

Chapter 2

Gottlieb and Whitehead Center Groups of Projective Spaces

By the use of Siegel's method and the classical results of homotopy groups of spheres and Lie groups, we determine in this chapter some Gottlieb groups of projective spaces or give the lower bounds of their orders. Furthermore, making use of the properties of Whitehead products, we determine some Whitehead center groups of projective spaces.

2.1 Preliminaries I

Let EX be the suspension of a space X and denote by $E : \pi_k(X) \rightarrow \pi_{k+1}(EX)$ the suspension homomorphism. Next, write $\sharp\alpha$ for the order of $\alpha \in \pi_k(X)$ and $\Delta = \Delta_{\mathbb{F}P} : \pi_k(\mathbb{F}P^n) \rightarrow \pi_{k-1}(\mathbb{S}^{d-1})$ for the connecting map. By [8, Theorem (2.1)] it holds:

$$\Delta(i_{\mathbb{F}*}E) = \text{id}_{\pi_{k-1}(\mathbb{S}^{d-1})} \text{ and } \pi_k(\mathbb{F}P^n) = \gamma_{n*}\pi_k(\mathbb{S}^{d(n+1)-1}) \oplus i_{\mathbb{F}*}E\pi_{k-1}(\mathbb{S}^{d-1}).$$

Especially,

$$\sharp\alpha = \sharp(i_{\mathbb{H}}E\alpha) \text{ for } \alpha \in \pi_{k-1}(\mathbb{S}^3). \quad (2.1)$$

According to [88], a map $f : X \rightarrow Y$ is *cyclic* if $f \vee \text{id}_Y : X \vee Y \rightarrow Y$ extends to $F : X \times Y \rightarrow Y$. The extension F is called an *associated map* for f [39]. We recall from [81, Lemma 2.1] and [88, Lemma 1.3]:

Lemma 2.1 (Siegel-Varadarajan). *Let $f : X \rightarrow Y$ be cyclic. Then, the composite $f \circ g : W \rightarrow Y$ is cyclic for any map $g : W \rightarrow X$.*

In particular, elements of $G_n(X)$ are cyclic. Notice that, in view of Chap. 1, an element $\alpha \in G_n(\mathbb{S}^m)$ if and only if $[\iota_m, \alpha] = 0$.

We say that a subgroup $A_n(X) \subseteq \pi_n(X)$ is *admissible* if $A_n(X) \circ \pi_k(\mathbb{S}^n) \subseteq A_k(X)$ for all $k \geq 1$. Further, an element $\alpha \in A_n(X)$ is called *$A_n(X)$ -admissible*.

Recall that given $\alpha \in \pi_k(\mathbb{S}^m)$ and $\beta \in \pi_l(\mathbb{S}^n)$ with $m, n > 0$, Lemma 1.2(1) yields

$$[\xi \circ \alpha, \eta \circ \beta] = 0 \text{ provided that } [\xi, \eta] = 0 \quad (2.2)$$

for $\xi \in \pi_m(X)$, $\eta \in \pi_n(X)$.

Then, (2.2) and Lemma 2.1 yield:

Corollary 2.2. *The subgroups $G_n(X), P_n(X) \subseteq \pi_n(X)$ are admissible for all $n \geq 1$.*

To state an analog of the formula from Lemma 1.2(2) for $m > n = 0$, we recall [92, Chap. III, Sect. 1] that given a space (X, x_0) with a base point and a path $u : I \rightarrow Y$ in a space Y , maps $f, g : X \rightarrow Y$ are called to be *freely homotopic along u* ($f \simeq_u g$) if there is such a homotopy $H : I \times X \rightarrow Y$ that $H(t, x_0) = u(t)$ for all $t \in I$.

Lemma 2.3. *If $f \simeq_u g$ then $f \circ f' \simeq_u g \circ f'$ for any map $f' : (X', x'_0) \rightarrow (X, x_0)$ of spaces with base points.*

Proof. Given a homotopy $H : I \times X \rightarrow Y$ with $H(t, x_0) = u(t)$ for all $t \in I$, consider $H' : I \times X' \rightarrow Y$ given by $H'(t, x') = H(t, f'(x'))$ for $(t, x') \in I \times X'$. Then, $H'(t, x'_0) = u(t)$ for all $t \in I$ and the result follows. $\square$

Now, let (X, x_0) be a space with *nondegenerate base point* (i.e., $(X, \{x_0\})$ is an ANR-pair [92, p. 22]), $f : (X, x_0) \rightarrow (Y, y_0)$, and $u : I \rightarrow Y$ such a path in Y that $f(x_0) = u(0)$. Then, by [92, Chap. III, (1.4)], there is such $g : (X, x_0) \rightarrow (Y, u(1))$ that $f \simeq_u g$.

Next, write η and ξ for the homotopy classes of u and f , respectively. Then, in view of [92, Chap. III, (1.5) and (1.6)], the homotopy class $\tau_\eta(\xi)$ of g is well defined. Consequently, Lemma 2.3 leads to

$$\tau_\eta(\xi \circ \xi') = \tau_\eta(\xi) \circ \xi' \quad (2.3)$$

for the homotopy class ξ' of a map $f' : (X', x'_0) \rightarrow (X, x_0)$.

On the other hand, for $\eta \in \pi_1(X)$ and $\xi \in \pi_{m+1}(X)$, the element $\tau_\eta(\xi) \in \pi_{m+1}(X)$ is the operation of η on ξ and, in view of [92, Chap. X, (7.6)], it holds

$$[\xi, \eta] = (-1)^{m+1}(\tau_\eta(\xi) - \xi). \quad (2.4)$$

Then, in virtue of (2.3), (2.4), and [92, Chap. X, (8.2) Theorem], we may state

$$[\xi \circ E\alpha, \eta] = (-1)^{k+m}[\xi, \eta] \circ E\alpha \quad (2.5)$$

for $\alpha \in \pi_k(\mathbb{S}^m)$, $\eta \in \pi_1(X)$, and $\xi \in \pi_{m+1}(X)$.

Let $q_n = q_{n,\mathbb{F}} : \mathbb{F}P^n \rightarrow \mathbb{S}^{dn}$ be the map pinching $\mathbb{F}P^{n-1}$ to the base point. Then, in view of [8] and [33, (2.10)(a)],

$$q_n \gamma_n = \begin{cases} (1 - (-1)^n) \iota_n & \text{for } \mathbb{F} = \mathbb{R}, \\ n \eta_{2n} & \text{for } \mathbb{F} = \mathbb{C}, \\ \pm n v_{4n}^+ & \text{for } \mathbb{F} = \mathbb{H}. \end{cases} \quad (2.6)$$

By (2.2), Proposition 1.6 and [8, (4.1-3)], we obtain a key formula determining the Whitehead center groups of $\mathbb{F}P^n$.

Lemma 2.4. *Let $h_0 \alpha \in \pi_k(\mathbb{S}^{2d(n+1)-3})$ be the 0-th Hopf–Hilton invariant for $\alpha \in \pi_k(\mathbb{S}^{d(n+1)-1})$. Then:*

1. $[\gamma_n \alpha, i_{\mathbb{R}}] = \begin{cases} 0 & \text{for odd } n, \\ (-1)^k \gamma_n (-2\alpha + [\iota_n, \iota_n] \circ h_0 \alpha) & \text{for even } n; \end{cases}$
2. $[\gamma_n \alpha, i_{\mathbb{C}}] = \begin{cases} 0 & \text{for odd } n, \\ \gamma_n (\eta_{2n+1} \circ E\alpha + [\iota_{2n+1}, \eta_{2n+1}] \circ Eh_0 \alpha) & \text{for even } n; \end{cases}$
3. $[\gamma_n \alpha, i_{\mathbb{H}}] = \pm(n+1) \gamma_n (v_{4n+3}^+ \circ E^3 \alpha + [\iota_{4n+3}, v_{4n+3}] \circ E^3 h_0 \alpha).$

We recall:

Lemma 2.5 (Mimura [49, Lemma 5.9]). *If $\alpha \in \pi_p(\mathbb{S}^n)$ and $\beta \in \pi_i(\mathbb{S}^p)$ then*

$$Eh_0(\alpha\beta) = (Eh_0(\alpha))(E\beta) + (E^n \alpha)(E^p \alpha)(Eh_0(\beta)).$$

Observe that [49, Corollary 5.8, p. 183] implies

$$H(\alpha) = h_0(\alpha) \quad (2.7)$$

for $\alpha \in \pi_k(\mathbb{S}^n)$ provided that $k \leq 3n - 3$.

Given a fibration $F \hookrightarrow E \xrightarrow{p} B$, write $\partial : \pi_{n+1}(B) \rightarrow \pi_n(F)$ for the connecting map. If $\partial(\alpha) = 0$ then there exists γ with $\alpha = p_* \gamma$ and $\partial(\alpha \circ \beta) = \partial p_*(\gamma \circ \beta) = 0$. Thus, we get that the subgroup

$$\text{Ker} \{ \partial : \pi_{k+1}(B) \rightarrow \pi_k(F) \} \subseteq \pi_{n+1}(B) \text{ is admissible} \quad (2.8)$$

and its elements are called ∂ -cyclic.

We also recall the formula

$$\partial(\alpha \circ E\beta) = \partial(\alpha) \circ \beta \quad [37, \text{Lemma 1}]. \quad (2.9)$$

We set

$$SO_{\mathbb{F}}(n) = \begin{cases} SO(n) & \text{for } \mathbb{F} = \mathbb{R}; \\ SU(n) & \text{for } \mathbb{F} = \mathbb{C}; \\ Sp(n) & \text{for } \mathbb{F} = \mathbb{H}. \end{cases}$$

Denote by $i_{n,\mathbb{F}} : SO_{\mathbb{F}}(n-1) \hookrightarrow SO_{\mathbb{F}}(n)$ and $p_{n,\mathbb{F}} : SO_{\mathbb{F}}(n) \rightarrow \mathbb{S}^{dn-1}$ the inclusion and projection maps, respectively. Then, we consider the exact sequence induced by the fibration $SO_{\mathbb{F}}(n+1) \xrightarrow{SO_{\mathbb{F}}(n)} \mathbb{S}^{d(n+1)-1}$:

$$(\mathcal{SF}_k^n) \cdots \rightarrow \pi_{k+1}(\mathbb{S}^{d(n+1)-1}) \xrightarrow{\Delta_{\mathbb{F}}} \pi_k(SO_{\mathbb{F}}(n)) \xrightarrow{i_*} \pi_k(SO_{\mathbb{F}}(n+1)) \xrightarrow{p_*} \pi_k(\mathbb{S}^{d(n+1)-1}) \rightarrow \cdots,$$

where $\Delta_{\mathbb{F}}$ is the connecting map, $i = i_{n+1,\mathbb{F}}$ and $p = p_{n+1,\mathbb{F}}$, respectively. A $\Delta_{\mathbb{F}}$ -cyclic element $\alpha \in \pi_k(\mathbb{S}^{d(n+1)-1})$ is called simply $\mathbb{F}$ -cyclic.

Notice that $\pi_3(SO(4)) = \{[\eta_2]_4, [\iota_3]\} \cong \mathbb{Z}^2$ and any element of $\pi_k(SO(4)) \cong \pi_k(SO(3)) \oplus \pi_k(\mathbb{S}^3)$ is uniquely represented by two elements of $\pi_k(\mathbb{S}^3)$:

$$[\eta_2]_4\alpha + [\iota_3]\beta \quad \text{for } \alpha, \beta \in \pi_k(\mathbb{S}^3),$$

where $[\eta_2]$ and $[\iota_3]$ are lifts of η_2 and ι_3 , respectively, defined by (1.56). Certainly, we can take $J([\iota_3]) = v_4^+$ and $J([\eta_2]) = v'^+$.

Next, define the subgroups of $\pi_k(\mathbb{S}^{d(n+1)-1})$ and $\pi_{k-1}(\mathbb{S}^{d-1})$ as follows:

$$M_k(\mathbb{S}^{d(n+1)-1}) = M_{k,\mathbb{F}}(\mathbb{S}^{d(n+1)-1}) = \{\alpha \in \pi_k(\mathbb{S}^{d(n+1)-1}) \mid [\gamma_n\alpha, i_{\mathbb{F}}] = 0\},$$

$$L'_{k-1}(\mathbb{S}^{d-1}) = L'_{k-1,n}(\mathbb{S}^{d-1}) = \{\beta \in \pi_{k-1}(\mathbb{S}^{d-1}) \mid [i_{\mathbb{F}}E\beta, i_{\mathbb{F}}] = 0\},$$

$$L''_{k-1}(\mathbb{S}^{d-1}) = L''_{k-1,n}(\mathbb{S}^{d-1}) = \{\beta \in \pi_{k-1}(\mathbb{S}^{d-1}) \mid [i_{\mathbb{F}}E\beta, \gamma_n] = 0\},$$

and

$$L_{k-1}(\mathbb{S}^{d-1}) = L_{k-1,n}(\mathbb{S}^{d-1}) = L'_{k-1,n}(\mathbb{S}^{d-1}) \cap L''_{k-1,n}(\mathbb{S}^{d-1}).$$

We also define the subgroup $Q_k(\mathbb{S}^3)$ of $\pi_k(\mathbb{S}^3)$ by

$$Q_k(\mathbb{S}^3) = \{\beta \in \pi_k(\mathbb{S}^3) \mid \langle \iota_3, \beta \rangle = 0\},$$

where $\langle -, - \rangle$ stands for the Samelson product.

The relation (2.2) and the result

$$\langle \iota_3, \beta \circ \delta \rangle = \langle \iota_3, \beta \rangle \circ E^3\delta \quad [34, (6.3)] \quad (2.10)$$

lead to:

Lemma 2.6. *The subgroups:*

1. $M_k(\mathbb{S}^{d(n+1)-1}) \subseteq \pi_k(\mathbb{S}^{d(n+1)-1})$;
2. $L'_{k-1}(\mathbb{S}^{d-1}), L''_{k-1}(\mathbb{S}^{d-1}), L_{k-1}(\mathbb{S}^{d-1}) \subseteq \pi_{k-1}(\mathbb{S}^{d-1})$;
3. $Q_k(\mathbb{S}^3) \subseteq \pi_k(\mathbb{S}^3)$

are admissible.

We set

$$\begin{aligned} L_{k-1}'^{d-1} &= L_{k-1}'(\mathbb{S}^{d-1}; 2), L_{k-1}''^{d-1} = L_{k-1}''(\mathbb{S}^{d-1}; 2), L_{k-1}^{d-1} = L_{k-1}(\mathbb{S}^{d-1}; 2), \\ P_k^{d(n+1)-1} &= P_k(\mathbb{S}^{d(n+1)-1}; 2), Q_{k-1}^{d-1} = Q_{k-1}(\mathbb{S}^{d-1}; 2), \text{ and} \\ M_k^{d(n+1)-1} &= M_k(\mathbb{S}^{d(n+1)-1}; 2). \end{aligned}$$

The covering $\rho : \mathbb{S}^3 \rightarrow SO(3)$ leads to the map $f : \mathbb{S}^3 \times \mathbb{S}^2 \rightarrow \mathbb{S}^2$ given by $f(x, y) = \rho(x)y$ for $(x, y) \in \mathbb{S}^3 \times \mathbb{S}^2$. Because $f|_{\mathbb{S}^3} = \eta_2$ and $f|_{\mathbb{S}^2} = \iota_2$, the results [12], [34, Sect. 9] and [82, p. 115] imply

$$J(\rho) = v'^+ = x\langle \iota_3, \iota_3 \rangle \text{ for an integer } 1 \leq x \leq 11 \text{ with } (x, 12) = 1. \quad (2.11)$$

Writing $(-, -)$ for the greatest common divisor, we can state:

Lemma 2.7. 1. $L_{3,n}''(\mathbb{S}^3) = \frac{24}{(24,n+1)}\pi_3(\mathbb{S}^3)$, $L_{4,n}''(\mathbb{S}^3) = \pi_4(\mathbb{S}^3)$ for $n \geq 1$;

2. $M_{k,\mathbb{F}}(\mathbb{S}^m) = \begin{cases} 0 & \text{for } \mathbb{F} = \mathbb{C}, k = 7 \text{ and } m = 5, \\ \pi_{12}(\mathbb{S}^9) & \text{for } \mathbb{F} = \mathbb{C}, k = 12 \text{ and } m = 9, \\ \frac{24}{(24,n+1)}\pi_{4n+3}(\mathbb{S}^{4n+3}) & \text{for } \mathbb{F} = \mathbb{H}, k = 4n + 3 \text{ and} \\ & m = 4n + 3 \text{ with } n \geq 1, \\ \frac{2}{(2,n+1)}\pi_{4n+6}(\mathbb{S}^{4n+3}) & \text{for } \mathbb{F} = \mathbb{H}, k = 4n + 6 \text{ and} \\ & m = 4n + 3 \text{ with } n \geq 1; \end{cases}$

3. $Q_k(\mathbb{S}^3) = \begin{cases} 12\pi_3(\mathbb{S}^3) & \text{for } k = 3, \\ 0 & \text{for } k = 4, 5, \\ 3\pi_6(\mathbb{S}^3) \cong \mathbb{Z}_4 & \text{for } k = 6. \end{cases}$

Proof. First, notice that Lemma 2.4(2) and (3) yields:

$$1. [\gamma_{2n}, i_{\mathbb{C}}] = \gamma_{2n}\eta_{4n+1} \text{ and } 2. [i_{\mathbb{H}}, \gamma_n] = \pm(n+1)\gamma_nv_{4n+3}^+. \quad (2.12)$$

Then, Lemma 1.2(2), (1.23), and (2.12)(2) lead to $[i_{\mathbb{H}}\eta_4, \gamma_n] = \pm(n+1)\gamma_nv_{4n+3}\eta_{4n+6} = 0$, and (1) follows.

By (1.25), (1.18), and (2.12)(1), $[\gamma_2\eta_5^2, i_{\mathbb{C}}] = \gamma_2\eta_5^3 = 4\gamma_2v_5 \neq 0$. Hence, $M_{7,\mathbb{C}}(\mathbb{S}^5) = 0$. Because, by (1.23) and (2.12)(1), $[\gamma_4v_9^+, i_{\mathbb{C}}] = \gamma_4\eta_9v_{10}^+ = 0$, we obtain $M_{12,\mathbb{C}}(\mathbb{S}^9) = \pi_{12}(\mathbb{S}^9)$. Next, Lemma 1.2(2), (1.8), and (2.12)(2) yield $[\gamma_nv_{4n+3}^+, i_{\mathbb{H}}] = \pm(n+1)\gamma_nv_{4n+3}^2$. Consequently, we obtain that $M_{4n+3,\mathbb{C}}(\mathbb{S}^{4n+3}) = \frac{24}{(24,n+1)}\pi_{4n+3}(\mathbb{S}^{4n+3})$, $M_{4n+6,\mathbb{C}}(\mathbb{S}^{4n+3}) = \frac{2}{(2,n+1)}\pi_{4n+6}(\mathbb{S}^{4n+3})$, and (2) is shown.

In view of (2.11), it holds $Q_3(\mathbb{S}^3) = 12\pi_3(\mathbb{S}^3)$. Further, (2.10) and (2.11) lead to $\langle \iota_3, \eta_3 \rangle = \langle \iota_3, \iota_3 \rangle \circ \eta_6 = v'\eta_6 \neq 0$ and $\langle \iota_3, \eta_3^2 \rangle = \langle \iota_3, \iota_3 \rangle \circ \eta_6^2 = v'\eta_6^2 \neq 0$. Thus, we deduce that $Q_4(\mathbb{S}^3) = 0$ and $Q_5(\mathbb{S}^3) = 0$. Finally, (1.25), (2.10), and (2.11) yield $\langle \iota_3, v'^+ \rangle = \langle \iota_3, \iota_3 \rangle \circ E^3v'^+ = 2xv'^+ \circ v_6^+ = \pm\alpha_1(3)\alpha_1(6)$. Then, (1.8) leads to $Q_6(\mathbb{S}^3) = 3\pi_6(\mathbb{S}^3) \cong \mathbb{Z}_4$ and the proof is complete. $\square$

Notice that for $\beta \in \pi_{k-1}(\mathbb{S}^3)$, by Lemma 1.2(2) and (2.12)(2), we get

$$[i_{\mathbb{H}}E\beta, \gamma_n] = \pm(n+1)\gamma_n v_{4n+3}^+ E^{4n+3}\beta. \quad (2.13)$$

Now, we show:

Lemma 2.8. 1. $L''_{k-1,n}(\mathbb{S}^3) = E^{-4n-3}(\text{Ker}(n+1)v_{4n+3}^+)$, where

$$v_{4n+3}^+ : \pi_{k+4n+2}(\mathbb{S}^{4n+6}) \rightarrow \pi_{k+4n+2}(\mathbb{S}^{4n+3})$$

is the induced homomorphism. In particular, $L'_{k-1,n}(\mathbb{S}^3) \subseteq L''_{k-1,n}(\mathbb{S}^3)$ for n odd;

2. $Q_{k-1}(\mathbb{S}^3) = L'_{k-1,n}(\mathbb{S}^3)$ for $n \geq 2$;
3. Let $n \geq 2$. Then, $L'_{k-1}(\mathbb{S}^3; 3) \subseteq L''_{k-1}(\mathbb{S}^3; 3)$ and $L'_{k-1}(\mathbb{S}^3; p) = L''_{k-1}(\mathbb{S}^3; p) = \pi_{k-1}(\mathbb{S}^3; p)$ for an odd prime $p \geq 5$;
4. $i_{\mathbb{H}*}P_k(\mathbb{S}^4) \subseteq i_{\mathbb{H}*}EQ_{k-1}(\mathbb{S}^3)$. Moreover, $i_{\mathbb{H}*}P_k(\mathbb{S}^4) = i_{\mathbb{H}*}EQ_{k-1}(\mathbb{S}^3)$ provided that $2E^4Q_{k-1}(\mathbb{S}^3) = 0$.

Proof. By (2.13), the first half of (1) follows. If $[E\beta, \iota_4] = 0$, then $[E\beta, v_4^+] = 0$ (2.2). This leads to the second one of (1) for $n = 1$. The fact that $i_{\mathbb{H}} \circ v_4^+ = 0$ for $n \geq 2$ and (1.20) yield to

$$[E\beta, \iota_4] = \pm(2v_4^+ - Ev'^+) \circ E^4\beta, \quad (2.14)$$

the relation $i_{\mathbb{H}}[E\beta, \iota_4] = 0$ implies $i_{\mathbb{H}}E(v'^+E^3\beta) = 0$, and so $v'^+E^3\beta = 0$. Hence,

$$2v_5^+E^5\beta = 0 \quad (2.15)$$

and $(n+1)v_{4n+3}^+E^{4n+3}\beta = 0$ for all odd n . This leads to the second one of (1). By means of (1.8) and (2.13), we obtain (2). Given $\beta \in L'_{k-1}(\mathbb{S}^3; 3)$, in view of (2.13) and (2.15), it holds $\beta \in L''_{k-1}(\mathbb{S}^3; 3)$ and the first half of (3) follows. The second one of (3) is a direct consequence of (2.13) and (2.14).

In view of (1.27)(2), we may suppose that $\beta = E\beta_1 + v_4^+\beta_2 \in P_k(\mathbb{S}^4)$ for $\beta_1 \in \pi_{k-1}(\mathbb{S}^3)$ and $\beta_2 \in \pi_k(\mathbb{S}^7)$. Then, by the relation $i_{\mathbb{H}}\beta = i_{\mathbb{H}}E\beta_1$, we obtain

$$0 = i_{\mathbb{H}}[\iota_4, \beta] = i_{\mathbb{H}}[\iota_4, E\beta_1] = \pm(i_{\mathbb{H}}E)(v'^+E^3\beta_1).$$

This implies the inclusion $G_k(\mathbb{S}^4) \subseteq EQ_{k-1}(\mathbb{S}^3) \oplus v_4^+ \pi_k(\mathbb{S}^7)$ and leads to the first half of (4). On the other hand, for $\beta \in Q_{k-1}(\mathbb{S}^3)$, we obtain $[\iota_4, E\beta] = 2v_4^+E^4\beta = 0$ by the assumption. This means $EQ_{k-1}(\mathbb{S}^3) \subseteq P_k(\mathbb{S}^4)$ and leads to the second half of (4), and the proof is complete. $\square$

The following has been suggested by K. Morisugi:

Theorem 2.9. $L''_{k-1,n}(\mathbb{S}^3) = \pi_{k-1}(\mathbb{S}^3)$ for $n \geq 1$, if $5 \leq k \leq 4n+2$.

Proof. Because $J([t_3]) = v_4^+$, in view of [85, Corollary 11.2], we have

$$v_{4n+3}^+ E^{4n+3} \beta = J([t_3]_{4n+3} \beta). \quad (2.16)$$

By the fact that $\pi_{k-1}(\mathbb{S}^3)$ is finite for $k \neq 4$ and

$$\pi_{k-1}(SO(4n+3)) \cong \begin{cases} \mathbb{Z} & \text{for } k \equiv 0 \pmod{4}; \\ \mathbb{Z}_2 & \text{for } k \equiv 1, 2 \pmod{8}; \\ 0 & \text{for } k \equiv 3, 5, 6, 7 \pmod{8} \end{cases}$$

if $k \leq 4n+2$ [14], we have the result for $k \neq 1, 2 \pmod{8}$.

Assume that $v_{4n+3} E^{4n+3} \beta \neq 0$ for $k \equiv 1 \pmod{8}$ (respectively $k \equiv 2 \pmod{8}$). Then, in view of [37, Lemma 2], we have $v_{4n+3} E^{4n+3} \beta = J([t_3]_{4n+3} \beta) = J(\delta \eta_{k-2})$ (respectively $v_{4n+3} E^{4n+3} \beta = J([t_3]_{4n+3} \beta) = J(\delta' \eta_{k-3}^2)$), where $\delta \in \pi_{k-2}(SO(4n+3))$ (respectively $\delta' \in \pi_{k-3}(SO(4n+3))$) is a generator. Since $J : \pi_{k-1}(SO(4n+3)) \rightarrow \pi_{k+4n+2}(\mathbb{S}^{4n+3})$ is a monomorphism [2, Theorem 1.1] (respectively [2, Theorem 1.3]), we obtain $\delta \eta_{k-2} = [t_3]_{4n+3} \beta$ (respectively $\delta' \eta_{k-3}^2 = [t_3]_{4n+3} \beta$). But, in view of [16, Theorem 1.1], the element $\delta \eta_{k-2}$ (respectively $\delta' \eta_{k-3}^2$) has the $SO(6)$ -of-origin. This contradiction leads to the statement for $k \equiv 1 \pmod{8}$ (respectively $k \equiv 2 \pmod{8}$) and the proof is complete. $\square$

Another Proof of Theorem 2.9. To consider the cases $k \equiv 1, 2 \pmod{8}$, set $j_{r,n}$ for the generator of $J\pi_r(SO(n))$ with $r \leq n-2$ and write (according to Adams' notations [2]) $j_r = j_{r,\infty}$. First, assume that $v_{4n+3} E^{4n+3} \beta \neq 0$ for $k \equiv 1 \pmod{8}$. Then, in view of [37, Lemma 2], we have $v_{4n+3} E^{4n+3} \beta = j_{k-2,4n+3} \eta_{k+4n+1}$ and $j_{k-2,4n+3} \eta_{k+4n+1}^2$ generates $J\pi_k(SO(4n+3))$. On the other hand, by [85, Proposition 3.1],

$$\beta \wedge \eta_2 = E^2(\beta \eta_{k-1}) = \eta_5 E^3 \beta \text{ and } v_4 \wedge \beta = v_7 E^7 \beta = (E^4 \beta) v_{k+3}.$$

Then, (1.23) implies a contradictory relation

$$j_{k-2,4n+3} \eta_{k+4n+1}^2 = v_{4n+3} (E^{4n+3} \beta) \eta_{k+4n+2} = v_{4n+3} \eta_{4n+6} E^{4n+4} \beta = 0.$$

Now, assume that $v_{4n+3} E^{4n+3} \beta \neq 0$ for $5 \leq k \leq 4n+2$ with $k \equiv 2 \pmod{8}$. Then, $v_{4n+3} E^{4n+3} \beta = j_{k-1,4n+3}$ and we get

$$\begin{aligned} < j_{k-1}, \eta, 2t > = < v E^\infty \beta, \eta, 2t > = < (E^\infty \beta) v, \eta, 2t > \\ & \supseteq (E^\infty \beta) < v, \eta, 2t > \equiv 0 \pmod{2\pi_{k+1}^s} \quad (1.102)(2) \end{aligned}$$

for the stable Toda bracket. On the other hand, by [2, Corollary 11.7] and [63, Theorem B. v)], we deduce that π_{k+1}^s contains the direct summand $\mathbb{Z}_8$ which is generated by $< j_{k-1}, j_3, 24t > \pmod{\text{odd components}}$.

Because $j_1 = \eta$, $j_3 = v^+$, and $j_{k-1} = j_{k-3} \circ \eta^2$ [37, Lemma 2], we derive

$$< j_{k-3}, j_3, 24\iota > \subseteq < j_{k-3}, 12j_3, 2\iota > = < j_{k-3}, \eta^3, 2\iota > \supseteq < j_{k-1}, \eta, 2\iota > .$$

Thus,

$$< j_{k-3}, j_3, 24\iota > \equiv < j_{k-1}, j_1, 2\iota > \pmod{2\pi_{k+1}^s}.$$

This is a contradiction with the above which leads to the assertion.

We write

$$P'_k(\mathbb{F}P^n) = P_k(\mathbb{F}P^n) \cap \gamma_n \pi_k(\mathbb{S}^{d(n+1)-1})$$

and

$$P''_k(\mathbb{F}P^n) = P_k(\mathbb{F}P^n) \cap i_{\mathbb{F}*}(E\pi_{k-1}(\mathbb{S}^{d-1})).$$

Notice that $P''_k(\mathbb{F}P^n) = 0$ if $d = 1, 2$ and $k \geq d + 1$.

Now, we show:

Proposition 2.10. $P'_k(\mathbb{F}P^n) = \gamma_n \pi_k(P_k(\mathbb{S}^{d(n+1)-1}) \cap M_k(\mathbb{S}^{d(n+1)-1}))$ and $P''_k(\mathbb{F}P^n) = i_{\mathbb{F}*}EL_{k-1}(\mathbb{S}^{d-1})$.

Proof. For $\alpha \in \pi_k(\mathbb{S}^{d(n+1)-1})$, it holds that $[\gamma_n \alpha, \gamma_n] = 0$ if and only if $[\alpha, \iota_{d(n+1)-1}] = 0$. This leads to the first half.

Next, for an element $\beta \in \pi_{k-1}(\mathbb{S}^{d-1})$, it holds that $i_{\mathbb{F}*}E\beta \in P''_k(\mathbb{F}P^n)$ if and only if $\beta \in L_{k-1}(\mathbb{S}^{d-1})$. This leads to the second half and completes the proof. $\square$

2.2 Preliminaries II

Since $H((2\iota_6)\bar{v}_6) = 4H(\bar{v}_6)$ [85, Proposition 2.2], we derive $(2\iota_6)\bar{v}_6 - 4\bar{v}_6 = x\varepsilon_6$ for some integer x . Suspending and using (1.67), we get $0 = -2\bar{v}_7 = x\varepsilon_7$ and $x \equiv 0 \pmod{2}$. Next, by [92, (8.5) Theorem, p. 534], $(2\iota_6)\bar{v}_6 = 2\bar{v}_6 + [\iota_6, \iota_6]h_0(\bar{v}_6)$. Consequently:

$$\begin{aligned} 1. (2\iota_6)\bar{v}_6 &= 4\bar{v}_6, \\ 2. [\iota_6, \iota_6]h_0(\bar{v}_6) &= 2\bar{v}_6. \end{aligned} \tag{2.17}$$

On the other hand, by [7], it holds $(2\iota_6)\bar{v}_6 = 2\bar{v}_6 + [\iota_6, v_6]$. Hence, (2.17)(1) improves (1.67) as follows:

$$2\bar{v}_6 = [\iota_6, v_6].$$

In view of [85, Proposition 2.2], it holds $H((2\iota_4)v_4^+) = 4H(v_4^+)$. Hence, $(2\iota_4)v_4^+ - 4v_4^+ = xEv'^+$ for some integer x . Suspending, we get $-2v_5^+ = 2xv_5^+$ with $x \equiv -1 \pmod{12}$. Thus, (1.20) yields:

$$\begin{aligned} 1. (2\iota_4)v_4 &= 4v_4 - Ev' = 2v_4 \pm [\iota_4, \iota_4], \\ 2. (2\iota_4)v_4^+ &= 4v_4^+ - Ev'^+ = 2v_4^+ \pm [\iota_4, \iota_4]. \end{aligned} \quad (2.18)$$

Because $H((2\iota_8)\sigma_8^+) = 4H(\sigma_8^+)$ [85, Proposition 2.2], we get $(2\iota_8)\sigma_8^+ - 4\sigma_8^+ = xE\sigma'^+$ for some integer x . Suspending, we get $-2\sigma_9^+ = 2x\sigma_9^+$ with $x \equiv -1 \pmod{120}$. Further, in view of [92, (8.5) Theorem, p. 534], it holds $(2\iota_8)\sigma_8^+ = 2\sigma_8^+ + [\iota_8, \iota_8]h_0(\sigma_8^+)$. Hence, (1.22) leads to:

$$\begin{aligned} 1. (2\iota_8)\sigma_8 &= 2\sigma_8 \pm [\iota_8, \iota_8] = 4\sigma_8 - E\sigma', \\ 2. (2\iota_8)\sigma_8^+ &= 2\sigma_8^+ \pm [\iota_8, \iota_8] = 4\sigma_8^+ - E\sigma'^+, \\ (3) [\iota_8, \iota_8]h_0(\sigma_8^+) &= 2\sigma_8^+ - E\sigma'^+. \end{aligned} \quad (2.19)$$

Now, we recall:

$$v'\zeta_6 = 0 \text{ [54, Lemma 3.3 (ii)] and [75, Proposition (2.4)(1)]}, \quad (2.20)$$

and

$$v'\bar{\varepsilon}_6 \neq 0 \text{ [85, Theorem 12.8]}. \quad (2.21)$$

There is $\bar{\varepsilon}' \in \pi_{20}^3$ such that

$$E\bar{\varepsilon}' = (Ev')\kappa_7 \text{ [85, Lemma 12.3]}. \quad (2.22)$$

Following [85, p. 103], choose $\rho^{IV} \in \{\sigma''', 2\iota_{12}, 8\sigma_{12}\}_1$, $\rho''' \in \{\sigma'', 4\iota_{13}, 4\sigma_{13}\}_1$, $\rho'' \in \{\sigma', 8\iota_{14}, 2\sigma_{14}\}_1$, and $\rho' \in \{\sigma_9, 16\iota_{16}, \sigma_{16}\}_1$. In the sequel, we write $\rho''^+ = \rho'' + \alpha_4(7) + \alpha_{2,5}(7)$.

There exists $\rho_{13} \in \pi_{28}^{13}$ such that

$$E^4\rho' = 2\rho_{13} \text{ [85, Lemma 10.9]}. \quad (2.23)$$

Denote that $\rho_n = E^{n-13}\rho_{13}$ for $n \geq 13$. Then,

$$v_{11}\rho_{14} = 0 \text{ [75, Proposition 2.17(7)] and [85, (12.23)]}. \quad (2.24)$$

By means of [49, Lemma 6.1] and its proof, we have

$$\eta_6\bar{\kappa}_7 \in \{v_6^2, 2\iota_{12}, \kappa_{12}\} \pmod{v_6^2 \circ \pi_{27}^{12} + \pi_{13}^6 \circ \kappa_{13} = v_6^2 \circ \{E^3\rho', \bar{\varepsilon}_{12}\} + \{\sigma''\} \circ \kappa_{13}}.$$

From the fact that $E^\infty(\sigma''\kappa_{13}) = (4\sigma)\kappa = 0$ (1.35)(1), (1.22) and (1.109), $E^\infty(v_6^2 E^3 \rho') = v^2(2\rho) = 0$ (2.23), and $v_6^2 \bar{e}_{12} = v_6^2 \eta_{12} \kappa_{13} = 0$ (1.108)(1), we have $E^\infty(\pi_{13}^6 \circ \kappa_{13} + v_6^2 \circ \pi_{27}^{12}) = 0$. But, in view of [49, Theorem A], it holds $\pi_{27}^6 = \{\eta_6 \bar{\kappa}_7\} \cong \mathbb{Z}_2$ and $\pi_{21}^s = \{\eta \bar{\kappa}, \sigma^3\} \cong (\mathbb{Z}_2)^2$. Hence, $E^\infty : \pi_{27}^6 \rightarrow \pi_{21}^s$ is a monomorphism and we obtain

$$\eta_6 \bar{\kappa}_7 = \{v_6^2, 2\iota_{12}, \kappa_{12}\}. \quad (2.25)$$

Next,

$$\eta_7 \sigma_8 \kappa_{15} = 0 \text{ [49, Lemma 8.1]}, \quad (2.26)$$

$$\varepsilon_5 \kappa_{13} = \eta_5^2 \bar{\kappa}_7 \text{ [73, I-Proposition 3.1(2)]} \quad (2.27)$$

and $\eta_9 \bar{\kappa}_{10} = \bar{\kappa}_9 \eta_{29}$ [85, Proposition 3.1] implies

$$\eta_8^2 \bar{\kappa}_{10} = (\eta_8 \bar{\kappa}_9) \eta_{29}. \quad (2.28)$$

Following [85, p. 153], choose $v_{16}^* \in \{\sigma_{16}, 2\sigma_{23}, v_{30}\}_1$ and denote $v_n^* = E^{n-16} v_{16}^*$ for $n \geq 16$. In view of [85, Lemma 12.18], there exists $\lambda \in \pi_{31}^{13}$ such that $H(\lambda) = v_{25}^2$ and $E^3 \lambda - 2v_{16}^* = \pm[\iota_{16}, v_{16}]$. Because $2v_{16}^* v_{34} = 0$ [49, (7.10)], we get $E^3(\lambda v_{31}) = [\iota_{16}, v_{16}^2]$ and the result [85, Corollary 12.25] implies

$$v_{16} \xi_{19} = v_{16} \circ ([\iota_{19}, \iota_{19}] - v_{19}^*) = [v_{16}, v_{16}] - v_{16} v_{19}^* = [\iota_{16}, v_{16}^2] - v_{16} v_{19}^* = E^3(\lambda v_{31}) - v_{16} v_{19}^*.$$

Consequently, (1.50) leads to $v_{15} v_{18}^* + v_{15} \xi_{18} - E^2(\lambda v_{31}) \in P\pi_{38}^{31} = P\{\sigma_{31}\} = 0$ and

$$E^2(\lambda v_{31}) = v_{15}(v_{18}^* + \xi_{18}). \quad (2.29)$$

Now, let $\alpha \in \pi_k^5$ and $\beta \in \pi_k^3$ be elements such that $2\iota_5 \circ \alpha = 0$ and $\{\eta_4, 2\iota_5, \alpha\} \ni E\beta$. Then, in view of (1.102)(1), $v_6 E^6 \beta \in v_6 \circ \{\eta_9, 2\iota_{10}, E^5 \alpha\} = [\iota_6, \iota_6] \circ E^6 \alpha$. Hence,

$$v_6 E^6 \beta = [\iota_6, \iota_6] \circ E^6 \alpha. \quad (2.30)$$

Next, for $\bar{\mu}_3 \in \{\mu_3, 2\iota_{12}, 8\sigma_{12}\}_1$ [85, Chap. XII, (i)], the result [85, Proposition 1.3] and (1.60) imply

$$\bar{\mu}_n \in \{\mu_n, 2\iota_{n+9}, 8\sigma_{n+9}\}_{n-2} \text{ for } n \geq 3. \quad (2.31)$$

Then,

$$v_6 \bar{\mu}_9 = 16P(\rho_{13}) \text{ [53, (16.6)]}. \quad (2.32)$$

Next, we follow [85, Chap. XII, (iii)] to choose $\xi_{12} \in \{\sigma_{12}, v_{19}, \sigma_{22}\}_1$. Then,

$$\sigma_9^3 = v_9 \xi_{12} \text{ [73, II, Proposition 2.1(2)]}. \quad (2.33)$$

From the fact that $\mu' \in \{\eta_3, 2\iota_4, \mu_4\}_1$, (1.102)(1) and (1.133), we have $\nu_6 E^6 \mu' \in \nu_6 \circ \{\eta_9, 2\iota_{10}, \mu_{10}\} = [\iota_6, \mu_6] = 0$. Hence,

$$\nu_6 E^6 \mu' = 0. \quad (2.34)$$

We recall $\pi_{21}^6 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$. Since $\pi_{22}^3 = \{\bar{\mu}', \nu' \mu_6 \sigma_{15}\} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, by (2.34) and the definition [85, p. 137] of $\bar{\mu}' \in \{\mu', 4\iota_{14}, 4\sigma_{14}\}_1$, we see that $\nu_6 E^6 \bar{\mu}' \in \circ \nu_6 \{E^6 \mu, 4\iota_{20}, \sigma_{20}\} \subseteq \{0, 4\iota_{20}, 4\sigma_{20}\} = \pi_{21}^6 \circ 4\sigma_{21} = 0$. This leads to

$$\nu_6 E^6 \bar{\mu}' = 0. \quad (2.35)$$

Next, for an add prime p , consider the exact sequence

$$\pi_i(\mathbb{S}^{2p-1}; p) \xrightarrow{G} \pi_{i+1}(\mathbb{S}^3; p) \xrightarrow{H} \pi_{i+1}(\mathbb{S}^{2p+1}; p) \xrightarrow{\Delta} \pi_{i-1}(\mathbb{S}^{2p-1}; p) \quad [85, \text{Proposition 13.3}] \quad (2.36)$$

for $i > 2p - 1$. If $p = 3$ and $i = 19$ then all groups in (2.36) are isomorphic to $\mathbb{Z}_3$ and $\pi_{k+18}(\mathbb{S}^{k+5}; 3) = \{\alpha_1(5+k)\beta_1(8+k)\}$ for $k = 0, 2$ [85, Theorem 13.10]. Since $\Delta E^2(\alpha_1(5)\beta_1(8)) = 3(\alpha_1(5)\beta_1(8)) = 0$ (1.9), the map H is an isomorphism and, consequently, G is trivial. Hence,

$$G(\beta) = \alpha_1(3)E\beta = 0 \text{ for } \beta \in \pi_{19}(\mathbb{S}^5; 3). \quad (2.37)$$

Now, we show:

Proposition 2.11. 1. $\pi_{17}(\mathbb{S}^3; 3) = \{\alpha_1(3)\alpha'_3(6)\} \cong \mathbb{Z}_3$ and

2. $\pi_{19}(\mathbb{S}^3; 3) = \{\alpha_{1,5}(3)\alpha_{1,5}(10)\} \cong \mathbb{Z}_5$;
3. $\pi_{21}(\mathbb{S}^3; 3) = \{\alpha_1(3)\alpha_4(6)\} \cong \mathbb{Z}_3$;
4. $\pi_{25}(\mathbb{S}^3; 3) = \{\alpha_1(3)\alpha_5(6)\} \cong \mathbb{Z}_3$.

Proof. We always use the exact sequence in [85, Proposition 13.3]. The first half of (1) follows from (1.10) and [85, Theorem 13.10(ii)], and [85, (13.6)'] leads to the second half of (1). By [85, Theorem 13.10(iv)], we obtain (2). Next, (3) follows from [85, Theorem 13.10(vi)] and [85, Theorem 13.10(vi);(vii)] with the facts $\pi_{24}(\mathbb{S}^3; 3) = 0$, $\pi_{25}(\mathbb{S}^3; 3) \cong \mathbb{Z}_3$ [87, p. 60, Table] yields (4) and the proof is complete. $\square$

Now, we show:

Lemma 2.12. 1. $M_k(\mathbb{S}^{d(2n+2)-1}) = \pi_k(\mathbb{S}^{d(2n+2)-1})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$;

2. (i) $E\pi_{k-1}(\mathbb{S}^{2n-1}) \cap \text{Ker } 2\iota_{2n*} \subseteq M_{k,\mathbb{R}}(\mathbb{S}^{2n})$, where $2\iota_{2n*} : \pi_k(\mathbb{S}^{2n}) \rightarrow \pi_k(\mathbb{S}^{2n})$ is the induced homomorphism; $M_{k,\mathbb{R}}(\mathbb{S}^{2n}) = \text{Ker } 2\iota_{2n*}$ for $k \leq 4n - 2$;
- (ii) $E\pi_{k-1}(\mathbb{S}^{4n}) \cap E^{-1}(\text{Ker } \eta_{4n+1*}) \subseteq M_{k,\mathbb{C}}(\mathbb{S}^{4n+1})$, where $\eta_{4n+1*} : \pi_{k+1}(\mathbb{S}^{4n+2}) \rightarrow \pi_{k+1}(\mathbb{S}^{4n+1})$; $M_{k,\mathbb{C}}(\mathbb{S}^{4n+1}) = E^{-1}(\text{Ker } \eta_{4n+1*})$ for $k \leq 8n$;

- (iii) $E\pi_{k-1}(\mathbb{S}^{4n+2}) \cap E^{-3}(\text{Ker}(n+1)v_{4n+3*}^+) \subseteq M_{k,\mathbb{H}}(\mathbb{S}^{4n+3})$, where $v_{4n+3*}^+ : \pi_{k+3}(\mathbb{S}^{4n+6}) \rightarrow \pi_{k+3}(\mathbb{S}^{4n+3})$; $M_{k,\mathbb{H}}(\mathbb{S}^{4n+3}) = E^{-3}(\text{Ker}(n+1)v_{4n+3*}^+)$ for $k \leq 8n+4$. In particular, $M_{k,\mathbb{H}}(\mathbb{S}^{4n+3}) = \pi_k(\mathbb{S}^{4n+3})$ for $n+1 \equiv 0 \pmod{24}$;
- (iv) $E\pi_{k-1}(\mathbb{S}^{8n+2}) \cap E^{-3}(\text{Ker}(2n+1)v_{8n+3*}^+) \subseteq M_{k,\mathbb{H}}(\mathbb{S}^{8n+3})$;
 $M_{k,\mathbb{H}}(\mathbb{S}^{8n+3}) = E^{-3}(\text{Ker}(2n+1)v_{8n+3*}^+)$ for $k \leq 16n+4$;
- (v) $M_{k,\mathbb{H}}(\mathbb{S}^{8n+7}) = E^{-3}(\text{Ker}(2(n+1)v_{8n+7*}^+))$;
- 3. $[\iota_{2n}, \iota_{2n}] \in M_{4n-1,\mathbb{R}}(\mathbb{S}^{2n})$;
- 4. $M_{k,\mathbb{R}}(\mathbb{S}^2) = \pi_k(\mathbb{S}^2)$ except $k=2$ and $M_{2,\mathbb{R}}(\mathbb{S}^2) = 0$;
- 5. $G_{k+2n}(\mathbb{S}^{2n}) \subseteq M_{k+2n,\mathbb{R}}(\mathbb{S}^{2n})$ for $k \leq 2n-2$;
- 6. (i) $M_{k,\mathbb{H}}(\mathbb{S}^{8n+3}; p) = G_k(\mathbb{S}^{8n+3}; p) = \pi_k(\mathbb{S}^{8n+3}; p)$ for an odd prime $p \geq 5$;
(ii) $M_{k,\mathbb{H}}(\mathbb{S}^{8n+3}; 3) = \pi_k(\mathbb{S}^{8n+3}; 3)$ for $n \equiv 1 \pmod{3}$;
- 7. $M_{k+8n+3,\mathbb{H}}^{8n+3} = \begin{cases} 8\pi_{8n+3}^{8n+3} & \text{for } k=0, \\ \pi_{k+8n+3}^{8n+3} & \text{for } k=1, 2, 4, 5, 7, 8, 9, 10, \\ 2\pi_{k+8n+3}^{8n+3} & \text{for } k=3, 6. \end{cases}$

Proof. (1) is a direct consequence of Lemma 2.4(1);(2).

By Lemma 2.4(1), $E\alpha \in M_{k,\mathbb{R}}(\mathbb{S}^{2n})$ if and only if $2E\alpha = 2\iota_{2n} \circ E\alpha = 0$. This leads to the first half of (2)-(i). The second half is obtained from the Freudenthal suspension theorem. In view of Lemma 2.4(2), $E\alpha \in M_{k,\mathbb{C}}(\mathbb{S}^{4n+1})$ if and only if $\eta_{4n+1}E^2\alpha = 0$ for $\alpha \in \pi_{k-1}(\mathbb{S}^{4n})$. This leads to (2)-(ii).

By means of Lemma 2.4(3), $E\alpha \in M_{k,\mathbb{H}}(\mathbb{S}^{4n+2})$ if and only if $(n+1)v_{4n+3}^+E^4\alpha = 0$. This leads to (2)-(iii). By the parallel argument, we have (2)-(iv). Since v_{8n+7} is cyclic (1.31), Lemma 2.4(3) leads to (2)-(v).

(3) is a direct consequence of [24, Proposition 2].

By Lemma 2.4(1), $[\gamma_2, i_{\mathbb{R}}] = -2\gamma_2$ and $[\gamma_2\eta_2, i_{\mathbb{R}}] = 0$. This implies $M_{2,\mathbb{R}}(\mathbb{S}^2) = 0$ and $\eta_2 \in M_{3,\mathbb{R}}(\mathbb{S}^2)$. By Lemma 2.6(1), $\pi_k(\mathbb{S}^2) = \eta_2 \circ \pi_k(\mathbb{S}^3) \subseteq M_{k,\mathbb{R}}(\mathbb{S}^2)$ and this leads to (4).

Suppose that $E\alpha \in P_{2n+k}(\mathbb{S}^{2n})$. Then, $0 = [\iota_{2n}, E\alpha] = [\iota_{2n}, \iota_{2n}] \circ E^{2n}\alpha$. In view of (1.66)(8), we have $0 = H[\iota_{2n}, \iota_{2n}] \circ E^{2n}\alpha = 2E^{2n}\alpha$. Since $E^{2n-1} : \pi_{2n+k}(\mathbb{S}^{2n}) \rightarrow \pi_{4n+k-1}(\mathbb{S}^{4n-1})$ is an isomorphism, we obtain $2E\alpha = 0$. Hence, (2)-(i) leads to (5).

Since $s[\iota_{8n+3}, \alpha] = 0$ for $s=2, p^i$ with some $i \geq 1$ and $t\alpha_1(8n+3)E^3\alpha = 0$ for $t=3, p^j$ with some $j \geq 1$, we get (6)-(i). The assumptions in (6)-(ii) imply that $2n+1 \equiv 0 \pmod{3}$ and $3^i v_{8n+3}^+ E^3\alpha = 3^i [\iota_{8n+3}, v_{8n+3}] \circ E^3 h_0(\alpha) = 0$ for some $i \geq 1$. Because $24v_{8n+3}^+ E^3\alpha = 2[\iota_{8n+3}, v_{8n+3}] \circ E^3 h_0(\alpha) = 0$, Lemma 2.4(3) leads to (6)-(ii).

Next, we show (7). By Lemma 2.4(3), we get that $\alpha \in M_{k+8n+3,\mathbb{H}}^{8n+3}$ if and only if $v_{8n+3}E^3\alpha = 0$. Hence, $M_{8n+3,\mathbb{H}}^{8n+3} = 8\pi_{8n+3}^{8n+3}$ and in view of the relation (1.23), we get that $M_{k+8n+3}^{8n+3} = \pi_{8n+k+3}^{8n+3}$ for $k=1, 2$.

Because of (1.19) and (1.34), we derive that $M_{8n+6,\mathbb{H}}^{8n+3} = 2\pi_{8n+6}^{8n+3}$. Evidently, $M_{8n+7,\mathbb{H}}^{8n+3} = M_{8n+8,\mathbb{H}}^{8n+3} = 0$.

The relation $\sharp v_{8n+3}^3 = 2$ and (1.34) imply $M_{8n+9, \mathbb{H}}^{8n+3} = 2\pi_{8n+9}^{8n+3} = 0$.

Because $\pi_{8n+10}^{8n+3} = \{\sigma_{8n+3}\} \cong \mathbb{Z}_{16}$, in view of (1.84), we deduce $M_{8n+10, \mathbb{H}}^{8n+3} = \pi_{8n+10}^{8n+3}$.

By means of (1.126), we obtain $v_7 \varepsilon_{10} = v_7 \bar{v}_{10} = 0$. So, Lemma 2.4(3) leads to $M_{8n+11, \mathbb{H}}^{8n+3} = \pi_{8n+11}^{8n+3}$.

The relations (1.23), (1.39), and (1.72) yield $v_{8n+3} \eta_{8n+6} \varepsilon_{8n+7} = 0$, $v_{8n+3}^4 = 0$ and $v_{8n+3} \mu_{8n+6} = 0$. Hence, again by Lemma 2.4(3), we get $M_{8n+12, \mathbb{H}}^{8n+3} = \pi_{8n+12}^{8n+3}$.

Finally, (1.23) leads to $M_{8n+13, \mathbb{H}}^{8n+3} = \pi_{8n+13}^{8n+3}$ and the proof is completed. $\square$

2.3 Whitehead Center Groups of Projective Spaces

First, notice that by [8, (2.3)] and (2.11), we get $[i_{\mathbb{H}}, i_{\mathbb{H}}] = i_{\mathbb{H}}[\iota_4, \iota_4] = \pm i_{\mathbb{H}}(E v'^+)$ and (2.1) leads to $\sharp[i_{\mathbb{H}}, i_{\mathbb{H}}] = 12$. Writing $[[-, -]]$ for the least common multiple, by Lemma 2.4(3) and the fact that $\sharp[i_{\mathbb{F}}, i_{\mathbb{F}}] = 1$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $\sharp[i_{\mathbb{H}}, \gamma_n] = \frac{24}{(24, n+1)}$, we have:

Example 2.13. If $n \geq 2$ then:

1. $P_1(\mathbb{R}P^n) = \frac{1+(-1)^{n-1}}{2} \pi_1(\mathbb{R}P^n)$;
2. $P_2(\mathbb{C}P^n) = \frac{3+(-1)^n}{2} \pi_2(\mathbb{C}P^n)$;
3. $P_4(\mathbb{H}P^n) = [[12, \frac{24}{(24, n+1)}]] \pi_4(\mathbb{H}P^n)$.

Next, we show the following result needed in the sequel.

Lemma 2.14. *Let G be an abelian group with $G = G_1 \oplus G_2$ and write $p_k : G \rightarrow G_k$ for the projections for $k = 1, 2$. If H is a subgroup of G such that*

$$p_k(H) < H \text{ for } k = 1, 2 \quad (2.38)$$

then $H = G_1 \cap H \oplus G_2 \cap H$.

Proof. Because $p_1(H) < G_1$ and $p_2(H) < G_2$, we have $H = p_1(H) \oplus p_2(H)$. Hence, by the assumption $p_k(H) < H$ for $k = 1, 2$, $p_1(H) < H \cap G_1$ and $p_2(H) < H \cap G_2$. Certainly, $H \cap G_1 < p_1(H)$ and $H \cap G_2 < p_2(H)$. Consequently, $H = H \cap G_1 \oplus H \cap G_2$ and the proof is complete. $\square$

Consider the homomorphisms $[-, i_{\mathbb{F}}] : \pi_k(\mathbb{F}P^n) \rightarrow \pi_{k+d-1}(\mathbb{F}P^n)$ and $[-, \gamma_n] : \pi_k(\mathbb{F}P^n) \rightarrow \pi_{k+d(n+1)-2}(\mathbb{F}P^n)$. By the bilinearity of the Whitehead products [92, Chap. X, (7.12) Corollary] and (2.2), we obtain

$$P_k(\mathbb{F}P^n) = \text{Ker } [-, i_{\mathbb{F}}] \cap \text{Ker } [-, \gamma_n].$$

Next, we show:

Lemma 2.15. *Let $n \geq 2$. Then*

1. $\text{Ker} [-, i_{\mathbb{F}}] = \gamma_n * M_k(\mathbb{S}^{d(n+1)-1}) \oplus i_{\mathbb{F}*} EL'_{k-1}(\mathbb{S}^{d-1})$.
2. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $k \geq d + 1$:
 - (i) $\text{Ker} [-, i_{\mathbb{F}}] = \gamma_n * M_k(\mathbb{S}^{d(n+1)-1})$;
 - (ii) $\text{Ker} [-, \gamma_n] = \gamma_n * G_k(\mathbb{S}^{d(n+1)-1})$.
3. For $\mathbb{F} = \mathbb{H}$:
 - (i) $(\text{Ker} [-, \gamma_n]; p) = \gamma_n * \pi_k(\mathbb{S}^{4n+3}; p) \oplus i_{\mathbb{H}*} EL''_{k-1}(\mathbb{S}^3; p)$ for an odd prime p ;
 - (ii) $(\text{Ker} [-, \gamma_n]; 2) = \gamma_n * G_k(\mathbb{S}^{4n+3}; 2) \oplus i_{\mathbb{H}*} EL''_{k-1}(\mathbb{S}^3; 2)$ provided that

$$[\iota_{4n+3}, \pi_k^{4n+3}] \cap (n+1)v_{4n+3}E^{4n+3}\pi_{k-1}^3 = 0. \quad (2.39)$$

Proof. 1. follows from Lemma 2.4 and the relation

$$0 = [\gamma_n \alpha + i_{\mathbb{F}} E\beta, i_{\mathbb{F}}] = [\gamma_n \alpha, i_{\mathbb{F}}] + [i_{\mathbb{F}} E\beta, i_{\mathbb{F}}].$$

In the quaternionic case, this is equivalent to the following two formulas:

$$(n+1)(v_{4n+3}^+ E^3 \alpha + [\iota_{4n+3}, v_{4n+3}] \circ E^3 h_0(\alpha)) = 0 \quad (2.40)$$

and, in view of (1.20),

$$v'^+ E^3 \beta = 0. \quad (2.41)$$

By the assumption of (2), $\pi_{k-1}(\mathbb{S}^{d-1}) = 0$. Hence, (2)-(i) follows from (1) and (2)-(ii) is obtained by the fact that $[\alpha, \iota_{d(n+1)-1}] = 0$ if and only if $[\gamma_n \alpha, \gamma_n] = 0$.

For $\gamma_n \alpha + i_{\mathbb{H}} E\beta \in \text{Ker} [-, \gamma_n]$, in view of (2.13), we see that

$$\begin{aligned} 0 &= [\gamma_n \alpha + i_{\mathbb{H}} E\beta, \gamma_n] \\ &= \gamma_n [\alpha, \iota_{4n+3}] + [i_{\mathbb{H}} E\beta, \gamma_n] = \gamma_n ([\alpha, \iota_{4n+3}] \pm (n+1)v_{4n+3}^+ E^{4n+3} \beta). \end{aligned}$$

That is,

$$[\alpha, \iota_{4n+3}] = \pm (n+1)v_{4n+3}^+ E^{4n+3} \beta. \quad (2.42)$$

Notice that in view of (1.1), Proposition 1.5, and (2.16), under the assumption that the J -homomorphism is a monomorphism, the relations (2.42) are equivalent to

$$\Delta(\alpha) = \pm (n+1)[\iota_3]_{4n+3} \beta.$$

Since $[\iota_{4n+3}, \pi_k(\mathbb{S}^{4n+3}; p)] = 0$ and $P_k(\mathbb{S}^{4n+3}; p) = \pi_k(\mathbb{S}^{4n+3}; p)$ for an odd prime p , (3)-(i) follows. The assumption (2.39) in (3)-(ii) implies that $\alpha \in P_k(\mathbb{S}^{4n+3}; 2)$ and $\beta \in L''_{k-1}(\mathbb{S}^3; 2)$. This leads to (3)-(ii). $\square$

Notice that the condition (2.39) in (3)-(ii) implies the assumption (2.38) of Lemma 2.14.

Now, we obtain:

- Proposition 2.16.** 1. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $k \geq d + 1$. Then, $P_k(\mathbb{F}P^n) = \gamma_{n*}(P_k(\mathbb{S}^{d(n+1)-1}) \cap M_k(\mathbb{S}^{d(n+1)-1}))$ for $n \geq 2$. In particular, $P_k(\mathbb{F}P^{2n+1}) = \gamma_{2n+1*}P_k(\mathbb{S}^{d(2n+2)-1})$ for $n \geq 1$;
2. $P_2(\mathbb{R}P^2) = 0$, $P_k(\mathbb{R}P^2) = \pi_k(\mathbb{R}P^2)$ for $k \geq 3$ and $P_{k+2n}(\mathbb{R}P^{2n}) = \gamma_{2n*}P_{k+2n}(\mathbb{S}^{2n})$ for $k \leq 2n - 2$;
3. $P_{k+4n+1}(\mathbb{C}P^{2n}) = \gamma_{2n*}(P_{k+4n+1}(\mathbb{S}^{4n+1}) \cap E^{-1}(\text{Ker } \eta_{4n+1*}))$ for $k \leq 4n - 1$.

Proof. By Lemmas 2.12(1) and 2.15(2), we obtain (1).

By (1), Lemma 2.12(1);(4);(5) and the fact that $P_k(\mathbb{S}^2) = \pi_k(\mathbb{S}^2)$ for $k \geq 3$, we obtain (2). By (1) and Lemma 2.12(2)-(ii), we obtain (3). This completes the proof. $\square$

In particular, by means of Lemma 2.12(1) and Example 2.13(2), we derive from Proposition 2.16(1) that $P_k(\mathbb{C}P^3) = \pi_k(\mathbb{C}P^3)$ for all $k \geq 1$ [39].

In the quaternionic case, we obtain:

- Proposition 2.17.** 1. $P_k(\mathbb{H}P^{2n+1}) = \gamma_{2n+1*}(G_k(\mathbb{S}^{8n+7}) \cap M_k(\mathbb{S}^{8n+7})) \oplus i_{\mathbb{H}*}EQ_{k-1}(\mathbb{S}^3)$. In particular, $P_k(\mathbb{H}P^{2n+1}) = \gamma_{2n+1*}G_k(\mathbb{S}^{8n+7}) \oplus i_{\mathbb{H}*}EQ_{k-1}(\mathbb{S}^3)$ for $n + 1 \equiv 0 \pmod{12}$;
2. (i) $P_k(\mathbb{H}P^{2n}; 3) = \gamma_{2n*}M_k(\mathbb{S}^{8n+3}; 3) \oplus i_{\mathbb{H}*}EQ_{k-1}(\mathbb{S}^3; 3)$. In particular, $P_k(\mathbb{H}P^{2n}; 3) = \gamma_{2n*}\pi_k(\mathbb{S}^{8n+3}; 3) \oplus i_{\mathbb{H}*}EQ_{k-1}(\mathbb{S}^3; 3)$ for $n \equiv 1 \pmod{3}$;
- (ii) $P_k(\mathbb{H}P^{2n}; p) = \gamma_{2n*}\pi_k(\mathbb{S}^{8n+3}; p) \oplus i_{\mathbb{H}*}E\pi_{k-1}(\mathbb{S}^3; p)$ for an odd prime $p \geq 5$;
- (iii) $P_k(\mathbb{H}P^{2n}; 2) = \gamma_{n*}(P_k^{8n+3} \cap M_k^{8n+3}) \oplus i_{\mathbb{H}*}EL_{k-1}^3$ provided that

$$[\iota_{8n+3}, M_k^{8n+3}] \cap v_{8n+3} \circ E^{8n+3} L_{k-1}^3 = 0; \quad (2.43)$$

3. $P_k(\mathbb{H}P^n) = i_{\mathbb{H}*}EQ_{k-1}(\mathbb{S}^3)$ for $5 \leq k \leq 4n + 2$ and $n \geq 2$.

Proof. 1. is a direct consequence of Lemmas 2.8(1);(2), 2.12(2)-(iii), and 2.15(1);(3) and the fact that

$$\text{Ker } ([-, \gamma_n] |_{\text{Ker } [-, i_{\mathbb{H}}]}) = \text{Ker } [-, i_{\mathbb{H}}] \cap \text{Ker } [-, \gamma_n].$$

Let n be even. Then, Lemmas 2.8(3) and 2.15(3)-(i) lead to the first half of (2)-(i). The second half follows from Lemma 2.12(6)-(ii). Lemmas 2.8(3), 2.12(6)-(i), and 2.15(4)-(i) imply (2)-(ii). By Lemma 2.15(1);(3)-(ii), we have (2)-(iii). By Lemmas 2.15(1);(3) and 2.8(2);(6), we obtain (3). This completes the proof. $\square$

Notice that in view of (1.1), Proposition 1.5, and (2.16), under the assumption that the J -homomorphism is a monomorphism, the relation (2.43) is equivalent to

$$\Delta(M_k^{8n+3}) \cap [\iota_3]_{8n+3} L'_{k-1} = 0. \quad (2.44)$$

Corollary 2.18. *Let $\alpha \in \pi_k^{4n+3}$ and $\beta \in \pi_{k-1}^3$ satisfy (2.40) and (2.41), and $E\alpha' \in M_l^{4n+3}$.*

1. *If $[\alpha, E\alpha'] \neq 0$ then $\gamma_n \alpha + i_{\mathbb{H}} E\beta \notin P_k(\mathbb{H}P^n)$;*
2. *Let n be even. Suppose that π_k^{4n+3} is a cyclic group generated by α with $[\alpha, E\alpha'] \neq 0$. Then, the condition (2.43) in Proposition 2.17(2)-(iii) holds.*

Proof. First, observe that in view of Lemma 2.4(3), we have $(n+1)v_{4n+3}E^4\alpha' = 0$.

1. Suppose that $\gamma_n \alpha + i_{\mathbb{H}} E\beta \in P_k(\mathbb{H}P^n)$. Hence, by (2.42),

$$[\alpha, E\alpha'] = [\alpha, \iota_{4n+3}] \circ E^k \alpha' = \pm(n+1)(E^{4n}\beta) \circ E^{k-4}(v_{4n+3}E^4\alpha') = 0.$$

2. Suppose that the condition (2.43) does not hold. Then, $[\alpha, \iota_{4n+3}] = (n+1)v_{4n+3}E^{4n+3}\beta$ for some $\beta \in L'_{k-1}$. Consequently, as in (1),

$$[\alpha, E\alpha'] = [\alpha, \iota_{4n+3}] \circ E^k \alpha' = \pm(n+1)(E^{4n}\beta) \circ E^{k-4}(v_{4n+3}E^4\alpha') = 0$$

and this completes the proof. $\square$

2.4 Some Whitehead Center Groups of Real Projective Spaces

In this section, we determine some P -groups of real projective spaces. First, we recall also that

$$\pi_k(\mathbb{S}^3) = \begin{cases} \{\alpha_1(3)\alpha_1(6)\} \cong \mathbb{Z}_3 & \text{for } k = 9, \\ \{\alpha_2(3), \alpha_{1,5}(3)\} \cong \mathbb{Z}_{15} & \text{for } k = 10, \\ \{\varepsilon', \eta_3\mu_4, \alpha_1(3)\alpha_2(6)\} \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_2 & \text{for } k = 13, \\ \{\mu', \varepsilon_3\nu_{11}, \nu'\varepsilon_6, \alpha_3(3), \alpha_{1,7}(3)\} \cong \mathbb{Z}_{84} \oplus (\mathbb{Z}_2)^2 & \text{for } k = 14. \end{cases}$$

Next, we notice that (2.5) yields

$$[\gamma_n \xi \circ E\alpha, i_{\mathbb{R}}] = (-1)^{k+m}[\gamma_n \xi, i_{\mathbb{R}}] \circ E\alpha \quad (2.45)$$

for $\xi \in \pi_{m+1}(\mathbb{S}^n)$, $\alpha \in \pi_k(\mathbb{S}^m)$. In particular, Lemma 2.4(1) leads to

$$[\gamma_{2n} E\alpha, i_{\mathbb{R}}] = (-1)^{k+1} 2E\alpha \text{ for } \alpha \in \pi_{k-1}(\mathbb{S}^{2n-1}).$$

This implies

$$E\alpha \in M_k(\mathbb{S}^{2n}) \text{ provided that } 2E\alpha = 0 \text{ for } \alpha \in \pi_{k-1}(\mathbb{S}^{2n-1}). \quad (2.46)$$

Now, we are ready to prove:

Theorem 2.19. *Let $n \geq 2$ and $k \leq 13$. Then, the equality $P_{k+n}(\mathbb{R}P^n) = \gamma_{n*}P_{n+k}(\mathbb{S}^n)$ holds except the pairs: $(k, n) = (4, 4), (5, 4), (6, 4), (8, 8), (9, 8), (10, 4), (10, 6), (10, 8), (10, 10), (12, 12), (13, 12)$, and $(8, 13)$.*

Furthermore, $P_8(\mathbb{R}P^4) = \gamma_{4*}\{(Ev')\eta_7\} \cong \mathbb{Z}_2$, $P_9(\mathbb{R}P^4) = \gamma_{4*}\{(Ev')\eta_7^2\} \cong \mathbb{Z}_2$ and $P_{10}(\mathbb{R}P^4) = \gamma_{4*}\{v_4^2\} \cong \mathbb{Z}_8$, $P_{16}(\mathbb{R}P^8) = \gamma_{8*}\{(E\sigma')\eta_{15}\} \cong \mathbb{Z}_2$, $P_{17}(\mathbb{R}P^8) = \gamma_{8*}\{(E\sigma')\eta_{15}^2\} \cong \mathbb{Z}_2$, $P_{14}(\mathbb{R}P^4) = \gamma_{4*}\{v_4^+\sigma'^+ + E\varepsilon', 2E\varepsilon'\} \cong \mathbb{Z}_{120} \oplus \mathbb{Z}_2$, $P_{16}(\mathbb{R}P^6) = \gamma_{6*}\{4v_6\sigma_9, \eta_6\mu_7\} \cong (\mathbb{Z}_2)^2$, $P_{18}(\mathbb{R}P^8) = \gamma_{8*}\{xv_8^+\sigma_{11}^+ + 2\sigma_8^+v_{15}^+\} \cong \mathbb{Z}_{24}$ for x as in (1.82), $P_{20}(\mathbb{R}P^{10}) = \gamma_{10*}\{2\sigma_{10}v_{17}, \eta_{10}\mu_{11}\} \cong (\mathbb{Z}_2)^2$, $P_{15}(\mathbb{R}P^4) = \gamma_{4*}\{v_4\sigma'\eta_{14}, 2E\mu', \varepsilon_4v_{12}, (Ev')\varepsilon_7\} \cong (\mathbb{Z}_2)^4$, $P_{16}(\mathbb{R}P^4) = \gamma_{4*}\{v_4\sigma'\eta_{14}^2, v_4^4, (Ev')\mu_7, (Ev')\eta_7\varepsilon_8\} \cong (\mathbb{Z}_2)^4$, $P_{24}(\mathbb{R}P^{12}) = \gamma_{12*}\{E\theta'\} \cong \mathbb{Z}_2$, $P_{17}(\mathbb{R}P^4) = \gamma_{4*}\{v_4^2\sigma_{10}, (Ev')\eta_7\mu_8\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2$, $P_{25}(\mathbb{R}P^{12}) = \gamma_{12*}\{(E\theta')\eta_{24}\} \cong \mathbb{Z}_2$ and $P_{21}(\mathbb{R}P^8) = \gamma_{8*}\{v_8\sigma_{11}v_{18}\} \cong \mathbb{Z}_2$.

That is:

$$\begin{aligned} 1. \quad P_n(\mathbb{R}P^n) &= \begin{cases} \pi_n(\mathbb{R}P^n) \cong \mathbb{Z} & \text{for } n = 3, 7, \\ 2\pi_n(\mathbb{R}P^n) \cong 2\mathbb{Z} & \text{for } n \equiv 1 \pmod{2}, \text{ unless } n = 3, 7, \\ 0 & \text{for } n \equiv 0 \pmod{2}; \end{cases} \\ 2. \quad P_{n+1}(\mathbb{R}P^n) &= \begin{cases} \pi_3(\mathbb{R}P^2) \cong \mathbb{Z} & \text{for } n = 2, \\ \pi_{n+1}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n = 6 \text{ or } n \equiv 3 \pmod{4}, \\ 0 & \text{for otherwise;} \end{cases} \\ 3. \quad P_{n+2}(\mathbb{R}P^n) &= \begin{cases} \pi_{n+2}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n = 5 \text{ or } n \equiv 2, 3 \pmod{4}, \\ 0 & \text{for otherwise;} \end{cases} \\ 4. \quad P_{n+3}(\mathbb{R}P^n) &= \begin{cases} \pi_5(\mathbb{R}P^2) \cong \mathbb{Z}_2 & \text{for } n = 2, \\ \pi_6(\mathbb{R}P^3) \cong \mathbb{Z}_{12} & \text{for } n = 3, \\ \gamma_{4*}\{3[\iota_4, \iota_4], 2Ev'\} \\ \cong 3\mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 4, \\ \pi_{n+3}(\mathbb{R}P^n) \cong \mathbb{Z}_{24} & \text{for } n \equiv 7 \pmod{8} \text{ or} \\ & n = 2^i - 3, i \geq 3, \\ 12\pi_{n+3}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n \equiv 1, 3, 5 \pmod{8}, n \geq 9 \\ & \text{unless } n = 2^i - 3, \\ 2\pi_{n+3}(\mathbb{R}P^n) \cong \mathbb{Z}_{12} & \text{for } n \equiv 2 \pmod{4}, n \geq 6 \text{ or } n = 12, \\ 0 & \text{for } n \equiv 0 \pmod{4}, n \geq 8 \text{ unless } n = 12; \end{cases} \end{aligned}$$

$$\begin{aligned}
5. \ P_{n+4}(\mathbb{R}P^n) &= \begin{cases} \pi_6(\mathbb{R}P^2) \cong \mathbb{Z}_{12} & \text{for } n = 2, \\ \pi_{n+4}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n = 3, 5, \\ \gamma_{4*}\{(Ev')\eta_7\} \cong \mathbb{Z}_2 & \text{for } n = 4, \\ \pi_{n+4}(\mathbb{R}P^n) = 0 & \text{for } n \geq 6; \end{cases} \\
6. \ P_{n+5}(\mathbb{R}P^n) &= \begin{cases} \pi_{n+5}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n = 2, 3, 5, \\ \gamma_{4*}\{(Ev')\eta_7^2\} \cong \mathbb{Z}_2 & \text{for } n = 4, \\ 3\pi_{11}(\mathbb{R}P^6) \cong 3\mathbb{Z} & \text{for } n = 6, \\ \pi_{n+5}(\mathbb{R}P^n) = 0 & \text{for } n \geq 7; \end{cases} \\
7. \ P_{n+6}(\mathbb{R}P^n) &= \begin{cases} \pi_{n+6}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n = 2, \ n \equiv 4, 5, 7 \pmod{8} \\ & \text{unless } n = 4 \text{ or } n = 2^i - 5, \ i \geq 4, \\ \pi_9(\mathbb{R}P^3) \cong \mathbb{Z}_3 & \text{for } n = 3, \\ \gamma_{4*}\{v_4^2\} \cong \mathbb{Z}_8 & \text{for } n = 4, \\ 0 & \text{for otherwise;} \end{cases} \\
8. \ P_{n+7}(\mathbb{R}P^n) &= \begin{cases} \pi_9(\mathbb{R}P^2) \cong \mathbb{Z}_3 & \text{for } n = 2, \\ \pi_{10}(\mathbb{R}P^3) \cong \mathbb{Z}_{15} & \text{for } n = 3, \\ \pi_{12}(\mathbb{R}P^5) \cong \mathbb{Z}_{30} & \text{for } n = 5, \\ \pi_{14}(\mathbb{R}P^7) \cong \mathbb{Z}_{120} & \text{for } n = 7, \\ \gamma_{8*}\{3[\iota_8, \iota_8], 4E\sigma'\} \\ \cong 3\mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 8, \\ \pi_{n+7}(\mathbb{R}P^n) \cong \mathbb{Z}_{240} & \text{for } n = 11 \text{ or } n \equiv 15 \pmod{16}, \\ 120\pi_{n+7}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n \text{ odd unless } n = 3, 5, 7, 11 \\ & \text{and } n \equiv 15 \pmod{16}, \\ 0 & \text{for } n = 4, 6 \text{ or } n \text{ even, } n \geq 10; \end{cases} \\
9. \ P_{n+8}(\mathbb{R}P^n) &= \begin{cases} \pi_{10}(\mathbb{R}P^2) \cong \mathbb{Z}_{15} & \text{for } n = 2, \\ \pi_{11}(\mathbb{R}P^3) \cong \mathbb{Z}_2 & \text{for } n = 3, \\ \pi_{14}(\mathbb{R}P^6) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2 & \text{for } n = 6, \\ \pi_{15}(\mathbb{R}P^7) \cong (\mathbb{Z}_2)^3 & \text{for } n = 7, \\ \gamma_{8*}\{(E\sigma')\eta_{15}\} \cong \mathbb{Z}_2 & \text{for } n = 8, \\ \gamma_{9*}\{[\iota_9, \iota_9]\} \cong \mathbb{Z}_2 & \text{for } n = 9, \\ \pi_{n+8}(\mathbb{R}P^n) \cong (\mathbb{Z}_2)^2 & \text{for } n=10 \text{ or } n \equiv 3 \pmod{4}, \ n \geq 11, \\ 0 & \text{for } n \equiv 0, 1 \pmod{4}, \ n \geq 4 \\ & \text{unless } n = 8, 9 \text{ or} \\ & \quad n \equiv 22 \pmod{32}, \ n \geq 54, \\ \gamma_{n*}\{\varepsilon_n\} \cong \mathbb{Z}_2 & \text{for } n \equiv 2 \pmod{8}, \ n \geq 18, \\ \gamma_{n*}\{\eta_n\sigma_{n+1}\} \cong \mathbb{Z}_2 & \text{for } n = 22, \ n \equiv 14 \pmod{16} \\ & \text{or } n \equiv 6 \pmod{32}, \ n \geq 14; \end{cases}
\end{aligned}$$

$$10. P_{n+9}(\mathbb{R}P^n) = \begin{cases} \pi_{11}(\mathbb{R}P^2) \cong \mathbb{Z}_2 & \text{for } n = 2, \\ \pi_{n+9}(\mathbb{R}P^n) \cong (\mathbb{Z}_2)^3 & \text{for } n = 3, 6, \\ \pi_{16}(\mathbb{R}P^7) \cong (\mathbb{Z}_2)^4 & \text{for } n = 7, \\ \pi_{n+9}(\mathbb{R}P^n) \cong (\mathbb{Z}_2)^3 & \text{for } n \equiv 3 \pmod{4}, n \geq 11, \\ \gamma_{8*}\{(\mathcal{E}\sigma')\eta_{15}^2\} \cong \mathbb{Z}_2 & \text{for } n = 8, \\ \gamma_{9*}\{\sigma_9\eta_{16}^2, \nu_9^3, \eta_9\varepsilon_{10}\} \cong (\mathbb{Z}_2)^3 & \text{for } n = 9, \\ \gamma_{10*}\{3[\iota_{10}, \iota_{10}], \nu_{10}^3, \eta_{10}\varepsilon_{11}\} \\ \cong 3\mathbb{Z} \oplus (\mathbb{Z}_2)^2 & \text{for } n = 10, \\ \gamma_{n*}\{\eta_n\varepsilon_{n+1}\} \cong \mathbb{Z}_2 & \text{for } n \equiv 1 \pmod{8}, n \geq 17 \\ & \text{unless } n = 2^i - 7, \\ 0 & \text{for } n \equiv 0 \pmod{8}, n \geq 16, \\ \gamma_{n*}\{\nu_n^3, \eta_n\varepsilon_{n+1}\} \cong (\mathbb{Z}_2)^2 & \text{for } n \equiv 2 \pmod{4}, n \geq 14, \\ & n = 2^i - 7, i \geq 5 \\ & \text{or } n \equiv 5 \pmod{8} \\ & \text{unless } n \equiv 53 \pmod{64}, \\ \gamma_{n*}\{\nu_n^3\} \cong \mathbb{Z}_2 & \text{for } n \equiv 4 \pmod{8} \text{ or} \\ & n \equiv 53 \pmod{64}, n \geq 117; \end{cases}$$

$$11. P_{n+10}(\mathbb{R}P^n) = \begin{cases} \pi_{12}(\mathbb{R}P^2) \cong (\mathbb{Z}_2)^2 & \text{for } n = 2, \\ \pi_{13}(\mathbb{R}P^3) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_2 & \text{for } n = 3, \\ \gamma_{4*}\{\nu_4^+\sigma'^+ + \mathcal{E}\varepsilon', 2\mathcal{E}\varepsilon'\} \cong \mathbb{Z}_{120} \oplus \mathbb{Z}_2 & \text{for } n = 4, \\ \pi_{15}(\mathbb{R}P^5) \cong \mathbb{Z}_{72} \oplus \mathbb{Z}_2 & \text{for } n = 5, \\ \gamma_{6*}\{4\nu_6\sigma_9, \eta_6\mu_7\} \cong (\mathbb{Z}_2)^2 & \text{for } n = 6, \\ \pi_{17}(\mathbb{R}P^7) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2 & \text{for } n = 7, \\ \gamma_{8*}\{x\nu_8^+\sigma_{11}^+ + 2\sigma_8^+\nu_{15}^+\} \cong \mathbb{Z}_{24} & \text{for } n = 8 \text{ and } x \text{ as in (1.82)}, \\ \gamma_{9*}\{\sigma_9\nu_{16}, \beta_1(9)\} \cong \mathbb{Z}_{24} & \text{for } n = 9, \\ \gamma_{10*}\{2\sigma_{10}\nu_{17}, \eta_{10}\mu_{11}\} \cong (\mathbb{Z}_2)^2 & \text{for } n = 10, \\ \pi_{21}(\mathbb{R}P^{11}) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2 & \text{for } n = 11, \\ \pi_{n+10}(\mathbb{R}P^n) \cong \mathbb{Z}_6 & \text{for } n \equiv 3 \pmod{4}, n \geq 15, \\ 0 & \text{for } n \equiv 0 \pmod{4}, n \geq 12, \\ \pi_{n+10}(\mathbb{R}P^n; 2) \cong \mathbb{Z}_2 & \text{for } n \equiv 2 \pmod{4}, n \geq 14, \\ \gamma_{n*}\{\beta_1(n)\} \cong \mathbb{Z}_3 & \text{for } n \equiv 1 \pmod{4}, n \geq 13; \end{cases}$$

$$12. P_{n+11}(\mathbb{R}P^n) = \begin{cases} \pi_{13}(\mathbb{R}P^2) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_2 & \text{for } n = 2, \\ \pi_{14}(\mathbb{R}P^3) \cong \mathbb{Z}_{84} \oplus (\mathbb{Z}_2)^2 & \text{for } n = 3, \\ \gamma_{4*}\{\nu_4\sigma'\eta_{14}, 2\mathcal{E}\mu', \varepsilon_4\nu_{12}, (\mathcal{E}\nu')\varepsilon_7\} \\ \cong (\mathbb{Z}_2)^4 & \text{for } n = 4, \\ \pi_{16}(\mathbb{R}P^5) \cong \mathbb{Z}_{504} \oplus (\mathbb{Z}_2)^2 & \text{for } n = 5, \end{cases}$$

$$\begin{aligned}
13. \quad P_{n+11}(\mathbb{R}P^n) &= \begin{cases} \gamma_6 * \{\bar{v}_6 v_{14}, 4\zeta_6\} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 & \text{for } n = 6, \\ \pi_{n+11}(\mathbb{R}P^n) \cong \mathbb{Z}_{504} \oplus \mathbb{Z}_2 & \text{for } n = 7, 9, \\ \gamma_8 * \{\bar{v}_8 v_{16}\} \cong \mathbb{Z}_2 & \text{for } n = 8, \\ \gamma_{12} * \{3[\iota_{12}, \iota_{12}]\} \cong 3\mathbb{Z} & \text{for } n = 12, \\ \pi_{n+11}(\mathbb{R}P^n) \cong \mathbb{Z}_{504} & \text{for } n \equiv 1 \pmod{2}, n \geq 11 \\ & \text{unless } n \equiv 115 \pmod{128}, \\ 0 & \text{for } n \equiv 0 \pmod{4}, n \geq 16, \\ 2\pi_{n+11}(\mathbb{R}P^n) \cong \mathbb{Z}_{252} & \text{for } n \equiv 115 \pmod{128}, n \geq 243, \\ 4\gamma_n * \pi_{n+11}^n \cong \mathbb{Z}_2 & \text{for } n \equiv 2 \pmod{4}, n \geq 10; \end{cases} \\
14. \quad P_{n+12}(\mathbb{R}P^n) &= \begin{cases} \pi_{14}(\mathbb{R}P^2) \cong \mathbb{Z}_{84} \oplus (\mathbb{Z}_2)^2 & \text{for } n = 2, \\ \pi_{15}(\mathbb{R}P^3) \cong (\mathbb{Z}_2)^2 & \text{for } n = 3, \\ \gamma_4 * \{v_4 \sigma' \eta_{14}^2, v_4^4, (Ev')\mu_7, \\ (Ev')\eta_7 \varepsilon_8\} \cong (\mathbb{Z}_2)^4 & \text{for } n = 4, \\ \pi_{17}(\mathbb{R}P^5; 2) \cong (\mathbb{Z}_2)^3 & \text{for } n = 5, \\ \pi_{18}(\mathbb{R}P^6) \cong \mathbb{Z}_{240} & \text{for } n = 6, \\ \pi_{n+12}(\mathbb{R}P^n) = 0 & \text{for } n = 7, 8, 9, \\ \gamma_{10} * \pi_{22}^{10} \cong \mathbb{Z}_4 & \text{for } n = 10, \\ \gamma_{12} * \{E\theta'\} \cong \mathbb{Z}_2 & \text{for } n = 12, \\ \pi_{n+12}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n = 11, 13, \\ \pi_{n+12}(\mathbb{R}P^n) = 0 & \text{for } n \geq 14; \end{cases} \\
15. \quad P_{n+13}(\mathbb{R}P^n) &= \begin{cases} \pi_{15}(\mathbb{R}P^2) \cong (\mathbb{Z}_2)^2 & \text{for } n = 2, \\ \pi_{n+13}(\mathbb{R}P^n) \cong \mathbb{Z}_6 & \text{for } n = 3, 7, 9, \\ P_{17}(\mathbb{R}P^4) = \gamma_4 * \{v_4^2 \sigma_{10}, \\ (Ev')\eta_7 \mu_8\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2 & \text{for } n = 4, \\ \pi_{n+13}(\mathbb{R}P^n) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2 & \text{for } n = 5, 11, \\ \pi_{n+13}(\mathbb{R}P^n) \cong \mathbb{Z}_2 & \text{for } n = 6, 10, \\ \gamma_8 * \{v_8 \sigma_{11} v_{18}\} \cong \mathbb{Z}_2 & \text{for } n = 8, \\ \gamma_{12} * \{(E\theta')\eta_{24}\} \cong \mathbb{Z}_2 & \text{for } n = 12, \\ \pi_{26}(\mathbb{R}P^{13}) \cong \mathbb{Z}_6 & \text{for } n = 13, \\ \gamma_{14} * \{3[\iota_{14}, \iota_{14}]\} \cong 3\mathbb{Z} & \text{for } n = 14, \\ \pi_{n+13}(\mathbb{R}P^n) \cong \mathbb{Z}_3 & \text{for } n \text{ odd}, n \geq 15, \\ 0 & \text{for } n \text{ even}, n \geq 16. \end{cases}
\end{aligned}$$

Proof. For n odd, the required equality holds by Proposition 2.16(1). For $n = 2$ or the case $k \leq n - 2$ with n even, the equality holds by Proposition 2.16(2).

Let $k \leq 7$. Then, it suffices to prove for the following pairs (k, n) :

- (i) (3, 4). In virtue of Lemma 2.12(3), $3[\iota_4, \iota_4] \in M_7(\mathbb{S}^4)$ and by (2.46), $[\gamma_4 Ev', i_{\mathbb{R}}] = -2\gamma_4 Ev'$. So, $P_7(\mathbb{S}^4) = \{3[\iota_4, \iota_4], 2Ev'\} \subseteq M_7(\mathbb{S}^4)$. Then, Proposition 2.16(1) leads to the equality $P_7(\mathbb{R}P^4) = \gamma_4 * P_7(\mathbb{S}^4)$.
- (ii) (4, 4). Recall that $P_8(\mathbb{S}^4) = \pi_8(\mathbb{S}^4) = \{v_4 \eta_7, (Ev')\eta_7\}$ Sect. 1.2.

The result [92, (8.5) Theorem, p. 534] yields $(2\iota_4)v_4^+ = 2v_4^+ + [\iota_4, \iota_4]h_0(v_4^+)$. Then, (2.18)(2) leads to

$$[\iota_4, \iota_4]h_0(v_4^+) = 2v_4^+ - E v'^+ \quad (2.47)$$

and, by Lemma 2.4(1), we obtain

$$[\gamma_4 v_4^+, i_{\mathbb{R}}] = -\gamma_4 E v'^+. \quad (2.48)$$

Observe that this yields

$$[\gamma_4 v_4, i_{\mathbb{R}}] = -\gamma_4 E v'. \quad (2.49)$$

Hence, in view of (2.45), we get $[\gamma_4 v_4 \eta_7, i_{\mathbb{R}}] = \gamma_4 (E v') \eta_7 \neq 0$. Further, (2.46) yields $[\gamma_4 (E v') \eta_7, i_{\mathbb{R}}] = 0$. Thus, the above implies $M_8(\mathbb{S}^4) = \{(E v') \eta_7\}$ and, by Proposition 2.16(1), we get $P_8(\mathbb{R}P^4) = \gamma_{4*}\{(E v') \eta_7\}$.

(iii) (5, 4), (5, 6). Recall from Sect. 1.2 that

$$G_{n+5}(\mathbb{S}^n) = \begin{cases} \pi_9(\mathbb{S}^4) = \{v_4 \eta_7^2, (E v') \eta_7^2\} & \text{for } n = 4, \\ 3\pi_{11}(\mathbb{S}^6) = \{3[\iota_6, \iota_6]\} & \text{for } n = 6. \end{cases}$$

The argument parallel to the above leads to $P_9(\mathbb{R}P^4) = \gamma_{4*}\{(E v') \eta_7^2\}$.

By Lemma 2.12(3), $M_{11}(\mathbb{S}^6) = \pi_{11}(\mathbb{S}^6)$. So, in view of Proposition 2.16(1), we obtain $P_{11}(\mathbb{R}P^6) = \gamma_{6*}P_{11}(\mathbb{S}^6) = 3\pi_{11}(\mathbb{R}P^6)$.

(iv) (6, 4), (6, 6). Recall from Sect. 1.2 that

$$G_{n+6}(\mathbb{S}^n) = \begin{cases} \pi_{10}(\mathbb{S}^4) = \{(v_4^+)^2, \alpha_1(4)\alpha_1(7)\} & \text{for } n = 4, \\ 0 & \text{for } n = 6. \end{cases}$$

The relations (1.28), (2.45) and (2.48) lead to

$$[\gamma_4 (v_4^+)^2, i_{\mathbb{R}}] = -2\gamma_4 \alpha_1(4)\alpha_1(7). \quad (2.50)$$

Next, (2.46) leads to

$$[\gamma_4 \alpha_1(4)\alpha_1(7), i_{\mathbb{R}}] = -2\gamma_4 \alpha_1(4)\alpha_1(7).$$

This implies $[\gamma_4 ((v_4^+)^2 - \alpha_1(4)\alpha_1(7)), i_{\mathbb{R}}] = 0$ and Proposition 2.16(1) yields $P_{10}(\mathbb{R}P^4) = \gamma_{4*}\{(v_4^+)^2\}$.

Because $G_{12}(\mathbb{S}^6) = 0$, Proposition 2.16(1) leads to $P_{12}(\mathbb{R}P^6) = 0$.

(v) (7, 6), (7, 8). Recall from Sect. 1.2 that

$$G_{n+7}(\mathbb{S}^n) = \begin{cases} 0 & \text{for } n = 6, \\ \{3[\iota_8, \iota_8], 4E\sigma'\} & \text{for } n = 8. \end{cases}$$

Because $G_{13}(\mathbb{S}^6) = 0$, Proposition 2.16(1) leads to $P_{13}(\mathbb{R}P^6) = 0$.

By Lemma 2.12(3) and (2.46) we get $P_{15}(\mathbb{S}^8) \subseteq M_{15}(\mathbb{S}^8)$. Then, Proposition 2.16(1) leads to the equality $P_{15}(\mathbb{R}P^8) = \gamma_{8*}P_{15}(\mathbb{S}^8)$.

Let $8 \leq k \leq 13$. Then, it is sufficient to check for the pairs (k, n) with n even:

- (i) $(8, 4), (8, 6), (8, 8)$. Recall from Sect. 1.6 that:

$$G_{n+8}(\mathbb{S}^n) = \begin{cases} 0 & \text{for } n = 4, \\ \pi_{14}(\mathbb{S}^6) = \{\bar{v}_6, \varepsilon_6, [\iota_6, \iota_6]\alpha_1(11)\} & \text{for } n = 6, \\ \{(E\sigma')\eta_{15}, \sigma_8\eta_{15} + \bar{v}_8 + \varepsilon_8\} & \text{for } n = 8. \end{cases}$$

Certainly, $P_{12}(\mathbb{R}P^4) = \gamma_{4*}G_{12}(\mathbb{S}^4)$. Notice that (2.46) leads to $\varepsilon_6 \in M_{14}(\mathbb{S}^6)$ and, in view of Lemma 2.12(3) and (2.45), we get $[\iota_6, \iota_6]\alpha_1(11) \in M_{14}(\mathbb{S}^6)$.

In view of Lemma 2.4(1) and (2.17)(2), we get

$$[\gamma_6\bar{v}_6, i_{\mathbb{R}}] = 0. \quad (2.51)$$

Consequently, $P_{14}(\mathbb{R}P^6) = \gamma_{6*}G_{14}(\mathbb{S}^6)$.

Next, Lemma 2.4(1) and (2.19)(3) lead to

$$[\gamma_8\sigma_8^+, i_{\mathbb{R}}] = \gamma_8 E\sigma'^+ \quad (2.52)$$

which implies

$$[\gamma_8\sigma_8, i_{\mathbb{R}}] = \gamma_8 E\sigma'. \quad (2.53)$$

Then, in view of (2.45), $[\gamma_8\sigma_8\eta_{15}, i_{\mathbb{R}}] = \gamma_8(E\sigma')\eta_{15} \neq 0$, and, by (2.46), it holds $[\gamma_8(E\sigma')\eta_{15}, i_{\mathbb{R}}] = 0$. Thus, we get $P_{16}(\mathbb{R}P^8) = \gamma_{8*}\{(E\sigma')\eta_{15}\}$.

- (ii) $(9, 4), (9, 6), (9, 8), (9, 10)$. Recall from Sect. 1.6 that:

$$G_{n+9}(\mathbb{S}^n) = \begin{cases} \{v_4^3\} & \text{for } n = 4, \\ \pi_{15}(\mathbb{S}^6) & \text{for } n = 6, \\ \{(E\sigma')\eta_{15}^2, \sigma_8\eta_{15}^2 + v_8^3 + \eta_8\varepsilon_9\} & \text{for } n = 8, \\ \{3[\iota_{10}, \iota_{10}], v_{10}^3, \eta_{10}\varepsilon_{11}\} & \text{for } n = 10. \end{cases}$$

Observe that $P_{n+9}(\mathbb{R}P^n) = \gamma_{n*}G_{n+9}(\mathbb{S}^n)$ for $n = 4, 6$. Further, (2.46) yields $(E\sigma')\eta_{15}^2 \in M_{17}(\mathbb{S}^8)$ and, in view of (2.45), (2.46), and (2.52), we get $[\gamma_8(\sigma_8\eta_{15}^2 + v_8^3 + \eta_8\varepsilon_9), i_{\mathbb{R}}] = \gamma_8(E\sigma')\eta_{15}^2 \neq 0$. Hence, $P_{17}(\mathbb{R}P^8) = \gamma_{8*}\{(E\sigma')\eta_{15}^2\}$. Next, (2.46) leads to $v_{10}^3, \eta_{10}\varepsilon_{11} \in M_{19}(\mathbb{S}^{10})$ and, by means of Lemma 2.12(3), we get $[\iota_{10}, \iota_{10}] \in M_{19}(\mathbb{S}^{10})$. Thus, $P_{19}(\mathbb{R}P^{10}) = \gamma_{10*}G_{19}(\mathbb{S}^{10})$ as well.

- (iii) $(10, 4), (10, 6), (10, 8), (10, 10)$. Recall from Sect. 1.6 that:

$$G_{n+10}(\mathbb{S}^n) = \begin{cases} \{\nu_4^+ \sigma'^+ \pm E\varepsilon', 2E\varepsilon', \alpha_1(4)\alpha_2(7), \\ \nu_4\alpha_2(7), \nu_4\alpha'_1(7)\}, & \text{if } n = 4; \\ \{\nu_6\sigma_9, \eta_6\mu_7, 3\beta_1(6)\} & \text{for } n = 6; \\ \{\sigma_8\nu_{15}, \nu_8\sigma_{11}, \sigma_8^+\alpha_1(15)\} & \text{for } n = 8; \\ \pi_{20}^{10} = \{\sigma_{10}\nu_{17}, \eta_{10}\mu_{11}\} & \text{for } n = 10. \end{cases}$$

Certainly, $[2\gamma_4 E\varepsilon', i_{\mathbb{R}}] = 0$ and $[\gamma_4 \nu_4^+ \alpha_{1,5}(7), i_{\mathbb{R}}] = -\gamma_4(E\nu'^+) \alpha_{1,5}(7) = 0$. Further, (2.46) yields $[\gamma_4 \alpha_1(4)\alpha_2(7), i_{\mathbb{R}}] = -2\gamma_4 \alpha_1(4)\alpha_2(7)$ and, in view of (2.45) and (2.48), it holds $[\gamma_4 \nu_4^+ \alpha_2(7), i_{\mathbb{R}}] = -\gamma_4(E\nu'^+) \alpha_2(7) = -2\gamma_4 \alpha_1(4)\alpha_2(7)$. Hence, $[\gamma_4(\alpha_1(4)\alpha_2(7) - \nu_4^+ \alpha_2(7)), i_{\mathbb{R}}] = 0$.

Next, in view of (1.66)(1),(2) and (2.7), it holds $H(\nu_4) = h_0(\nu_4) = \iota_7$ and $H(\sigma') = h_0(\sigma') = \eta_{13}$. Then, the monomorphism $E : \pi_{14}(\mathbb{S}^7) \rightarrow \pi_{15}(\mathbb{S}^8)$, (1.23) and Lemma 2.5 lead to $h_0(\nu_4\sigma') = \sigma'$. Consequently, (1.20) yields

$$[\iota_4, \iota_4]h_0(\nu_4\sigma') = (2\nu_4 - E\nu')\sigma'. \quad (2.54)$$

The proof of [92, Theorem (8.6), p. 534] yields that the Hopf–Hilton invariants $h_j(\sigma') \in \pi_{14}(\mathbb{S}^{6k+1}) = 0$ with $k \geq 3$ for $j \geq 1$. Thus, $2h_0(\sigma') = 0$ and [92, (8.6) Theorem, p. 534] imply

$$(2\nu_4 - E\nu')\sigma' = 2\nu_4\sigma' - (E\nu')\sigma' - (2[\nu_4, E\nu'])h_0(\sigma') = 2\nu_4\sigma' - (E\nu')\sigma'.$$

Hence, the relation $(E\nu')\sigma' = 2E\varepsilon' = \eta_4^2\varepsilon_6$ [54, (3.2)] and (2.54) yield

$$[\iota_4, \iota_4]h_0(\nu_4\sigma') = 2\nu_4\sigma' - 2E\varepsilon'.$$

Then, Lemma 2.4(1) leads to

$$[\gamma_4 \nu_4 \sigma', i_{\mathbb{R}}] = -2\gamma_4 E\varepsilon' \quad (2.55)$$

and

$$[\gamma_4(\nu_4\sigma' + E\varepsilon'), i_{\mathbb{R}}] = \gamma_4(-2\nu_4\sigma' - 2E\varepsilon' + 2\nu_4\sigma' - 2E\varepsilon') = 0.$$

Consequently, in view of the presentations of σ'^+ and ν_4^+ , we get $P_{14}(\mathbb{R}P^4) = \gamma_{4*}\{\nu_4\sigma' + E\varepsilon', 2E\varepsilon', \alpha_1(4)\alpha_2(7) - \nu_4^+ \alpha_2(7), \nu_4^+ \alpha_{1,5}(7)\} = \gamma_{4*}\{\nu_4^+ \sigma'^+ + E\varepsilon', 2E\varepsilon'\}$.

Notice that $[\gamma_6 \nu_6 \sigma_9, i_{\mathbb{R}}] = -2\gamma_6 \nu_6 \sigma_9$ and $[3\gamma_6 \beta_1(6), i_{\mathbb{R}}] = -6\gamma_6 \beta_1(6) \neq 0$. Hence, (2.46) yields $P_{16}(\mathbb{R}P^6) = \gamma_{6*}\{4\nu_6\sigma_9, \eta_6\mu_7\}$.

By (1.9), (2.45), (2.52), and [85, Proposition 3.1], we get that $[\gamma_8 \sigma_8^+ \alpha_1(15), i_{\mathbb{R}}] = \gamma_8(E\sigma'^+) \alpha_1(15) = \gamma_8 \alpha_2(8) \alpha_1(15) = \gamma_8 \alpha_1(8) \alpha_2(11) = 0$. Next, in view of Lemma 2.12(3), $[\gamma_8(\nu_8 \sigma_{11} - 2x \sigma_8 \nu_{15}), i_{\mathbb{R}}] = 0$ for x as in (1.82). Thus, we have $P_{18}(\mathbb{R}P^8) = \gamma_{8*}\{\nu_8 \sigma_{11} - 2x \sigma_8 \nu_{15} + \sigma_8^+ \alpha_1(15)\}$. One can also easily observe that $P_{18}(\mathbb{R}P^8) = \gamma_{8*}\{x \nu_8^+ \sigma_{11}^+ + 2\sigma_8^+ \nu_{15}^+\}$ for x as in (1.82).

Further, the relation $[\gamma_{10}\sigma_{10}\nu_{17}, i_{\mathbb{R}}] = -2\gamma_{10}\sigma_{10}\nu_{17}$ and (2.46) lead to $P_{20}(\mathbb{R}P^{10}) = \gamma_{10*}\{2\sigma_{10}\nu_{17}, \eta_{10}\mu_{11}\}$.

(iv) (11, 4), (11, 6), (11, 8), (11, 10), (11, 12). Recall from Sect. 1.6 that:

$$G_{n+11}(\mathbb{S}^n) = \begin{cases} \{\nu_4\sigma'\eta_{14}, \nu_4\bar{\nu}_7, \nu_4\varepsilon_7, \\ 2E\mu', \varepsilon_4\nu_{12}, (Ev')\varepsilon_7\} & \text{for } n = 4; \\ \{\bar{\nu}_6\nu_{14}, 4\zeta_6\} & \text{for } n = 6; \\ \{\bar{\nu}_8\nu_{16}\} & \text{for } n = 8; \\ 4\pi_{21}^{10} = \{4\zeta_{10}\} & \text{for } n = 10; \\ \{3[\iota_{12}, \iota_{12}]\} & \text{for } n = 12. \end{cases}$$

Certainly, by (2.46), it holds $P_{n+11}(\mathbb{R}P^n) = \gamma_{n*}G_{n+11}(\mathbb{S}^n)$ for $n = 8, 10$.

In view of (2.45) and (2.55), we get $[\gamma_4\nu_4\sigma'\eta_{14}, i_{\mathbb{R}}] = -2\gamma_4(E\varepsilon')\eta_{14} = 0$. The relations (1.122), (2.45), (2.49), and [85, Theorem 7.4] lead to $[\gamma_4\nu_4\bar{\nu}_7, i_{\mathbb{R}}] = -\gamma_4(E\nu')\bar{\nu}_7 = \gamma_4\varepsilon_4\nu_{12} \neq 0$ and $[\gamma_4\nu_4\varepsilon_7, i_{\mathbb{R}}] = -\gamma_4(E\nu')\varepsilon_7 \neq 0$. Because (2.46) leads to $[2\gamma_4E\mu', i_{\mathbb{R}}] = [\gamma_4\varepsilon_4\nu_{12}, i_{\mathbb{R}}] = [\gamma_4(E\nu')\varepsilon_7, i_{\mathbb{R}}] = 0$, we deduce that

$$P_{15}(\mathbb{R}P^4) = \gamma_{4*}\{\nu_4\sigma'\eta_{14}, 2E\mu', \varepsilon_4\nu_{12}, (Ev')\varepsilon_7\}.$$

Further, (2.45) and (2.51) lead to $[\gamma_6\bar{\nu}_6\nu_{14}, i_{\mathbb{R}}] = 0$ and (2.46) to $4\zeta_6 \in M_{17}(\mathbb{S}^6)$. Hence, $P_{17}(\mathbb{R}P^6) = \gamma_{6*}G_{17}(\mathbb{S}^6)$ and, by Lemma 2.12(3), $P_{23}(\mathbb{R}P^{12}) = \gamma_{12*}G_{23}(\mathbb{S}^{12})$.

(v) (12, 4), (12, 6), (12, 8), (12, 10), (12, 12). Recall from Sect. 1.6 that:

$$G_{n+12}(\mathbb{S}^n) = \begin{cases} \pi_{16}(\mathbb{S}^4) = \{\nu_4\sigma'\eta_{14}^2, \nu_4^4, \nu_4\mu_7, \\ \nu_4\eta_7\varepsilon_8, (Ev')\mu_7, (Ev')\eta_7\varepsilon_8\} & \text{for } n = 4; \\ \pi_{18}(\mathbb{S}^6) = \{[\iota_6, \iota_6]\sigma_{11}^+\} & \text{for } n = 6; \\ \pi_{20}(\mathbb{S}^8) = 0 & \text{for } n = 8; \\ \pi_{22}^{10} = \{[\iota_{10}, \nu_{10}]\} & \text{for } n = 10; \\ \pi_{24}(\mathbb{S}^{12}) = \{\theta, E\theta'\} & \text{for } n = 12. \end{cases}$$

Certainly, $P_{20}(\mathbb{R}P^8) = \pi_{20}(\mathbb{R}P^8) = 0$.

In view of (2.45) and (2.55), we get $[\gamma_4\nu_4\sigma'\eta_{14}^2, i_{\mathbb{R}}] = -2\gamma_4(E\varepsilon')\eta_{14}^2 = 0$. The relations (1.28), (2.45), and (2.49) lead to $[\gamma_4\nu_4^4, i_{\mathbb{R}}] = -(Ev')\nu_7^3 = 0$. Again, (1.133), (2.45), and (2.49) yield $[\gamma_4\nu_4\mu_7, i_{\mathbb{R}}] = -\gamma_4(E\nu')\mu_7 \neq 0$, $[\gamma_4\nu_4\eta_7\varepsilon_8, i_{\mathbb{R}}] = -\gamma_4(E\nu')\eta_7\varepsilon_8 \neq 0$ and $[\gamma_4\nu_4(\mu_7 + \eta_7\varepsilon_8), i_{\mathbb{R}}] = \gamma_4(E\nu')(\mu_7 + \eta_7\varepsilon_8) \neq 0$. Because (2.46) leads to $[\gamma_4(E\nu')\mu_7, i_{\mathbb{R}}] = [\gamma_4(E\nu')\eta_7\varepsilon_8, i_{\mathbb{R}}] = 0$, we deduce that $P_{16}(\mathbb{R}P^4) = \gamma_{4*}\{\nu_4\sigma'\eta_{14}^2, \nu_4^4, (Ev')\mu_7, (Ev')\eta_7\varepsilon_8\}$.

Next, (2.46) leads to $[\gamma_{12}E\theta', i_{\mathbb{R}}] = 0$. Because $H(\theta) = \eta_{23}$ (1.66)(7), in view of Lemma 2.4(1), (1.15), (1.127), and (2.7), it holds

$$[\gamma_{12}\theta, i_{\mathbb{R}}] = \gamma_{12}E\theta' \neq 0 \quad (2.56)$$

and $P_{24}(\mathbb{R}P^{12}) = \gamma_{12*}\{E\theta'\}$. Further, Lemma 1.2(2) and Lemma 2.12(3) yield $P_{n+12}(\mathbb{R}P^n) = \gamma_{n*}\pi_{n+12}^n$ for $n = 6, 10$ as well.

(vi) $(13, 4), (13, 6), (13, 8), (13, 10), (13, 12), (13, 14)$. Recall from Sect. 1.6 that:

$$G_{n+13}(\mathbb{S}^n) = \begin{cases} \pi_{17}^4 = \{v_4^2\sigma_{10}, v_4\eta_7\mu_8, (Ev')\eta_7\mu_8\} & \text{for } n = 4; \\ \pi_{19}^6 = \{v_6\sigma_9v_{16}\} & \text{for } n = 6, \\ \pi_{21}^8 = \{\sigma_8v_{15}^2, v_8\sigma_{11}v_{18}\} & \text{for } n = 8, \\ \pi_{23}^{10} = \{\sigma_{10}v_{17}^2\} & \text{for } n = 10, \\ \pi_{25}^{12} = \{\theta\eta_{24}, (E\theta')\eta_{24}\} & \text{for } n = 12, \\ \pi_{27}^{14} = \{3[l_{14}, l_{14}]\} & \text{for } n = 14. \end{cases}$$

Certainly, $P_{n+13}(\mathbb{R}P^n) = \gamma_{n*}G_{n+13}(\mathbb{S}^n)$ for $n = 6, 10$. The relations (1.82) and (2.45) and (2.53) lead to $[\gamma_8\sigma_8v_{15}^2, i_{\mathbb{R}}] = \gamma_8(E\sigma')v_{15}^2 = \gamma_8v_8\sigma_{11}v_{18} \neq 0$. Because, in view of (2.46), it holds $[\gamma_8v_8\sigma_{11}v_{18}, i_{\mathbb{R}}] = 0$, we deduce $P_{21}(\mathbb{R}P^8) = \gamma_{8*}\{v_8\sigma_{11}v_{18}\}$.

Further, (2.46) leads to $[\gamma_{12}(E\theta')\eta_{24}, i_{\mathbb{R}}] = 0$ and, in view of (1.16), (2.45), and (2.56), it holds $[\gamma_{12}\theta\eta_{24}, i_{\mathbb{R}}] = \gamma_{12}[l_{12}, \eta_{12}^2] \neq 0$. Hence, $P_{25}(\mathbb{R}P^{12}) = \gamma_{12*}\{(E\theta')\eta_{24}\}$.

By (1.28), (2.45), and (2.49), we get $[\gamma_4v_4^2\sigma_{10}, i_{\mathbb{R}}] = -\gamma_4(Ev')v_7\sigma_{10} = 0$. Again, (2.45), (2.49), and [85, Theorem 7.7] yield $[\gamma_4v_4\eta_7\mu_8, i_{\mathbb{R}}] = -\gamma_4(Ev')\eta_7\mu_8 \neq 0$. Because, by (2.46), it holds $[\gamma_4(Ev')\eta_7\mu_8, i_{\mathbb{R}}] = 0$, we derive that $P_{17}(\mathbb{R}P^4) = \gamma_{4*}\{\gamma_4v_4^2\sigma_{10}, \gamma_4(Ev')\eta_7\mu_8\}$.

Finally, in view of Lemma 2.12(3), it holds $P_{27}(\mathbb{R}P^{14}) = \gamma_{14*}G_{27}(\mathbb{S}^{14})$.

Then, in light of the above and Chap. 1, the formulae (1)–(14) follow and the proof is complete. $\square$

2.5 Some Whitehead Center Groups of Complex and Quaternionic Projective Spaces

First, we show:

Theorem 2.20. *Let $n \geq 2$. Then, $P_{k+2n+1}(\mathbb{C}P^n) = \gamma_{n*}G_{k+2n+1}(\mathbb{S}^{2n+1})$ for $0 \leq k \leq 13$ except $(k, n) = (2, 2), (9, 2), (10, 2), (10, 4), (13, 4)$ and $P_7(\mathbb{C}P^2) = 0$, $P_{14}(\mathbb{C}P^2) = \gamma_{2*}\{v_5^3\} \cong \mathbb{Z}_2$, $P_{15}(\mathbb{C}P^2) = \gamma_{2*}\{v_5\sigma_8, \beta_1(5)\} \cong \mathbb{Z}_{24}$, $P_{19}(\mathbb{C}P^4) = \gamma_{4*}\{\beta_1(9)\} \cong \mathbb{Z}_3$, $P_{22}(\mathbb{C}P^4) = \gamma_{4*}\{\alpha_1(9)\beta_1(12)\} \cong \mathbb{Z}_3$, that is:*

1. $P_{2n+1}(\mathbb{C}P^n) = \begin{cases} \pi_{2n+1}(\mathbb{C}P^n) \cong \mathbb{Z} & \text{for } n = 3, \\ 2\pi_{2n+1}(\mathbb{C}P^n) \cong 2\mathbb{Z} & \text{for } n \geq 2 \text{ unless } n = 3; \end{cases}$
2. $P_{2n+k+1}(\mathbb{C}P^n) = \begin{cases} 0 & \text{for even } n, \\ \pi_{2n+k+1}(\mathbb{C}P^n) \cong \mathbb{Z}_2 & \text{for odd } n \end{cases} \quad \text{with } k = 1, 2;$

3. $P_{2n+4}(\mathbb{C}P^n) = \begin{cases} \pi_{2n+4}(\mathbb{C}P^n) \cong \mathbb{Z}_{24} & \text{for } n \equiv 3 \pmod{4} \text{ or } n = 2^i - 2 \geq 2, \\ 2\pi_{2n+4}(\mathbb{C}P^n) \cong \mathbb{Z}_{12} & \text{for otherwise;} \end{cases}$
4. $P_{2n+k+1}(\mathbb{C}P^n) = \begin{cases} 0 & \text{for } n \geq 3, \\ \pi_{2n+k+1}(\mathbb{C}P^n) \cong \mathbb{Z}_2 & \text{for } n = 2 \end{cases}$ with $k = 4, 5$;
5. $P_{2n+7}(\mathbb{C}P^n) = \begin{cases} \pi_{2n+7}(\mathbb{C}P^n) \cong \mathbb{Z}_2 & \text{for } n \equiv 2, 3 \pmod{4} \text{ or } n = 2^i - 3 \geq 5, \\ 0 & \text{for otherwise;} \end{cases}$
6. $P_{2n+8}(\mathbb{C}P^n) = \begin{cases} \pi_{12}(\mathbb{C}P^2) \cong \mathbb{Z}_{30} & \text{for } n = 2, \\ \pi_{2n+8}(\mathbb{C}P^n) \cong \mathbb{Z}_{240} & \text{for } n = 5 \text{ or } n \equiv 7 \pmod{8}, \\ 2\pi_{2n+8}(\mathbb{C}P^n) \cong \mathbb{Z}_{120} & \text{for otherwise;} \end{cases}$
7. $P_{2n+9}(\mathbb{C}P^n) = \begin{cases} \pi_{13}(\mathbb{C}P^2) \cong \mathbb{Z}_2 & \text{for } n = 2, \\ \pi_{15}(\mathbb{C}P^3) \cong (\mathbb{Z}_2)^3 & \text{for } n = 3, \\ \gamma_{4*}\{[\iota_9, \iota_9]\} \cong \mathbb{Z}_2 & \text{for } n = 4, \\ 0 & \text{for } n \equiv 0 \pmod{2} \text{ unless } n = 2, 4, \\ \pi_{2n+9}(\mathbb{C}P^n) \cong (\mathbb{Z}_2)^2 & \text{for } n \equiv 1 \pmod{2} \text{ unless } n = 3; \end{cases}$
8. $P_{2n+10}(\mathbb{C}P^n) = \begin{cases} \gamma_{2*}\{v_5^3\} \cong \mathbb{Z}_2 & \text{for } n = 2, \\ \pi_{16}(\mathbb{C}P^3) \cong (\mathbb{Z}_2)^4 & \text{for } n = 3, \\ \gamma_{4*}\{\sigma_9\eta_{16}^2, v_9^3, \eta_9\varepsilon_{10}\} \cong (\mathbb{Z}_2)^3 & \text{for } n = 4, \\ \pi_{2n+10}(\mathbb{C}P^n) \cong (\mathbb{Z}_2)^3 & \text{for } n \equiv 1 \pmod{2}, n \geq 5, \\ \gamma_{n*}\{v_{2n+1}^3, \eta_{2n+1}\varepsilon_{2n+2}\} \cong (\mathbb{Z}_2)^2 & \text{for } n \equiv 2 \pmod{4}, n \geq 6 \\ & \text{or } n = 2^i - 4, i \geq 4 \\ & \text{unless } n \equiv 26 \pmod{32}, \\ \gamma_{n*}\{v_{2n+1}^3\} \cong \mathbb{Z}_2 & \text{for } n \equiv 26 \pmod{32}, \\ \gamma_{n*}\{\eta_{2n+1}\varepsilon_{2n+2}\} \cong \mathbb{Z}_2 & \text{for } n \equiv 0 \pmod{4}, n \geq 8 \\ & \text{unless } n = 2^i - 4; \end{cases}$
9. $P_{2n+11}(\mathbb{C}P^n) = \begin{cases} \gamma_{2*}\{v_5\sigma_8, \beta_1(5)\} \cong \mathbb{Z}_{24} & \text{for } n = 2, \\ \pi_{17}(\mathbb{C}P^3) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2 & \text{for } n = 3, \\ \pi_{21}(\mathbb{C}P^5) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2 & \text{for } n = 5, \\ \gamma_{n*}\{\beta_1(2n+1)\} \cong \mathbb{Z}_3 & \text{for } n \equiv 0 \pmod{2}, n \geq 4, \\ \pi_{2n+11}(\mathbb{C}P^n) \cong \mathbb{Z}_6 & \text{for } n \equiv 1 \pmod{2}, n \geq 7; \end{cases}$
10. $P_{2n+12}(\mathbb{C}P^n) = \begin{cases} \pi_{2n+12}(\mathbb{C}P^n) \cong \mathbb{Z}_{504} \oplus (\mathbb{Z}_2)^2 & \text{for } n = 2, \\ \pi_{2n+12}(\mathbb{C}P^n) \cong \mathbb{Z}_{504} \oplus \mathbb{Z}_2 & \text{for } n = 3, 4, \\ \pi_{2n+12}(\mathbb{C}P^n) \cong \mathbb{Z}_{504} & \text{for } n \geq 5 \text{ unless } n \equiv 57 \pmod{62}, \\ 2\pi_{2n+12}(\mathbb{C}P^n) \cong \mathbb{Z}_{252} & \text{for } n \equiv 57 \pmod{62}; \end{cases}$
11. $P_{2n+13}(\mathbb{C}P^n) = \begin{cases} \pi_{17}(\mathbb{C}P^2) \cong (\mathbb{Z}_2)^3 & \text{for } n = 2, \\ \pi_{2n+13}(\mathbb{C}P^n) = 0 & \text{for } n = 3, 4 \text{ or } n \geq 7, \\ \pi_{2n+13}(\mathbb{C}P^n) \cong \mathbb{Z}_2 & \text{for } n = 5, 6; \end{cases}$
12. $P_{2n+14}(\mathbb{C}P^n) = \begin{cases} \pi_{2n+14}(\mathbb{C}P^n) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2 & \text{for } n = 2, 5, \\ \pi_{2n+14}(\mathbb{C}P^n) \cong \mathbb{Z}_6 & \text{for } n = 3, 6, \\ \gamma_{4*}\{\alpha_1(9)\beta_1(12)\} \cong \mathbb{Z}_3 & \text{for } n = 4, \\ \pi_{2n+14}(\mathbb{C}P^n) \cong \mathbb{Z}_3 & \text{for } n \geq 7. \end{cases}$

Proof. We make use of Lemma 2.4(2) to find $M_{k+2n+1, \mathbb{C}}(\mathbb{S}^{2n+1})$ for $1 \leq k \leq 13$ and $n = 2, 4, 6$ with $k > 2n - 1$. First, we consider $M_{k+5, \mathbb{C}}(\mathbb{S}^5)$ for $4 \leq k \leq 13$. By the relation (1.23), we obtain $M_{k+5, \mathbb{C}}(\mathbb{S}^5) = \pi_{k+5}(\mathbb{S}^5)$ for $4 \leq k \leq 6$ and $k = 12, 13$. Moreover, in view of $h_0(\sigma''') = H(\sigma''') = 4v_9$ (1.66)(3) and (2.7), the relations Lemma 1.2(2) and (1.23) imply $[\iota_5, \eta_5] \circ Eh_0(\sigma''') = 0$. Then, in view of (1.36), we obtain $M_{12, \mathbb{C}}(\mathbb{S}^5) = \pi_{12}(\mathbb{S}^5)$. Because $\eta_5^2 \varepsilon_7 = 4v_5 \sigma_8 \neq 0$ (1.81) and $\eta_n \mu_{n+1} \neq 0$ for $n \geq 2$ (1.62), we deduce that $M_{13, \mathbb{C}}(\mathbb{S}^5) = 0$, $M_{14, \mathbb{C}}(\mathbb{S}^5) = \{v_5^3\}$, and $M_{15, \mathbb{C}}(\mathbb{S}^5) = \{v_5 \sigma_8, \beta_1(5)\}$. The relation (1.140) yields $M_{16, \mathbb{C}}(\mathbb{S}^5) = \pi_{16}(\mathbb{S}^5)$.

To find $M_{k+9, \mathbb{C}}(\mathbb{S}^9)$ for $8 \leq k \leq 13$, we recall [85, Theorems 7.1–7.4 and 7.6–7.7]:

$$\pi_{k+9}(\mathbb{S}^9) = \begin{cases} \{\sigma_9 \eta_{16}, \bar{v}_9, \varepsilon_9\} & \text{for } k = 8, \\ \{\sigma_9 \eta_{16}^2, v_9^3, \mu_9, \eta_9 \varepsilon_{10}\} & \text{for } k = 9, \\ \{\sigma_9 v_{16}, \eta_9 \mu_{10}, \beta_1(9)\} & \text{for } k = 10, \\ \{\zeta_9 + \alpha'_3(9) + \alpha_{1,7}(9), \bar{v}_9 v_{17}\} & \text{for } k = 11, \\ 0 & \text{for } k = 12, \\ \{\sigma_9 v_{16}^2, \alpha_1(9) \beta_1(12)\} & \text{for } k = 13. \end{cases}$$

In view of [85, Theorem 7.2], (1.69)(1), and (1.72), we get $\eta_9 \sigma_{10} \eta_{17} = v_9^3 + \eta_9 \varepsilon_{10} \neq 0$, $\eta_9 \bar{v}_{10} = v_9^3 \neq 0$, $\eta_9 \varepsilon_{10} \neq 0$, and $\eta_9(E[\iota_9, \iota_9]) = 0$. Consequently, by means of (1.70), it holds $M_{17, \mathbb{C}}(\mathbb{S}^9) = \{\sigma_9 \eta_{16} + \bar{v}_9 + \varepsilon_9\} = \{[\iota_9, \iota_9]\}$.

By (1.82) and (1.81), it holds $\eta_9^2 \varepsilon_{11} = 4v_9 \sigma_{12} = 0$. Then, in view of (1.18) and (1.69)(1), we get $\eta_9 \sigma_{10} \eta_{17}^2 = \eta_9^3 \sigma_{12} = 4v_9 \sigma_{12} = 0$. Next, $\eta_9 v_{10}^3 = 0$ (1.23) and $\eta_9 \mu_{10} \neq 0$ (1.62). Hence, $M_{18, \mathbb{C}}(\mathbb{S}^9) = \{\sigma_9 \eta_{16}^2, v_9^3, \eta_9 \varepsilon_{10}\}$. In view of (1.125) and (1.64)(2), it holds $M_{19, \mathbb{C}}(\mathbb{S}^9) = \{\beta_1(9)\}$.

By $\eta_9 \zeta_{10} = 0$ (1.140) and $\bar{v}_{10} v_{18} = 0$ (1.129), we get $M_{20, \mathbb{C}}(\mathbb{S}^9) = \pi_{20}(\mathbb{S}^9)$. Certainly, $\pi_{21}(\mathbb{S}^9) = 0$ implies $M_{21, \mathbb{C}}(\mathbb{S}^9) = 0$. The relation (1.109)(2) yields $M_{22, \mathbb{C}}(\mathbb{S}^9) = \{\alpha_1(9) \beta_1(12)\}$.

In virtue of (1.128), we get that $M_{25, \mathbb{C}}(\mathbb{S}^{13}) = \pi_{25}(\mathbb{S}^{13})$ and $M_{26, \mathbb{C}}(\mathbb{S}^{13}) = \pi_{26}(\mathbb{S}^{13})$.

Notice that for $k \leq 4n - 1$, we have

$$E^{-1}(\text{Ker } \eta_{4n+1*}) = \begin{cases} 2\pi_{4n+k+1}(\mathbb{S}^{4n+1}) & \text{for } k = 0, 7, \\ \{v_{4n+1}^3, \eta_{4n+1} \varepsilon_{4n+2}\} & \text{for } k = 9, \\ 0 & \text{for } k = 1, 2, 8, 10, 12, \\ \pi_{4n+k+1}(\mathbb{S}^{4n+1}) & \text{for } 3 \leq k \leq 6, k = 11, 13. \end{cases}$$

This, (1.2)–(1.50), Proposition 2.16(1);(3) and Chap. 1 lead to the assertion and the proof is complete. $\square$

We recall:

$$\pi_k(\mathbb{S}^3) = \begin{cases} \{v' \eta_6^{k-6}\} \cong \mathbb{Z}_2 & \text{for } k = 7, 8, \\ \{\varepsilon_3\} \cong \mathbb{Z}_2 & \text{for } k = 11, \\ \{\eta_3 \varepsilon_4, \mu_3\} \cong (\mathbb{Z}_2)^2 & \text{for } k = 12. \end{cases}$$

We also need:

$$\pi_k(\mathbb{S}^3) = \begin{cases} \{v'\mu_6, v'\eta_6\varepsilon_7\} \cong (\mathbb{Z}_2)^2 & \text{for } k = 15, \\ \{v'\eta_6\mu_7, \alpha_1(3)\beta_1(6)\} \cong \mathbb{Z}_6 & \text{for } k = 16, \\ \{\varepsilon_3v_{11}^2, \alpha_1(3)\alpha'_3(6), \alpha_{1,5}(3)\alpha_{1,5}(10)\} \cong \mathbb{Z}_{30} & \text{for } k = 17, \\ \{\bar{\varepsilon}_3, \alpha_4(3), \alpha_{2,5}(3)\} \cong \mathbb{Z}_{30} & \text{for } k = 18. \end{cases}$$

Notice that (2.10) and Lemma 2.6(3) lead to:

$$v' \circ \pi_k^6 \subseteq Q_k^3. \quad (2.57)$$

Further, we show:

- Lemma 2.21.** 1. $Q_k(\mathbb{S}^3) = 0$ for $k = 11, 12$;
 2. $Q_k(\mathbb{S}^3) = \pi_k(\mathbb{S}^3) \cong \mathbb{Z}_2$ for $k = 7, 8$ and $Q_9(\mathbb{S}^3) = \pi_9(\mathbb{S}^3) \cong \mathbb{Z}_3$;
 3. $Q_{10}(\mathbb{S}^3) = 3\pi_{10}(\mathbb{S}^3) \cong \mathbb{Z}_5$ and $Q_{13}(\mathbb{S}^3) = \{\varepsilon', \alpha_1(3)\alpha_2(6)\} \cong \mathbb{Z}_{12}$;
 4. $Q_{14}(\mathbb{S}^3) = \pi_{14}(\mathbb{S}^3) \cong \mathbb{Z}_{84} \oplus (\mathbb{Z}_2)^2$;
 5. $Q_{15}(\mathbb{S}^3) = \pi_{15}(\mathbb{S}^3) \cong (\mathbb{Z}_2)^2$, $Q_{16}(\mathbb{S}^3) = \pi_{16}^3 = \{v'\eta_6\mu_7\} \cong \mathbb{Z}_2$, $Q_{17}(\mathbb{S}^3) = \pi_{17}(\mathbb{S}^3) \cong \mathbb{Z}_{30}$ and $Q_{18}(\mathbb{S}^3) = 6\pi_{18}(\mathbb{S}^3) \cong \mathbb{Z}_5$.

Proof. It is known that $v'E^3\alpha \neq 0$ for $\alpha = \varepsilon_3, \eta_3\varepsilon_4, \mu_3$, and $\eta_3\mu_4$. This, (2.10), and (2.11) lead to (1) and the fact that $Q_{13}(\mathbb{S}^3) \not\cong \eta_3\mu_4$.

Notice that $\pi_{k+6}^3 = v' \circ \pi_{k+6}^6$ for $k = 1, 2$ and $\alpha_1(3)\alpha_1(6)\alpha_1(9) = 0$ (1.8). Hence, (2.10), (2.11), and (2.57) imply (2).

Because $\alpha_1(3)\alpha_2(6) \neq 0$ and $\alpha_1(3)\alpha_{1,5}(6) = 0$, the relations (2.10) and (2.11) lead to $Q_{10}(\mathbb{S}^3) = \{\alpha_{1,5}(3)\} \cong \mathbb{Z}_5$.

In view of (1.9), we get $\alpha_1(3)\alpha_1(6)\alpha_2(9) = -3\alpha_1(3)\beta_1(6) = 0$. Then, (2.10), (2.11), (1.124), and (2.57) yield (3).

Next, $\pi_{14}^3 = \{\mu', \varepsilon_3v_{11}, v'\varepsilon_6\} \cong \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$. In view of (1.18), (1.64), and (1.140), we obtain $v'E^3\mu' = \eta_3^3\zeta_6 = 0$. By means of (1.122), the relation (2.57) implies $Q_{14}^3 = \pi_{14}^3$. Then, (2.10), (2.11), and $\alpha_1(3)\alpha_{1,7}(6) = 0$, $\alpha_1(3)\alpha_3(6) = 3\alpha_1(3)\alpha'_3(6) = 0$ (1.10) lead to (4).

Observe that the relations (2.10), (2.11), and (1.9) lead to $\alpha_1(3)\beta_1(6) \notin Q_{16}(\mathbb{S}^3)$. Because $v' \circ \pi_k^6 = \pi_k^3$ for $k = 15, 16$, the relation (2.57) leads to the first and second one of (5).

Because $\pi_{17}^3 = v' \circ \pi_{17}^6$ (1.122), the relation (2.57) yields $Q_{17}^3 = \pi_{17}^3$. Then, (2.10), (2.11), and (2.37) lead to the third one of (5).

We know that $\alpha_1(3)\alpha_4(6) \neq 0$ Proposition 2.11(3). This, $\alpha_1(3)\alpha_{2,5}(6) = 0$, (2.10), (2.11), and (2.21) lead to the last of (5), and the proof is complete. $\square$

Notice that by (1.1), Propositions 1.6, 1.5, and [85, Proposition 2.5],

$$[\iota_n, \pi_k^n] = J(\Delta\pi_k^n) = P(E^{n+1}\pi_k^n).$$

We show:

Proposition 2.22. *Let $4 \leq k \leq 22$. Then, $\nu_6 E^6 \pi_k^3 \subseteq P \pi_{k+8}^{13}$, $\nu_7 E^7 \pi_k(\mathbb{S}^3) = 0$ for $4 \leq k \leq 21$, and $\nu_k E^k \pi_{22}(\mathbb{S}^3) \cong \mathbb{Z}_3$ for $5 \leq k \leq 9$.*

Proof. We know $\nu_5 \circ E^5 \nu' = 2\nu_5^2 = 0$ (1.25). This implies $\nu_6 E^6 \pi_k^3 = 0$ for $4 \leq k \leq 10$.

The relations (2.30), (1.126), (1.39), and (2.34) imply $\nu_6 E^6 \pi_{11}^3 = P \pi_{19}^{13}$, $\nu_6 E^6 \pi_{12}^3 = 8P \pi_{20}^{13}$, and $\nu_6 E^6 \mu' = 0$. By $\pi_{18}^5 \cong (\mathbb{Z}_2)^2$, we obtain $\nu_5 E^5 \varepsilon' \in \{\nu_5, 2\nu_8, \nu_{11}\} \circ 2\nu_{15} \subseteq 2\pi_{18}^5 = 0$. Consequently, $\nu_6 E^6 (\nu' \varepsilon_6) = 0$. The relation (1.126) implies $\nu_6 \varepsilon_9 \nu_{17} = 2\bar{\nu}_6 \circ \nu_{14}^2 = 0$. In view of (1.18), (1.122), and the relation (2.20), we get

$$\pi_k^3 = \nu' \circ \pi_k^6 \text{ for } 15 \leq k \leq 17.$$

Therefore, we obtain $\nu_6 E^6 \pi_k^3 = 0$ for $15 \leq k \leq 17$.

Since $\pi_{18}^3 = \{\bar{\varepsilon}_3\} \cong \mathbb{Z}_2$, (1.108) implies $\nu_6 E^6 \pi_{18}^3 = 0$.

Further $\pi_{19}^3 = \{\eta_3 \bar{\varepsilon}_4, \mu_3 \sigma_{12}\} \cong (\mathbb{Z}_2)^2$, $\nu_6 \eta_9 \bar{\varepsilon}_{10} = 0$ (1.23), and $\nu_6 \mu_9 \sigma_{18} = 8P(\sigma_{13}^2)$ (1.39) lead to $\nu_6 E^6 \pi_{18}^3 = 8\{P(\sigma_{13}^2)\}$.

Next, we recall

$$\pi_{20}^3 = \{\bar{\varepsilon}', \bar{\mu}_3, \eta_3 \mu_4 \sigma_{13}\} \cong \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2.$$

Then, the relations (2.32) and (2.22) imply $\nu_6 E^6 \pi_{20}^3 = 16\{P(\rho_{13})\}$.

From the fact that $\pi_{21}^3 = \{\mu' \sigma_{14}, \nu' \bar{\varepsilon}_6, \eta_3 \bar{\mu}_4\}$ and (2.34), we obtain $\nu_6 E^6 \pi_{21}^3 = 0$. The relation (2.35) implies $\nu_6 E^6 \pi_{22}^3 = 0$.

Next, we examine the case for odd primes. Those cases appear for $k = 6, 9, 10, 13, 14$ and $16 \leq k \leq 22$. By (1.8), the assertion

$$\alpha_1(7) E^7 \pi_k(\mathbb{S}^3; 3) = 0 \quad (2.58)$$

holds for $k = 6, 9, 13, 16$.

By (1.9) and (1.10), (2.58) holds for $k = 10, 14$.

By (1.8) and Proposition 2.11(1), (2.58) holds for $k = 17$.

For $k = 18$, we have $\alpha_1(7) E^7 \pi_{18}(\mathbb{S}^3; 3) = \{\alpha_1(7) \alpha_4(10)\} \subseteq E^2 \pi_{23}(\mathbb{S}^5; 3) = 0$ [85, p. 185].

By [85, Theorem 13.10(iv);(v)], $E^2 \pi_{l+19}(\mathbb{S}^3; 3) = 0$ for $l = 0, 1$, and hence, (2.58) holds for $k = 19, 20$.

By [85, p. 185], $E^2 \pi_{21}(\mathbb{S}^3; 3) = 0$, and hence, (2.58) for $k = 21$ holds.

Finally, for $k = 22$, by Proposition 2.11(4) and [87, p. 56], $\alpha_1(9) \alpha_5(12) \neq 0$. This leads to the last result and completes the proof. $\square$

We propose:

Problem 2.23. If $\nu' E^3 \beta = 0$ for $\beta \in \pi_{k-1}^3$ then $\nu_4 \wedge \beta = 0$?

Lemma 2.8(1) and Proposition 2.22 yield:

Corollary 2.24. $L''_{k-1,n}(\mathbb{S}^3) = \pi_{k-1}(\mathbb{S}^3)$ for $n \geq 2$ and $5 \leq k \leq 22$.

Writing R_k^n for the subgroup of $\pi_k(SO(n))$ consisting of its free part and $\pi_k(SO(n); 2)$, we get from $(\mathcal{SF}_k^n)$ the exact sequence

$$(\mathcal{R}_k^n) \cdots \longrightarrow \pi_{k+1}^n \xrightarrow{\Delta} R_k^n \xrightarrow{i_*} R_k^{n+1} \xrightarrow{p_*} \cdots$$

Now, we show:

Theorem 2.25. $P_{4n+3+k}(\mathbb{H}P^n) = P'_{4n+3+k}(\mathbb{H}P^n) \oplus P''_{4n+3+k}(\mathbb{H}P^n)$ for $0 \leq k \leq 10$, where:

1. $P'_{4n+3}(\mathbb{H}P^n) = [[\frac{24}{(24,n+1)}, 2]]\gamma_n * \pi_{4n+3}(\mathbb{S}^{4n+3})$ for $n \geq 2$;
2. $P'_{4n+3+k}(\mathbb{H}P^n) = \gamma_n * \pi_{4n+k+3}(\mathbb{S}^{4n+3})$ for $k = 1, 2, 4, 5, 8, 9, 10$;
3. $P'_{4n+6}(\mathbb{H}P^n) = \frac{3+(-1)^n}{2}\gamma_n * \pi_{4n+6}(\mathbb{S}^{4n+3})$;
4. $P'_{4n+9}(\mathbb{H}P^n) = \frac{1-(-1)^n}{2}\gamma_n * \pi_{4n+9}(\mathbb{S}^{4n+3})$;
5. $P'_{4n+10}(\mathbb{H}P^n) = \begin{cases} 2\gamma_n * \pi_{4n+10}(\mathbb{S}^{4n+3}), & \text{for } n \equiv 0, 1, 2 \pmod{4}; \\ \gamma_n * \pi_{4n+10}(\mathbb{S}^{4n+3}), & \text{for } n \equiv 3 \pmod{4} \text{ or } n = 2. \end{cases}$

Proof. First, recall that in view of Proposition 2.10, it holds $P'_k(\mathbb{H}P^n) = \gamma_n * (G_k(\mathbb{S}^{4n+3}) \cap M_k(\mathbb{S}^{4n+3}))$. Because the groups $G_{4n+k+3}(\mathbb{S}^{4n+3})$ for $0 \leq k \leq 10$ have been determined in Sects. 1.2 and 1.6, Proposition 2.10 and Lemma 2.12(2)–(iii) yield

$$P'_{4n+k+3}(\mathbb{H}P^n) = \begin{cases} [[\frac{24}{(24,n+1)}, 2]]\gamma_n * \pi_{4n+3}(\mathbb{S}^{4n+3}) & \text{for } k = 0; \\ \gamma_n * \pi_{4n+k+3}(\mathbb{S}^{4n+3}) & \text{for } k = 1, 2, 4, 5; \\ \frac{3+(-1)^n}{2}\gamma_n * \pi_{4n+6}(\mathbb{S}^{4n+3}) & \text{for } k = 3; \\ \frac{3+(-1)^n}{2}\gamma_n * \pi_{4n+6}(\mathbb{S}^{4n+3}) & \text{for } k = 6; \\ 2\gamma_n * \pi_{4n+10}(\mathbb{S}^{4n+3}) & \text{for } k = 7 \text{ with } n \equiv 0, 1, 2 \pmod{4}; \\ \gamma_n * \pi_{4n+10}(\mathbb{S}^{4n+3}) & \text{for } k = 7 \text{ with } n \equiv 3 \pmod{4} \text{ or } n = 2; \\ \gamma_n * \pi_{4n+k+3}(\mathbb{S}^{4n+3}) & \text{for } k = 8, 9, 10 \end{cases}$$

and (1)–(5) follow.

Next, notice that by means of Lemma 2.8(1)–(2), Proposition 2.10, and Proposition 2.17(1), we get

$$P_k(\mathbb{H}P^{2n+1}) = P'_k(\mathbb{H}P^{2n+1}) \oplus P''_k(\mathbb{H}P^{2n+1}).$$

Now, we show that

$$P_{8n+3+k}(\mathbb{H}P^{2n}) = P'_{8n+3+k}(\mathbb{H}P^{2n}) \oplus P''_{8n+3+k}(\mathbb{H}P^{2n})$$

for $0 \leq k \leq 10$. By Lemma 2.8(2)–(3) and Proposition 2.17(2)(i)–(ii), we see that it holds for p -primary components with an odd prime p . To show that for the 2-primary component, we check the condition (2.43) in Proposition 2.17(2)–(iii). Let n be even and we work in the 2-primary component. The assertion is a direct consequence of the fact that $[\iota_{4n+3}, \pi_{4n+3+k}^{4n+3}] = 0$ for $k = 1, 2, 8, 9, 10$ [20, Proposition 1.3].

For $k = 0, 3, 6$, the condition (2.43) holds because $2[\iota_{4n+3}, \iota_{4n+3}] = 2[\iota_{4n+3}, \nu_{4n+3}] = 2[\iota_{4n+3}, \nu_{4n+3}^2] = 0$.

We know $[\iota_{4n+3}, \sigma_{4n+3}] = 0$ if and only if $n = 2$ or $n \equiv 3 \pmod{4}$ (1.50). So, for the case $n \neq 2$ and $n \equiv 0, 2 \pmod{4}$, suppose that there exists $\beta \in \pi_{4n+9}^3$, such that $[\iota_{4n+3}, \sigma_{4n+3}] = \nu_{4n+3} E^{4n+3} \beta$. By [10] and [30],

$$R_{4n+9}^{4n+k} \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 & \text{for } k = 1; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_4 & \text{for } k = 2; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } k = 3; \\ \mathbb{Z}_2 & \text{for } k = 4. \end{cases}$$

Here, the first direct summand $\mathbb{Z}_2$ corresponds to the Bott result: $R_{4n+9}^\infty \cong \mathbb{Z}_2$. Since $n \equiv 0, 2 \pmod{4}$, in view of [2, Theorem 1.3], the map $J : R_{4n+9}^\infty \rightarrow \pi_{4n+9}^S$ is a monomorphism. But, by means of (1.50), it holds $[\iota_{4n+3}, \sigma_{4n+3}] \neq 0$ and this element does not correspond to the Bott one. Hence, the map $J : R_{4n+9}^{4n+3} \rightarrow \pi_{4n+9}^{4n+3}$ is also a monomorphism and, by means of (2.44), it holds $\Delta(\sigma_{4n+3}) = [\iota_3]_{4n+3} \beta$.

In the exact sequence $(\mathcal{R}_{4n+9}^{4n+k-1})$ ($k = 2, 3$):

$$R_{4n+9}^{4n+k-1} \xrightarrow{i_*} R_{4n+9}^{4n+k} \xrightarrow{P_*} \pi_{4n+9}^{4n+k-1},$$

we know $\sharp[\iota_{4n+2}, \sigma_{4n+2}] = 16$ (1.50) and $[\iota_{4n+1}, \eta_{4n+1} \sigma_{4n+2}] \neq 0$, $[\iota_{4n+1}, \bar{\nu}_{4n+1}] \neq 0$, and $[\iota_{4n+1}, \varepsilon_{4n+1}] \neq 0$ [20, Lemma 4.3]. So, $i_* : R_{4n+9}^{4n+k-1} \rightarrow R_{4n+9}^{4n+k}$ are epimorphisms for $k = 2, 3$, respectively. This shows that $[\iota_3]_{4n+1} \beta$ generates the direct summand $\mathbb{Z}_8$ in R_{4n+9}^{4n+1} and contradicts the fact that $4\beta = 0$ [32, Corollary (1.22)]. Hence, we have the assertion for $k = 7$ and this completes the proof. $\square$

For n even, since $[\iota_{8n+3}, \sigma_{8n+3}^2] \neq 0$ [66, Table 1], the assertion of Theorem 2.25 for $k = 7$, $n \equiv 0 \pmod{4}$ is obtained from Corollary 2.18(2).

Since ζ_n is cyclic except for $n \not\equiv 115 \pmod{128}$ [20, p. 426], we obtain:

Remark 2.26. The assertion of Theorem 2.25 for $k = 11$ holds and $P'_{4n+14}(\mathbb{H}P^n) = \gamma_n * \pi_{4n+14}(\mathbb{S}^{4n+3})$, for $n \not\equiv 115 \pmod{128}$.

Since the nontrivial Whitehead product cannot be desuspended too much, we add:

Conjecture 2.27. The assertion $P_{4n+3+k}(\mathbb{H}P^n) = P'_{4n+3+k}(\mathbb{H}P^n) \oplus P''_{4n+3+k}(\mathbb{H}P^n)$ holds all $k \geq 1$.

Now, we show:

Lemma 2.28. 1. $\text{Ker}\{E^4 : \pi_{k-1}(\mathbb{S}^3) \rightarrow \pi_{k+3}(\mathbb{S}^7)\} \subseteq (i_{\mathbb{H}}E)_*^{-1} P_k(\mathbb{H}P^n)$ for $k, n \geq 1$. In particular, $i_{\mathbb{H}}E(v'\alpha) \in P''_k(\mathbb{H}P^n)$ provided that $\sharp\alpha = 2$ for $\alpha \in \pi_{k-1}^7$.
2. If $i_{\mathbb{H}}E(v'^+)(E^4\alpha) \neq 0$ for $\alpha \in \pi_{n-1}(\mathbb{S}^3)$ then $i_{\mathbb{H}}(E\alpha) \notin P_n(\mathbb{H}P^2)$.

Proof. (1) is obtained from the formula

$$[i_{\mathbb{H}}E\alpha, \beta] = [i_{\mathbb{H}}, \beta] \circ E^m\alpha \text{ for } \beta \in \pi_m(\mathbb{H}P^n) \text{ Lemma 1.2(2).} \quad (2.59)$$

In view of (2.14), it holds $\pm[i_{\mathbb{H}}(E\alpha), i_{\mathbb{H}}] = i_{\mathbb{H}}(Ev'^+)(E^4\alpha)$ for $\alpha \in \pi_{n-1}(\mathbb{S}^3)$ and (2) follows. $\square$

Let $n \geq 2$. Then, immediately, we get:

Example 2.29.

$$P_k(\mathbb{H}P^n) \ni \begin{cases} i_{\mathbb{H}}E(v'\eta_6) & \text{for } k = 8, \\ i_{\mathbb{H}}E(v'\eta_6^2) & \text{for } k = 9, \\ i_{\mathbb{H}}E(\alpha_1(3)\alpha_1(6)) & \text{for } k = 10, \\ i_{\mathbb{H}}E(v'\varepsilon_6) & \text{for } k = 15, \\ i_{\mathbb{H}}E(v'\mu_6), i_{\mathbb{H}}E(v'\eta_6\varepsilon_7) & \text{for } k = 16, \\ i_{\mathbb{H}}E(v'\eta_6\mu_7) & \text{for } k = 17, \\ i_{\mathbb{H}}E(\varepsilon_3v_{11}^2) & \text{for } k = 18. \end{cases}$$

By Proposition 2.17, Lemma 2.21, Theorem 2.25, and the above, we obtain:

Proposition 2.30. 1. Let $n \geq 2$. Then:

$$\begin{aligned} P_k(\mathbb{H}P^n) &= \begin{cases} 0 & \text{for } k = 5, 6, \\ i_{\mathbb{H}*}E\{v'\} \cong \mathbb{Z}_4 & \text{for } k = 7, \\ i_{\mathbb{H}*}E\{v'\eta_6\} \cong \mathbb{Z}_2 & \text{for } k = 8, \\ i_{\mathbb{H}*}E\{v'\eta_6^2\} \cong \mathbb{Z}_2 & \text{for } k = 9, \\ i_{\mathbb{H}*}E\{\alpha_1(3)\alpha_1(6)\} \cong \mathbb{Z}_3 & \text{for } k = 10; \end{cases} \\ 2. \quad P_k(\mathbb{H}P^2) &= \begin{cases} \{8\gamma_2\} \oplus i_{\mathbb{H}*}E\{\alpha_{1,5}(3)\} \cong 8\mathbb{Z} \oplus \mathbb{Z}_5 & \text{for } k = 11, \\ \{\gamma_2\eta_{11}\} \cong \mathbb{Z}_2 & \text{for } k = 12, \\ \{\gamma_2\eta_{11}^2\} \cong \mathbb{Z}_2 & \text{for } k = 13; \end{cases} \\ 3. \quad P_k(\mathbb{H}P^3) &= \begin{cases} \{6\gamma_3\} \oplus i_{\mathbb{H}*}E\{\mu'^+, \varepsilon_3v_{11}, v'\varepsilon_6\} \\ \cong 6\mathbb{Z} \oplus \mathbb{Z}_{84} \oplus (\mathbb{Z}_2)^2 & \text{for } k = 15, \\ \{\gamma_3\eta_{15}\} \oplus i_{\mathbb{H}*}E\{v'\mu_6, v'\eta_6\varepsilon_7\} \cong (\mathbb{Z}_2)^3 & \text{for } k = 16, \\ \{\gamma_3\eta_{15}^2\} \oplus i_{\mathbb{H}*}E\{v'\eta_6\mu_7\} \cong (\mathbb{Z}_2)^2 & \text{for } k = 17, \\ \{\gamma_3v_{15}, \gamma_3\alpha_1(15)\} \oplus i_{\mathbb{H}*}E\{\varepsilon_3v_{11}^2, \\ \alpha_1(3)\alpha'_3(6), \alpha_{1,5}(3)\alpha_{1,5}(10)\} \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_{30} & \text{for } k = 18, \end{cases} \text{ where} \\ \mu'^+ &= \mu' + \alpha_3(3) + \alpha_{1,7}(3); \\ 4. \quad \text{Let } n \geq 3. \text{ Then:} \end{aligned}$$

$$P_k(\mathbb{H}P^n) = \begin{cases} i_{\mathbb{H}*}E\{\alpha_{1,5}(3)\} \cong \mathbb{Z}_5 & \text{for } k = 11, \\ 0 & \text{for } k = 12, 13, \\ i_{\mathbb{H}*}E\{\varepsilon', \alpha_1(3)\alpha_2(6)\} \cong \mathbb{Z}_{12} & \text{for } k = 14. \end{cases}$$

2.6 Gottlieb Groups of Real Projective Spaces

Hereafter, we set

$$G'_k(\mathbb{F}P^n) = G_k(\mathbb{F}P^n) \cap \gamma_{n*}\pi_k(\mathbb{S}^{d(n+1)-1}).$$

Notice that

$$G'_k(\mathbb{F}P^n) = G_k(\mathbb{F}P^n) \text{ if } \mathbb{F} = \mathbb{R}, \mathbb{C} \text{ and } k \geq d + 1. \quad (2.60)$$

First of all, we show:

Proposition 2.31. $G'_k(\mathbb{F}P^n) \subseteq \gamma_{n*}G_k(\mathbb{S}^{d(n+1)-1}).$

Proof. The real case is just [22, Theorem 6-2]. For any element $\gamma_n\alpha \in G'_k(\mathbb{F}P^n)$, we have $0 = [\gamma_n\alpha, \gamma_n] = \gamma_n \circ [\alpha, \iota_{d(n+1)-1}]$. Since $\gamma_{n*} : \pi_{k+d(n+1)-2}(\mathbb{S}^{d(n+1)-1}) \rightarrow \pi_{k+d(n+1)-2}(\mathbb{F}P^n)$ is a monomorphism, $\alpha \in P_k(\mathbb{S}^{d(n+1)-1}) = G_k(\mathbb{S}^{d(n+1)-1})$. $\square$

Let K be a closed subgroup of a Lie group H and write H/K for the left coset. We recall [39, Theorem II.5] and [81, Example 2.2] which are directly obtained by Lemma 2.1 and the fact that the usual pairing $H \times H/K \rightarrow H/K$ is an associated map for the projection $p : H \rightarrow H/K$:

Lemma 2.32. *Let K be a closed subgroup of a Lie group H and H/K the left coset. Then, the projection $p : H \rightarrow H/K$ is cyclic and $p_*\pi_n(H) \subseteq G_n(H/K)$ for $n \geq 1$.*

Write $i'_{n,\mathbb{F}} : O_{\mathbb{F}}(n-1) \times O_{\mathbb{F}}(1) \hookrightarrow O_{\mathbb{F}}(n)$ for the inclusion map and $p'_{n,\mathbb{F}} : O_{\mathbb{F}}(n) \rightarrow \mathbb{F}P^{n-1}$ for the quotient one. Now, we consider the exact sequence induced by the fibration $O_{\mathbb{F}}(n+1) \xrightarrow{O_{\mathbb{F}}(n) \times O_{\mathbb{F}}(1)} \mathbb{F}P^n$:

$$(\mathcal{F}\mathcal{P}_k^n) \cdots \rightarrow \pi_k(\mathbb{F}P^n) \xrightarrow{\Delta'_{\mathbb{F}}} \pi_{k-1}(O_{\mathbb{F}}(n) \times O_{\mathbb{F}}(1)) \xrightarrow{i'_*} \pi_{k-1}(O_{\mathbb{F}}(n+1)) \xrightarrow{p'_*} \cdots,$$

where $i' = i'_{n+1,\mathbb{F}}$ and $p' = p'_{n+1,\mathbb{F}}$. Then, by Lemma 2.32, we have

$$\text{Ker } \Delta'_{\mathbb{F}} = p'_*\pi_k(O_{\mathbb{F}}(n+1)) \subseteq G_k(\mathbb{K}P^n). \quad (2.61)$$

Next, we consider the natural map from $(\mathcal{S}\mathcal{F}_k^n)$ to $(\mathcal{F}\mathcal{P}_k^n)$:

$$\begin{array}{ccccc} \pi_k(SO_{\mathbb{F}}(n+1)) & \xrightarrow{p_*} & \pi_k(\mathbb{S}^{d(n+1)-1}) & \xrightarrow{\Delta_{\mathbb{F}}} & \pi_{k-1}(SO_{\mathbb{F}}(n)) \\ \parallel & & \downarrow \gamma_{n*} & & \downarrow \cap \\ \pi_k(O_{\mathbb{F}}(n+1)) & \xrightarrow{p'_*} & \pi_k(\mathbb{F}P^n) & \xrightarrow{\Delta'_{\mathbb{F}}} & \pi_{k-1}(O_{\mathbb{F}}(n) \times O_{\mathbb{F}}(1)). \end{array} \quad (2.62)$$

We show a key lemma determining $G_k(\mathbb{F}P^n)$:

- Lemma 2.33.** 1. $\text{Ker}\{\Delta_{\mathbb{F}} : \pi_k(\mathbb{S}^{d(n+1)-1}) \rightarrow \pi_{k-1}(SO_{\mathbb{F}}(n))\} \subseteq \gamma_n^{-1} G_k(\mathbb{F}P^n)$.
In particular, it holds $\gamma_n \pi_k(\mathbb{S}^{d(n+1)-1}; p) \subseteq G_k(\mathbb{F}P^n)$ provided $\pi_{k-1}(SO_{\mathbb{F}}(n); p) = 0$ for a prime p .
2. Let $k \geq d + 1$. If $E^{d-1} \circ J_{\mathbb{F}} \mid_{\Delta_{\mathbb{F}}(\pi_k(\mathbb{S}^{d(n+1)-1}))} : \Delta_{\mathbb{F}} \pi_k(\mathbb{S}^{d(n+1)-1}) \rightarrow \pi_{k+d(n+1)-2}(\mathbb{S}^{d(n+1)-1})$ is a monomorphism then $\gamma_n \star G_k(\mathbb{S}^{d(n+1)-1}) \subseteq G_k(\mathbb{F}P^n)$. In particular, under the assumption,

$$G_k(\mathbb{F}P^n) = \gamma_n \star G_k(\mathbb{S}^{d(n+1)-1}) \text{ for } \mathbb{F} = \mathbb{R}, \mathbb{C} \text{ and } G'_k(\mathbb{H}P^n) = \gamma_n \star G_k(\mathbb{S}^{4n+3}).$$

Proof. By the commutativity of the right square of the diagram (2.62), $\text{Ker } \Delta_{\mathbb{F}} = \gamma_n^{-1}(\text{Ker } \Delta'_{\mathbb{F}})$. Hence, (2.61) implies (1).

(2) By the assumption, (1.1), (1.49), and Proposition 1.5 we get $G_k(\mathbb{S}^{d(n+1)-1}) = \text{Ker}(J \circ \Delta) = \text{Ker } \Delta_{\mathbb{F}}$. So, (1) leads to $\gamma_n \star G_k(\mathbb{S}^{d(n+1)-1}) \subseteq G_k(\mathbb{F}P^n)$ and, consequently, we obtain $\gamma_n \star G_k(\mathbb{S}^{d(n+1)-1}) \subseteq G'_k(\mathbb{F}P^n)$. On the other hand, Proposition 2.31 yields $G'_k(\mathbb{F}P^n) \subseteq \gamma_n \star G_k(\mathbb{S}^{d(n+1)-1})$. This and (2.60) complete the proof. $\square$

By Proposition 2.31, Lemma 2.33(1), and the fact that

$$\pi_{k-1}(SO(2)) = 0 \text{ for } k \geq 3, \quad (2.63)$$

we obtain:

Corollary 2.34. $\text{Ker}\{\Delta : \pi_k(\mathbb{S}^n) \rightarrow \pi_{k-1}(SO(n))\} \subseteq \gamma_n^{-1} G_k(\mathbb{R}P^n) \subseteq G_k(\mathbb{S}^n)$.

In particular:

1. $G_k(\mathbb{R}P^n) = \pi_k(\mathbb{R}P^n)$ provided that $\Delta \pi_k(\mathbb{S}^n) = 0$ for $k \geq 2$;
2. $G_k(\mathbb{S}^2) = \pi_k(\mathbb{S}^2)$ and $G_k(\mathbb{R}P^2) = \pi_k(\mathbb{R}P^2)$ for $k \geq 3$.

By [21], we know:

Theorem 2.35 (Gottlieb). $G_1(\mathbb{R}P^n) = \begin{cases} 0 & \text{for even } n; \\ \pi_1(\mathbb{R}P^n) & \text{for odd } n. \end{cases}$

In view of (1.3) and $2\pi_{n-1}(SO(n)) = 0$ for n odd [10], the relation (1.52) and Corollary 2.34 yield the result [77]:

Theorem 2.36 (Pak-Woo).

$$G_n(\mathbb{R}P^n) = \gamma_n \star G_n(\mathbb{S}^n) = \begin{cases} 0 & \text{for even } n; \\ \pi_n(\mathbb{R}P^n) & \text{for } n = 1, 3, 7; \\ 2\pi_n(\mathbb{R}P^n) & \text{for odd } n \text{ and } n \neq 1, 3, 7. \end{cases}$$

Notice that Example 2.13(1), Theorems 2.19 and 2.35, and (2.36) imply $G_k(\mathbb{R}P^n) = P_k(\mathbb{R}P^n)$ for $k = 1, n$.

Hereafter, a $\Delta_{\mathbb{F}}$ -cyclic element is simply called $\mathbb{F}$ -cyclic.

Now, we examine the equality

$$(*)_{\alpha} \quad \sharp \Delta \alpha = \sharp [\iota_n, \alpha] \text{ for } \alpha \in \pi_{n+k}(\mathbb{S}^n).$$

Notice that (1.1) and Proposition 1.5 implies the divisibility $\sharp[l_n, \alpha] \mid \sharp\Delta\alpha$ for $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ and the J -homomorphism restricted to $\{\Delta\alpha\} \subseteq \pi_{k-1}(SO(n))$ is a monomorphism if and only if the equality $(*)_\alpha$ holds.

Lemma 2.37. 1. If $\alpha \in \pi_{2n+k}(\mathbb{S}^{2n})$ then $2\Delta E\alpha = 0$.

2. The equality $(*)_\alpha$ holds for $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ provided that:

- (i) α is $\mathbb{R}$ -cyclic;
- (ii) $\sharp\alpha = \sharp[l_n, \alpha]$. In particular, for $[l_n, \alpha] \neq 0$ with $\sharp\alpha = 2$.

3. The equality $(*)_{E\alpha}$ holds for $\alpha \in \pi_{2n+k}(\mathbb{S}^{2n})$ with $[l_{2n+1}, E\alpha] \neq 0$.

4. Let p be an odd prime. The equality $(*)_{E\alpha}$ holds for $\alpha \in \pi_{2n+k-1}(\mathbb{S}^{2n-1}; p)$ with $\sharp\alpha = \sharp E^{2n}\alpha$ for p an odd prime.

Proof. 1. Let $j_n : \mathbb{R}P^{n-1} \rightarrow SO(n)$ be the reflection map. Because $q_n = p_{n+1} \circ j_{n+1}$, the relation (2.6) yields $2\iota_{2n+1} = q_{2n+1} \circ \gamma_{2n+1} = p_{2n+2} \circ j_{2n+2} \circ \gamma_{2n+1}$. This leads to $\Delta(2\iota_{2n+1}) \in \Delta p_{2n+2*} \pi_{2n+1}(SO(2n+2)) = 0$.

Now, given $\alpha \in \pi_{2n+k}(\mathbb{S}^{2n})$, the relations (2.9) yield that $2\Delta E\alpha = \Delta(2E\alpha) = \Delta((2\iota_{2n+1})E\alpha) = \Delta(2\iota_{2n+1})\alpha = 0$.

2. (i);(ii) follows directly from (1.1) and Proposition 1.5.

3. In view of (2.2) and (1.2), we obtain $2[l_{2n+1}, E\alpha] = 0$ for $\alpha \in \pi_{2n+k}(\mathbb{S}^{2n})$ and (1) leads to $\sharp\Delta E\alpha = \sharp[l_{2n+1}, E\alpha] = 2$.

4. Given $\alpha \in \pi_{2n+k-1}(\mathbb{S}^{2n-1}; p)$, owing to Lemma 1.2(2), we get $[l_{2n}, E\alpha] = [l_{2n}, \iota_{2n}] \circ E^{2n}\alpha$. Serre's isomorphism (1.7) leads to $\sharp[l_{2n}, E\alpha] = \sharp E^{2n}\alpha$ and, by $\sharp\alpha = \sharp E^{2n}\alpha$, (1.1), and Proposition 1.5, the result follows. $\square$

In particular, Lemma 2.37(4) leads to:

Corollary 2.38. The equality $(*)_{E\alpha}$ holds for α equal to $\alpha_1(2m+1) \in \pi_{2m+4}(\mathbb{S}^{2m+1}; 3)$, $\alpha_2(2m+1) \in \pi_{2m+8}(\mathbb{S}^{2m+1}; 3)$, $\alpha_{1,5}(2m+1) \in \pi_{2m+8}(\mathbb{S}^{2m+1}; 5)$ for $m \geq 1$, $\alpha_{1,7}(2m+1) \in \pi_{2m+12}(\mathbb{S}^{2m+1}; 7)$ for $m \geq 1$, $\alpha'_3(2m+1) \in \pi_{2m+12}(\mathbb{S}^{2m+1}; 3)$ for $m \geq 2$, or $\beta_1(2m+1) \in \pi_{2m+11}(\mathbb{S}^{2m+1}; 3)$ for $m \geq 3$.

To state the next result, we show:

Proposition 2.39. 1. $\sharp\Delta\iota_n = \begin{cases} 1 & \text{for } n = 1, 3, 7, \\ 2 & \text{for odd } n \text{ and } n \neq 1, 3, 7, \\ \infty & \text{for even } n; \end{cases}$

2. $\sharp\Delta E\nu' = 4$;

3. $\sharp\Delta\nu_n = \begin{cases} 1 & \text{for } n \equiv 7 \pmod{8} \text{ or } n = 5, \\ 2 & \text{for } n \equiv 1, 3, 5 \pmod{8} \text{ with } n \geq 9, \\ 4 & \text{for } n \equiv 2 \pmod{4} \text{ with } n \geq 6 \text{ or } n = 4, 12, \\ 8 & \text{for } n \equiv 0 \pmod{4} \text{ with } n \geq 8 \text{ unless } n = 12; \end{cases}$

4. $\sharp\Delta(\nu_4\eta_7) = 2$;

5. $\sharp\Delta(E\nu')\eta_7 = 2$;

6. $\sharp\Delta(\nu_5\eta_8) = 1$;

7. $\sharp\Delta(\nu_4\eta_7^2) = 2$;

8. $\sharp\Delta(E\nu')\eta_7^2 = 2$;

9. $\sharp\Delta(v_5\eta_8^2) = 1$;
10. $\sharp\Delta[\iota_6, \iota_6] = 30$;
11. $\sharp\Delta v_n^2 = \begin{cases} 1 & \text{for } n \equiv 4, 5, 7 \pmod{8}, \\ 2 & \text{for otherwise;} \end{cases}$
12. $\sharp\Delta\sigma''' = 1$;
13. $\sharp\Delta\sigma'' = 4$;
14. $\sharp\Delta E\sigma' = 8$;
15. $\sharp\Delta\sigma_n = \begin{cases} 16 & \text{for } n \equiv 0 \pmod{2} \text{ with } n \geq 10, \\ 8 & \text{for } n = 8, \\ 1 & \text{for } n \equiv 15 \pmod{16}, \\ 2 & \text{for } n \equiv 1 \pmod{2} \text{ with } n \geq 9, n \not\equiv 15 \pmod{16}. \end{cases}$

Proof. 1. By Lemma 2.37(2)-(ii), we have

$$\sharp\Delta\iota_n = \begin{cases} 2 & \text{for odd } n \text{ and } n \neq 1, 3, 7, \\ \infty & \text{for even } n. \end{cases}$$

2. First, in the exact sequence $(\mathcal{R}_3^4)$:

$$\pi_4(\mathbb{S}^4) \xrightarrow{\Delta} \pi_3(SO(4)) \xrightarrow{i_*} \pi_3(SO(5)) \rightarrow 0,$$

by [82, Theorem 23.6],

$$\pm \Delta\iota_4 = 2[\iota_3] - [\eta_2]_4. \quad (2.64)$$

Next, recall [51] that there is an isomorphism

$$\pi_k(\mathbb{S}^3) \oplus \pi_{k-1}(SO(3)) \xrightarrow{\cong} \pi_k(SO(4)) \quad (2.65)$$

given by the correspondence $(\alpha, \beta) \mapsto [\iota_3]\alpha + [\eta_2]_4\beta$.

Note that (1.20) is regarded as the J -image of (2.64).

This, in view of (2.9), implies $\pm\Delta E v' = \pm\Delta\iota_4 \circ v' = 2[\iota_3]v' - [\eta_2]_4v'$ and hence, $\sharp\Delta E v' = 4$.

3. We recall that

$$\pi_7(SO(5)) \cong \mathbb{Z}, \quad (2.66)$$

$\pi_k(SO(5)) = 0$ for $k = 8, 9$ and $J(\Delta v_4) = \pm[\iota_4, v_4] = \pm 2v_4^2$ by (1.27)(4) and Proposition 1.5. So, from the sequence $(\mathcal{R}_6^4)$, we get that

$$\Delta v_4 \equiv \pm[\iota_3]v' \pmod{[\eta_2]_4v'}$$

and the equality $\sharp\Delta v_4 = 4$ holds.

By (2.66), we have

$$v_5 \text{ is } \mathbb{R}\text{-cyclic}. \quad (2.67)$$

Moreover, [63, Theorem (4.3.2)] yields that

$$v_{8n-1} \text{ is } \mathbb{R}\text{-cyclic for } n \geq 1. \quad (2.68)$$

In the sequence $(\mathcal{R}_{8n-1}^{8n-3})$ for $n \geq 2$:

$$\pi_{8n}^{8n-3} \xrightarrow{\Delta} R_{8n-1}^{8n-3} \xrightarrow{i_*} R_{8n-1}^{8n-2},$$

we know $R_{8n-1}^{8n-3} \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $R_{8n-1}^{8n-2} \cong \mathbb{Z}$ [37, p. 161, Table]. So, we obtain

$$\sharp \Delta v_{8n-3} = 2.$$

By (1.31) and Lemma 2.37(1), we also get that $\sharp \Delta v_n = 2$ for $n \equiv 1, 3 \pmod{8}$ with $n \geq 9$. In view of $(\mathcal{R}_{2n+2}^{2n})$, [37, Table, p. 161], and the fact that $\pi_{14}(SO(12)) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{24}$ [46, App. A, Table VII, Topology, p. 1745], it holds

$$\sharp \Delta v_{2n} = \begin{cases} 8 & \text{for } n \text{ even and } n \geq 4; \\ 4 & \text{for } n \text{ odd and } n \geq 3 \text{ or } n = 6. \end{cases}$$

- 4–9. By the fact that $\pi_k(SO(5)) = 0$ for $k = 8, 9$ [37, Table, p. 162] and (2.65), the map $\Delta : \pi_k(\mathbb{S}^4) \rightarrow \pi_{k-1}(SO(4)) \cong \pi_{k-1}(SO(3)) \oplus \pi_{k-1}(\mathbb{S}^3) \cong (\mathbb{Z}_2)^2$ is an isomorphism and $\Delta : \pi_{k+1}(\mathbb{S}^5) \rightarrow \pi_k(SO(5))$ is trivial. This implies $\sharp \Delta(v_4 \eta_7) = \sharp \Delta E(v' \eta_6) = \sharp \Delta(v_4 \eta_7^2) = \sharp \Delta E(v' \eta_6^2) = 2$ and $\sharp \Delta(v_5 \eta_7) = \sharp \Delta(v_5 \eta_8^2) = 1$.
10. By $(\mathcal{R}_{10}^6)$ and the fact that $\pi_{10}(SO(6)) \cong \mathbb{Z}_{120} \oplus \mathbb{Z}_2$, $\pi_{10}(SO(7)) \cong \mathbb{Z}_8$ [37, Table, p. 162], and $\pi_{10}(\mathbb{S}^6) = 0$, we obtain $\sharp \Delta[\iota_6, \iota_6] = 30$.
11. In view of (2.68), we have that v_{8n-1}^2 is $\mathbb{R}$ -cyclic for $n \geq 1$ and, by [20, Lemma 3.1], v_{8n-k}^2 are $\mathbb{R}$ -cyclic for $n \geq 1$ and $k = 3, 4$. Notice that Theorem 1.14 yields

$$\sharp \Delta v_n^2 = 2 \text{ for } n \not\equiv 4, 5, 7 \pmod{8} \text{ and } n \neq 2^i - 5 \text{ with } i \geq 4.$$

For $n = 2^i - 5$ with $i \geq 4$, we consider the commutative diagram

$$\begin{array}{ccccccc} & & & \pi_{n+6}(\mathbb{S}^n) & & & \\ & & & \downarrow i'_* & \searrow \Delta & & \\ \pi_{n+6}(SO(n)) & \xrightarrow{i_{n,n+3}*} & \pi_{n+6}(SO(n+3)) & \xrightarrow{p_*} & \pi_{n+6}(V_{n+3,3}) & \xrightarrow{\Delta'} & \pi_{n+5}(SO(n)) \\ & & & & \downarrow p'_* & & \\ & & & & \pi_{n+6}(V_{n+3,2}), & & \end{array} \quad (2.69)$$

where the horizontal and vertical sequences are exact.

By [37, Table, p. 161], $\pi_{n+6}(SO(n+3)) \cong \mathbb{Z}_2$ for $n \geq 11$. So, in view of [14], we get $i_{n+3, \infty*} : \pi_{n+6}(SO(n+3)) \rightarrow \pi_{n+6}(SO)$ is an isomorphism. Hence, from the fact that the generator of $\pi_{n+6}(SO)$ has the $SO(6)$ -of-origin [16, Theorem 1.1], we see that $i_{n, n+3*}$ is an epimorphism.

Denote by $\mathbb{R}P_k^n = \mathbb{R}P^n / \mathbb{R}P^{k-1}$ for $k \leq n$ the stunted real projective space. We obtain $\pi_{n+6}(V_{n+3,3}) \cong \pi_{n+6}(\mathbb{R}P_n^{n+2}) \cong \pi_9^8(\mathbb{R}P_3^5) \cong \pi_9^8(\mathbb{R}P_3^4) \cong \mathbb{Z}_2$ and $\pi_{n+6}(V_{n+3,2}) \cong \pi_{n+6}(\mathbb{S}^{n+1} \vee \mathbb{S}^{n+2}) = 0$. Hence, by the diagram (2.69) $\Delta v_n^2 \neq 0$ and so $\sharp \Delta v_n^2 = 2$.

12. By the relation (2.67) the element v_5 is $\mathbb{R}$ -cyclic and $\pi_8(SO(5)) = 0$ [37, Table, p. 162]. Then, (1.17)(3) and Proposition 1.12 lead to

$$\Delta(\sigma''') \in \Delta\{v_5, 8\iota_8, v_8\}_1 \subseteq \{0, 8\iota_7, v_7\} = \pi_8(SO(5)) \circ v_8 = 0.$$

13. The equation $\sharp \Delta \sigma'' = 4$ follows from (1.39).

14. First, we need the formula

$$\pm \Delta \iota_8 = 2[\iota_7] - [\eta_6]_8. \quad (2.70)$$

Here, we use the fact that $\pi_7(SO(7)) = \{[\eta_6]\} \cong \mathbb{Z}$ and $\pi_7(SO(8)) \cong \pi_7(\mathbb{S}^7) \oplus \pi_7(SO(7)) \cong \mathbb{Z}^2$ is generated by $[\iota_7]$ and $[\eta_6]_8$ [35, Table 2]. Next, using [35, Table 2], [36, Lemma 1.2(i)], and (2.70), we obtain $\sharp \Delta E\sigma' = 8$.

15. The relation (1.50) and Lemma 2.37(2)(ii) lead to

$$\sharp \Delta \sigma_n = 16 \text{ for } n \equiv 0 \pmod{2} \text{ with } n \geq 10.$$

Next, consider the exact sequence

$$(\mathcal{SR}_{14}^8) \cdots \longrightarrow \pi_{15}(\mathbb{S}^8) \xrightarrow{\Delta} \pi_{14}(SO(8)) \xrightarrow{i_*} \pi_{14}(SO(9)) \xrightarrow{p_*} \cdots$$

Then, applying [35, Table 2] and [36, Lemma 1.2(ii)], we get $\sharp \Delta \sigma_8 = 8$.

By [36, Theorem 1(iii) and Lemma 1.1], it holds $\sharp \Delta \sigma_{11} = 2$. In view of (1.50) and Lemma 2.37(3), we also get that $\sharp \Delta \sigma_n = 2$ for $n \equiv 1 \pmod{2}$ with $n \geq 9$ and $n \neq 11$. Moreover, [63, Theorem (4.3.2)] yields that

$$\sigma_{16n-1} \text{ is } \mathbb{R}\text{-cyclic for } n \geq 1$$

and the proof is complete. $\square$

Now, we can derive:

Corollary 2.40. *If $k \leq 7$ and $\alpha \in \pi_{n+k}^n$ then:*

1. *the equality $(*)_\alpha$ holds except $\alpha = Ev', v_{2i-3}$ with $i \geq 4$, $(Ev')\eta_7$, $v_4\eta_7$, $(Ev')\eta_7^2$, $v_4\eta_7^2$, $[\iota_6, \iota_6]$, v_{2i-5}^2 with $i \geq 4$, $E\sigma'$, σ_8 , and σ_{11} ;*
2. *$\sharp \Delta \alpha = 2\sharp[\iota_n, \alpha]$ for $\alpha = Ev', v_{2i-3}$ with $i \geq 4$, $(Ev')\eta_7$, $v_4\eta_7$, $(Ev')\eta_7^2$, $v_4\eta_7^2$, v_{2i-5}^2 with $i \geq 4$, $E\sigma'$, and σ_{11} ;*
3. *$\sharp \Delta[\iota_6, \iota_6] = 10\sharp[\iota_6, [\iota_6, \iota_6]]$;*
4. *$\sharp \Delta \sigma_8^+ = 21\sharp[\iota_8, \sigma_8^+]$.*

Proof. 1. By (1.53), (1.55), (1.2)–(1.16), and Proposition 2.39(1), the equality $(*)_\alpha$ holds for $\alpha = \iota_n, \eta_n, \eta_n^2$.

The relation (1.31) and Proposition 2.39(3) yield that the equality $(*)_{v_n}$ holds for $n \neq 2^i - 3$ with $i \geq 4$.

Proposition 2.39(5),(8) leads to the equality $(*)_\alpha$ for $\alpha = v_5\eta_8, v_5\eta_8^2$.

In view of Theorem 1.14 and Proposition 2.39(10), the equality $(*)_{v_n^2}$ holds for $n \neq 2^i - 5$ with $i \geq 4$.

Lemma 2.37(2)(i)–(ii) and Proposition 2.39(11)–(12) lead to the equalities $(*)_{\sigma'''} and $(*)_{\sigma''}$, respectively.$

By means of (1.50) and Proposition 2.39(15), the equality $(*)_{\sigma_n}$ holds for $n \neq 8, 11$.

2. The relation (1.29) and Proposition 2.39(2) yield $\sharp\Delta E v' = 2\sharp[\iota_4, E v'] = 4$.

In view of (1.31) and Proposition 2.39(3), we get $\sharp\Delta v_{2^i-3} = 2\sharp[\iota_{2^i-3}, v_{2^i-3}] = 2$ with $i \geq 4$.

By (2.2), (1.25), and (1.23), we obtain $[\iota_4, v_4\eta_7] = [\iota_4, (E v')\eta_7] = 0$ and $[\iota_4, (E v')\eta_7^2] = [\iota_4, v_4\eta_7^2] = 0$. Then, Proposition 2.39(4)–(5),(7)–(8) leads to:

$$\begin{aligned}\sharp\Delta(v_4\eta_7) &= 2\sharp[\iota_4, v_4\eta_7] = 2, \\ \sharp\Delta(E v')\eta_7 &= 2\sharp[\iota_4, (E v')\eta_7] = 2, \\ \sharp\Delta(v_4\eta_7^2) &= 2\sharp[\iota_4, v_4\eta_7^2] = 2, \\ \sharp\Delta(E v')\eta_7^2 &= 2\sharp[\iota_4, (E v')\eta_7^2] = 2.\end{aligned}$$

In view of Theorem 1.14 and Proposition 2.39(11), it holds $\sharp\Delta v_{2^i-5} = 2\sharp[\iota_{2^i-5}, v_{2^i-5}] = 2$ with $i \geq 4$.

The relation (1.42) and Proposition 2.39(14) yield $\sharp\Delta E \sigma' = 2\sharp[\iota_8, E \sigma'] = 8$.

By means of (1.50) and Proposition 2.39(15), it holds $\sharp\Delta \sigma_{11} = 2\sharp[\iota_{11}, \sigma_{11}] = 2$.

3. In view of the relation (1.4) and Proposition 2.39(10), we obtain $\sharp\Delta[\iota_6, \iota_6] = 10\sharp[\iota_6, [\iota_6, \iota_6]] = 30$.

4. Consider the exact sequence

$$(\mathcal{SR}_{14}^8) \cdots \longrightarrow \pi_{15}(\mathbb{S}^8) \xrightarrow{\Delta} \pi_{14}(SO(8)) \xrightarrow{i_*} \pi_{14}(SO(9)) \xrightarrow{p_*} \cdots$$

In virtue of [51, p. 132, Table], $\pi_{14}(SO(7)) \cong \mathbb{Z}_{2520} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$, $\pi_{14}(SO(8)) \cong \pi_{14}(SO(7)) \oplus \pi_{14}(\mathbb{S}^7) \cong \mathbb{Z}_{2520} \oplus \mathbb{Z}_{120} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$, and $\pi_{14}(SO(9)) \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2$. Because $\Delta v_8^2 \neq 0$ Proposition 2.39(13), we conclude that $i_* : \pi_{14}(SO(8)) \rightarrow \pi_{14}(SO(9))$ is an epimorphism. Further, by means of Corollary 2.38 and Proposition 2.39(14), the element $\Delta E \sigma'^+$ generates $\mathbb{Z}_{120}$ in $\pi_{14}(SO(8))$.

Consequently, Proposition 2.39(15) yields that $\sharp\Delta \sigma_8^+ = 2520$. Then, the relation (1.51) leads to $\sharp\Delta \sigma_8^+ = 21\sharp[\iota_8, \sigma_8^+]$ and the proof is complete. $\square$

In virtue of Proposition 2.31, $\gamma_{n*}G_k(\mathbb{S}^n)$ is an upper bound of $G_k(\mathbb{R}P^n)$ for $k \geq 2$. Now, we show the following result:

Theorem 2.41. *If $k \leq 7$ then the equality $G_{k+n}(\mathbb{R}P^n) = \gamma_{n*}G_{k+n}(\mathbb{S}^n)$ holds except the following pairs: $(k, n) = (3, 4), (3, 2^i - 3)$ with $i \geq 4, (4, 4), (5, 4), (6, 4), (5, 6), (6, 2^i - 5)$ with $i \geq 5, (7, 8)$ and $(7, 11)$.*

Furthermore:

1. $G_7(\mathbb{R}P^4) \supseteq 12\pi_7(\mathbb{R}P^4)$;
2. $G_{2^i}(\mathbb{R}P^{2^i-3}) \supseteq 2\pi_{2^i}(\mathbb{R}P^{2^i-3})$ for $i \geq 4$;
3. $G_{10}(\mathbb{R}P^4) \supseteq 3\pi_{10}(\mathbb{R}P^4)$;
4. $G_{11}(\mathbb{R}P^6) \supseteq 30\pi_{11}(\mathbb{R}P^6)$;
5. $G_{15}(\mathbb{R}P^8) \supseteq 2520\pi_{15}(\mathbb{R}P^8)$;
6. $G_{18}(\mathbb{R}P^{11}) \supseteq 2\pi_{18}(\mathbb{R}P^{11})$.

Proof. Corollary 2.40 concludes that the restriction map $J|_{\Delta\pi_{n+k}(\mathbb{S}^n)}: \Delta\pi_{n+k}(\mathbb{S}^n) \rightarrow \pi_{k+2n-1}(\mathbb{S}^n)$ is not a monomorphism if and only if $(k, n) = (3, 4), (3, 2^i - 3)$ with $i \geq 4, (4, 4), (5, 4), (6, 4), (5, 6), (6, 2^i - 5)$ with $i \geq 5, (7, 8)$ and $(7, 11)$. In the light of Lemmas 2.2 and 2.33, Proposition 2.16(1), Chap. 1, and [82], we obtain the equality.

By Proposition 2.39(3), we get $\sharp\Delta\nu_4^+ = 12$. This and (2.61) imply (1): $G_7(\mathbb{R}P^4) \supseteq p'_*\pi_7(SO(5)) = 12\pi_7(\mathbb{R}P^4)$.

(2) follows from the fact that $\text{Ker}\{\Delta: \pi_{2^i}(\mathbb{S}^{2^i-3}) \rightarrow \pi_{2^i-1}(SO(2^i-3))\} = 2\pi_{2^i}(\mathbb{S}^{2^i-3})$ (Proposition 2.39(3)).

In view of Proposition 2.39(11), it holds $\sharp\Delta(\nu_4^+)^2 = 3$. This and (2.61) imply (3): $G_{10}(\mathbb{R}P^4) \supseteq p'_*\pi_{10}(SO(5)) = 3\pi_{10}(\mathbb{R}P^4)$. By means of Proposition 2.39(10), we know that $\sharp\Delta[\iota_6, \iota_6] = 30$. Hence, Lemma 2.33(1) yields (4).

By Corollary 2.40(4), it holds

$$\text{Ker}\{\Delta: \pi_{15}(\mathbb{S}^8) \rightarrow \pi_{14}(SO(8))\} = 2520\pi_{15}(\mathbb{S}^8)$$

and this leads to (5).

(6) follows from the fact that $\text{Ker}\{\Delta: \pi_{18}(\mathbb{S}^{11}) \rightarrow \pi_{17}(SO(11))\} = 2\pi_{18}(\mathbb{S}^{11})$ (Proposition 2.39(15)). This completes the proof. $\square$

To state the next result, we recall that

$$\varepsilon_{4n+3} \text{ and } \mu_{4n+3} \text{ are } \mathbb{R}\text{-cyclic [20, Example 3.2].} \quad (2.71)$$

By Lemma 2.37(1), we get

$$\Delta E\pi_{2n+k}(\mathbb{S}^{2n}; p) = 0 \text{ for an odd prime } p. \quad (2.72)$$

Proposition 2.42. $G_{n+k}(\mathbb{R}P^n) = \pi_{n+k}(\mathbb{R}P^n)$ if $n \equiv 3 \pmod{4}$ and $k = 8, 9$ or $n \equiv 3 \pmod{4}$ with $n \geq 7$ and $k = 10$, or $n \equiv 1 \pmod{4}$, $n \equiv 7 \pmod{8}$ with $n \geq 13$ and $k = 11$, or $n \equiv 1 \pmod{2}$ with $n \geq 15$ and $k = 13$.

Proof. By (2.9) and (1.53), we get $\Delta(\eta_n\sigma_{n+1}) = \Delta(\eta_n\mu_{n+1}) = 0$ for $n \equiv 3 \pmod{4}$. This and (2.71) imply $\Delta\pi_{n+k}(\mathbb{S}^n) = \Delta\pi_{n+10}^n = 0$ for $k = 8, 9$ and $n \equiv 3 \pmod{4}$. By (2.72), $\Delta\pi_{n+10}(\mathbb{S}^n; 3) = 0$ for n odd with $n \geq 7$.

Next, by [30, Table 2] and the diagram in [20, p. 402], $\Delta\xi_n = 0$ for $n \equiv 1 \pmod{4}$, $n \equiv 7 \pmod{8}$, and $n \geq 13$. This, Corollary 2.34, and (2.72) imply

$\gamma_n \star \pi_{n+11}(\mathbb{S}^n) \subseteq G_{n+11}(\mathbb{R}P^n)$ for $n \equiv 1 \pmod{4}$, $n \equiv 7 \pmod{8}$ with $n \geq 13$. Because $\pi_{n+13}(\mathbb{S}^n) \cong \mathbb{Z}_3$ for $n \geq 15$, in view of (2.72), we have $\Delta\pi_{n+13}(\mathbb{S}^n) = 0$ for $n \equiv 1 \pmod{2}$ with $n \geq 15$. Then, by means of Corollary 2.34, the proof is complete. $\square$

2.7 Gottlieb Groups of Complex and Quaternionic Projective Spaces

By [23, Theorem 1], we obtain

$$G_2(\mathbb{C}P^n) = 0 \text{ for } n \geq 1$$

proved in [19, 39] and [40] as well.

To consider $G_k(\mathbb{C}P^n)$ for $k > 2$, we recall the group structures of $\pi_{2n+k-1}(SU(n))$ for $0 \leq k \leq 7$ from [13, 14, 37], [46, 47, App. A, Table VII, Topology, pp. 1745–1746], and [48].

$$\pi_{2n-1}(SU(n)) \cong \mathbb{Z}; \quad \pi_{2n}(SU(n)) \cong \mathbb{Z}_n! \text{ for } n \geq 2;$$

$$\pi_{2n+1}(SU(n)) \cong \begin{cases} \mathbb{Z}_2 & \text{for even } n, \\ 0 & \text{for odd } n; \end{cases}$$

$$\pi_{2n+2}(SU(n)) \cong \begin{cases} \mathbb{Z}_{(n+1)!} \oplus \mathbb{Z}_2 & \text{for even } n \geq 4, \\ \mathbb{Z}_{\frac{(n+1)!}{2}} & \text{for odd } n \geq 3; \end{cases}$$

$$\pi_{2n+3}(SU(n)) \cong \begin{cases} \mathbb{Z}_{(24,n)} & \text{for even } n, \\ \mathbb{Z}_{\frac{(24,n+3)}{2}} & \text{for odd } n \geq 3; \end{cases}$$

$$\pi_{2n+4}(SU(n)) \cong \begin{cases} \mathbb{Z}_{\frac{(n+2)!(24,n)}{48}} & \text{for even } n \geq 4, \\ \mathbb{Z}_{\frac{(n+2)!(24,n+3)}{24}} & \text{for odd } n \geq 3; \end{cases}$$

$$\pi_{2n+5}(SU(n)) \cong \pi_{2n+5}(SU(n+1)) \cong \begin{cases} \mathbb{Z}_{\frac{(24,n+4)}{2}} & \text{for even } n, \\ \mathbb{Z}_{(24,n+1)} & \text{for odd } n \geq 3; \end{cases}$$

$$\pi_{2n+6}(SU(n)) \cong \begin{cases} \pi_{2n+6}(SU(n+1)) & \text{for } n \equiv 2, 3 \pmod{4} \text{ with } n \geq 3, \\ \pi_{2n+6}(SU(n+1)) \oplus \mathbb{Z}_2 & \text{for } n \equiv 0, 1 \pmod{4} \text{ with } n \geq 4. \end{cases}$$

Write $\omega_n = \omega_{n,\mathbb{C}}$ and notice that $\Delta_{\mathbb{C}} t_{2n+1} = \omega_n = x j_n E \gamma_{n-1}$ with $(x, n!) = 1$ is a generator of $\pi_{2n}(SU(n))$, where $j_n : E\mathbb{C}P^{n-1} \hookrightarrow SU(n)$ is the canonical inclusion satisfying $p_n j_n = E q_{n-1}$. Further, by (2.6), we get

$$p_n \omega_n = (n-1) \eta_{2n-1}. \quad (2.73)$$

In view of [60, Proposition 6.5 (i)], there exists such $q_n \in \pi_{2n+2}(E\mathbb{C}P^{n-1})$ that

$$(Ei_n'')_{q_n} = \begin{cases} E\gamma_n & \text{for even } n, \\ 2E\gamma_n & \text{for odd } n, \end{cases}$$

where $i_n'' : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ is the inclusion map.

To search a generator of $\pi_{2n+4}(SU(n))$, notice that, by means of (2.73), the result [60, Proposition 7.3 i)] yields the trivial map $i_{n+2*} : \pi_{2n+5}(SU(n+1)) \longrightarrow \pi_{2n+5}(SU(n+2))$. Hence, $i_{n,n+2,*} : \pi_{2n+5}(SU(n)) \longrightarrow \pi_{2n+5}(SU(n+2))$ is trivial as well. Then, the fibration $p_{n+2}'' : SU(n+2) \xrightarrow{SU(n)} SU(n+2)/SU(n)$ leads to a monomorphism $p_{n+2*}'' : \pi_{2n+5}(SU(n+2)) \longrightarrow \pi_{2n+5}(SU(n+2)/SU(n))$. Because, in view of [18], it holds

$$\pi_{2n+5}(SU(n+2)/SU(n)) \cong \begin{cases} \mathbb{Z}_2 & \text{for even } n \geq 4, \\ 0 & \text{for odd } n \geq 3, \end{cases}$$

we deduce that $p_{n+2*}'' : \pi_{2n+5}(SU(n+2)) \longrightarrow \pi_{2n+5}(SU(n+2)/SU(n))$ is an isomorphism for $n \geq 3$. Hence, $i_{n,n+2,*} : \pi_{2n+4}(SU(n)) \longrightarrow \pi_{2n+4}(SU(n+2))$ is a monomorphism. Consequently, there exists such a generator $\omega_n' \in \pi_{2n+4}(SU(n))$ that

$$i_{n,n+2}\omega_n' = \begin{cases} \frac{48}{(24,n)}\omega_{n+2} & \text{for even } n \geq 4, \\ \frac{24}{(24,n+3)}\omega_{n+2} & \text{for odd } n \geq 3. \end{cases}$$

The exact sequences (SC_{2n+k}^n) for $k = 5, 6$ lead to the isomorphism

$$i_* : \pi_{2n+5}(SU(n)) \xrightarrow{\cong} \pi_{2n+5}(SU(n+1))$$

and the epimorphism

$$\pi_{2n+6}(SU(n)) \xrightarrow{i_*} \pi_{2n+6}(SU(n+1)) \rightarrow 0,$$

respectively. Hence, there exist such generators $\omega_n'' \in \pi_{2n+5}(SU(n))$ and $\omega_n''' \in \pi_{2n+6}(SU(n))$ that $i_{n+1}\omega_n'' = \omega_{n+1}v_{2n+2}^+$ and $i_{n+1}\omega_n''' = \omega_{n+1}'$ for $n > 2$.

Making use of the results above and [60, Propositions 3.2, 7.2, and 7.3], we obtain the following result which overlaps with those of [37, 47, pp. 163–165] and [48]:

Lemma 2.43.

$$\pi_{2n+k}(SU(n)) = \begin{cases} \{\omega_n\} \cong \mathbb{Z}_{n!} & \text{for } k = 0 \text{ and } n \geq 2, \\ \{\omega_n \eta_{2n}\} \cong \mathbb{Z}_2 & \text{for } k = 1 \text{ and even } n, \\ \{j_n Q_n, \omega_n \eta_{2n}^2\} \cong \mathbb{Z}_{(n+1)!} \oplus \mathbb{Z}_2 & \text{for } k = 2 \text{ and even } n \geq 4, \\ \{j_n Q_n\} \cong \mathbb{Z}_{\frac{(n+1)!}{2}} & \text{for } k = 2 \text{ and odd } n \geq 3, \\ \{\omega_n v_{2n}^+\} \cong \mathbb{Z}_{(24,n)} & \text{for } k = 3 \text{ and even } n, \\ \{\omega_n v_{2n}^+\} \cong \mathbb{Z}_{\frac{(24,n+3)}{2}} & \text{for } k = 3 \text{ and odd } n \geq 3, \\ \{\omega'_n\} \cong \mathbb{Z}_{\frac{(n+2)!(24,n)}{48}} & \text{for } k = 4 \text{ and even } n, \\ \{\omega'_n\} \cong \mathbb{Z}_{\frac{(n+2)!(24,n+3)}{24}} & \text{for } k = 4 \text{ and odd } n \geq 3, \\ \{\omega''_n\} \cong \mathbb{Z}_{\frac{(24,n+4)}{2}} & \text{for } k = 5 \text{ and even } n \geq 4, \\ \{\omega''_n\} \cong \mathbb{Z}_{(24,n+1)} & \text{for } k = 5 \text{ and odd } n \geq 3, \end{cases}$$

$$\begin{cases} \{\omega'''_n, \omega_n v_{2n}^2\} \cong \mathbb{Z}_{\frac{(n+3)!(24,n+4)}{24}} \oplus \mathbb{Z}_2 & \text{for } k = 6 \text{ and } n \equiv 0 \pmod{4} \text{ with } n \geq 4, \\ \{\omega'''_n, \omega_n v_{2n}^2\} \cong \mathbb{Z}_{\frac{(n+3)!(24,n+1)}{48}} \oplus \mathbb{Z}_2 & \text{for } k = 6 \text{ and } n \equiv 1 \pmod{4} \text{ with } n \geq 5, \\ \{\omega'''_n\} \cong \mathbb{Z}_{\frac{(n+3)!(24,n+4)}{24}} & \text{for } k = 6 \text{ and } n \equiv 2 \pmod{4} \text{ with } n \geq 4, \\ \{\omega'''_n\} \cong \mathbb{Z}_{\frac{(n+3)!(24,n+1)}{48}} & \text{for } k = 6 \text{ and } n \equiv 3 \pmod{4} \text{ with } n \geq 3. \end{cases}$$

Further:

1. $\#\Delta_{\mathbb{C}} t_{2n+1} = n!$;
2. $\#\Delta_{\mathbb{C}} \eta_{2n+1} = \begin{cases} 2 & \text{for even } n, \\ 1 & \text{for odd } n; \end{cases}$
3. $\#\Delta_{\mathbb{C}} \eta_{2n+1}^2 = \begin{cases} 2 & \text{for even } n, \\ 1 & \text{for odd } n; \end{cases}$
4. $\#\Delta_{\mathbb{C}} v_{2n+1}^+ = \begin{cases} (24, n) & \text{for even } n, \\ (24, n+3)/2 & \text{for odd } n \geq 3; \end{cases}$
5. $\#\Delta_{\mathbb{C}} v_{2n+1}^2 = \begin{cases} 1 & \text{for } n \equiv 2, 3 \pmod{4} \text{ and } n \geq 2, \\ 2 & \text{for } n \equiv 0, 1 \pmod{4} \text{ and } n \geq 4. \end{cases}$

By Lemmas 2.33(1) and 2.43(1), we obtain [39, Theorem III.8]:

Theorem 2.44 (Lang). $n! \pi_{2n+1}(\mathbb{C}P^n) \subseteq G_{2n+1}(\mathbb{C}P^n)$.

By Theorems 2.20(1) and 2.44, it holds

$$G_5(\mathbb{C}P^2) = P_5(\mathbb{C}P^2) = 2\pi_5(\mathbb{C}P^2).$$

In view of Lemma 2.43(2),(4), it holds:

$$\begin{aligned} (1) \quad & \eta_{4n+3} \text{ and} \\ (2) \quad & (12, n+2)v_{4n+3}^+ \end{aligned} \tag{2.74}$$

are $\mathbb{C}$ -cyclic.

Now, we show:

Theorem 2.45. 1. Let $k = 1, 2$. Then

$$G_{k+2n+1}(\mathbb{C}P^n) = \begin{cases} 0 & \text{for even } n, \\ \pi_{2n+k+1}(\mathbb{C}P^n) \cong \mathbb{Z}_2 & \text{for odd } n. \end{cases}$$

$$2. G_{2n+4}(\mathbb{C}P^n) \supseteq \begin{cases} (24, n)\pi_{2n+4}(\mathbb{C}P^n) \cong \mathbb{Z}_{\frac{24}{(24, n)}} & \text{for even } n, \\ \frac{(24, n+3)}{2}\pi_{2n+4}(\mathbb{C}P^n) \cong \mathbb{Z}_{\frac{48}{(24, n+3)}} & \text{for odd } n. \end{cases}$$

In particular, $G_{2n+4}(\mathbb{C}P^n) = 2\pi_{2n+4}(\mathbb{C}P^n)$ for $n \equiv 2, 10 \pmod{12}$ with $n \geq 10$ except $n = 2^{i-1} - 2$ or $n \equiv 1, 17 \pmod{24}$ for $n \geq 17$ and $G_{2n+4}(\mathbb{C}P^n) = \pi_{2n+4}(\mathbb{C}P^n)$ for $n \equiv 7, 11 \pmod{12}$.

$$3. G_{2n+6}(\mathbb{C}P^n) = \pi_{2n+6}(\mathbb{C}P^n) \cong \begin{cases} 0 & \text{for } n \geq 3, \\ \mathbb{Z}_2 & \text{for } n = 2. \end{cases}$$

$$4. G_{2n+7}(\mathbb{C}P^n) = \pi_{2n+7}(\mathbb{C}P^n) \text{ for } n \equiv 2, 3 \pmod{4}.$$

Proof. By Proposition 2.16(1) and the fact that $G_{k+2n+1}(\mathbb{S}^{2n+1}) = 0$ if $k = 1, 2$ and n is even, we have $G_{k+2n+1}(\mathbb{C}P^n) = 0$ in this case. By Lemmas 2.33(1) and 2.43(2) and (3), $G_{2n+k+1}(\mathbb{C}P^n) = \pi_{2n+k+1}(\mathbb{C}P^n)$ for n odd. This leads to (1).

In view of Corollary 2.34, Lemmas 2.33(1) and 2.43(4) imply the first half of (2).

As it is easily seen (1.31), for n even, $(24, n) = \sharp[\iota_{2n+1}, v_{2n+1}^+] = 2$ if and only if $n \equiv 2, 10 \pmod{12}$ for $n \geq 10$ and $n \neq 2^{i-1} - 2$. For n odd, $\frac{(24, n+3)}{2} = \sharp[\iota_{2n+1}, v_{2n+1}^+] = 2$ if and only if $n \equiv 1, 17 \pmod{24}$ for $n \geq 17$. Moreover, $\frac{(24, n+3)}{2} = \sharp[\iota_{2n+1}, v_{2n+1}^+] = 1$ if and only if $n \equiv 7, 11 \pmod{12}$. This and Lemma 2.33(2) lead to the second half of (2).

Since $\Delta_{\mathbb{C}} : \pi_{10}(\mathbb{S}^5) \rightarrow \pi_9(SU(2)) \cong \mathbb{Z}_3$ is trivial, we have $G_{10}(\mathbb{C}P^2) = \pi_{10}(\mathbb{C}P^2) \cong \mathbb{Z}_2$ and this leads to (3).

(4) follows from Lemmas 2.33(1) and 2.43(5), and the proof is complete. $\square$

Next, we present:

Proposition 2.46. $G_k(\mathbb{C}P^3; 2) = \pi_k(\mathbb{C}P^3; 2)$ for $8 \leq k \leq 24$ unless $k = 15, 21$. Further, $\gamma_{3, \mathbb{C}*}\{\sigma'\eta_{14}, \bar{v}_7 + \varepsilon_7\} \subseteq G_{15}(\mathbb{C}P^3; 2)$ and $\gamma_{3, \mathbb{C}*}\{\sigma'\sigma_{14}\} \subseteq G_{21}(\mathbb{C}P^3; 2)$.

Proof. We make use of Lemma 2.33(1) and show that $\Delta_{\mathbb{C}}|_{\pi_{k+1}^7} : \pi_{k+1}^7 \rightarrow \pi_k(SU(3); 2)$ is trivial for $7 \leq k \leq 22$ unless $k = 15, 21$. First, we recall:

$$\pi_k(\mathbb{S}^7) = \begin{cases} \{\eta_7\} \cong \mathbb{Z}_2 & \text{for } k = 8; \\ \{\eta_7^2\} \cong \mathbb{Z}_2 & \text{for } k = 9; \\ \{v_7^+\} \cong \mathbb{Z}_{24} & \text{for } k = 10; \\ 0 & \text{for } k = 11; \\ 0 & \text{for } k = 12; \\ \{v_7^2\} \cong \mathbb{Z}_2 & \text{for } k = 13; \\ \{\sigma'^+\} \cong \mathbb{Z}_{120} & \text{for } k = 14; \\ \{\sigma'\eta_{14}, \bar{v}_7, \varepsilon_7\} \cong (\mathbb{Z}_2)^3 & \text{for } k = 15; \\ \{\sigma'\eta_{14}^2, v_7^3, \mu_7, \eta_7\varepsilon_8\} \cong (\mathbb{Z}_2)^4 & \text{for } k = 16; \\ \{v_7\sigma_{10}, \eta_7\mu_8, \beta_1(7)\} \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2 & \text{for } k = 17; \\ \{\xi_7^+, \bar{v}_7v_{15}\} \cong \mathbb{Z}_{504} \oplus \mathbb{Z}_2 & \text{for } k = 18; \\ 0 & \text{for } k = 19; \\ \\ \{v_7\sigma_{10}v_{17}, \alpha_1(7)\beta_1(10)\} \cong \mathbb{Z}_6 & \text{for } k = 20; \\ \{\sigma'\sigma_{14}, \kappa_7, \alpha'\} \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_4 & \text{for } k = 21; \\ \{\rho''^+, \sigma'\bar{v}_{14}, \sigma'\varepsilon_{14}, \bar{\varepsilon}_7\} \cong \mathbb{Z}_{120} \oplus (\mathbb{Z}_2)^3 & \text{for } k = 22; \\ \{\sigma'\mu_{14}, E\xi', \mu_7\sigma_{16}, \eta_7\bar{\varepsilon}_8\} \cong (\mathbb{Z}_2)^4 & \text{for } k = 23; \\ \{\sigma'\eta_{14}\mu_{15}, v_7\kappa_{10}, \bar{\mu}_7, \eta_7\mu_8\sigma_{17}\} \cong (\mathbb{Z}_2)^4 & \text{for } k = 24, \end{cases} \quad (2.75)$$

where $E\alpha' = [[\iota_8, \iota_8], \iota_8]$.

Further, we make use of the relation

$$\Delta_{\mathbb{C}} = i_{3, \mathbb{C}*} \Delta_{\mathbb{H}} \quad (2.76)$$

determined by (1.48)(2). Since $\pi_6(Sp(2)) = \{\omega\}$ and $\pi_6(Sp(2)) = 0$ [55, Theorem 5.1], we have $\Delta_{\mathbb{H}}(\iota_7) = x\omega$ for an integer x with $(x, 12) = 1$. Hence, (2.76) yields

$$\Delta_{\mathbb{C}}(\iota_7) = xi_{3, \mathbb{C}*}\omega. \quad (2.77)$$

But, $\pi_4(SU(3)) = 0$ [54, Theorem 4.1] leads to $i_{3, \mathbb{C}*}\eta_3 = 0$. Then, the relations (2.9), (1.33), and (2.77) imply $\Delta_{\mathbb{C}}(\eta_7) = i_{3, \mathbb{C}*}(\omega\eta_6) = (i_{3, \mathbb{C}*}\eta_3)v_4 = 0$. Next, (2.9) and (1.28) lead to $\Delta_{\mathbb{C}}(v_7) = i_{3, \mathbb{C}*}(\omega v_6) = i_{3, \mathbb{C}*}(v'v_6) = 0$. Consequently, in view of (2.9), $\Delta_{\mathbb{C}}(\eta_7 E\alpha) = \Delta_{\mathbb{C}}(v_7 E\beta) = 0$ for any $\alpha \in \pi_n^7$ and $\beta \in \pi_n^9$.

Because $\pi_8(SU(3); 2) \cong \mathbb{Z}_4$ [54, Theorem 4.1], using the fact that $\Delta_{\mathbb{C}}(v_7^3) = 0$, the relations (1.35) and (1.60) and Proposition 1.12 yield $\Delta_{\mathbb{C}}(\mu_7) \in \Delta_{\mathbb{C}}\{\eta_7, 2\iota_8, E^3\sigma'''\}_3 \subseteq \{0, 2\iota_7, E^2\sigma'''\}_2 = \pi_8(SU(3); 2) \circ 4E\sigma' = 0$. Hence, (2.9) implies $\Delta_{\mathbb{C}}(\mu_7 E\delta) = 0$ for any $\delta \in \pi_n^{15}$.

In view of $\pi_{16}(SU(3); 2) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$ [85, Lemma 5.14], (2.31), and Proposition 1.12, we get $\Delta_{\mathbb{C}}(\bar{\mu}_7) \in \Delta_{\mathbb{C}}\{\mu_7, 2\iota_{16}, 8\sigma_{16}\}_5 \subseteq \{0, 2\iota_{15}, 8\sigma_{15}\}_4 = \pi_{16}(SU(3); 2) \circ 8\sigma_{16} = 0$. Further, (2.20) and (2.77) lead to $\Delta_{\mathbb{C}}(\xi_7) = 0$.

Because $\Delta_{\mathbb{H}}(\sigma') = 2\varepsilon'$, $\Delta_{\mathbb{H}}(\kappa_7) = \pm\bar{\varepsilon}'$ and $\Delta_{\mathbb{H}}(\rho'') = 0$ [54, Proposition 3.2 ii)], we apply [54, Theorem 4.1] and (2.76) to get $\Delta_{\mathbb{C}}(\sigma') = 2i_{3, \mathbb{C}*}(\varepsilon') = 0$,

$\Delta_{\mathbb{C}}(\kappa_7) = i_{3,\mathbb{C}*}(\bar{\varepsilon}') \neq 0$, and $\Delta_{\mathbb{C}}(\rho'') = 0$. Further, (2.9) yields $\Delta_{\mathbb{C}}(\sigma'E\gamma) = 0$ for any $\gamma \in \pi_n^{13}$. In view of (1.106), the above leads to $\Delta_{\mathbb{H}}(E\zeta') = v'\zeta' = 2\varepsilon'\eta_{13}\varepsilon_{14} = 0$ and $\Delta_{\mathbb{C}}(E\zeta') = 0$.

Next, (2.9), (1.122)(1), (2.77), and [54, (4.1)] imply $\Delta_{\mathbb{C}}(\varepsilon_7) = \Delta_{\mathbb{C}}(\bar{v}_7) = i_{3,\mathbb{C}*}(v'\varepsilon_6) = i_{3,\mathbb{C}*}(\varepsilon_3 v_{11}) = 2([v_5^2] \circ v_{11}) \neq 0$ and $\Delta_{\mathbb{C}}(\bar{v}_7 + \varepsilon_7) = 0$. Finally, (1.123) leads to $\Delta_{\mathbb{C}}(\bar{v}_7 v_{15}) = i_{3,\mathbb{C}*}(v'\varepsilon_6 v_{14}) = 0$ and the proof follows. $\square$

Then, we obtain:

- Corollary 2.47.** 1. $G_k(\mathbb{C}P^2) = \pi_k(\mathbb{C}P^2)$ for $10 \leq k \leq 12$;
 2. $G_k(\mathbb{C}P^2; p) = \pi_k(\mathbb{C}P^2; p)$ for an odd prime p ;
 3. $G_k(\mathbb{C}P^{2n+1}) \supseteq \gamma_{2n+1}\eta_{4n+3}^m \circ \pi_k^{4n+m+3}$ for $m = 1, 2$;
 4. (i) $G_k(\mathbb{C}P^{2n}) \supseteq 2(12, n)\gamma_{2n}v_{4n+1}^+ \circ \pi_k(\mathbb{S}^{4n+4})$;
 (ii) $G_k(\mathbb{C}P^{2n+1}) \supseteq (12, n+2)\gamma_{2n+1}v_{4n+3}^+ \circ \pi_k(\mathbb{S}^{4n+6})$;
 5. $G_k(\mathbb{C}P^{4n+2}) \supseteq \gamma_{4n+2}v_{8n+5}^2 \circ \pi_k^{8n+11}$;
 6. $G_{8n+11}(\mathbb{C}P^{4n+1}) = \pi_{8n+11}(\mathbb{C}P^{4n+1})$ for $n \geq 2$;
 7. $G_{8n+k}(\mathbb{C}P^{4n+3}) = \pi_{8n+k}(\mathbb{C}P^{4n+3})$ for $k = 28, 29$ with $n \geq 2$.

Proof. By [54, Lemma 3.1(i)], we get $\Delta_{\mathbb{C}}\pi_k(\mathbb{S}^5) = 0$ for $10 \leq k \leq 12$. Hence, Lemma 2.33(1) yields (1).

Corollary 2.2 and Theorem 2.45 yield (2)–(5).

The relations (1.57), (2.9), and (2.74)(1) lead to

$$\begin{aligned} \Delta_{\mathbb{C}}\varepsilon_{4n+3} &= (\Delta_{\mathbb{C}}\iota_{4n+3}) \circ \varepsilon_{4n+2} \in \{\omega_{2n+1}, \eta_{4n+2}, v_{4n+3}\} \circ 2v_{4n+7} \\ &\subseteq \pi_{4n+7}(SU(2n+1)) \circ 2v_{4n+7}. \end{aligned}$$

Then, by the group structure $\pi_{4n+7}(SU(2n+1)) \cong \mathbb{Z}_{2(12,n+1)}$, we obtain that ε_{4n+3} is $\mathbb{C}$ -cyclic for even n . Hence, by (2.9), (1.68), and (2.74), the element $\bar{v}_{4n+3}$ is $\mathbb{C}$ -cyclic for even n as well. Because $\pi_8^s = \{\bar{v}, \varepsilon\} \cong (\mathbb{Z}_2)^2$ [85, Theorem 7.1], this yields (6).

Notice that (2.9) and (2.74) imply the $\mathbb{C}$ -cyclicity of the elements: $\eta_{8n+7}\bar{\kappa}_{8n+8}$, $v_{8n+7}\xi_{8n+10}$, $\eta_{8n+7}^2\bar{\kappa}_{8n+8}$, and $v_{8n+7}\bar{\sigma}_{8n+10}$. Further, recall from [49] that $\pi_{21}^s = \{\eta\bar{\kappa}, \sigma^3\} \cong (\mathbb{Z}_2)^2$ and $\pi_{22}^s = \{\varepsilon\kappa, v\bar{\sigma}\} \cong (\mathbb{Z}_2)^2$. Then, (2.33) and (2.27) lead to (7) for $k = 28, 29$ and $n \geq 2$, and the proof is complete. $\square$

Now, we consider the case of the quaternionic projective space. By [23, Theorem 1], we obtain

$$G_4(\mathbb{H}P^n) = 0 \quad \text{for } n \geq 1$$

proved in [19, 39] and [40] as well.

By Proposition 2.30(1);(4) and Chap. 1,

$$G_k(\mathbb{H}P^n) = 0 \quad \text{for } n \geq 1 \text{ and } k = 5, 6;$$

$$G_k(\mathbb{H}P^n) = 0 \quad \text{for } n \geq 3 \text{ and } k = 12, 13.$$

We recall the group structures $\pi_{4n+k-1}(Sp(n))$ for $-1 \leq k \leq 6$ from [14], [46, App. A, Table VII, Topology, p. 1746], [50], and [55]:

$$\pi_{4n-2}(Sp(n)) = 0; \quad \pi_{4n-1}(Sp(n)) \cong \mathbb{Z};$$

$$\pi_{4n}(Sp(n)) \cong \begin{cases} 0 & \text{for even } n, \\ \mathbb{Z}_2 & \text{for odd } n; \end{cases}$$

$$\pi_{4n+1}(Sp(n)) \cong \begin{cases} 0 & \text{for even } n, \\ \mathbb{Z}_2 & \text{for odd } n; \end{cases}$$

$$\pi_{4n+2}(Sp(n)) \cong \begin{cases} \mathbb{Z}_{(2n+1)!} & \text{for even } n, \\ \mathbb{Z}_{2(2n+1)!} & \text{for odd } n; \end{cases}$$

$$\pi_{4n+3}(Sp(n)) \cong \mathbb{Z}_2;$$

$$\pi_{4n+4}(Sp(n)) \cong \begin{cases} (\mathbb{Z}_2)^2 & \text{for even } n, \\ \mathbb{Z}_2 & \text{for odd } n; \end{cases}$$

$$\pi_{4n+5}(Sp(n)) \cong \begin{cases} \mathbb{Z}_{(24,n+2)} \oplus \mathbb{Z}_2 & \text{for even } n; \\ \mathbb{Z}_{(24,n+2)} & \text{for odd } n. \end{cases}$$

Write θ_{4n-1} for a generator on $\pi_{4n-1}(Sp(n))$. Then, [55] yields

$$p_*(\theta_{4n+3}) = \begin{cases} (2n+1)!l_{4n+3} & \text{for even } n, \\ 2(2n+1)!l_{4n+3} & \text{for odd } n. \end{cases}$$

Denoting $\omega_n = \omega_{n,\mathbb{H}}$ and making use of the exact sequences $(S\mathcal{H}_{4n+k}^n)$ for $k = 2, 3, 4, 5$, (2.9), [50, 55], [85, Proposition 1.8], and the results above, we obtain:

Lemma 2.48.

$$\pi_{4n+k}(Sp(n)) = \begin{cases} \{\theta_{4n-1}\eta_{4n-1}\} \cong \mathbb{Z}_2 & \text{for } k = 0 \text{ and odd } n, \\ \{\theta_{4n-1}\eta_{4n-1}^2\} \cong \mathbb{Z}_2 & \text{for } k = 1 \text{ and odd } n, \\ \{\omega_n\} \cong \mathbb{Z}_{(2n+1)!} & \text{for } k = 2 \text{ and even } n, \\ \{\omega_n\} \cong \mathbb{Z}_{2(2n+1)!} & \text{for } k = 2 \text{ and odd } n, \\ \{\omega_n\eta_{4n+2}\} \cong \mathbb{Z}_2 & \text{for } k = 3 \text{ and any } n, \\ \{\omega_n\eta_{4n+2}^2, \alpha\} \cong (\mathbb{Z}_2)^2 & \text{for } k = 4 \text{ and even } n, \\ \{\omega_n\eta_{4n+2}^2\} \cong \mathbb{Z}_2 & \text{for } k = 4 \text{ and odd } n, \\ \{\omega_nv_{4n+2}^+, \alpha\eta_{4n+4}\} \cong \mathbb{Z}_{(24,n+2)} \oplus \mathbb{Z}_2 & \text{for } k = 5 \text{ and even } n, \\ \{\omega_nv_{4n+2}^+\} \cong \mathbb{Z}_{(24,n+2)} & \text{for } k = 5 \text{ and odd } n, \end{cases}$$

where $\alpha \in -\{\omega_n, (2n+1)!l_{4n+2}, \eta_{4n+2}\}$ with $i_*(\alpha) = \theta_{4n-1} \circ \eta_{4n-1}$.

Further:

1. $\sharp\Delta_{\mathbb{H}\iota_{4n+3}} = \begin{cases} (2n+1)! & \text{for even } n, \\ 2(2n+1)! & \text{for odd } n; \end{cases}$
2. $\sharp\Delta_{\mathbb{H}\eta_{4n+3}^k} = 2$ for $k = 1, 2$;
3. $\sharp\Delta_{\mathbb{H}v_{4n+3}^+} = (24, n+2)$.

By Lemma 2.48(3), we get that

$$(24, n+2)v_{4n+3}^+ \text{ is } \mathbb{H}\text{-cyclic.} \quad (2.78)$$

By Lemmas 2.2, 2.33(1), and 2.48(1);(3), we obtain:

Theorem 2.49. 1. $G_{4n+3}(\mathbb{H}P^n) \supseteq \begin{cases} (2n+1)!\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3}) & \text{for even } n, \\ 2(2n+1)!\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3}) & \text{for odd } n; \end{cases}$
 2. $G_k(\mathbb{H}P^n) \supseteq (24, n+2)\gamma_n v_{4n+3}^+ \circ \pi_k(\mathbb{S}^{4n+6})$. In particular, we derive $G_{4n+6}(\mathbb{H}P^n) \supseteq (24, n+2)\gamma_{n*}\pi_{4n+6}(\mathbb{S}^{4n+3}) \cong \mathbb{Z}_{\frac{24}{(24,n+2)}}$ for $n \geq 2$.

Observe that Theorem 2.49 improves [40, Corollary 2.7]. Namely, we obtain:

Corollary 2.50. 1. $G'_{8n+9}(\mathbb{H}P^{2n}) = 0$ and $G'_{8n+10}(\mathbb{H}P^{2n+1}) = \gamma_{2n+1*}\pi_{8n+10}(\mathbb{S}^{8n+7})$ for $n \not\equiv 0 \pmod{3}$.
 2. $G'_{4n+14}(\mathbb{H}P^n) = \gamma_{n*}\pi_{4n+14}(\mathbb{S}^{4n+3})$ for $n \equiv 5, 9 \pmod{12}$ with $n \geq 5$ and $n \equiv 15, 23 \pmod{24}$.

Proof. (1) is a direct consequence of Theorem 2.25(4) and Theorem 2.49(2).

We know from [56] and [57] that

$$\pi_{4n+13}(Sp(n)) \cong \begin{cases} \mathbb{Z}_8 & \text{for } n \equiv 1 \pmod{4} \text{ and } n \geq 5, \\ \mathbb{Z}_{(128, 4(n-3))} & \text{for } n \equiv 3 \pmod{4}. \end{cases}$$

So, for n satisfying $(24, n+2) = 1$, $n \equiv 1 \pmod{4}$, or $n \equiv 7 \pmod{8}$, in view of (1.138), (2.9), and (2.78), we obtain

$$\begin{aligned} \Delta_{\mathbb{H}}\zeta_{4n+3} &= (\Delta_{\mathbb{H}}\iota_{4n+3}) \circ \zeta_{4n+2} \in \{\omega_n, v_{4n+2}, \sigma_{4n+5}\} \circ 16\iota_{4n+13} \\ &\subseteq \pi_{4n+13}(Sp(n)) \circ 16\iota_{4n+13} = 0. \end{aligned}$$

Because $\pi_{4n+13}(Sp(n); 2) = \pi_{4n+13}(Sp(n))$, Lemma 2.33 leads to (2) and the proof is established. $\square$

By the use of the exact sequence $(\mathcal{SH}_{11}^2)$, Lemma 2.48(2), and the fact that $\pi_{12}(Sp(2)) = \{i_{2, \mathbb{H}}\varepsilon_3\} \cong \mathbb{Z}_2$ [54, Theorem 5.1], we obtain

$$\Delta_{\mathbb{H}}\eta_{11} = i_{2, \mathbb{H}}\varepsilon_3.$$

Then, by the relation (1.71)(2), we get:

Example 2.51. $G_{18+k}(\mathbb{H}P^2) \supseteq \{\gamma_2\eta_{11}^k\sigma_{11+k}\} \cong \mathbb{Z}_2$ for $k = 1, 2$.

The result [33, (2.10)(b)] implies

$$p_n * \omega_n = \pm(n+1)v_{4n-1}^+, \text{ where } p_n = p_{n, \mathbb{H}} \text{ and } n \geq 2. \quad (2.79)$$

Then, we obtain:

Lemma 2.52. *The elements $\eta_{16n-1}\bar{\kappa}_{16n}$ are $\mathbb{H}$ -cyclic for $n \geq 1$.*

Proof. By (2.24) and (2.25), it holds

$$\{v_n^2, 2\iota_{n+6}, \kappa_{n+6}\} \ni \eta_n \bar{\kappa}_{n+1} \pmod{\sigma_n \kappa_{n+7}} \text{ for } n \geq 9.$$

By Lemma 2.48(3), $\sharp(\omega_{4n} v_{16n+2}^+) = 2(12, 2n+1)$. So, we can define the Toda bracket

$$\{\omega_{4n} v_{16n+2}^+, 2(12, 2n+1)\iota_{16n+5}, \kappa_{16n+5}\} \subseteq \pi_{16n+20}(Sp(4n)).$$

In view of (2.24) and the fact that $\pi_{16n+20}^{16n+5} = \{\rho_{16n+5}, \eta_{16n+5}\kappa_{16n+6}\}$, we have $(v_{16n-1}^2)^+ \circ \pi_{16n+20}^{16n+5} = 0$. Hence, by (2.79), we obtain

$$\begin{aligned} p_{4n} * \{\omega_{4n} v_{16n+2}^+, 2(12, 2n+1)\iota_{16n+5}, \kappa_{16n+5}\} &\subseteq \{(v_{16n-1}^2)^+, 2(12, 2n+1) \\ &\iota_{16n+5}, \kappa_{16n+5}\} \ni \eta_{16n-1} \bar{\kappa}_{16n} \pmod{\sigma_{16n-1} \kappa_{16n+6}}. \end{aligned}$$

Then, the relation $\sigma_{16n-1} \kappa_{16n+6} = 0$ for $n \geq 1$ [49, Proposition 7.2] yields

$$p_{4n} * \{\omega_{4n} v_{16n+2}^+, 2(12, 2n+1)\iota_{16n+5}, \kappa_{16n+5}\} = \eta_{16n-1} \bar{\kappa}_{16n}.$$

This completes the proof. $\square$

We show:

Corollary 2.53. $G'_{16n+k}(\mathbb{H}P^{4n-1}) = \gamma_{4n-1} * \pi_{16n+k}(\mathbb{S}^{16n-1})$ for $k = 20, n \geq 1$ and $k = 21, n \geq 2$.

Proof. By Corollary 2.2, (2.29), and (2.78), the element $E^2(\lambda v_{31}) = v_{15} v_{18}^* + v_{15} \xi_{18}$ is $\mathbb{H}$ -cyclic. Further, (2.33) and (2.78) yield $\Delta_{\mathbb{H}} \sigma_{16n-1}^3 = 0$ and Lemma 2.52 implies $\Delta_{\mathbb{H}}(\eta_{16n-1} \bar{\kappa}_{16n}) = 0$.

By means of (2.78), we get that $v_{8n-1} \bar{\sigma}_{8n+2}$ is $\mathbb{H}$ -cyclic. In view of (2.9), (2.28), and Lemma 2.52, it holds $\Delta_{\mathbb{H}}(\eta_{16n-1}^2 \bar{\kappa}_{16n+1}) = \Delta_{\mathbb{H}}(\eta_{16n-1} \bar{\kappa}_{16n}) \eta_{16n+19} = 0$.

This, Lemma 2.33, and the group structures of $\pi_{16n+k}(\mathbb{S}^{16n-1})$ for $k = 20, n \geq 1$ and $k = 21, n \geq 2$ [49, Theorems A and B] complete the proof. $\square$

Finally, we show:

Proposition 2.54. 1. $G_{18}(\mathbb{H}P^2) \supseteq 40\gamma_{2*} \pi_{18}(\mathbb{S}^{11})$;

2. $G_{21}(\mathbb{H}P^2) \supseteq 2\gamma_{2*} \pi_{21}(\mathbb{S}^{11})$;

3. $G_{22}(\mathbb{H}P^2) \supseteq 8\gamma_{2*} \pi_{22}(\mathbb{S}^{11}) \cong \mathbb{Z}_{63}$;

4. $G_{22}(\mathbb{H}P^3) \supseteq 4\gamma_{3*} \pi_{22}(\mathbb{S}^{15}) \cong \mathbb{Z}_{60}$.

Proof. By [54, Theorem 5.1] and [55, Lemmas 3.1–4], we see that:

1. $\text{Ker}\{\Delta_{\mathbb{H}} : \pi_{18}(\mathbb{S}^{11}) \rightarrow \pi_{17}(Sp(2))\} = 40\pi_{18}(\mathbb{S}^{11});$
2. $\text{Ker}\{\Delta_{\mathbb{H}} : \pi_{21}(\mathbb{S}^{11}) \rightarrow \pi_{20}(Sp(2))\} = 2\pi_{21}(\mathbb{S}^{11});$
3. $\text{Ker}\{\Delta_{\mathbb{H}} : \pi_{22}(\mathbb{S}^{11}) \rightarrow \pi_{21}(Sp(2))\} = 8\pi_{22}(\mathbb{S}^{11});$
4. $\text{Ker}\{\Delta_{\mathbb{H}} : \pi_{22}(\mathbb{S}^{15}) \rightarrow \pi_{21}(Sp(3))\} = 4\pi_{22}(\mathbb{S}^{15}).$

Hence, by Lemma 2.33(1), we obtain the assertion and this completes the proof. $\square$

We close this section with:

Question 2.55. What about $G_k(\mathbb{H}P^n) \cap i_{\mathbb{H}*}E\pi_{k-1}(\mathbb{S}^3)$?

2.8 The Case of the Cayley Projective Plane

Let the homotopy cofiber $\mathbb{K}P^2 = \mathbb{S}^8 \cup_{\sigma_8^+} e^{16}$ of the map $\sigma_8^+ : \mathbb{S}^{15} \rightarrow \mathbb{S}^8$ be the Cayley projective plane and write $i_{\mathbb{K}} : \mathbb{S}^8 \hookrightarrow \mathbb{K}P^2$ for the inclusion map. By [23, Theorem 1], we obtain

$$G_8(\mathbb{K}P^2) = 0$$

proved in [19] and [40] as well.

We recall the fibration

$$p : F_4 \longrightarrow F_4/Spin(9) = \mathbb{K}P^2$$

and the induced exact sequence:

$$\cdots \longrightarrow \pi_n(\mathbb{K}P^2) \xrightarrow{\Delta_{\mathbb{K}}} \pi_{n-1}(Spin(9)) \xrightarrow{i_*} \pi_{n-1}(F_4) \xrightarrow{p_*} \cdots. \quad (2.80)$$

Since $\pi_7(F_4) = 0$, $\pi_7(Spin(9)) \cong \mathbb{Z}$ and $\pi_8(\mathbb{K}P^2) \cong \mathbb{Z}$ [51, p. 132, Table], it follows that

$$\Delta_{\mathbb{K}} : \pi_8(\mathbb{K}P^2) \longrightarrow \pi_7(Spin(9))$$

is an isomorphism.

Identifying $\pi_n(Spin(9))$ with $\pi_n(SO(9))$ for $n \geq 2$, we may set $\Delta_{\mathbb{K}}(i_{\mathbb{K}}) = \pm[\iota_7]_9$. Then, in view of (2.9), we have $\Delta_{\mathbb{K}}(i_{\mathbb{K}}E\alpha) = \pm[\iota_7]_9\alpha$ for $\alpha \in \pi_{n-1}(\mathbb{S}^7)$ which implies:

Lemma 2.56. *If $\pi_k(Spin(9))$ is a 2-primary group and p is an odd prime then $i_{\mathbb{K}}E\pi_k(\mathbb{S}^7; p) \subseteq G_{k+1}(\mathbb{K}P^2)$.*

The groups $\pi_n(\mathbb{K}P^2)$ have been determined in [51] for $n \leq 23$ and $\pi_n(\mathbb{K}P^2; 2)$ for $n \leq 38$ in [29]. Let

$$Y = \mathbb{S}^7 \cup_{\sigma'\sigma_{14} + \alpha'} e^{22}$$

be the homotopy cofiber of the map $\sigma'\sigma_{14} + \alpha' : \mathbb{S}^{21} \rightarrow \mathbb{S}^7$, where $E\alpha' = [[\iota_8, \iota_8], \iota_8] \neq 0$ (1.4). By [51, Proposition 7.1], the 2- and 3-primary components of $\pi_n(\mathbb{K}P^2)$ are obtained by determining $\pi_{n-1}(Y)$ for $n \leq 28$, and the cell complex Y is the 28-skeleton of $\Omega(\mathbb{K}P^2)$.

Recall from [85, Chap. I] that a map $f : EX_1 \rightarrow X_2$ determines $\Omega_0 f : X_1 \rightarrow \Omega X_2$ defined by $\Omega_0 f = (\Omega f) \circ i$, where $i : X_1 \rightarrow \Omega(EX_1)$ is the adjoint map of id_{EX_1} . Thus, we get the factorization

$$\begin{array}{ccc} \mathbb{S}^7 & \xrightarrow{\Omega_0 i_{\mathbb{K}}} & \Omega(\mathbb{K}P^2) \\ i_Y \downarrow & \nearrow & \\ Y & & \end{array} \quad (2.81)$$

Further, the associated with Y sequence

$$\cdots \rightarrow \pi_k(\mathbb{S}^7) \xrightarrow{i_Y^*} \pi_k(Y) \xrightarrow{p_Y^*} \pi_k(\mathbb{S}^{22}) \xrightarrow{\Delta} \pi_{k-1}(\mathbb{S}^7) \rightarrow \cdots \quad (2.82)$$

is exact for $k \leq 27$ [51, (7.6)], where $i_Y : \mathbb{S}^7 \rightarrow Y$ and $p_Y : Y \rightarrow \mathbb{S}^{22}$ are the inclusion and quotient maps, respectively. We need:

- Lemma 2.57.** 1. $\pi_n(\mathbb{K}P^2) = i_{\mathbb{K}*}E\pi_{n-1}(\mathbb{S}^7) \cong \pi_{n-1}(\mathbb{S}^7)$ for $n \leq 21$ or $n = 25, 27, 28$;
 2. $\pi_{22}(\mathbb{K}P^2) = i_{\mathbb{K}*}\{\kappa_8\} \cong \mathbb{Z}_4$ and $i_{\mathbb{K}}(E\sigma')\sigma_{15} = 0$;
 3. $\pi_{23}(\mathbb{K}P^2) = \{\widetilde{8\iota_{22}}\} \oplus i_{\mathbb{K}*}\{E\rho''^+, \bar{\varepsilon}_8, (E\sigma')\varepsilon_{15}\} \cong \mathbb{Z} \oplus \mathbb{Z}_{120} \oplus (\mathbb{Z}_2)^2$, where $\widetilde{8\iota_{22}} \in \{i_{\mathbb{K}}, (E\sigma')\sigma_{15}, 8\iota_{22}\}_1$ and $i_{\mathbb{K}}(E\sigma')\varepsilon_{15} = i_{\mathbb{K}}(E\sigma')\bar{\nu}_{15}$;
 4. $\pi_{24}(\mathbb{K}P^2) = i_{\mathbb{K}*}\{(E\sigma')\mu_{15}, \mu_8\sigma_{17}, \eta_8\bar{\varepsilon}_9\} \cong (\mathbb{Z}_2)^3$ and $i_{\mathbb{K}}\eta_8\bar{\varepsilon}_9 = i_{\mathbb{K}}(E\sigma')\nu_{15}^3 = i_{\mathbb{K}}(E\sigma')(\eta_{15}\varepsilon_{16} + \sigma_{15}\eta_{22}^2) = i_{\mathbb{K}}(E\sigma')\eta_{15}\varepsilon_{16}$;
 5. there is the short exact sequence

$$0 \rightarrow \mathbb{Z}_{24} \oplus \mathbb{Z}_2 \rightarrow \pi_{26}(\mathbb{K}P^2) \rightarrow \mathbb{Z}_{24} \rightarrow 0$$

and $\pi_{26}(\mathbb{K}P^2; 2) = \{\eta_8\bar{\mu}_9, \bar{\nu}_{22}\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{64}$ for $\bar{\nu}_{22} \in \{i_{\mathbb{K}}, (E\sigma')\sigma_{15}, \nu_{22}\}$ with $8\bar{\nu}_{22} = \pm i_{\mathbb{K}}\zeta_8\sigma_{19}$;

6. $\pi_{29}(\mathbb{K}P^2; 2) = \{\bar{\nu}_{22}\nu_{26}, i_{\mathbb{K}}\eta_8\bar{\kappa}_9, i_{\mathbb{K}}(E\sigma')\kappa_{15}\} \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^2$.

Proof. First, the homotopy exact sequence

$$\cdots \rightarrow \pi_n(\mathbb{S}^8) \xrightarrow{i_{\mathbb{K}*}} \pi_n(\mathbb{K}P^2) \rightarrow \pi_n(\mathbb{K}P^2, \mathbb{S}^8) \xrightarrow{\partial} \pi_{n-1}(\mathbb{S}^8) \rightarrow \cdots$$

of the pair $(\mathbb{K}P^2, \mathbb{S}^8)$ and (1.27)(3) implies $i_{\mathbb{K}*}E\pi_{n-1}(\mathbb{S}^7) \subseteq \pi_n(\mathbb{K}P^2)$. Blakers–Massey theorem leads to isomorphism $\pi_n(\mathbb{K}P^2, \mathbb{S}^8) \xrightarrow{\cong} \pi_n(\mathbb{S}^{16})$ for $n \leq 23$. Hence, those yield $\pi_n(\mathbb{K}P^2) = i_{\mathbb{K}*}E\pi_{n-1}(\mathbb{S}^7) \cong \pi_{n-1}(\mathbb{S}^7)$ for $n \leq 21$.

Next, we make use of (2.82) to describe the groups $\pi_n(\mathbb{K}P^2)$ for $n = 23, 24, 26$. By Blakers–Massey theorem, $\pi_{22}(Y, \mathbb{S}^7) \cong \mathbb{Z}$ and $\pi_{23}(Y, \mathbb{S}^7) \cong \pi_{23}(\mathbb{S}^{22})$. Since $(\sigma'\sigma_{14} + \alpha')\eta_{21} = \sigma'(\bar{v}_{14} + \varepsilon_{14})$, we obtain the group $\pi_{22}(Y)$. Further, in view of (2.82), we have $\Delta(\iota_{22})_{(2)} = \sigma'\sigma_{14}$. Because id_Y is regarded as an extension of the inclusion map $i_Y : \mathbb{S}^7 \rightarrow Y$, [85, Proposition 1.7] yields a coextension $-\widetilde{8\iota_{21}} \in \{i_Y, \sigma'\sigma_{14}, 8\iota_{21}\}$. On the other hand, in view of the factorization (2.81), the result [85, Proposition 1.3] leads to the formula

$$-\Omega_0\{i_{\mathbb{K}}, (E\sigma')\sigma_{15}, 8\iota_{22}\}_1 = \{i_Y, \sigma'\sigma_{14}, 8\iota_{21}\}$$

which yields the corresponding coextension $\widetilde{8\iota_{22}} \in \{i_{\mathbb{K}}, (E\sigma')\sigma_{15}, 8\iota_{22}\}_1$ generating the direct summand $\mathbb{Z}$ of the group $\pi_{23}(\mathbb{K}P^2)$.

By the parallel argument, we have the group $\pi_{23}(Y)$ and the assertions for $n = 23, 24$ follow.

We know that $\pi_{26}(\mathbb{S}^{22}) = 0$, $\pi_{25}(\mathbb{S}^7) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2$, $\pi_{25}(\mathbb{S}^{22}) \cong \mathbb{Z}_{24}$, and the relation $(\sigma'\sigma_{14} + \alpha')v_{21} = 0$ (1.85). So, the short exact sequence (2.82) leads to the short exact one

$$0 \rightarrow \mathbb{Z}_{24} \oplus \mathbb{Z}_2 \rightarrow \pi_{26}(\mathbb{K}P^2) \rightarrow \mathbb{Z}_{24} \rightarrow 0$$

and, as above, to a coextension $\tilde{v}_{22} \in \{i_{\mathbb{K}}, (E\sigma')\sigma_{15}, v_{22}\}$. In view of (1.65),

$$8\{i_{\mathbb{K}}, (E\sigma')\sigma_{15}, v_{22}\} = -i_{\mathbb{K}*}\{(E\sigma')\sigma_{15}, v_{22}, 8\iota_{25}\} \supseteq -i_{\mathbb{K}}(E\sigma')\{\sigma_{15}, v_{22}, 8\iota_{25}\} \ni$$

$-i_{\mathbb{K}}(E\sigma')\zeta_{15} = x i_{\mathbb{K}}\zeta_8\sigma_{19} \pmod{8i_{\mathbb{K}*}\pi_{26}^8 = 0}$ for x as in (1.65). Consequently, we derive that $8\tilde{v}_{22} = \pm i_{\mathbb{K}}\zeta_8\sigma_{19}$ and (5) follows.

(6) is determined by [29, Theorem 5.3] and the proof is complete. $\square$

Now, we show:

- Lemma 2.58.** 1. $\text{Ker}\{E^8 : \pi_{n-1}(\mathbb{S}^7) \rightarrow \pi_{n+7}(\mathbb{S}^{15})\} \subseteq (i_{\mathbb{K}}E)_*^{-1}P_n(\mathbb{K}P^2)$. In particular, $i_{\mathbb{K}}E(\sigma'\alpha) \in P_n(\mathbb{K}P^2)$ provided that $\sharp\alpha = 2$ for $\alpha \in \pi_{n-1}^{14}$;
2. If $\sharp(i_{\mathbb{K}}(E\sigma')(E^8\alpha)) = 2^m$ with $m > 0$ for $\alpha \in \pi_{n-1}^7$ then $2^k i_{\mathbb{K}}(E\alpha) \notin P_n(\mathbb{K}P^2)$ provided that $k < m$.

Proof. (1) is obtained from the formula parallel to (2.59). (2) follows from the formula

$$\pm [i_{\mathbb{K}}(E\alpha), i_{\mathbb{K}}] = i_{\mathbb{K}}(E\sigma')(E^8\alpha) \quad (2.83)$$

determined by Lemma 1.2(2) and (1.22) for $\alpha \in \pi_{n-1}(\mathbb{S}^7)$. $\square$

Hence, we deduce:

Corollary 2.59.

$$P_n(\mathbb{K}P^2) \ni \begin{cases} i_{\mathbb{K}}E(\sigma'\eta_{14}) & \text{for } n = 16, \\ i_{\mathbb{K}}E(\sigma'\eta_{14}^2) & \text{for } n = 17, \\ i_{\mathbb{K}}E(\sigma'\nu_{14}) = xi_{\mathbb{K}}E(\nu_7\sigma_{10}) & \text{for an odd integer } x \text{ and } n = 18, \\ i_{\mathbb{K}}E(\bar{\nu}_7\nu_{15}) & \text{for } n = 19, \\ i_{\mathbb{K}}E(\nu_7\sigma_{10}\nu_{17}) & \text{for } n = 21, \\ i_{\mathbb{K}}E(\bar{\nu}_7\nu_{15}^2) & \text{for } n = 22, \\ i_{\mathbb{K}}E(\sigma'\varepsilon_{14}) = i_{\mathbb{K}}E(\sigma'\bar{\nu}_{14}) & \text{for } n = 23, \\ i_{\mathbb{K}}E(\sigma'\mu_{14}), i_{\mathbb{K}}E(\eta_7\bar{\varepsilon}_8) & \text{for } n = 24, \\ i_{\mathbb{K}}E(\sigma'\eta_{14}\mu_{15}) & \text{for } n = 25. \end{cases}$$

Proof. Because, by Lemma 2.57(3), $i_{\mathbb{K}}E(\sigma'\varepsilon_{14}) = i_{\mathbb{K}}E(\sigma'\bar{\nu}_{14})$ and $E^2\sigma' = 2\sigma_9$ (1.22), Lemma 2.58(1) leads to:

$$P_n(\mathbb{K}P^2) \ni \begin{cases} i_{\mathbb{K}}E(\sigma'\eta_{14}) & \text{for } n = 16, \\ i_{\mathbb{K}}E(\sigma'\eta_{14}^2) & \text{for } n = 17, \\ i_{\mathbb{K}}E(\sigma'\varepsilon_{14}) = i_{\mathbb{K}}E(\sigma'\bar{\nu}_{14}) & \text{for } n = 23, \\ i_{\mathbb{K}}E(\sigma'\mu_{14}) & \text{for } n = 24, \\ i_{\mathbb{K}}E(\sigma'\eta_{14}\mu_{15}) & \text{for } n = 25. \end{cases}$$

By means of (1.82), $i_{\mathbb{K}}E(\sigma'\nu_{14}) = xi_{\mathbb{K}}E(\nu_7\sigma_{10})$ for an odd integer x . Hence, (1.84) and Lemma 2.58(1) lead to $i_{\mathbb{K}}E(\sigma'\nu_{14}) = xi_{\mathbb{K}}E(\nu_7\sigma_{10}) \in P_{18}(\mathbb{K}P^2)$. Next, Lemma 2.58(1) and the relation (1.129) yield $i_{\mathbb{K}}E(\bar{\nu}_7\nu_{15}) \in P_{19}(\mathbb{K}P^2)$. Because $i_{\mathbb{K}}E(\nu_7\sigma_{10}) \in P_{18}(\mathbb{K}P^2)$, in view of Corollary 2.2, the element $i_{\mathbb{K}}E(\nu_7\sigma_{10}\nu_{17}) \in P_{21}(\mathbb{K}P^2)$. In virtue of (1.84) and (1.107)(1), Lemma 2.58(1) leads to $i_{\mathbb{K}}E(\eta_7\bar{\varepsilon}_8) \in P_{24}(\mathbb{K}P^2)$ and the proof is complete. $\square$

Next, we show:

- Theorem 2.60.** 1. $P_n(\mathbb{K}P^2) = 0$ for $n = 9, 10, 12, 13, 14, 20$;
 2. $G_{11}(\mathbb{K}P^2) = P_{11}(\mathbb{K}P^2) = 8\pi_{11}(\mathbb{K}P^2) \cong \mathbb{Z}_3$;
 3. $8\pi_{15}(\mathbb{K}P^2) \subseteq G_{15}(\mathbb{K}P^2)$;
 4. $i_{\mathbb{K}*}E\{\sigma'\eta_{14}\} \subseteq P_{16}(\mathbb{K}P^2) \subseteq i_{\mathbb{K}*}E\{\sigma'\eta_{14}, \eta_7\sigma_8\}$;
 5. $i_{\mathbb{K}*}E\{\sigma'\eta_{14}^2\} \subseteq P_{17}(\mathbb{K}P^2) \subseteq i_{\mathbb{K}*}E\{\sigma'\eta_{14}^2, \nu_7^3 + \eta_7\varepsilon_8\} = i_{\mathbb{K}*}E\{\sigma'\eta_{14}, \eta_7\sigma_8\} \circ \eta_{16}$;

6. $8\pi_{18}(\mathbb{K}P^2) \subseteq G_{18}(\mathbb{K}P^2) \subseteq P_{18}(\mathbb{K}P^2) = i_{\mathbb{K}*}\{\nu_8\sigma_{11}, \beta_1(8)\} \cong \mathbb{Z}_{24}$;
7. $P_{19}(\mathbb{K}P^2; 2) = i_{\mathbb{K}*}E\{\bar{\nu}_7\nu_{15}\} \cong \mathbb{Z}_2$;
8. $2\pi_{21}(\mathbb{K}P^2) \subseteq G_{21}(\mathbb{K}P^2) \subseteq P_{21}(\mathbb{K}P^2) = \pi_{21}(\mathbb{K}P^2)$.
9. $P_{22}(\mathbb{K}P^2) = 2\pi_{22}(\mathbb{K}P^2)$.

Proof. The results about the lower bounds of $P_n(\mathbb{K}P^2)$ are obtained in Lemma 2.58.

Since, by (2.75) and Lemma 2.57(1), $\pi_n(\mathbb{K}P^2) \cong \pi_{n-1}(\mathbb{S}^7) = 0$ for $n = 12, 13, 20$, the assertion for $n = 12, 13, 20$ of (1) follows.

By Lemma 2.57(1),

$$i_{\mathbb{K}}(E\sigma')\eta_{15} \neq 0, i_{\mathbb{K}}(E\sigma')\eta_{15}^2 \neq 0,$$

$$i_{\mathbb{K}}(E\sigma')\nu_{15}^2 = i_{\mathbb{K}}\nu_8\sigma_{11}\nu_{18} \neq 0.$$

This and Lemma 2.58(2) lead to the assertion for $n = 9, 10, 14$ of (1).

Then, by the facts that $\pi_{11}(\mathbb{K}P^2) = i_{\mathbb{K}*}\{\nu_8^+\} \cong \mathbb{Z}_{24}$ Lemma 2.57(1), $\pi_{10}(Spin(9)) \cong \mathbb{Z}_8$ [51, p. 132, Table], Lemma 2.56 implies

$$i_{\mathbb{K}}\alpha_1(8) \in G_{11}(\mathbb{K}P^2). \quad (2.84)$$

Moreover, $\sharp((i_{\mathbb{K}}(E\sigma')\nu_{18}) = 8$ (1.82) and Lemma 2.57(1). This and Lemma 2.58(2) yield $P_{11}(\mathbb{K}P^2; 2) = 0$, and (2) follows.

Because $\pi_{14}(\mathbb{S}^7) = \{\sigma'^+\} \cong \mathbb{Z}_{120}$ (2.75), we get $\pi_{15}(\mathbb{K}P^2) = \{i_{\mathbb{K}}(E\sigma'^+)\} \cong \mathbb{Z}_{120}$. Then, in view of Lemma 2.56 and Lemma 2.57(1), $8\pi_{15}(\mathbb{K}P^2) \subseteq G_{15}(\mathbb{K}P^2)$ which yields (3).

We recall from (2.75) that $\pi_{15}(\mathbb{S}^7) = \{\sigma'\eta_{14}, \bar{\nu}_7, \varepsilon_7\} \cong (\mathbb{Z}_2)^3$. By Lemma 2.57(1),

$$\pi_{16}(\mathbb{K}P^2) = i_{\mathbb{K}*}E\{\sigma'\eta_{14}, \bar{\nu}_7, \varepsilon_7\}.$$

Next, by Lemma 2.57(3), $i_{\mathbb{K}}(E\sigma')\bar{\nu}_{15} = i_{\mathbb{K}}(E\sigma')\varepsilon_{15} \neq 0$ and (1.69)(1) yields $i_{\mathbb{K}}(E\sigma')(\bar{\nu}_{15} + \varepsilon_{15}) = i_{\mathbb{K}}(E\sigma')(\eta_{15}\sigma_{16}) = 0$. This leads to (4).

We recall from (2.75) that

$$\pi_{16}(\mathbb{S}^7) = \{\sigma'\eta_{14}^2, \nu_7^3, \mu_7, \eta_7\varepsilon_8\} \cong (\mathbb{Z}_2)^4.$$

By Lemma 2.57(4), $i_{\mathbb{K}}(E\sigma')\nu_{15}^3 = i_{\mathbb{K}}\eta_8\bar{\varepsilon}_9 \neq 0$, $i_{\mathbb{K}}(E\sigma')\mu_{15} \neq 0$.

Then, by Lemma 2.58(2), $i_{\mathbb{K}}\nu_8^3 \notin P_{17}(\mathbb{K}P^2)$ and $i_{\mathbb{K}}\mu_8 \notin P_{17}(\mathbb{K}P^2)$. Moreover, by (1.69)(1) and (1.72), we obtain $i_{\mathbb{K}}(E\sigma')(\eta_{15}\varepsilon_{16}) = i_{\mathbb{K}}(E\sigma')(\nu_{15}^3) \neq 0$ and $i_{\mathbb{K}}(E\sigma')(\nu_{15}^3 + \eta_{15}\varepsilon_{16}) = i_{\mathbb{K}}(E\sigma')(\sigma_{15}\eta_{22}^2) = 0$. This leads to (5).

We recall from (2.75) that $\pi_{17}(\mathbb{S}^7) = \{\nu_7\sigma_{10}, \eta_7\mu_8, \beta_1(7)\} \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2$. By Lemma 2.57(1),

$$\pi_{18}(\mathbb{K}P^2) = i_{\mathbb{K}*}E\{\nu_7\sigma_{10}, \eta_7\mu_8, \beta_1(7)\} \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_2.$$

Since $\pi_{17}(Spin(9)) \cong \mathbb{Z}_8 \oplus (\mathbb{Z}_2)^2$ [51, p. 132, Table], Lemma 2.56 implies $i_{\mathbb{K}}\beta_1(8) \in G_{18}(\mathbb{K}P^2)$. In view of Corollary 2.59, the element $i_{\mathbb{K}}E(v_7\sigma_{10}) \in P_{18}(\mathbb{K}P^2)$. By Lemma 2.57(1) and the fact from (2.75) that

$$\pi_{24}(\mathbb{S}^7) = \{\sigma'\eta_{14}\mu_{15}, v_7\kappa_{10}, \bar{\mu}_7, \eta_7\mu_8\sigma_{17}\} \cong (\mathbb{Z}_2)^4,$$

we see that $i_{\mathbb{K}}E(\sigma'\eta_{14}\mu_{15}) \neq 0$ and, in view of Lemma 2.58(2), $i_{\mathbb{K}}E(\eta_7\mu_8) \notin P_{18}(\mathbb{K}P^2)$. This leads to (6).

Next, we recall from (2.75) that

$$\pi_{18}^7 = \{\zeta_7, \bar{v}_7v_{15}\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2.$$

But, $i_{\mathbb{K}}(E\sigma')\zeta_{15} = xi_{\mathbb{K}}\zeta_8\sigma_{19}$ for x as in (1.65), and $\sharp(i_{\mathbb{K}}(\zeta_8\sigma_{19})) = 8$ (Lemma 2.57(5)), so Lemma 2.58(2) yields $2^k i_{\mathbb{K}}(E\zeta_7) \notin P_{19}(\mathbb{K}P^2)$ with $k < 3$. In view of Corollary 2.59, the element $i_{\mathbb{K}}E(\bar{v}_7v_{15}) \in P_{19}(\mathbb{K}P^2)$ and this leads to (7).

By Lemma 2.57(1) and the fact from (2.75) that $\pi_{20}(\mathbb{S}^7) = \{v_7\sigma_{10}v_{17}, \alpha_1(7)\beta_1(10)\} \cong \mathbb{Z}_6$, we obtain $\pi_{21}(\mathbb{K}P^2) = i_{\mathbb{K}*}\{v_8\sigma_{11}v_{18}, \alpha_1(8)\beta_1(11)\}$. In view of (2.84), we get

$$i_{\mathbb{K}}\alpha_1(8)\beta_1(11) \in G_{21}(\mathbb{K}P^2)$$

and, by means of Corollary 2.59, the element $i_{\mathbb{K}}E(v_7\sigma_{10}v_{17}) \in P_{21}(\mathbb{K}P^2)$. Those lead to (8).

Recall $\pi_{22}(\mathbb{K}P^2) = \{i_{\mathbb{K}}\kappa_8\} \cong \mathbb{Z}_4$ Lemma 2.57(2). By Lemma 2.57(6) and (2.83), $[i_{\mathbb{K}}\kappa_8, i_{\mathbb{K}}] = i_{\mathbb{K}}(E\sigma')\kappa_{15} \neq 0$. Hence, in view of (1.109) and Corollary 2.59, (9) follows and the proof is complete. $\square$

We show:

Proposition 2.61. *If $\sharp(\sigma_9 E^9 \alpha) = m > 1$ for $\alpha \in \pi_{n-1}^7$ then $k(i_{\mathbb{K}}E\alpha) \notin \text{Ker}\{\Delta_{\mathbb{K}} : \pi_n(\mathbb{K}P^2) \rightarrow \pi_{n-1}(Spin(9))\}$ unless $k \equiv 0 \pmod{m}$.*

Proof. Suppose that $\Delta_{\mathbb{K}}(ki_{\mathbb{K}}E\alpha) = 0$ with $k \not\equiv 0 \pmod{m}$ for $\sharp(\sigma_9 E^9 \alpha) = m$. Then, $0 = kJ\Delta_{\mathbb{K}}(i_{\mathbb{K}}E\alpha) = \pm kJ([\iota_7]_9\alpha) = \pm k\sigma_9 E^9 \alpha$ leads to a contradiction. $\square$

Using (2.75), we obtain:

Example 2.62. The following $\alpha \in \pi_{n-1}^7$ subject:

1. $\sharp(\sigma_9 E^9 \alpha) = 2$ for: $\alpha = \eta_7, \eta_7^2, v_7^2, \eta_7\sigma_8, \bar{v}_7, \varepsilon_7, v_7^3, \mu_7, \eta_7\varepsilon_8, \eta_7\mu_8, v_7\kappa_{10}, \bar{\mu}_7, \mu_7\sigma_{16};$
2. $\sharp(\sigma_9 E^9 \alpha) = 8$ for: $\alpha = v_7, \sigma', \zeta_7.$

Owing to Example 2.62 and Theorem 2.60, we propose:

Question 2.63. $G_n(\mathbb{K}P^2; 2) = 0?$

Gottlieb and Whitehead Center Groups of Spheres,
Projective and Moore Spaces

Golasinski, M.; Mukai, J.

2014, XVII, 132 p. 7 illus., Hardcover

ISBN: 978-3-319-11516-0