Chapter 2
Ascoli–Arzela Theory

The first chapter illustrated the usefulness of the Ascoli–Arzela theorem in proving analytic theorems. In this chapter, we'll find out what this result is.

In the topological proof of the Cauchy–Peano theorem, the Ascoli–Arzela theorem was used to conclude that a set A, in a space $C[x_0 - \alpha, x_0 + \alpha]$ of functions, was compact. Analysis requires us to move out of the euclidean, that is, finite-dimensional, world into the infinite-dimensional world of spaces of functions. We will find that we can make the transition successfully if we carry enough compactness along with us. We could build compactness into our hypotheses, but we must be careful because we don't want theorems with such strong hypotheses that no interesting examples can satisfy them. Fortunately, there is a certain amount of compactness that we do not have to hypothesize. A Theorem (2.7) that identifies the compactness built into certain function spaces, a theorem that depends on the Ascoli–Arzela theorem, is the ultimate goal of this section because we will use it several times later in the book. That's enough of generalities; now let's find out just what we've been talking about.

A set in a metric space is *complete* if every Cauchy sequence in the set converges. If any of the words in the previous sentence aren't part of your working vocabulary, I assume you'll go look them up in a real analysis book. There you will also be reminded that the complete subsets of $\mathbf{R}^n$ are precisely the closed ones.

For $\epsilon > 0$, an ϵ*-net* S in a metric space X is a subset of X with the property that every point of X is within ϵ of some point of S. To say the same thing more precisely, for $x \in X$, define

$$B(x;\epsilon) = \{y \in X : d(x,y) < \epsilon\},$$

where d is the metric of X, and say that a subset S of X is an ϵ*-net* if X is the union of all $B(s;\epsilon)$ for $s \in S$. A metric space X is *totally bounded* (some people prefer the term *precompact*) if given an $\epsilon > 0$, there is a finite ϵ-net for X. For instance, every

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bounded subset of $\mathbf{R}^n$ has this property. Notice that a subset of a totally bounded set is also totally bounded. Furthermore, it is not hard to see that if a subset of a space is totally bounded, so is the closure of that subset.

It is easy to demonstrate that if X is a compact metric space, then it is totally bounded: given $\epsilon > 0$, form the cover $\{B_x = B(x;\epsilon)\}_{x \in X}$ of X and, for a finite subcover $\{B_{x_j}\}$, the set $\{x_j\}$ is an ϵ-net for X. The first step on the road toward the Ascoli–Arzela theorem is to show that, in complete metric spaces, the converse is also true.

Theorem 2.1. *Let X be a complete metric space. If X is totally bounded, then it is compact.*

Proof. Suppose that the complete, totally bounded space X is not compact, in other words, that there is an open cover of X that has no finite subcover. We'll see that this leads to a contradiction. The existence of the cover will allow us to build a sequence $\{x_n\}$ in X with the following two properties: (1) $d(x_n, x_{n+1}) < 2^{-n+1}$; (2) the subset $B(x_n; 2^{-n+1})$ cannot be covered by a finite subfamily of the cover. Before constructing the sequence, let's see why it proves the theorem. First of all, condition (1) implies that the sequence is Cauchy. To verify the Cauchy property, suppose we are given $\epsilon > 0$, then choose natural numbers m and n with, say, $m \leq n$ and large enough so that $2^{-m+2} < \epsilon$, then $d(x_m, x_n) < \epsilon$ because

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k=m}^{n-1} 2^{-k+1} \leq 2^{-m+2}.$$

Let x be the limit of $\{x_n\}$ and let U be a member of the cover that contains x. Choose ϵ small enough so that $B(x; 2\epsilon) \subseteq U$. Then choose n large enough so that $2^{-n+1} < \epsilon$ and also, by convergence of the sequence, $d(x_n, x) < \epsilon$. But that tells us that $B(x_n; 2^{-n+1}) \subseteq U$ so $B(x_n; 2^{-n+1})$ certainly can be covered by a finite subfamily of the cover, in fact by the single member U, contrary to (2). Thus once we construct the sequence $\{x_n\}$ we know it will lead us to a contradiction, as we had hoped, so we now construct it—inductively. To identify x_1, use the total boundedness of X to construct a finite 1-net $\{y_1, y_2, \ldots, y_m\}$. If each $B(y_j; 1)$ could be covered by a finite subcover of the cover, we would have a finite subcover that covered all of X; therefore there is a y_j such that $B(y_j; 1)$ cannot be so covered and we set x_1 equal to that y_j. Supposing that $\{x_1, x_2, \ldots, x_n\}$ have been found which satisfy (1) and (2), we need to produce x_{n+1}. To do that, we note that $B(x_n; 2^{-n+1})$ is totally bounded since X is and concentrate now on the subspace to locate x_{n+1}, essentially in the same way we found x_1. The subspace has a finite 2^{-n}-net $\{y_1, y_2, \ldots, y_m\}$. Since (2) holds by assumption, that says $B(x_n; 2^{-n+1})$ cannot be covered by any finite subfamily of the given cover. Therefore there is a y_j such that $B(y_j; 2^{-n})$ has the same property and we let $x_{n+1} = y_j$, which gives us property (2), and of course (1) holds because this all takes place within $B(x_n; 2^{-n+1})$. □

It often happens that we don't need a set within a space to be compact itself, but we do want the set to lie in a compact subset of the space. When a subset S in a space X is contained in a compact subset K of X, then S is called a *relatively compact* subset of X. Since K is closed in X, it contains the closure $\overline{S}$ of S, as a closed subset, and thus $\overline{S}$ is itself compact. Therefore an equivalent definition of a relatively compact subset is one whose closure is compact.

Theorem 2.1 gives us a lot of relatively compact subsets of complete metric spaces. If S is a totally bounded subset of a complete metric space X, then $\overline{S}$ is also totally bounded and it is complete because it is closed in X. Thus, by Theorem 2.1, we have

Corollary 2.2. *A totally bounded subset S of a complete metric space X is relatively compact.*

It follows that bounded subsets of euclidean spaces are relatively compact, but we knew that anyway. What we really want is a nice way to identify some relatively compact subsets of another class of complete metric spaces which we describe next.

Let E be a metric space with metric denoted by d and let $u : E \to \mathbf{R}$ be a real-valued function which, for the moment, need not be continuous. However, we do want u to be *bounded*, that is, $u(E)$ must be a bounded subset of the reals. Then we can define the *supremum norm* (known familiarly as the *sup norm*, pronounced "soup") $\|u\|$ of u to be the least upper bound (also known as the supremum) of the set $\{|u(x)| : x \in E\}$. The set $B(E)$ of all bounded real-valued functions on E is a metric space with distance between functions u and v given by $\|u - v\|$. Another name for $\|u\|$ is the *uniform norm* and that emphasizes the fact that convergence of a sequence of functions in $B(E)$ is what real analysis texts call uniform convergence of the sequence.

The space $B(E)$ is complete because it inherits that property from the reals. What I mean by this is that if you start with a sequence $\{u_n\}$ in $B(E)$ that is Cauchy, then for each $x \in E$ the sequence of reals $u_n(x)$ is Cauchy and therefore has a limit we'll call $u(x)$. This defines a function u and that function is bounded, so the limit is still in $B(E)$.

The space we are really interested in, though, is $C(E)$, the bounded, *continuous* real-valued functions on E. Since convergence in $B(E)$ is uniform convergence, the limit of a sequence of continuous functions in $B(E)$ must also be continuous. In more topological language, $C(E)$ is a closed subset of the space $B(E)$. Therefore, $C(E)$ is also a complete metric space.

The main result of this section identifies a useful class of relatively compact subsets of $C(E)$. By Corollary 2.2, since $C(E)$ is complete, all totally bounded subsets will be relatively compact. However, total boundedness of a set of real-valued functions isn't an easy hypothesis to check, so we'll replace it by two other hypotheses. One of the hypotheses is boundedness: a set A in $C(E)$ is *bounded* if there is a number β such that $\|u\| < \beta$ for all functions $u \in A$. To state the other hypothesis, a set A in $C(E)$ is said to be *equicontinuous at* $x \in E$ if given $\epsilon > 0$, there exists $\delta_x > 0$ such that if $x, y \in E$ with $d(x, y) < \delta_x$, then $|u(x) - u(y)| < \epsilon$ for all $u \in A$. (Note that the functions u are real-valued and we are using the usual notion of distance in $\mathbf{R}$ when we write $|u(x)-u(y)|$.) Call the set A *equicontinuous* if it is equicontinuous at all $x \in E$. Surprisingly, these hypotheses are reasonably convenient to check, and that's how we'll find relatively compact subsets of $C(E)$ when E itself is a compact space (a hypothesis we haven't used up to this point) because of the following.

Theorem 2.3 (Ascoli–Arzela Theorem). *Let E be a compact metric space. If A is an equicontinuous, bounded subset of $C(E)$, then A is relatively compact.*

Proof. The idea is to prove that A is totally bounded by starting with a given ϵ and use it in conjunction with the definition of equicontinuity to produce a finite ϵ-net (of functions, remember) for A. Specifically, for each $x \in E$ we use δ_x from that definition, chosen so that if $d(x,y) < \delta_x$ then $|u(x)-u(y)| < \frac{\epsilon}{4}$ for all $u \in A$. This gives us a cover $\{B(x;\delta_x)\}$ of E and here is where we use the compactness of the space E, to extract a finite subcover which we denote by $\{B(x_j;\delta_j)\}$ for $j = 1,2,\ldots,n$. (In a sense, we want the finite set $\{x_j\}$ to substitute for the points of E.) For some x_j (just one of them) look at the set of points $\{u(x_j)\}$ for all $u \in A$, which we denote by $A[x_j]$. The set $A[x_j]$ is bounded in $\mathbf{R}$ since we assumed that A is a bounded subset of $C(E)$, so there is a finite $\frac{\epsilon}{4}$-net

$$\{z_1(x_j), z_2(x_j), \ldots, z_{k(j)}(x_j)\}$$

for the set $A[x_j]$. This just means that given $u \in A$, there is some $z_s(x_j)$ for which $|u(x_j) - z_s(x_j)| < \frac{\epsilon}{4}$ and we want the net to substitute in some way for $A[x_j]$. Let $\mu = \{\mu_1, \mu_2, \ldots, \mu_n\}$ denote an n-tuple of integers with the property $1 \le \mu_s \le k(s)$. For each μ, let v_μ be a function in A such that

$$|v_\mu(x_j) - z_{\mu_j}(x_j)| < \frac{\epsilon}{4}$$

for all $j = 1,2,\ldots,n$. It may be that there is no function in A with this property for some of the n-tuples μ and in that case there is no corresponding v_μ defined. In this way, we obtain a set of at most $k(1)k(2)\cdots k(n)$ functions v_μ in A, and we claim that these functions form an ϵ-net for A. To prove it, we take any $u \in A$ and first we find a candidate for the v_μ that we believe is within ϵ of it. For each $j = 1,2,\ldots,n$ there is an integer μ_j such that

$$|u(x_j) - z_{\mu_j}(x_j)| < \frac{\epsilon}{4}$$

so, not surprisingly, we'll use the corresponding v_μ. The function v_μ must exist for that μ since u itself satisfies the defining property, though of course it isn't likely to be the v_μ we chose for the net. To prove that $\|u - v_\mu\| < \epsilon$, we take any $x \in E$; then by the cover property, x is in some $B(x_j;\delta_j)$ and, by the triangle inequality,

$$\begin{aligned}|u(x) - v_\mu(x)| &\le |u(x) - u(x_j)| + |u(x_j) - z_{\mu_j}(x_j)| \\ &\quad + |z_{\mu_j}(x_j) - v_\mu(x_j)| + |v_\mu(x_j) - v_\mu(x)| < \epsilon\end{aligned}$$

because the first and last terms in the sum are less than $\frac{\epsilon}{4}$ by equicontinuity and the other terms are each less than $\frac{\epsilon}{4}$ by the net property of the z's. □

If you look back to the topological proof of the Cauchy–Peano theorem in Chap. 1, you'll see that we have already met the concepts of total boundedness and equicontinuity, in the space $C[x_0 - \alpha, x_0 + \alpha]$. These properties are implied by the conditions defining the subset A:

$$(1)\ |u(x) - y_0| \leq a \quad \text{for all} \quad x \in [x_0 - \alpha, x_0 + \alpha]$$

$$(2)\ |u(x_1) - u(x_2)| \leq M|x_1 - x_2| \quad \text{for all} \quad x_1, x_2 \in [x_0 - \alpha, x_0 + \alpha]$$

Condition (1) gives us a bound

$$\beta = \max(|y_0 + a|, |y_0 - a|)$$

for all $\|u\|$. Equicontinuity follows from (2) because, given $\epsilon > 0$, we can use $\delta_x = \frac{\epsilon}{M}$ for all $x \in [x_0 - \alpha, x_0 + \alpha]$. The Ascoli–Arzela theorem then tells us that A has a compact closure and, since A is closed, it is compact.

The way we used the Ascoli–Arzela in the approximation proof of the Cauchy–Peano theorem was to conclude that a certain sequence has a convergent subsequence. That depends on some information from introductory topology, namely

Theorem 2.4. *A metric space X is compact if and only if every sequence in X contains a convergent subsequence.*

There is a proof of this fact that fits so nicely into the circle of ideas we have been discussing, I can't resist the temptation to include it, but I've placed it at the end of this chapter so it will be easy to skip if you wish.

Now suppose that in the Ascoli–Arzela Theorem 2.3, the bounded, equicontinuous set A is a sequence of functions $A = \{u_1, u_2, \ldots, u_n, \ldots\}$. Since the closure of A is compact, every sequence in it, so in particular A itself, contains a convergent subsequence. (Keep in mind, however, that the convergence is in $C(E)$, not necessarily in A.) Therefore we have the following important special case of the Ascoli–Arzela theorem.

Corollary 2.5. *Let E be a compact metric space. Every bounded, equicontinuous sequence in $C(E)$ has a subsequence that converges in $C(E)$.*

The approximation proof of the Cauchy–Peano Existence theorem depended on Corollary 2.5 in the following way. The sequence of approximate solutions ϕ_n can be shown to be bounded and equicontinuous. Therefore, the corollary furnishes a convergent subsequence and it was its limit function ϕ that turned out to be the solution to the initial-value problem.

At the beginning of this chapter I promised you some compactness properties for function spaces that you don't have to include in hypotheses. Let's next become acquainted with some terminology from analysis which is used in describing those properties.

A *normed (real) linear space* is a linear space X along with a *norm* $\|\cdot\| : X \to \mathbf{R}$ which has the properties (i) $\|x\| \geq 0$ for all $x \in X$, (ii) $\|cx\| = |c|\|x\|$ for all real c (where $|c|$ is the absolute value of c) and $x \in X$, and (iii) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ for all $x_1, x_2 \in X$. The definition $d(x, y) = \|x - y\|$ produces a metric on X under which addition and scalar multiplication are continuous. Thus we can view the normed linear space X as a metric space with this metric, when it is convenient to do so. We've seen some examples of normed linear spaces: euclidean spaces, and also $C(E)$ (or $B(E)$) with the sup norm. A set S in X is *bounded* if there is a number β such that $\|s\| < \beta$ for all $s \in S$. (That's the same definition we used in $C(E)$.)

It's customary to call a continuous function a *map*, and that's the term I'll generally use. Let $f : X \to Y$ be a map where X and Y are normed linear spaces. The map f is said to be *compact* if the image $f(X)$ is a relatively compact subset of Y. A less restrictive but closely related concept is: the map f is *completely continuous* if for any bounded subset S of X, the set $f(S)$ is relatively compact. Thus a completely continuous map is one that is a compact map when you restrict it to any bounded subset of its domain. Theorem 2.4 tells us how these maps behave with respect to sequences in the domain. If $\{x_n\}$ is a sequence in X and $f : X \to Y$ is a compact map, then the sequence $\{f(x_n)\}$ converges to some $y \in Y$, though not necessarily to a point of the image $f(X)$. If f is only completely continuous, then we can draw the same conclusion about a sequence $\{x_n\}$ provided that the sequence is bounded.

A useful property of completely continuous maps is the way their nice behavior is preserved when we compose them with continuous functions, as follows.

Theorem 2.6. *Suppose that X, Y, and Z are normed linear spaces, $f : X \to Y$ is a completely continuous map, and $g : Y \to Z$ is a map; then $gf : X \to Z$ is completely continuous.*

Proof. Let S be a bounded subset of X and let $\{z_n\}$ be a sequence in $gf(S) \subseteq Z$. We can therefore find a sequence $\{x_n\}$ in S such that $z_n = gf(x_n)$ for all n. The complete continuity of f gives us a convergent subsequence of the sequence $\{f(x_n)\}$; denote it by $\{f(x_{n_k})\}$. Since g is a continuous function, it takes convergent sequences to convergent sequences, and it follows that $gf(x_{n_k}) = z_{n_k}$ is a subsequence of $\{z_n\}$ that converges (in Z), so $gf(S)$ is relatively compact. □

You may have noticed that nothing in the definition of compact or completely continuous map depended on the linear space structure of X and Y. For instance, you can make sense of the idea of a bounded set in any metric space. That's certainly true, but we will meet these concepts only in the case of maps between linear spaces. What is more to the point, the ideas and terminology have a close (but, as we shall see, rather awkward) relationship with a subject that does make essential use of the linear structures, as follows. For normed linear spaces X and Y, a function $L : X \to Y$ is *linear* if it preserves the addition and scalar multiplication operations, that is $L(x_1 + x_2) = L(x_1) + L(x_2)$ for $x_1, x_2 \in X$ and $L(cx) = cL(x)$ for $c \in \mathbf{R}$ and $x \in X$. Thus $f(X)$ is a linear subspace of Y and it certainly cannot be relatively compact unless L is the constant function. However, there is a concept of a *compact linear* map $L : X \to Y$, namely, if S is a bounded subset of X, then $L(X)$ is relatively compact in Y. In other words, in the setting of linear maps, "compact" means what "completely continuous" means in the more general setting. That's a pretty messy situation, but the terminology is firmly established so I won't try to change it. If you are aware of the problem and distinguish clearly between the linear and nonlinear contexts, it will not cause trouble.

We next introduce some important normed linear spaces. Let $[a, b]$ be a closed interval in $\mathbf{R}$, let $u : [a, b] \to \mathbf{R}$ be a function, and let $u^{(j)}$ denote the jth derivative of u, with $u^{(0)} = u$. Let $C^k[a, b]$ be the set of all functions $u : [a, b] \to \mathbf{R}$ such that for each of $j = 0, 1, \ldots, k$, the function $u^{(j)}$ is well defined and continuous on $[a, b]$. Then $C^k[a, b]$ is a linear space and we can define a norm $\|.\|_k$ on it, called the *C^k norm*, by

$$\|u\|_k = \sum_{j=0}^{k} \|u^{(j)}\|,$$

where $\|u^{(j)}\|$ is the sup norm of $u^{(j)}$. Notice that the C^0 norm is the sup norm and that then $C^0[a, b]$ is $C[a, b]$.

A function in $C^{k+1}[a, b]$ is also in $C^k[a, b]$, so there is an inclusion $j : C^{k+1}[a, b] \to C^k[a, b]$. We will state the compactness property we have been promising in terms of a property of j because it will be useful in that form. However, it will help to know that the property can also be viewed as a comparison of the C^k and C^{k+1} norms. We will see that if a sequence in $C^{k+1}[a, b]$ is bounded (with respect to the C^{k+1} norm of course), then, when we view the same sequence as lying in $C^k[a, b]$, it has a strong property: it contains a convergent subsequence, but in the sense of convergence in the larger space. Thus requiring boundedness of the sequence in terms of first $k + 1$ derivatives implies convergence (a form of compactness—compare Theorem 2.4) in terms of the first k derivatives. The property is stated in the terminology of not-necessarily linear functions, because that is the way it will be used, though the inclusion of one linear space in another certainly is a linear map.

Theorem 2.7. *The inclusion $j : C^{k+1}[a, b] \to C^k[a, b]$ is a completely continuous map.*

Proof. We have to show that every bounded subset of $C^{k+1}[a, b]$ is relatively compact as a subset of $C^k[a, b]$. This means every sequence in the set contains a convergent subsequence, so we can replace that bounded set by a sequence $\{u_n\}$ which is bounded in $C^{k+1}[a, b]$ and we set out to prove that the sequence contains a subsequence that converges in $C^k[a, b]$. We claim that for each $j \leq k$, the set $\{u_n^{(j)}\}$ in $C[a, b]$ is equicontinuous. The sequence $\{u_n\}$ is bounded and that means there is a number β such that $\|u_n\|_{k+1} < \beta$ for all n. From the definition of the C^k norm, we can see that this implies $|u_n^{(j)}(x)| < \beta$ for all $j \leq k + 1$ and all

$x \in [a,b]$. Given $\epsilon > 0$, we will show that $\delta = \frac{\epsilon}{\beta}$ will establish all the equicontinuity required. In other words, for any $x \in [a,b]$, we will see that if $|x-y| < \delta$, then $|u_n^{(j)}(x) - u_n^{(j)}(y)| < \epsilon$ for any n and any $j \le k$. The reason is the mean value theorem which implies that

$$\left| \frac{u_n^{(j)}(x) - u_n^{(j)}(y)}{x-y} \right| = |u_n^{(j+1)}(c)| < \beta$$

for some $c \in [a,b]$. The Ascoli–Arzela theorem (specifically its Corollary 2.5) gives us a function $g_{<0>}$ in $C[a,b]$ which is the limit of a subsequence of $\{u_n\}$. The derivatives of the functions in the subsequence are still bounded and equicontinuous, so they also have a subsequence which converges to a function in $C[a,b]$ that we will call $g_{<1>}$. If we differentiate the functions in that subsequence, we can repeat the argument to obtain a continuous function $g_{<2>}$. We continue in this way until we have the function $g_{<k>}$. To avoid some potentially disastrous notational problems replace the original sequence, using only those terms that were used in the last step (which are a subset of those used in every previous one), but still call the sequence $\{u_n\}$. In this way, we have constructed functions $g_{<0>}, g_{<1>}, g_{<2>}, \ldots, g_{<k>}$ such that $u_n^{(j)} \to g_{<j>}$ for $j = 0, 1, 2, \ldots, k$, where the arrow indicates uniform convergence. Letting $u = g_{<0>}$, we will show that $g_{<j>} = u^{(j)}$ for $j \le k$, which implies that $u \in C^k[a,b]$ and also that $\{u_n\}$ converges to u in that space, that is, we can make $\|u_n - u\|_k$ as small as we want by choosing n big enough. For $1 \le j \le k$, use the fundamental theorem of calculus to write

$$u_n^{(j-1)}(x) = \int_a^x u_n^{(j)}(t)\, dt + u_n^{(j-1)}(a).$$

Taking the limit, on the left-hand side we have $g_{<j-1>}(x)$ and on the other side we get

$$\int_a^x g_{<j>}(t)\, dt + g_{<j-1>}(a).$$

By uniqueness of the limit, we see that $g_{<j>}$ is the derivative of $g_{<j-1>}$. □

We now have a useful tool for extracting convergent sequences, although I'm afraid you'll have to wait a while to see it employed. The difficulty is that you can't do much with it alone, so we'll have to build more machinery in order to get a good payoff from all the work we did in this section. On the other hand, a lot of the concepts we met here will turn up again.

To conclude the chapter, here is the proof I promised earlier.

Proof of Theorem 2.4 (A metric space is compact if and only if every sequence has a convergent subsequence). If we assume we have a sequence $\{x_n\}$ in a compact space X, it's not hard to extract a convergent subsequence; here's how to do it.

If for each $x \in X$ we could find $\epsilon_x > 0$ such that $B(x;\epsilon_x)$ contained only a finite subset of $\{x_n\}$, then a finite subcover of $\{B(x;\epsilon_x)\}$ would fail to cover all of the sequence, so it could hardly cover X. Thus there exists $x \in X$ such that $B(x;\frac{1}{n})$ contains infinitely many points of the sequence, so we can choose a different point $x_{j(n)} \in B(x;\frac{1}{n})$ in the sequence for each n and that subsequence will do the job.

The proof in the other direction uses the ideas of this chapter. We will show that if X is a space in which every sequence has a convergent subsequence (called a *sequentially compact* space) and X is metric, with metric d, then X is complete and totally bounded, so it is compact by Theorem 2.1. We can get completeness out of the way very quickly. Take a Cauchy sequence $\{x_j\}$ in the sequentially compact metric space X, so there is a subsequence $\{x_{j_k}\}$ converging to some point x, but we want the entire sequence to converge to x. Given $\epsilon > 0$, there exists $N_1 > 0$ such that $j_k \geq N_1$ implies $d(x_{j_k}, x) < \frac{\epsilon}{2}$. By the Cauchy property, there exists $N_2 > 0$ such that $j, j_k \geq N_2$ implies $d(x_j, x_{j_k}) < \frac{\epsilon}{2}$. Therefore, if j is greater than both N_1 and N_2, we get $d(x_j, x) < \epsilon$. To prove total boundedness and finish this chapter at last, we use a contrapositive argument. That is, we suppose that X is not totally bounded so there exists $\epsilon > 0$ for which X has no finite ϵ-net and show that X is therefore not sequentially compact, by constructing a sequence in X that has no convergent subsequence. Let any point of X be x_1, but choose x_2 so it is not within ϵ of x_1, where the ϵ is the one for which there is no finite net. If there were no such x_2, that would mean x_1 by itself was an ϵ-net for X. Choose x_3 to be a point of X at a distance of more than ϵ from both x_1 and x_2. Continuing in this way, we choose $x_n \in X$ that is in the complement of the union of the $B(x_j;\epsilon)$ for $j = 1, 2, \ldots, n-1$. The next point of the sequence has to exist since otherwise the previous points would constitute a finite ϵ-net for X. By construction, no subsequence of $\{x_n\}$ converges. $\square$

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