

that is

$$f(x_\lambda) \leq \frac{1}{b-a}\int_a^b f(s)\diamondsuit_\alpha s.$$

The same arguments are used to prove the case when f is nondecreasing.

Theorem 2.6.7 *Let $\mathbb{T}$ be a time scale, α, $\lambda \in [0,1]$ and a, $b \in \mathbb{T}$. Let $f : [a,b]_\mathbb{T} \to \mathbb{R}$ be a continuous convex function. Then*

(i). if f is nondecreasing on $[a,b]_\mathbb{T}$, then for all $\alpha \in [0,\lambda]$ one has

$$f(x_\lambda) \leq \frac{1}{b-a}\int_a^b f(t)\diamondsuit_\alpha t, \tag{2.6.8}$$

and for all $\alpha \in [\lambda, 1]$, one has

$$\frac{1}{b-a}\int_a^b f(t)\diamondsuit_\alpha t \leq \frac{b-x_\lambda}{b-a}f(a) + \frac{x_\lambda - a}{b-a}f(b). \tag{2.6.9}$$

(ii). *if f is nonincreasing on $[a,b]_{\mathbb{T}}$, then for all $\alpha \in [0,\lambda]$ one has the inequality (2.6.9), and for all $\alpha \in [\lambda, 1]$, one has the inequality (2.6.8).*

Now we prove an inequality of Hermite–Hadamard type with a weight function.

Theorem 2.6.8 *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a continuous convex function and let $w : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a continuous function such that $w(t) \geq t$ for all $t \in \mathbb{T}$ and $\int_a^b w(t)\Diamond_\alpha t > 0$. Then*

$$\begin{aligned} f(x_{w,\alpha}) &\leq \frac{1}{\int_a^b w(t)\Diamond_\alpha t} \int_a^b f(t)w(t)\Diamond_\alpha t \\ &\leq \frac{b - x_{w,\alpha}}{b-a} f(a) + \frac{x_{w,\alpha} - a}{b-a} f(b), \end{aligned} \tag{2.6.10}$$

where $x_{w,\alpha} = \int_a^b tw(t)\Diamond_\alpha t / \int_a^b w(t)\Diamond_\alpha t$.

Proof. For the convex function $f(t)$, we have

$$f(t) \leq f(a) + \frac{f(b)-f(a)}{b-a}(t-a).$$

Multiplying this inequality by $w(t)$ which is nonnegative, we get after integration that

$$\begin{aligned} \int_a^b w(t)f(t)\Diamond_\alpha t &\leq f(a)\int_a^b w(t)\Diamond_\alpha t \\ &\quad + \frac{f(b)-f(a)}{b-a}\left[\int_a^b tw(t)\Diamond_\alpha t - a\int_a^b w(t)\Diamond_\alpha t\right], \end{aligned}$$

that is

$$\frac{1}{\int_a^b w(t)\Diamond_\alpha t}\int_a^b f(t)\Diamond_\alpha t \leq \frac{b-x_{w,\alpha}}{b-a} f(a) + \frac{x_{w,\alpha}-a}{b-a} f(b),$$

which is the right-hand side of (2.6.10). For the left-hand side, we use Theorem 2.2.6, by taking $g(s) = s$ and $h(t) = w(t)$ and $F = f$ to get that

$$f\left(\frac{\int_a^b w(s)s\Diamond_\alpha s}{\int_a^b w(s)\Diamond_\alpha s}\right) \leq \frac{\int_a^b f(s)w(s)\Diamond_\alpha s}{\int_a^b w(s)s\Diamond_\alpha s}.$$

Hence, we have

$$f(x_{w,\alpha}) \leq \frac{1}{\int_a^b w(t)\Diamond_\alpha t}\int_a^b w(t)f(t)\Diamond_\alpha t,$$

which is the left-hand side of (2.6.10). The proof is complete. ■

Remark 2.6.4 *If we consider concave functions instead of the convex functions, the inequalities (2.6.6), (2.6.8)–(2.6.10) are reversed.*

2.7 Čebyšev Inequalities

The Čebyšev inequality (see [110]) is given by

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx, \quad (2.7.1)$$

where $f,\ g : [a,b] \to \mathbb{R}$ are integrable functions both increasing or both decreasing and $p : [a,b] \to \mathbb{R}^+$ is an integrable function. If one of the functions f or g is nonincreasing and the other nondecreasing then the inequality in (2.7.1) is reversed. The special case of (2.7.1), when $p = 1$ is given by

$$\int_a^b f(x)g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx. \quad (2.7.2)$$

For each of the above inequalities there exists a corresponding discrete analogue. The discrete version of (2.7.1) is given by

$$\sum_{i=1}^n p(i) \sum_{i=1}^n p(i)a(i)b(i) \geq \sum_{i=1}^n p(i)a(i) \sum_{i=1}^n p(i)g(i), \quad (2.7.3)$$

where $a = (a(1),\ a(2), \ldots, a(n))$, $b = (b(1),\ b(2), \ldots, b(n))$ are two nondecreasing (or nonincreasing) sequences and $p = (p(1),\ p(2), \ldots, p(n))$ is a nonnegative sequence with equality if and only if at least one of the sequences a or b is constant. The discrete version of (2.7.2) is given by

$$\sum_{i=1}^n p(i)a(i)b(i) \geq \frac{1}{n}\sum_{i=1}^n a(i) \sum_{i=1}^n g(i), \quad (2.7.4)$$

and is also called the discrete Čebyšev's inequality.

In this section we obtain Čebyšev's type inequalities on time scales which as special cases contain the above continuous and discrete inequalities. The results are adapted from [26, 156].

Theorem 2.7.1 *Suppose that* $p \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$. *Let* $f_1,\ f_2,\ k_1,\ k_2 \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ *satisfy the following two conditions:*

(C_1). $f_2(x)k_2(x) > 0$ *on* $[a,b]_{\mathbb{T}}$,

(C_2). $\frac{f_1(x)}{f_2(x)}$ *and* $\frac{k_1(x)}{k_2(x)}$ *are similarly ordered (or oppositely ordered), that is, for all* $x,\ y \in [a,b]_{\mathbb{T}}$

$$\left(\frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)}\right)\left(\frac{k_1(x)}{k_2(x)} - \frac{k_1(y)}{k_2(y)}\right) \geq 0 \ (or\ \leq 0).$$

Then

$$\begin{aligned} & \frac{1}{2}\int_a^b\int_a^b p(x)p(y)\begin{vmatrix} f_1(x) & f_1(y) \\ f_2(x) & f_2(y)\end{vmatrix}\begin{vmatrix} k_1(x) & k_1(y) \\ k_2(x) & k_2(y)\end{vmatrix}\Delta x\Delta y \\ = & \begin{vmatrix} \int_a^b p(x)f_1(x)k_1(x)\Delta x & \int_a^b p(x)f_1(x)k_2(x)\Delta x \\ \int_a^b p(x)f_2(x)k_1(x)\Delta x & \int_a^b p(x)f_2(x)k_2(x)\Delta x \end{vmatrix} \geq 0 \ (\leq 0). \end{aligned} \tag{2.7.5}$$

Proof. Let $x, y \in [a, b]_{\mathbb{T}}$. Then it follows from (C_1), (C_2) and the identity

$$\begin{aligned} & p(x)p(y)\begin{vmatrix} f_1(x) & f_1(y) \\ f_2(x) & f_2(y)\end{vmatrix}\begin{vmatrix} k_1(x) & k_1(y) \\ k_2(x) & k_2(y)\end{vmatrix} \\ = \ & p(x)p(y)f_2(x)f_2(y)k_2(x)k_2(y)\left(\frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)}\right)\left(\frac{k_1(x)}{k_2(x)} - \frac{k_1(y)}{k_2(y)}\right), \end{aligned}$$

that (2.7.5) holds. The proof is complete. ■

Putting $f_1(x) = f(x)$, $k_1(x) = g(x)$ and $f_2(x) = k_2(x) = 1$ in Theorem 2.7.1, we have the following delta Čebyšev's type inequality on time scales.

Corollary 2.7.1 *Suppose that p, f, $g \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ with $p(x) > 0$ on $[a, b]_{\mathbb{T}}$. Let $f(x)$ and $g(x)$ be similarly ordered (or oppositely ordered). Then*

$$\int_a^b p(x)\Delta x\int_a^b p(x)f(x)g(x)\Delta x \geq (\leq)\int_a^b p(x)f(x)\Delta x\int_a^b p(x)g(x)\Delta x. \tag{2.7.6}$$

Remark 2.7.1 *Let p, $\gamma \in C_{rd}([a, b]_{\mathbb{T}}, [0, \infty))$. If $f(x)$ and $g(x)$ are similarly ordered (or oppositely ordered), then it follows from (2.7.6) that*

$$\begin{aligned} & \int_a^b p(x)\Delta x\int_a^b p(x)f(\gamma(x))g(\gamma(x))\Delta x \\ \geq \ & (\leq)\int_a^b p(x)f(\gamma(x))\Delta x\int_a^b p(x)g(\gamma(x))\Delta x. \end{aligned}$$

Remark 2.7.2 *Let p, $f_i \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ for $i = 1, 2, \ldots, n$ with $p(x) > 0$ on $[a, b]_{\mathbb{T}}$. Suppose that $f_1(x)$, $f_2(x), \ldots, f_n(x)$ are similarly ordered. Then we have from (2.7.6) that*

$$
\begin{aligned}
&\left(\int_a^b p(x)\Delta x\right)^{n-1}\int_a^b p(x)(f_1(x)f_2(x)\ldots.f_n(x))\Delta x \\
=\;&\left(\int_a^b p(x)\Delta x\right)^{n-2}\left(\int_a^b p(x)\Delta x\right)\left(\int_a^b p(x)(f_1(x)f_2(x)\ldots.f_n(x))\Delta x\right) \\
\geq\;&\left(\int_a^b p(x)\Delta x\right)^{n-2}\left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)(f_2(x)\ldots.f_n(x))\Delta x\right) \\
\geq\;&\left(\int_a^b p(x)\Delta x\right)^{n-3}\left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)f_2(x)\Delta x\right) \\
&\times\left(\int_a^b p(x)(f_3(x)\ldots.f_n(x))\Delta x\right) \\
\geq\;&\ldots\geq\left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)f_2(x)\Delta x\right)\ldots\left(\int_a^b p(x)f_n(x)\Delta x\right).
\end{aligned}
$$

This gives us that

$$
\begin{aligned}
&\left(\int_a^b p(x)\Delta x\right)^{n-1}\int_a^b p(x)(f_1(x)f_2(x)\ldots.f_n(x))\Delta x\geq\left(\int_a^b p(x)f_1(x)\Delta x\right) \\
&\times\left(\int_a^b p(x)f_2(x)\Delta x\right)\ldots\left(\int_a^b p(x)f_n(x)\Delta x\right).
\end{aligned}
\tag{2.7.7}
$$

In particular, if $f_1 = f_2 = \ldots = f_n$, *then*

$$
\left(\int_a^b p(x)\Delta x\right)^{n-1}\int_a^b p(x)(f_n(x))^n\Delta x\geq\left(\int_a^b p(x)f(x)\Delta x\right)^n.
$$

Putting $f(x) = \frac{f_1(x)}{f_2(x)}$, $g(x) = \frac{g_1(x)}{g_2(x)}$ and $p(x) = f_2(x)g_2(x)$ in (2.7.6), we have the following delta Čebyšev's type inequality on time scales.

Corollary 2.7.2 *Suppose that* f_1, f_2, g_1, $g_2 \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$ *with* $f_2(x)g_2(x) > 0$ *on* $[a,b]_{\mathbb{T}}$. *If* $\frac{f_1(x)}{f_2(x)}$ *and* $\frac{g_1(x)}{g_2(x)}$ *are both increasing or both decreasing, then*

$$
\int_a^b f_1(x)g_1(x)\Delta x\int_a^b f_2(x)g_2(x)\Delta x\geq\int_a^b f_1(x)g_2(x)\Delta x\int_a^b f_2(x)g_1(x)\Delta x.
\tag{2.7.8}
$$

If one of $\frac{f_1(x)}{f_2(x)}$ *or* $\frac{g_1(x)}{g_2(x)}$ *is nonincreasing and the other nondecreasing then the inequality in (2.7.8) is reversed.*

We notice that if $f_1(x) = f(x)f_2(x)$, $g_1(x) = g(x)g_2(x)$ and $p(x) = f_2(x)g_2(x)$, then the inequality (2.7.8) reduces to the inequality (2.7.6).

Theorem 2.7.2 *Let $f \in C_{rd}([a,b]_{\mathbb{T}},[0,\infty))$ be decreasing (or increasing) with $\int_a^b xp(x)f(x)\Delta x > 0$ and $\int_a^b p(x)f(x)\Delta x > 0$. Then*

$$\frac{\int_a^b xp(x)f^2(x)\Delta x}{\int_a^b xp(x)f(x)\Delta x} \geq (\leq) \frac{\int_a^b p(x)f^2(x)\Delta x}{\int_a^b p(x)f(x)\Delta x}. \tag{2.7.9}$$

Proof. Clearly, for any x, $y \in [a,b]_{\mathbb{T}}$,

$$\int_a^b \int_a^b f(x)f(y)p(x)p(y)(y-x)(f(x)-f(y))\Delta x \Delta y \geq (\leq)\, 0,$$

which implies inequality (2.7.9). The proof is complete. ■

Remark 2.7.3 *Let $f \in C_{rd}([a,b]_{\mathbb{T}},[0,\infty))$ and n be a positive integer. If p and g are replaced by p/f and f^n respectively, then the Čebyšev inequality (2.7.6) is reduced to the inequality*

$$\int_a^b p(x)\left(f(x)\right)^n \Delta x \int_a^b \frac{p(x)}{f(x)}\Delta x \geq \int_a^b p(x)\Delta x \int_a^b p(x)\left(f(x)\right)^{n-1}\Delta x,$$

which implies that

$$\begin{aligned} & \int_a^b p(x)\left(f(x)\right)^n \Delta x \left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^2 \\ \geq\ & \int_a^b p(x)\Delta x \int_a^b p(x)\left(f(x)\right)^{n-1}\Delta x \int_a^b \frac{p(x)}{f(x)}\Delta x \\ \geq\ & \left(\int_a^b p(x)\Delta x\right)^2 \int_a^b p(x)\left(f(x)\right)^{n-2}\Delta x, \end{aligned}$$

provided f and f^n are similarly ordered. Proceeding we get

$$\int_a^b p(x)\left(f(x)\right)^n \Delta x \left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^n \geq \left(\int_a^b p(x)\Delta x\right)^{n+1}.$$

Theorem 2.7.3 *If p, $f \in C_{rd}([a,b]_{\mathbb{T}},[0,\infty))$ with $f(x)>0$ on $[a,b]_{\mathbb{T}}$ and n a positive integer, then*

$$\left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^n \left(\int_a^b p(x)f^n(x)\Delta x\right) \geq \left(\int_a^b p(x)\Delta x\right)^n. \tag{2.7.10}$$

Proof. It follows from $f(x) > 0$ on $[a,b]_{\mathbb{T}}$ that $f^n(x)$ and $1/f(x)$ are oppositely ordered on $[a,b]_{\mathbb{T}}$. Hence by (2.7.6) we have

$$\begin{aligned}
&\int_a^b p(x)\left(f(x)\right)^n \Delta x \left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^n \\
\geq\ & \int_a^b p(x)\Delta x \left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^{n-1} \int_a^b p(x)\left(f(x)\right)^{n-1}\Delta x \\
\geq\ & \left(\int_a^b p(x)\Delta x\right)^2 \left(\int_a^b \frac{p(x)}{f(x)}\Delta x\right)^{n-2} \int_a^b p(x)\left(f(x)\right)^{n-2}\Delta x \\
\geq\ & \ldots \geq \left(\int_a^b p(x)\Delta x\right)^n,
\end{aligned}$$

which is the desired inequality (2.7.10). The proof is complete. ■

Theorem 2.7.4 *Let $g_1, g_2, \ldots, g_n \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ and p, $h_1, h_2, \ldots, h_{n-1} \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$ with $g_n(x) > 0$ on $[a,b]_{\mathbb{T}}$. If*

$$\frac{g_1(x)g_2(x)\ldots g_{n-1}(x)}{h_1(x)h_2(x)\ldots h_{n-1}(x)} \text{ and } \frac{h_{n-1}(x)}{g_n(x)},$$

are similarly ordered (or oppositely ordered), then

$$\begin{aligned}
&\int_a^b p(x)g_n(x)\Delta x \int_a^b \frac{p(x)g_1(x)g_2(x)\ldots g_{n-1}(x)}{h_1(x)h_2(x)\ldots h_{n-1}(x)}\Delta x \\
\geq\ & (\leq) \int_a^b p(x)h_{n-1}(x)\Delta x \int_a^b \frac{p(x)g_1(x)g_2(x)\ldots g_n(x)}{h_1(x)h_2(x)\ldots h_{n-1}(x)}\Delta x.
\end{aligned} \tag{2.7.11}$$

Proof. Taking

$$f_1(x) = \frac{g_1(x)g_2(x)\ldots g_{n-1}(x)}{h_1(x)h_2(x)\ldots h_{n-1}(x)},\ k_1(x) = h_{n-1}(x),\ f_2(x) = 1, \text{ and } k_2(x) = g_n(x),$$

in Theorem 2.7.1, we get the desired inequality (2.7.11). The proof is complete. ■

Theorem 2.7.5 *Let p, f_1, $f_2, \ldots, f_n \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$ and g_1, $g_2, \ldots$, $g_n \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$. If the functions $f_1, \frac{f_2}{g_1}, \ldots, \frac{f_n}{g_{n-1}}$ are similarly ordered and for each pair $\frac{f_k}{g_{k-1}}$, g_{k-1} is oppositely ordered for $k = 2, 3, \ldots, n$, then*

$$\begin{aligned}
&\int_a^b p(x)f_1(x)\frac{f_2(x)f_3(x)\ldots f_n(x)}{g_1(x)g_2(x)\ldots g_{n-1}(x)}\Delta x \\
&\geq \frac{\int_a^b p(x)f_1(x)\Delta x \int_a^b p(x)f_2(x)\Delta x \ldots \int_a^b p(x)f_n(x)\Delta x}{\int_a^b p(x)g_1(x)\Delta x \int_a^b p(x)g_2(x)\Delta x \ldots \int_a^b p(x)g_n(x)\Delta x}.
\end{aligned} \tag{2.7.12}$$

Proof. Let f_1, $f_2, \ldots, f_n$ be replaced by f_1, $\frac{f_2}{g_1}, \ldots, \frac{f_n}{g_{n-1}}$ in (2.7.7), and we obtain

$$\begin{aligned}&\left(\int_a^b p(x)\Delta x\right)^{n-1}\int_a^b p(x)f_1(x)\frac{f_2(x)f_3(x)\ldots f_n(x)}{g_1(x)g_2(x)\ldots g_{n-1}(x)}\Delta x\\ \geq\;&\left(\int_a^b p(x)f_1(x)\Delta x\right)\prod_{k=2}^{n}\int_a^b p(x)\frac{f_k(x)}{g_{k-1}(x)}\Delta x.\end{aligned}\tag{2.7.13}$$

Also, since $\frac{f_k}{g_{k-1}}$, g_{k-1} is oppositely ordered for $k = 2, 3, \ldots, n$, it follows from (2.7.6), that

$$\int_a^b p(x)\Delta x\left(\int_a^b p(x)f_k(x)\Delta x\right)\leq\left(\int_a^b p(x)g_{k-1}(x)\Delta x\right)\int_a^b p(x)\frac{f_k(x)}{g_{k-1}(x)}\Delta x.$$

Thus

$$\int_a^b p(x)\frac{f_k(x)}{g_{k-1}(x)}\Delta x\geq\frac{\int_a^b p(x)\Delta x\left(\int_a^b p(x)f_k(x)\Delta x\right)}{\int_a^b p(x)g_{k-1}(x)\Delta x.}$$

This and (2.7.13) imply (2.7.12). The proof is complete. ■

Theorem 2.7.6 *Let p, f_1, $f_2, \ldots, f_n \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$ and k_1, $k_2, \ldots,$ $k_{n-1} \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$. If*

$$\frac{f_1(x)f_2(x)\ldots f_{i-1}(x)}{k_1(x)k_2(x)\ldots k_{i-1}(x)}\;\textit{and}\;\frac{k_{i-1}(x)}{f_i(x)},$$

are similarly ordered (or oppositely ordered) for $i = 2, 3, .., n$, then

$$\begin{aligned}&\left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)f_2(x)\Delta x\right)\ldots\left(\int_a^b p(x)f_n(x)\Delta x\right)\\ \geq\;(\leq)&\left(\int_a^b p(x)k_1(x)\Delta x\right)\left(\int_a^b p(x)k_2(x)\Delta x\right)\ldots\left(\int_a^b p(x)k_{n-1}(x)\Delta x\right)\\ &\times\int_a^b p(x)\frac{f_1(x)f_2(x)\ldots f_n(x)}{k_1(x)k_2(x)\ldots k_{n-1}(x)}\Delta x.\end{aligned}\tag{2.7.14}$$

Proof. If $f_1(x)$, $k_1(x)$, $f_2(x)$ and $k_2(x)$ are replaced by $f_1(x)$, 1, $k_1(x)$, $\frac{f_2(x)}{k_1(x)}$ in Theorem 2.7.1, then we obtain

$$\int_a^b p(x)f_1(x)\Delta x\int_a^b p(x)f_2(x)\Delta x\geq(\leq)\int_a^b p(x)k_1(x)\Delta x\int_a^b p(x)\frac{f_1(x)f_2(x)}{k_1(x)}\Delta x.$$

Thus the theorem holds for $n = 2$. Suppose that the theorem holds for $n-1$, that is

$$\begin{aligned} &\left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)f_2(x)\Delta x\right)\dots\left(\int_a^b p(x)f_{n-1}(x)\Delta x\right) \\ \geq \quad &(\leq)\left(\int_a^b p(x)k_1(x)\Delta x\right)\left(\int_a^b p(x)k_2(x)\Delta x\right)\dots\left(\int_a^b p(x)k_{n-2}(x)\Delta x\right) \\ &\times\int_a^b p(x)\frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{k_1(x)k_2(x)\dots k_{n-2}(x)}\Delta x, \end{aligned} \tag{2.7.15}$$

if

$$\frac{f_1(x)f_2(x)\dots f_{i-1}(x)}{k_1(x)k_2(x)\dots k_{i-1}(x)} \text{ and } \frac{k_{i-1}(x)}{f_i(x)},$$

are similarly ordered (or oppositely ordered) for $i = 2, 3, .., n-1$. Multiplying both sides of (2.7.15) by $\int_a^b p(x)f_n(x)\Delta x$, we get that

$$\begin{aligned} &\int_a^b p(x)f_1(x)\Delta x\int_a^b p(x)f_2(x)\Delta x\dots\int_a^b p(x)f_{n-1}(x)\Delta x\int_a^b p(x)f_n(x)\Delta x \\ \geq \quad &(\leq)\left(\int_a^b p(x)k_1(x)\Delta x\right)\left(\int_a^b p(x)k_2(x)\Delta x\right)\dots\left(\int_a^b p(x)k_{n-2}(x)\Delta x\right) \\ &\times\int_a^b p(x)\frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{k_1(x)k_2(x)\dots k_{n-2}(x)}\Delta x\int_a^b p(x)f_n(x)\Delta x. \end{aligned} \tag{2.7.16}$$

It follows from Theorem 2.7.5 that

$$\begin{aligned} &\int_a^b p(x)\frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{k_1(x)k_2(x)\dots k_{n-2}(x)}\Delta x\int_a^b p(x)f_n(x)\Delta x \\ \geq \quad &(\leq)\int_a^b p(x)\frac{f_1(x)f_2(x)\dots f_n(x)}{k_1(x)k_2(x)\dots k_{n-1}(x)}\Delta x\int_a^b p(x)k_{n-1}(x)\Delta x. \end{aligned}$$

This and (2.7.16) imply

$$\begin{aligned} &\int_a^b p(x)f_1(x)\Delta x\int_a^b p(x)f_2(x)\Delta x\dots\int_a^b p(x)f_{n-1}(x)\Delta x\int_a^b p(x)f_n(x)\Delta x \\ \geq \quad &(\leq)\left(\int_a^b p(x)k_1(x)\Delta x\right)\left(\int_a^b p(x)k_2(x)\Delta x\right)\dots\left(\int_a^b p(x)k_{n-1}(x)\Delta x\right) \\ &\times\int_a^b p(x)\frac{f_1(x)f_2(x)\dots f_n(x)}{k_1(x)k_2(x)\dots k_{n-1}(x)}\Delta x. \end{aligned}$$

Then, by induction we have the desired inequality (2.7.14). The proof is complete. ∎

Remark 2.7.4 *Let $k_n \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$. If $f_1(x)$, $f_2(x), \dots, f_n(x)$ and $k_1(x), k_2(x).,\dots, k_{n-1}(x)$ are replaced by $f_1(x)f_2(x)\dots f_n(x)$, $k_1(x)k_2(x)\dots k_n(x)$, $f_1(x)k_2(x)\dots k_n(x)$, $k_1(x)f_2(x)k_3(x)\dots k_n(x), \dots,$ $k_1(x)k_2(x)\dots k_{n-2}(x)f_{n-1}(x)k_n(x)$ in Theorem 2.7.6, respectively, then*

$$\begin{aligned}
&\int_a^b p(x)f_1(x)f_2(x)\dots f_n(x)\Delta x\left(\int_a^b p(x)k_1(x)k_2(x)\dots k_n(x)\Delta x\right)^{-1}\\
\geq\ &\left(\int_a^b p(x)f_1(x)k_2(x)\dots k_n(x)\Delta x\right)\left(\int_a^b p(x)k_1(x)f_2(x)k_3(x)\dots k_n(x)\Delta x\right)\\
&\dots\int_a^b p(x)k_1(x)k_2(x)\dots k_{n-1}(x)f_n(x)\Delta x, \qquad (2.7.17)
\end{aligned}$$

if $\frac{f_i(x)}{k_i(x)} > 0$ for $i = 1, 2, \dots, n$ and $k_1(x)k_2(x)\dots k_{n-1}(x) > 0$ on $[a,b]_{\mathbb{T}}$.

Remark 2.7.5 *Letting $f_1(x) = f_2(x) = \dots = f_n(x) = f(x)$ and $k_1(x) = k_2(x) = \dots = k_n(x) = k^{\frac{1}{n-1}}(x)$ in (2.7.17) with $k(x) > 0$ on $[a,b]_{\mathbb{T}}$, we obtain a Hölder type inequality on time scales*

$$\left(\int_a^b p(x)f(x)k(x)\Delta x\right)^n \leq \int_a^b p(x)\left(f(x)\right)^n \Delta x\left(\int_a^b p(x)k^{\frac{n}{n-1}}(x)\Delta x\right)^{n-1}.$$

Remark 2.7.6 *Let p, f, $g \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$. Putting $f_1(x) = (f(x))^n g(x)$, $f_2(x) = f_3(x) = \dots = f_n(x) = g(x)$, and $k_1(x) = k_2(x) = \dots = k_{n-1}(x) = f(x)g(x)$ in (2.7.14), we see that*

$$\left(\int_a^b p(x)f(x)g(x)\Delta x\right)^n \leq \int_a^b p(x)\left(f(x)\right)^n g(x)\Delta x\left(\int_a^b p(x)g(x)\Delta x\right)^{n-1}.$$

Remark 2.7.7 *Taking $k_1(x) = k_2(x) = \dots = k_{n-1}(x) = (f_1(x)f_2(x)\dots f_n(x))^{\frac{1}{n}}$ in (2.7.14), we obtain*

$$\begin{aligned}
&\left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)f_2(x)\Delta x\right)\dots\left(\int_a^b p(x)f_n(x)\Delta x\right)\\
\geq\ &\left(\int_a^b p(x)\left(f_1(x)f_2(x)\dots f_n(x)\right)^{\frac{1}{n}}\Delta x\right)^n,
\end{aligned}$$

if $f_i > 0$ on $[a,b]_{\mathbb{T}}$ and $\frac{1}{f_i(x)}\left(f_1(x)f_2(x)\dots f_n(x)\right)^{\frac{1}{n}}$ $(i = 1, 2, \dots, n)$ are similarly ordered.

Remark 2.7.8 *Taking $k_1(x) = k_2(x) = \ldots = k_{n-1}(x) = 1$ in (2.7.14), we get the Čebyšev type inequality*

$$\begin{aligned}&\left(\int_a^b p(x)f_1(x)\Delta x\right)\left(\int_a^b p(x)f_2(x)\Delta x\right)\ldots\left(\int_a^b p(x)f_n(x)\Delta x\right)\\ \leq\ &\left(\int_a^b p(x)\Delta x\right)^{n-1}\int_a^b p(x)f_1(x)f_2(x)\ldots f_n(x)\Delta x,\end{aligned}$$

if $f_i > 0$ on $[a,b]_{\mathbb{T}}$ and $f_i(x)$ $(i = 1, 2, \ldots, n)$ are similarly ordered.

We end this section by considering the Čebyšev inequality in the case of nabla integrals; see [26].

Theorem 2.7.7 *Let f and g be both increasing or both decreasing in $[a,b]_{\mathbb{T}}$. Then*

$$\int_a^b f(t)g(t)\nabla t \geq \frac{1}{b-a}\int_a^b f(t)\nabla t\int_a^b g(t)\nabla t. \tag{2.7.18}$$

If one of the functions is increasing and the other is decreasing, then the inequality is reversed.

Now, we give some applications of Theorem 2.7.7.

Theorem 2.7.8 *Assume that $f^{\nabla^{n+1}}$ is monotonic on $[a,b]_{\mathbb{T}}$ and let*

$$\check{R}_{n,f}(t,s) = f(s) - \sum_{k=0}^{n}\hat{h}_k(t,s)f^{\nabla^k}(s) = \int_t^s \hat{h}_n(s,\rho(\xi))f^{\nabla^{n+1}}(\xi)\nabla\xi.$$

(i). If $f^{\nabla^{n+1}}$ is increasing, then

$$\begin{aligned}&\int_a^b \check{R}_{n,f}(a,t)\nabla t - \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a}\right]\hat{h}_{n+2}(b,a)\\ \geq\ &\left[f^{\nabla^{n+1}}(a) - f^{\nabla^{n+1}}(b)\right]\hat{h}_{n+2}(b,a).\end{aligned} \tag{2.7.19}$$

(ii). If $f^{\nabla^{n+1}}$ is decreasing, then

$$\begin{aligned}&\int_a^b \check{R}_{n,f}(a,t)\nabla t - \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a}\right]\hat{h}_{n+2}(b,a)\\ \leq\ &\left[f^{\nabla^{n+1}}(a) - f^{\nabla^{n+1}}(b)\right]\hat{h}_{n+2}(b,a).\end{aligned}$$

Proof. The proof of (ii) is analogous to that of (i) so we will just consider (i). Let $F(t) = f^{\nabla^{n+1}}(t)$ and $G(t) = \hat{h}_n(b,\rho(t))$. Then F is increasing and G is decreasing by assumption. From inequality (2.7.18), we see that

$$\int_a^b F(t)G(t)\nabla t \leq \frac{1}{b-a}\int_a^b F(t)\nabla t\int_a^b G(t)\nabla t. \tag{2.7.20}$$

By Corollary 2.6.1, we see that

$$\int_a^b F(t)G(t)\nabla t = \int_a^b \hat{h}_{n+1}(b,\rho(t))f^{\nabla^{n+1}}(t)\nabla t = \int_a^b \check{R}_{n,f}(a,t)\nabla t.$$

We also have

$$\int_a^b F(t)\nabla t = f^{\nabla^n}(b) - f^{\nabla^n}(a), \text{ and } \int_a^b G(t)\nabla t = \int_a^b \hat{h}_{n+1}(b,\rho(t))\nabla t = \hat{h}_{n+2}(b,a).$$

Thus the inequality (2.7.20) implies that

$$\int_a^b \check{R}_{n,f}(a,t)\nabla t \leq \frac{1}{b-a}\left(f^{\nabla^n}(b) - f^{\nabla^n}(a)\right)\hat{h}_{n+2}(b,a).$$

Since $f^{\nabla^{n+1}}$ is increasing on $[a,b]_{\mathbb{T}}$,

$$\begin{aligned} f^{\nabla^{n+1}}(a)\hat{h}_{n+2}(b,a) &\leq \frac{1}{b-a}\left(f^{\nabla^n}(b) - f^{\nabla^n}(a)\right)\hat{h}_{n+2}(b,a) \\ &\leq f^{\nabla^{n+1}}(b)\hat{h}_{n+2}(b,a), \end{aligned}$$

and, we have

$$\begin{aligned} &\int_a^b \check{R}_{n,f}(a,t)\nabla t - \frac{1}{b-a}\left(f^{\nabla^n}(b) - f^{\nabla^n}(a)\right)\hat{h}_{n+2}(b,a) \\ \geq &\int_a^b \check{R}_{n,f}(a,t)\nabla t - f^{\nabla^{n+1}}(b)\hat{h}_{n+2}(b,a). \end{aligned}$$

Now Corollary 2.6.1 and $f^{\nabla^{n+1}}$ is increasing imply that

$$f^{\nabla^{n+1}}(b)\int_a^b \hat{h}_{n+1}(b,\rho(t))\nabla t \geq \int_a^b \check{R}_{n,f}(a,t)\nabla t \geq f^{\nabla^{n+1}}(a)\int_a^b \hat{h}_{n+1}(b,\rho(t))\nabla t,$$

which simplifies to

$$f^{\nabla^{n+1}}(b)\hat{h}_{n+2}(b,a) \geq \int_a^b \check{R}_{n,f}(a,t)\nabla t \geq f^{\nabla^{n+1}}(a)\int_a^b \hat{h}_{n+2}(b,a)\nabla t.$$

We now have inequality (2.7.19). The proof is complete. ■

Theorem 2.7.9 *Assume that $f^{\nabla^{n+1}}$ is monotonic on $[a,b]_{\mathbb{T}}$.*

(i) If $f^{\Delta^{n+1}}$ is increasing, then

$$\begin{aligned} &0 \leq (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t - \left[\frac{f^{\Delta^n}(b) - f^{\Delta^n}(a)}{b-a}\right] g_{n+2}(b,a) \\ \leq &\left[f^{\Delta^{n+1}}(b) - f^{\Delta^{n+1}}(a)\right] g_{n+2}(b,a). \end{aligned} \qquad (2.7.21)$$

(ii). If $f^{\Delta^{n+1}}$ is decreasing, then

$$\begin{aligned} 0 &\geq (-1)^{n+1}\int_a^b R_{n,f}(b,t)\nabla t - \left[\frac{f^{\Delta^n}(b) - f^{\Delta^n}(a)}{b-a}\right] g_{n+2}(b,a) \\ &\geq \left[f^{\Delta^{n+1}}(b) - f^{\Delta^{n+1}}(a)\right] g_{n+2}(b,a). \end{aligned}$$

Proof. The proof of (ii) is analogous to that of (i) so we only consider (i). Let $F(t) = f^{\Delta^{n+1}}(t)$ and $G(t) = (-1)^{n+1}h_{n+1}(a,\sigma(t))$. Then F and G are increasing. Inequality (2.7.6) with $p = 1$, $f = F$ and $g = G$, gives

$$\int_a^b F(t)G(t)\Delta t \geq \frac{1}{b-a}\int_a^b F(t)\Delta t \int_a^b G(t)\Delta t. \tag{2.7.22}$$

By Lemma 2.6.2 with $t = a$,

$$\begin{aligned} \int_a^b F(t)G(t)\Delta t &= (-1)^{n+1}\int_a^b h_{n+1}(a,\sigma(t)) f^{\Delta^{n+1}}(t)\Delta t \\ &= (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t. \end{aligned}$$

We also have $\int_a^b F(t)\Delta t = f^{\Delta^n}(b) - f^{\Delta^n}(b)$ and

$$\int_a^b G(t)\Delta t = (-1)^{n+1}\int_a^b h_{n+1}(a,\sigma(t))\Delta t = g_{n+2}(b,a).$$

Thus by (2.7.22), we have

$$0 \leq (-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t - \frac{1}{b-a}\left[f^{\Delta^n}(b) - f^{\Delta^n}(b)\right] g_{n+2}(b,a).$$

Since $f^{\Delta^{n+1}}$ is increasing on $[a,b]_{\mathbb{T}}$,

$$f^{\Delta^{n+1}}(a) g_{n+2}(b,a) \leq \frac{1}{b-a}\left[f^{\Delta^n}(b) - f^{\Delta^n}(b)\right] g_{n+2}(b,a) \leq f^{\Delta^{n+1}}(b) g_{n+2}(b,a),$$

and we have

$$\begin{aligned} &(-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t - f^{\Delta^{n+1}}(a) g_{n+2}(b,a) \\ \geq\ &(-1)^{n+1}\int_a^b R_{n,f}(b,t)\Delta t - \frac{\left[f^{\Delta^n}(b) - f^{\Delta^n}(b)\right]}{b-a} g_{n+2}(b,a). \end{aligned}$$

Now, from Definition 1.4.1, since

$$g_n(t,s) = (-1)^n h_n(s,t),$$

we have by Lemma 2.6.2 with $t = a$ that

$$(-1)^{n+1} \int_a^b R_{n,f}(b,t)\Delta t = \int_a^b g_{n+1}(\sigma(t),a) f^{\Delta^{n+1}}(t)\Delta t.$$

Since $f^{\Delta^{n+1}}$ is increasing, we get that

$$\begin{aligned} f^{\Delta^{n+1}}(b) \int_a^b g_{n+1}(\sigma(t),a)\Delta t &\geq (-1)^{n+1} \int_a^b R_{n,f}(b,t)\Delta t \\ &\geq f^{\Delta^{n+1}}(a) \int_a^b g_{n+1}(\sigma(t),a)\Delta t \end{aligned}$$

which simplifies to

$$f^{\Delta^{n+1}}(b) g_{n+1}(b,a) \geq (-1)^{n+1} \int_a^b R_{n,f}(b,t)\Delta t \geq f^{\Delta^{n+1}}(a) g_{n+2}(b,a).$$

We now have (2.7.21). The proof is complete. ■

Remark 2.7.9 *In Theorem 2.7.8 (i), if $n = 0$, we obtain*

$$\int_a^b f(t)\nabla t \leq (b-a)f(a) + \frac{\hat{h}_2(b,a)}{b-a}(f(b)-f(a)). \tag{2.7.23}$$

Theorem 2.7.10 *Assume that f is nabla convex on $[a,b]_{\mathbb{T}}$, that is, $f^{\nabla^2} \geq 0$ on $[a,b]_{\mathbb{T}}$. Then*

$$\int_a^b f^{\rho}(t)(t-a)\nabla t \leq (b-a)f(b) - \frac{\hat{h}_2(b,a)}{b-a}(f(b)-f(a)). \tag{2.7.24}$$

Proof. If $F = f^{\nabla}$ and $G = t - a = \hat{h}_1(t,a)$, then both F and G are increasing functions. By Čebyšev's inequality we see that

$$\int_a^b f^{\rho}(t)(t-a)\nabla t \geq \frac{1}{b-a} \int_a^b f^{\nabla}(t)\nabla t \int_a^b \hat{h}_1(t,a)\nabla t.$$

Using nabla integration by parts on the left-hand side we get the desired inequality (2.7.24). The proof is complete. ■

The following result is a Hermite–Hadamard type inequality for time scales and is obtained by a combination of (2.7.23) and (2.7.24).

Corollary 2.7.3 *Let f be nabla convex on $[a,b]_{\mathbb{T}}$. Then*

$$\frac{1}{b-a} \int_a^b \frac{f^{\rho}(t) + f(t)}{2}\nabla t \leq \frac{f(a)+f(b)}{2}.$$

http://www.springer.com/978-3-319-11001-1

Dynamic Inequalities On Time Scales
Agarwal, R.; O'Regan, D.; Saker, S.
2014, X, 256 p., Hardcover
ISBN: 978-3-319-11001-1