# Chapter 5
# A Bound in Terms of Fusion Systems

In this chapter we obtain more inequalities on the invariants of a block by using local data. This time the fusion system of the block plays a role. The exposition appeared in [114].

Brauer proved Olsson's Conjecture for 2-blocks with dihedral defect groups using a Galois action on the generalized decomposition numbers (see [41]). We put his approach into an abstract framework. Let $B$ be a $p$-block of a finite group $G$ with defect group $D$, and let $(u, b_u)$ be a subsection for $B$. Let $p^k$ be the order of $u$, and let $\zeta := \zeta_{p^k}$ be a primitive $p^k$-th root of unity. Then there exist integral vectors $a_i^\varphi := (a_i^\varphi(\chi))_{\chi\in\mathrm{Irr}(B)} \in \mathbb{Z}^{k(B)}$ such that

$$d^u_{\chi\varphi} = \sum_{i=0}^{\varphi(p^k)-1} a_i^\varphi(\chi)\zeta^i \tag{5.1}$$

(see Sect. 1.6).

Let $\mathscr{G}$ be the Galois group of the cyclotomic field $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$. Then $\mathscr{G} \cong \mathrm{Aut}(\langle u\rangle) \cong (\mathbb{Z}/p^k\mathbb{Z})^\times$ and we will often identify these groups. We will also interpret the elements of $\mathscr{G}$ as integers in $\{1, \ldots, p^k\}$ by a slight abuse of notation. Then $(u^\gamma, b_u)$ for $\gamma \in \mathscr{G}$ is also a (*algebraically conjugate*) subsection and

$$\gamma(d^u_{\chi\varphi}) = d^{u^\gamma}_{\chi\varphi} = \sum_{i=0}^{\varphi(p^k)-1} a_i^\varphi(\chi)\zeta^{i\gamma}.$$

Now the situation splits naturally into characteristic 2 and odd characteristic, since the structure of the corresponding Galois groups differs significantly.



B. Sambale, *Blocks of Finite Groups and Their Invariants*, Lecture Notes in Mathematics 2127, DOI 10.1007/978-3-319-12006-5_5

## 5.1 The Case $p = 2$

Let $p = 2$, and let $\mathscr{F}$ be the fusion system of $B$. Then by Lemma 1.34 we may assume that $\langle u\rangle$ is fully $\mathscr{F}$-normalized and $\mathrm{C}_D(u)$ is a defect group of $b_u$. As before, $\langle u\rangle$ is also fully $\mathscr{F}$-centralized and

$$\mathrm{Aut}_{\mathscr{F}}(\langle u\rangle) = \mathrm{Aut}_D(\langle u\rangle) = \mathrm{N}_D(\langle u\rangle)/\,\mathrm{C}_D(u).$$

We begin with a refinement of the orthogonality relations. For a subsection $(u, b_u)$ with $\mathrm{IBr}(b_u) = \{\varphi\}$ we set $a_i := a_i^{\varphi}$ for all $i$. Moreover, if $u, v \in D$ are conjugate in $D$, we write $u \sim_D v$.

**Proposition 5.1** *Let $B$ be a 2-block of a finite group with defect group $D$ and fusion system $\mathscr{F}$. Let $(u, b_u)$ be a $B$-subsection such that $l(b_u) = 1$ and $\langle u\rangle \neq 1$ is fully $\mathscr{F}$-normalized of order $2^k$. Then*

$$(a_i, a_j) = \begin{cases} 2|\mathrm{N}_D(\langle u\rangle) \cap \mathrm{C}_D(u^i)/\langle u\rangle| & \text{if } u^j \sim_D u^i \nsim_D u^{j+2^{k-1}}, \\ -2|\mathrm{N}_D(\langle u\rangle) \cap \mathrm{C}_D(u^i)/\langle u\rangle| & \text{if } u^j \nsim_D u^i \sim_D u^{j+2^{k-1}}, \\ 0 & \text{otherwise} \end{cases}$$

*for $i, j \in \{0, \ldots, 2^{k-1} - 1\}$. In particular, $(a_0, a_0) = 2|\mathrm{N}_D(\langle u\rangle)/\langle u\rangle|$.*

*Proof* We set $d^u := (d^u_{\chi\varphi} : \chi \in \mathrm{Irr}(B))$ and $|\mathrm{N}_D(\langle u\rangle) \cap \mathrm{C}_D(u^i)/\,\mathrm{C}_D(u)| = 2^r$. Then

$$\frac{1}{2^{k-1}} \sum_{\gamma\in\mathscr{G}} d^{u^\gamma} \zeta^{-i\gamma} = \frac{1}{2^{k-1}} \sum_{l=0}^{2^{k-1}-1} \sum_{\gamma\in\mathscr{G}} a_l \zeta^{(l-i)\gamma} = a_i$$

for $i = 0, \ldots, 2^{k-1} - 1$. Hence,

$$(a_i, a_j) = 2^{2(1-k)} \sum_{\gamma,\delta\in\mathscr{G}} \big(d^{u^\gamma}, d^{u^\delta}\big) \zeta^{j\delta - i\gamma}.$$

If $u^\gamma$ and $u^\delta$ are conjugate under $\mathrm{Aut}_{\mathscr{F}}(\langle u\rangle)$, we have $\big(d^{u^\gamma}, d^{u^\delta}\big) = 2^d$ by Theorem 1.14. If we regard $\mathrm{Aut}_{\mathscr{F}}(\langle u\rangle)$ as a subgroup of $\mathscr{G}$, this means $\gamma\delta^{-1} \in \mathrm{Aut}_{\mathscr{F}}(\langle u\rangle)$. Therefore,

$$(a_i, a_j) = 2^{2(1-k)+d} \sum_{\gamma\in\mathscr{G}} \sum_{\delta\in\mathrm{Aut}_{\mathscr{F}}(\langle u\rangle)} \zeta^{(j\delta-i)\gamma} = 2^{2(1-k)+d} \sum_{\delta\in\mathrm{Aut}_{\mathscr{F}}(\langle u\rangle)} \sum_{\gamma\in\mathscr{G}} \zeta^{(j\delta-i)\gamma}.$$

Observe that if $|\langle u^i\rangle| \neq |\langle u^j\rangle|$, then $(a_i, a_j) = 0$. If $u^i$ is $\mathscr{F}$-conjugate to $u^j$, then there is a $\delta \in \mathrm{Aut}_{\mathscr{F}}(\langle u\rangle)$ such that $j\delta - i \equiv 0 \pmod{2^k}$. In this case there are precisely $2^r$ such elements and the corresponding sum contributes $2^{r+k-1}$. Similarly, if $u^i$ is $\mathcal{F}$-conjugate to $u^{j+2^{k-1}}$, we get the contribution $-2^{r+k-1}$ in the sum. All other summands vanish. This shows the result. $\square$

**Theorem 5.2** *Let $B$ be a 2-block of a finite group $G$ with defect group $D$ and fusion system $\mathcal{F}$, and let $(u, b_u)$ be a $B$-subsection such that $\langle u\rangle \neq 1$ is fully $\mathcal{F}$-normalized and $b_u$ has Cartan matrix $C_u = (c_{ij})$. Let $\mathrm{IBr}(b_u) = \{\varphi_1, \ldots, \varphi_{l(b_u)}\}$ such that $\varphi_1, \ldots, \varphi_m$ are stable under $\mathrm{N}_D(\langle u\rangle)$ and $\varphi_{m+1}, \ldots, \varphi_{l(b_u)}$ are not. Then $m \geq 1$. Suppose further that $u$ is conjugate to $u^{-5^n}$ for some $n \in \mathbb{Z}$ in $D$. Then*

$$k_0(B) \leq \frac{2|\mathrm{N}_D(\langle u\rangle)/\,\mathrm{C}_D(u)|}{|\langle u\rangle|} \sum_{1\leq i\leq j\leq m} q_{ij}c_{ij} \tag{5.2}$$

*for every positive definite, integral quadratic form $q(x_1, \ldots, x_m) = \sum_{1\leq i\leq j\leq m} q_{ij}x_i x_j$. In particular if $l(b_u) = 1$, we get*

$$k_0(B) \leq 2|\mathrm{N}_D(\langle u\rangle)/\langle u\rangle|. \tag{5.3}$$

*Proof* Let $\chi \in \mathrm{Irr}_0(B)$ and $|\langle u\rangle| = 2^k$ for some $k \geq 1$. We write $d^u_\chi := (d^u_{\chi\varphi_1}, \ldots, d^u_{\chi\varphi_l})$, where $l := l(b_u)$. Then

$$d^u_{\chi\varphi_i} \equiv \gamma(d^u_{\chi\varphi_i}) \equiv \sum_{j=0}^{2^{k-1}-1} a^i_j(\chi) \pmod{\mathrm{Rad}\,\mathcal{O}}$$

for $\gamma \in \mathcal{G}$. In particular $d^u_{\chi\varphi_i} \equiv \overline{d^u_{\chi\varphi_i}} \pmod{\mathrm{Rad}\,\mathcal{O}}$. We write $|\mathrm{C}_D(u)|C_u^{-1} = (\tilde{c}_{ij})$. Then it follows from Proposition 1.36 that

$$\begin{aligned} 0 \not\equiv m^u_{\chi\chi} &\equiv \sum_{1\leq i,j\leq l} \tilde{c}_{ij} d^u_{\chi\varphi_i}\overline{d^u_{\chi\varphi_j}} \equiv \sum_{1\leq i\leq l} \tilde{c}_{ii}(d^u_{\chi\varphi_i})^2 \\ &\equiv \sum_{1\leq i\leq l} \tilde{c}_{ii} \sum_{j=0}^{2^{k-1}-1} a^i_j(\chi)^2 \equiv \sum_{1\leq i\leq l} \tilde{c}_{ii} \sum_{j=0}^{\varphi(2^k)-1} a^i_j(\chi) \pmod{\mathrm{Rad}\,\mathcal{O}}. \end{aligned}$$

Now every $g \in \mathrm{N}_D(\langle u\rangle)$ induces a permutation on $\mathrm{IBr}(b_u)$. Let $P_g$ be the corresponding permutation matrix. Then $g$ also acts on the rows $d^u_i := (d^u_{\chi\varphi_i} : \chi \in \mathrm{Irr}(B))$ for $i = 1, \ldots, l$, and it follows that $C_uP_g = P_gC_u$. Hence, we also have $C_u^{-1}P_g = P_gC_u^{-1}$ for all $g \in \mathrm{N}_D(\langle u\rangle)$. If $\{\varphi_{m_1}, \ldots, \varphi_{m_2}\}$ $(m < m_1 < m_2 \leq l)$ is an orbit under $\mathrm{N}_D(\langle u\rangle)$, it follows that $d^u_{\chi\varphi_{m_1}} \equiv \ldots \equiv d^u_{\chi\varphi_{m_2}} \pmod{\mathrm{Rad}\,\mathcal{O}}$ and $\tilde{c}_{m_1m_1} = \ldots = \tilde{c}_{m_2m_2}$. Since the length of this orbit is even, we get

$$\sum_{1\leq i\leq m} \tilde{c}_{ii} \sum_{j=0}^{2^{k-1}-1} a^i_j(\chi) \not\equiv 0 \pmod 2.$$

In particular, $m \ge 1$. In case $|\langle u\rangle| = 2$ this simplifies to

$$\sum_{1\le i\le m} \tilde{c}_{ii} a_0^i(\chi) \not\equiv 0 \pmod 2.$$

We show that this holds in general. Thus, let $k \ge 2$ and $i \in \{1,\dots,m\}$. Since $(u, b_u)$ is conjugate to $(u^{-5^n}, b_u)$ and $\varphi_i$ is stable, we have

$$\sum_{j=0}^{2^{k-1}-1} a_j^i(\chi)\zeta^j = d^u_{\chi\varphi_i} = d^{u^{-5^n}}_{\chi\varphi_i} = \sum_{j=0}^{2^{k-1}-1} a_j^i(\chi)\zeta^{-5^n j}.$$

Moreover, for every $j \in \{0,\dots,2^{k-1}-1\}$ there is some $j_1 \in \{0,\dots,\varphi(2^k)-1\}$ such that $\zeta^{-5^n j} = \pm\zeta^{j_1}$. In order to compare coefficients observe that

$$\begin{aligned}\zeta^j = \zeta^{-5^n j} &\implies j \equiv -5^n j \pmod{2^k} \implies 1 \equiv -5^n \pmod{2^k/\gcd(2^k, j)}\\ &\implies j = 0.\end{aligned}$$

Hence, the set $\{\pm\zeta^j : j = 1,\dots,2^{k-1}-1\}$ splits under the action of $\langle -5^n + 2^k\mathbb{Z}\rangle$ into orbits of even length. This shows $\sum_{j=0}^{2^{k-1}-1} a_j^i(\chi) \equiv a_0^i(\chi) \pmod 2$. Hence,

$$\sum_{1\le i\le m} \tilde{c}_{ii} a_0^i(\chi) \not\equiv 0 \pmod 2 \tag{5.4}$$

for every $\chi \in \mathrm{Irr}_0(B)$. In particular, there is an $i \in \{1,\dots,m\}$ such that $a_0^i(\chi) \ne 0$. This gives

$$k_0(B) \le \sum_{1\le i\le j\le m} q_{ij}(a_0^i, a_0^j)$$

(see proof of Theorem 4.2).

Now let $k$ again be arbitrary. Observe that $a_0^i = 2^{1-k}\sum_{\gamma\in\mathcal{G}} \gamma(d_i^u)$ for $i \in \{1,\dots,m\}$. By the orthogonality relations for generalized decomposition numbers we have $(d_i^{u^\gamma}, d_j^{u^\delta}) = c_{ij}$ for $\gamma,\delta \in \mathcal{G}$ if $u^\gamma$ and $u^\delta$ are conjugate under $\mathrm{N}_D(\langle u\rangle)$. Otherwise we have $(d_i^{u^\gamma}, d_j^{u^\delta}) = 0$. This implies

$$(a_0^i, a_0^j) = 2^{2(1-k)} \sum_{\gamma,\delta\in\mathcal{G}} (d_i^{u^\gamma}, d_j^{u^\delta}) = \frac{2|\mathrm{N}_D(\langle u\rangle)/\mathrm{C}_D(u)|}{2^k} c_{ij},$$

and (5.2) follows. In case $l = 1$ we have $C = (|\mathrm{C}_D(u)|)$, and (5.3) is also clear. □

In the situation of Theorem 5.2 we have $u \in \mathrm{Z}(\mathrm{C}_G(u))$. Hence, all Cartan invariants $c_{ij}$ are divisible by $|\langle u\rangle|$. This shows that the right hand side of (5.2) is always an integer. It is also known that $k_0(B)$ is divisible by 4 unless $|D| \le 2$.

Observe that the subsection $(u, b_u)$ in Theorem 5.2 cannot be major unless $|\langle u\rangle| \le 2$, since then $u$ would be contained in $\mathrm{Z}(D)$.

If $D$ is rational of nilpotency class (at most) 2, Gluck's Conjecture would imply $m = l(b_u)$ in Theorem 5.2. In this case it suffices to know the Cartan matrix $C_u$ only up to basic sets. For, changing the basic set is essentially the same as taking another quadratic form $q$ (see [172]). This must always hold in case $l(b_u) = 2$. Here we get the following simpler result.

**Theorem 5.3** *Let $p = 2$, and let $(u, b_u)$ be a $B$-subsection such that $\langle u\rangle$ is fully $\mathcal{F}$-normalized and $u$ is conjugate to $u^{-5^n}$ for some $n \in \mathbb{Z}$ in $D$. If $l(b_u) \le 2$, then*

$$k_0(B) \le 2|\mathrm{N}_D(\langle u\rangle)/\langle u\rangle|.$$

*Proof* We use the notation of the proof of Theorem 5.2. We may assume that $l = 2 = m$. Here we can use (5.4) in a stronger sense. Since $|\mathrm{C}_D(u)|$ occurs as elementary divisor of $C_u$ exactly once, we see that the rank of $\frac{|\mathrm{C}_D(u)|}{\det C_u} C_u \pmod 2$ is 1. Hence, $\frac{|\mathrm{C}_D(u)|}{\det C_u} C_u \pmod 2$ has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pmod 2, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2, \qquad \text{or} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \pmod 2.$$

Now it is easy to see that we may change the basic set for $b_u$ such that $|\mathrm{C}_D(u)|c_{11}/\det C_u$ is even and as small as possible. Then we also have to replace the rows $d_1^u$ and $d_2^u$ by linear combinations of each other. This gives rows $\hat{d}_i^u$ and $\hat{a}_j^i$ for $i = 1, 2$ and $j = 0, \ldots, \varphi(2^k) - 1$. Observe that the contributions do not depend on the basic set for $C_u$. Moreover, $\tilde{c}_{11}$ is odd and $\tilde{c}_{22}$ is even. Hence, (5.4) takes the form

$$\hat{a}_0^1(\chi) \not\equiv 0 \pmod 2$$

for all $\chi \in \mathrm{Irr}_0(B)$. Since both $\varphi_1$ and $\varphi_2$ are stable under $\mathrm{N}_D(\langle u\rangle)$, we have $\gamma(\hat{d}_1^u) = \hat{d}_1^u$ for all $\gamma \in \mathrm{Aut}_{\mathcal{F}}(\langle u\rangle)$. Hence,

$$k_0(B) \le (\hat{a}_0^1, \hat{a}_0^1) = \frac{|\mathrm{N}_D(\langle u\rangle)/\,\mathrm{C}_D(u)|c_{11}}{\varphi(2^k)}$$

as above. It remains to show that $c_{11} \le |\mathrm{C}_D(u)|$. The reduction theory of quadratic forms gives an equivalent matrix $C_u' = (c_{ij}')$ such that $0 \le 2c_{12}' \le \min(c_{11}', c_{22}')$ (see Chap. 3). In case $c_{12}' = 0$ we may assume $c_{11} \le c_{11}' = |\mathrm{C}_D(u)|$, since $|\mathrm{C}_D(u)|$ is the largest elementary divisor of $C_u'$. Hence, let $c_{12}' > 0$. Since the entries of $C_u$ and thus also of $C_u'$ are divisible by $\alpha := \det C_u / |\mathrm{C}_D(u)|$, we even have $c_{12}' \geq \alpha$. It follows that

$$3\alpha^2 \leq 3(c_{12}')^2 \leq c_{11}'c_{22}' - (c_{12}')^2 = \det C_u' \leq \frac{|\mathrm{C}_D(u)|^2}{2}$$

and $\alpha \leq |\mathrm{C}_D(u)|/4$. From Eq. (3.1) on page 28 we obtain

$$\begin{aligned}\max(c_{11}', c_{22}') &\leq c_{11}' + c_{22}' - c_{12}' \leq c_{11}' + c_{22}' - \alpha \\ &\leq \alpha \frac{|\mathrm{C}_D(u)|/\alpha + 3}{2} = \frac{|\mathrm{C}_D(u)| + 3\alpha}{2} \leq |\mathrm{C}_D(u)|.\end{aligned}$$

If $\alpha^{-1}c_{11}'$ or $\alpha^{-1}c_{22}'$ is even, the result follows from the minimality of $c_{11}$. Otherwise we replace $C_u'$ by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} C_u' \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} c_{11}' + c_{22}' - 2c_{12}' & c_{12}' - c_{22}' \\ c_{12}' - c_{22}' & c_{22}' \end{pmatrix}.$$

Then $c_{11} \leq c_{11}' + c_{22}' - 2c_{12}' \leq |\mathrm{C}_D(u)|$. This finishes the proof. □

If in the situation of Theorem 5.2 we have $m < l(b_u)$, we really need to know the "exact" Cartan matrix $C_u$ which is unknown in most cases. For $p > 2$ there are not always stable characters in $\mathrm{IBr}(b_u)$ (see Proposition (2E)(ii) and the example following it in [154]).

Let us come back to our initial example. Let $D$ be a (non-abelian) 2-group of maximal class. Then there is an element $x \in D$ such that $|D : \langle x \rangle| = 2$ and $x$ is conjugate to $x^{-5^n}$ for some $n \in \{0, |\langle x \rangle|/8\}$ under $D$. Since $\langle x \rangle \trianglelefteq D$, the subgroup $\langle x \rangle$ is fully $\mathcal{F}$-normalized, and $b_x$ has cyclic defect group $\mathrm{C}_D(x) = \langle x \rangle$. Since, $e(b_x) = 1$, we get $l(b_x) = 1$. Hence, Theorem 5.2 shows Olsson's Conjecture $k_0(B) \leq 4 = |D : D'|$. This was already proved in [41, 212].

## 5.2 The Case $p > 2$

Now we turn to the case where $B$ is a $p$-block of $G$ for an odd prime $p$. We fix some notation for this section. As before $(u, b_u)$ is a $B$-subsection such that $|\langle u \rangle| = p^k$. Moreover, $\zeta \in \mathbb{C}$ is a primitive $p^k$-th root of unity. Since the situation is more complicated for odd primes, we assume further that $l(b_u) = 1$. We write $\mathrm{IBr}(b_u) = \{\varphi_u\}$. Then the generalized decomposition numbers $d^u_{\chi\varphi_u}$ for $\chi \in \mathrm{Irr}(B)$ form a column $d(u)$. Let $d$ be the defect of $b_u$. Since $u \in \mathrm{Z}(\mathrm{C}_G(u))$, $u$ is contained in every defect group of $b_u$. In particular, $k \leq d$. As in the case $p = 2$ we can write

$$d(u) = \sum_{i=0}^{\varphi(p^k)-1} a_i^u \zeta^i$$

with $a_i^u \in \mathbb{Z}^{k(B)}$ (change of notation!). We define the following matrix

$$A := \big(a_i^u(\chi) : i = 0, \ldots, \varphi(p^k) - 1,\ \chi \in \mathrm{Irr}(B)\big) \in \mathbb{Z}^{\varphi(p^k)\times k(B)}.$$

The proof of the main theorem of this section is an application of the next proposition.

**Proposition 5.4** *For every positive definite, integral quadratic form*

$$q(x_1, \ldots, x_{\varphi(p^k)}) = \sum_{1\leq i\leq j\leq\varphi(p^k)} q_{ij} x_i x_j$$

*we have*

$$k_0(B) \leq \sum_{1\leq i\leq j\leq\varphi(p^k)} q_{ij}(a_{i-1}^u, a_{j-1}^u). \tag{5.5}$$

*If $(u, b_u)$ is major, we can replace $k_0(B)$ by $\sum_{i=0}^{\infty} p^{2i} k_i(B)$ in (5.5).*

*Proof* By Lemma 1.37(i) every column $a^u(\chi)$ of $A$ corresponding to a character $\chi$ of height 0 does not vanish. Hence, we have

$$\begin{aligned} k_0(B) &\leq \sum_{\chi\in\mathrm{Irr}(B)} q(a^u(\chi)) = \sum_{\chi\in\mathrm{Irr}(B)} \sum_{1\leq i\leq j\leq\varphi(p^k)} q_{ij} a_{i-1}^u(\chi) a_{j-1}^u(\chi) \\ &= \sum_{1\leq i\leq j\leq\varphi(p^k)} q_{ij}(a_{i-1}^u, a_{j-1}^u). \end{aligned}$$

If $(u, b_u)$ is major and $\chi \in \mathrm{Irr}(B)$, then $p^{-h(\chi)} a^u(\chi)$ is a non-vanishing integral column by Lemma 1.37(ii). In this case we have

$$\sum_{i=0}^{\infty} p^{2i} k_i(B) \leq \sum_{\chi\in\mathrm{Irr}(B)} p^{2h(\chi)} q(p^{-h(\chi)} a^u(\chi)) = \sum_{1\leq i\leq j\leq\varphi(p^k)} q_{ij}(a_{i-1}^u, a_{j-1}^u).$$

The second claim follows. □

Notice that we have used only a weak version of Lemma 1.37 in the proof above.

In order to find a suitable quadratic form it is often very useful to replace $A$ by $UA$ for some integral matrix $U \in \mathrm{GL}(\varphi(p^k), \mathbb{Q})$ (observe that the argument in the proof of Proposition 5.4 remains correct).

However, we need a more explicit expression of the scalar products $(a_i^u, a_j^u)$. For this reason we introduce an auxiliary lemma about inverses of Vandermonde matrices. Let $\mathcal{G} = \{\sigma_1, \ldots, \sigma_{\varphi(p^k)}\}$. For an integer $i \in \mathbb{Z}$ there is $i' \in \{1, \ldots, p^{k-1}\}$ such that $-i \equiv i' \pmod{p^{k-1}}$. We will use this notation for the rest of the section.

**Lemma 5.5** *The inverse of the Vandermonde matrix* $V := \big(\sigma_i(\zeta)^{j-1}\big)_{i,j=1}^{\varphi(p^k)}$ *is given by*

$$V^{-1} = p^{-k}\big(\sigma_j(t_{i-1})\big)_{i,j=1}^{\varphi(p^k)},$$

*where* $t_i = \zeta^{-i} - \zeta^{i'}$.

*Proof* For $i, j \in \{0, \ldots, \varphi(p^k) - 1\}$ we have

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(t_i)\sigma_l(\zeta)^j = \sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j-i} - \zeta^{j+i'}).$$

Assume first that $i = j$. Then $\zeta^{j-i} = 1$ and $j + i' = i + i'$ is divisible by $p^{k-1}$ but not by $p^k$. Hence, $\zeta^{j+i'}$ is a primitive $p$-th root of unity. Since the second coefficient of the $p$-th cyclotomic polynomial $\Phi_p(X) = X^{p-1} + X^{p-2} + \ldots + X + 1$ is 1, we get $\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j+i'}) = -p^{k-1}$. This shows that

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(1 - \zeta^{i+i'}) = \varphi(p^k) + p^{k-1} = p^k.$$

Now let $i \neq j$. Then $j - i \not\equiv 0 \pmod{p^k}$ and $j + i' \not\equiv 0 \pmod{p^k}$. Moreover, $j - i \equiv j + i' \pmod{p^{k-1}}$, since $i + i' \equiv 0 \pmod{p^{k-1}}$. Assume first that $j - i \not\equiv 0 \pmod{p^{k-1}}$. Then $\zeta^{j-i}$ is a primitive $p^s$-th root of unity for some $s \geq 2$. Since the second coefficient of the $p^s$-th cyclotomic polynomial $\Phi_{p^s}(X) = X^{(p-1)p^{s-1}} + X^{(p-2)p^{s-1}} + \ldots + X^{p^{s-1}} + 1$ (see Lemma I.10.1 in [204]) is 0, we have $\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j-i}) = 0$. The same holds for $j + i'$. Finally let $j - i \equiv 0 \pmod{p^{k-1}}$. Then we have (as in the first part of the proof)

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j-i} - \zeta^{j+i'}) = -p^{k-1} + p^{k-1} = 0.$$

This proves the claim. □

Now let $\mathscr{A} := \mathrm{Aut}_{\mathscr{F}}(\langle u\rangle) \le \mathscr{G}$. The next proposition shows that the scalar products $(a_i^u, a_j^u)$ only depend on $p$, $k-d$ and $\mathscr{A}$.

**Proposition 5.6** *We have*

$$p^{k-d}(a_i^u, a_j^u) = |\{\tau \in \mathscr{A} : p^k \mid i - j\tau\}| - |\{\tau \in \mathscr{A} : p^k \mid i + j'\tau\}| \\ + |\{\tau \in \mathscr{A} : p^k \mid i' - j'\tau\}| - |\{\tau \in \mathscr{A} : p^k \mid i' + j\tau\}|. \tag{5.6}$$

*Proof* Let $W := \big(d_{\chi\varphi_u}^{\sigma_i(u)} : i = 1, \ldots, \varphi(p^k),\ \chi \in \mathrm{Irr}(B)\big)$ be a part of the generalized decomposition matrix. If $V$ is the Vandermonde matrix in Lemma 5.5, we have $VA = W$ and $A = V^{-1}W$. This shows

$$\big((a_{i-1}^u, a_{j-1}^u)\big)_{i,j=1}^{\varphi(p^k)} = AA^{\mathrm{T}} = V^{-1}WW^{\mathrm{T}}V^{-\mathrm{T}} = V^{-1}W\overline{W}^{\mathrm{T}}\overline{V}^{-\mathrm{T}}.$$

Now let $S := (s_{ij})_{i,j=1}^{\varphi(p^k)}$, where

$$s_{ij} := \begin{cases} 1 & \text{if } \sigma_i\sigma_j^{-1} \in \mathscr{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the orthogonality relations (see proof of Theorem 5.2) imply $W\overline{W}^{\mathrm{T}} = p^d S$. It follows that

$$p^{2k-d}(a_i^u, a_j^u) = \sum_{l=1}^{\varphi(p^k)} \sigma_l(t_i) \sum_{m=1}^{\varphi(p^k)} s_{lm}\sigma_m(\overline{t_j}) = \sum_{l=1}^{\varphi(p^k)} \sum_{\tau\in\mathscr{A}} \sigma_l(t_i\tau(\overline{t_j}))$$
$$= \sum_{\tau\in\mathscr{A}} \sum_{l=1}^{\varphi(p^k)} \sigma_l((\zeta^{-i} - \zeta^{i'})\tau(\zeta^j - \zeta^{-j'}))$$
$$= \sum_{\tau\in\mathscr{A}} \sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j\tau-i} + \zeta^{i'-j'\tau} - \zeta^{-i-j'\tau} - \zeta^{i'+j\tau}). \tag{5.7}$$

As in the proof of Lemma 5.5 we have

$$\sum_{l=1}^{\varphi(p^k)} \sigma_l(\zeta^{j\tau-i}) = \begin{cases} \varphi(p^k) & \text{if } p^k \mid j\tau - i, \\ 0 & \text{if } p^{k-1} \nmid j\tau - i, \\ -p^{k-1} & \text{otherwise.} \end{cases}$$

This can be combined to

$$\sum_{\tau\in\mathscr{A}}\sum_{l=1}^{\varphi(p^k)}\sigma_l(\zeta^{j\tau-i}) = p^k|\{\tau\in\mathscr{A} : p^k \mid j\tau-i\}| - p^{k-1}|\{\tau\in\mathscr{A} : p^{k-1} \mid j\tau-i\}|.$$

We get similar expressions for the other numbers $i'-j'\tau$, $-i-j'\tau$ and $i'+j\tau$. Since $i+i' \equiv j+j' \equiv 0 \pmod{p^{k-1}}$, we have $j\tau-i \equiv i'-j'\tau \equiv -i-j'\tau \equiv i'+j\tau \pmod{p^{k-1}}$. Thus, the terms of the form $p^{k-1}|\{\ldots\}|$ in (5.7) cancel out each other. This proves the proposition. □

Since the group $\mathrm{Aut}(\langle u\rangle)$ is cyclic, $\mathscr{A}$ is uniquely determined by its order. We introduce a notation.

**Definition 5.7** Let $\mathscr{A}$ be as in Proposition 5.6. Then we define $\Gamma(d,k,|\mathscr{A}|)$ as the minimum of the expressions

$$\sum_{1\le i\le j\le\varphi(p^k)} q_{ij}(a^u_{i-1}, a^u_{j-1})$$

where $q$ ranges over all positive definite, integral quadratic forms. By Proposition 5.4 we have $k_0(B) \le \Gamma(d,k,|\mathscr{A}|)$, and $\sum_{i=0}^{\infty} p^{2i}k_i(B) \le \Gamma(d,k,|\mathscr{A}|)$ if $(u,b_u)$ is major.

We will calculate $\Gamma(d,k,|\mathscr{A}|)$ by induction on $k$. First we collect some easy facts.

**Lemma 5.8** *Let $\mathscr{H} \le (\mathbb{Z}/p^k\mathbb{Z})^\times$ where we regard $\mathscr{H}$ as a subset of $\{1,\ldots,p^k\}$. Then $|\{\sigma\in\mathscr{H} : \sigma\equiv 1 \pmod{p^j}\}| = \gcd(|\mathscr{H}|, p^{k-j})$ for $1\le j\le k$.*

*Proof* The canonical epimorphism $(\mathbb{Z}/p^k\mathbb{Z})^\times \to (\mathbb{Z}/p^j\mathbb{Z})^\times$ has kernel $\mathscr{K}$ of order $p^{k-j}$. Hence, $|\{\sigma\in\mathscr{H} : \sigma\equiv 1 \pmod{p^j}\}| = |\mathscr{H}\cap\mathscr{K}| = \gcd(|\mathscr{H}|, p^{k-j})$, since the $p$-subgroups of the cyclic group $(\mathbb{Z}/p^k\mathbb{Z})^\times$ are totally ordered by inclusion. □

**Lemma 5.9** *We have*

$$(a^u_0, a^u_0) = \big(|\mathscr{A}| + |\mathscr{A}|_p\big)p^{d-k}$$

*and*

$$\frac{p^{k-d}}{\gcd(|\mathscr{A}|_p, j)}(a^u_i, a^u_j) \in \{0, \pm1, \pm2\}$$

*for $i+j > 0$. If $a^u_i \ne 0$ for some $i \ge 1$, then $(a^u_i, a^u_i) = 2p^{d-k}\gcd(|\mathscr{A}|_p, i)$. Moreover, $(a^u_i, a^u_j) = 0$ whenever $\gcd(i, p^{k-1}) \ne \gcd(j, p^{k-1})$.*

*Proof* For $i = j = 0$ we have $i + j'\tau = p^{k-1}\tau \not\equiv 0 \pmod{p^k}$ and $i' + j\tau = p^{k-1} \not\equiv 0 \pmod{p^k}$ for all $\tau \in \mathscr{A}$. Moreover, by Lemma 5.8 there are precisely $|\mathscr{A}|_p$ elements $\tau \in \mathscr{A}$ such that $i' - j'\tau = p^{k-1}(1-\tau) \equiv 0 \pmod{p^k}$. The first claim follows from Proposition 5.6.

Now let $i + j > 0$ and $\tau \in \mathscr{A}$ such that $i \equiv j\tau \pmod{p^k}$. Then we have $j \neq 0$. Assume that also $\tau_1 \in \mathscr{A}$ satisfies $i \equiv j\tau_1 \pmod{p^k}$. Then $j(\tau - \tau_1) \equiv 0 \pmod{p^k}$ and $\tau^{-1}\tau_1 \equiv 1 \pmod{p^k / \gcd(p^k, j)}$. Thus, Lemma 5.8 implies

$$|\{\tau \in \mathscr{A} : p^k \mid i - j\tau\}| \in \{0, \gcd(|\mathscr{A}|_p, j)\}.$$

The same argument also works for the other summands in (5.6), since $\gcd(|\mathscr{A}|_p, j) = \gcd(|\mathscr{A}|_p, j')$. This gives

$$p^{k-d}(a_i^u, a_j^u) \in \{0, \pm\gcd(|\mathscr{A}|_p, j), \pm 2\gcd(|\mathscr{A}|_p, j)\}$$

whenever $i + j > 0$.

Suppose $i \geq 1$ and $i \equiv i\tau \pmod{p^k}$ for some $\tau \in \mathscr{A}$. Then $\tau \equiv 1 \pmod{p}$ and thus $i \equiv i\tau - (i + i')(\tau - 1) \equiv -i'\tau + i + i' \pmod{p^k}$. Hence, $i' \equiv i'\tau \pmod{p^k}$. This shows $|\{\tau \in \mathscr{A} : p^k \mid i - i\tau\}| = |\{\tau \in \mathscr{A} : p^k \mid i' - i'\tau\}|$. Moreover, we have $|\{\tau \in \mathscr{A} : p^k \mid i + i'\tau\}| = |\{\tau \in \mathscr{A} : p^k \mid i\tau^{-1} + i'\}| = |\{\tau \in \mathscr{A} : p^k \mid i' + i\tau\}|$. This shows $a_i^u = 0$ or $(a_i^u, a_i^u) = 2p^d \gcd(|\mathscr{A}|_p, i)/p^k$.

Finally suppose that $\gcd(i, p^{k-1}) \neq \gcd(j, p^{k-1})$. Then $i \not\equiv j\tau \pmod{p^{k-1}}$ and thus $p^k \nmid i - j\tau$ for all $\tau \in \mathscr{A}$. The same holds for the other terms in (5.6), since $i + i' \equiv j + j' \equiv 0 \pmod{p^{k-1}}$. The last claim follows. ◻

**Proposition 5.10** *We have*

$$\Gamma(d, 1, |\mathscr{A}|) = \big(|\mathscr{A}| + (p-1)/|\mathscr{A}|\big)p^{d-1}.$$

*Proof* Since $|\mathscr{A}| \mid p-1$, we have $|\mathscr{A}|_p = 1$. Hence, $(a_0^u, a_0^u) = (|\mathscr{A}|+1)p^{d-1}$ and $(a_i^u, a_j^u) \in \{0, \pm p^{d-1}, \pm 2p^{d-1}\}$ for $i + j > 0$ by Lemma 5.9. First we determine the indices $i$ such that $a_i^u = 0$. For this we use Proposition 5.6. Observe that we always have $i' = 1$. In particular for all $i, j$ we have $p \mid i' - j'\tau$ for $\tau = 1$. It follows that $a_i^u = 0$ if and only if $-i \equiv \tau \pmod{p}$ for some $\tau \in \mathscr{A}$. We write this condition in the form $-i \in \mathscr{A}$. This gives exactly $|\mathscr{A}| - 1$ vanishing rows and columns. Thus, all the scalar products $(a_i^u, a_j^u)$ with $-i \in \mathscr{A}$ or $-j \in \mathscr{A}$ vanish. Hence, assume that $-i \notin \mathscr{A}$ and $-j \notin \mathscr{A}$. Then $(a_i^u, a_j^u) \in \{p^{d-1}, 2p^{d-1}\}$ for $i + j > 0$. In case $(a_i^u, a_j^u) = 2p^{d-1}$ we have $a_i^u = a_j^u$. This happens exactly when $j \neq 0$ and $ij^{-1} \in \mathscr{A}$. Since $-i \notin \mathscr{A}$, the coset $i\mathscr{A}$ in $\mathscr{G}$ does not contain $-1$. Hence, there are precisely $|\mathscr{A}|$ choices for $j$ such that $ij^{-1} \in \mathscr{A}$.

Hence, we have shown that the rows $a_i^u$ for $i = 1, \ldots, p-2$ split into $|\mathscr{A}| - 1$ zero rows and $(p-1)/|\mathscr{A}| - 1$ groups consisting of $|\mathscr{A}|$ equal rows each. If we replace the matrix $A$ by $UA$ for a suitable matrix $U \in \mathrm{GL}(p-1, \mathbb{Z})$, we get a new matrix with exactly $(p-1)/|\mathscr{A}|$ non-vanishing rows (this is essentially the same as taking another (positive definite) quadratic form in (5.5), see [172]). After leaving out the zero rows we get a $(p-1)/|\mathscr{A}| \times (p-1)/|\mathscr{A}|$ matrix

$$AA^{\mathrm{T}} = p^{d-1}\begin{pmatrix} |\mathscr{A}|+1 & 1 & \ldots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \ldots & 1 & 2 \end{pmatrix}.$$

Now we can apply the quadratic form $q$ corresponding to the Dynkin diagram $A_{(p-1)/|\mathscr{A}|}$ in Eq. (5.5). This gives

$$\Gamma(d,1,|\mathscr{A}|) \le \big(|\mathscr{A}| + (p-1)/|\mathscr{A}|\big)p^{d-1}.$$

On the other hand, $p^{1-d}AA^{\mathrm{T}}$ is the square of the matrix

$$\begin{pmatrix} 1 \cdots 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & & & 1 \end{pmatrix}$$

which has exactly $|\mathscr{A}|+(p-1)/|\mathscr{A}|$ columns. This shows that $\Gamma(d,1,|\mathscr{A}|)$ cannot be smaller. □

The next proposition gives an induction step.

**Proposition 5.11** *If* $|\mathscr{A}|_p \ne 1$, *then*

$$\Gamma(d,k,|\mathscr{A}|) = \Gamma(d,k-1,|\mathscr{A}|/p).$$

*Proof* Since $|\mathscr{A}|_p \ne 1$, we have $k \ge 2$. Let $i \in \{1,\ldots,\varphi(p^k)-1\}$ such that $\gcd(i,p) = 1$. We will see that $(a_i^u, a_i^u) = 0$ and thus $a_i^u = 0$. By Lemma 5.9 and Eq. (5.6) it suffices to show that there is some $\tau \in \mathscr{A}$ such that $p^k \mid i' + i\tau$. We can write this in the form $-i^{-1}i' \in \mathscr{A}$, since $i$ represents an element of $(\mathbb{Z}/p^k\mathbb{Z})^\times$. Now let $-i' = i + \alpha p^{k-1}$ for some $\alpha \in \mathbb{Z}$. Then $-i^{-1}i' = 1 + i^{-1}\alpha p^{k-1}$ is an element of order $p$ in $\mathscr{G}$. Since $\mathscr{G}$ has only one subgroup of order $p$, it follows that $-i^{-1}i' \in \mathscr{A}$.

Hence, in order to apply Proposition 5.4 it remains to consider the indices which are divisible by $p$. Let $\overline{\mathscr{A}}$ be the image of the canonical map $(\mathbb{Z}/p^k\mathbb{Z})^\times \to (\mathbb{Z}/p^{k-1}\mathbb{Z})^\times$ under $\mathscr{A}$. Then $|\overline{\mathscr{A}}| = |\mathscr{A}|/p$ (cf. Lemma 5.8). If $i$ and $j$ are divisible by $p$, we have

$$|\{\tau \in \mathscr{A} : p^k \mid i + j\tau\}| = p \cdot |\{\tau \in \overline{\mathscr{A}} : p^{k-1} \mid (i/p) + (j/p)\tau\}|.$$

A similar equality holds for the other summands in (5.6). Here observe that $(i/p)' = i'/p$, where the dash on the left refers to the case $p^{k-1}$. Thus, the remaining matrix is just the matrix in case $p^{k-1}$. Hence, $\Gamma(d,k,|\mathscr{A}|) = \Gamma(d,k-1,|\overline{\mathscr{A}}|) = \Gamma(d,k-1,|\mathscr{A}|/p)$. □

Now we are in a position to prove the main theorem of this section.

**Theorem 5.12** *Let $B$ be a $p$-block of a finite group where $p$ is an odd prime, and let $(u,b_u)$ be a $B$-subsection such that $l(b_u)=1$ and $b_u$ has defect $d$. Moreover, let $\mathscr{F}$ be the fusion system of $B$, and let $|\mathrm{Aut}_{\mathscr{F}}(\langle u\rangle)| = p^s r$ where $p \nmid r$ and $s \geq 0$. Then we have*

$$k_0(B) \leq \frac{|\langle u\rangle| + p^s(r^2-1)}{|\langle u\rangle|\cdot r}p^d. \tag{5.8}$$

*If (in addition) $(u,b_u)$ is major, we can replace $k_0(B)$ by $\sum_{i=0}^{\infty} p^{2i}k_i(B)$ in (5.8).*

*Proof* As before let $|\langle u\rangle| = p^k$. We will prove by induction on $k$ that

$$\Gamma(d,k,p^s r) = \frac{p^k + p^s(r^2-1)}{p^k r}p^d.$$

By Proposition 5.10 we may assume $k \geq 2$. By Proposition 5.11 we can also assume that $s = 0$. As before we consider the matrix $A$. Like in the proof of Proposition 5.11 it is easy to see that the indices divisible by $p$ form a block of the matrix $AA^{\mathrm{T}}$ which contributes $\Gamma(d,k-1,r)/p$ to $\Gamma(d,k,r)$. It remains to deal with the matrix $\tilde{A} := \big(a_i^u : \gcd(i,p)=1\big)$. By Lemma 5.9 the entries of $p^{k-d}\tilde{A}\tilde{A}^{\mathrm{T}}$ lie in $\{0,\pm1,\pm2\}$. Moreover, if $\gcd(i,p)=1$ we have $(a_i^u,a_i^u) = 2p^{d-k}$ (see proof of Proposition 5.11).

With the notation of the proof of Proposition 5.6 we have $VA = W$. In particular $\mathrm{rk}\,AA^{\mathrm{T}} = \mathrm{rk}\,A = \mathrm{rk}\,W = |\mathscr{G} : \mathscr{A}|$. If we set $A_1 := \big(a_i^u : p \mid i\big)$, it also follows that $\mathrm{rk}\,A_1A_1^{\mathrm{T}} = \mathrm{rk}\,A_1 = \varphi(p^{k-1})/r$. Since the rows of $\tilde{A}$ are orthogonal to the rows of $A_1$ (see Lemma 5.9), we see that $\mathrm{rk}\,\tilde{A} = (\varphi(p^k)-\varphi(p^{k-1}))/r = p^{k-2}(p-1)^2/r$.

Now we will find $p^{k-2}(p-1)^2/r$ linearly independent rows of $\tilde{A}$. For this observe that $\mathscr{A}$ acts on $\Omega := \{i : 1 \leq i \leq p^{k-1},\ \gcd(i,p)=1\}$ by ${}^\tau i := \tau\cdot i \pmod{p^{k-1}}$ for $\tau \in \mathscr{A}$. Since $p \nmid r$, every orbit has length $r$ (see Lemma 5.8). We choose a set of representatives $\Delta$ for these orbits. Then $|\Delta| = p^{k-2}(p-1)/r$. Finally for $i \in \Delta$ we set $\Delta_i := \{i + jp^{k-1} : j = 0,\ldots,p-2\}$. We claim that the rows $a_i^u$ with $i \in \bigcup_{j\in\Delta}\Delta_j$ are linearly independent. We do this in two steps.

**Step 1:** $(a_i^u,a_j^u) = 0$ for $i,j \in \Delta$, $i \neq j$.

We will show that all summands in (5.6) vanish. First assume that $i \equiv j\tau \pmod{p^k}$ for some $\tau \in \mathscr{A}$. Then of course we also have $i \equiv j\tau \pmod{p^{k-1}}$ which contradicts the choice of $\Delta$. Exactly the same argument works for the other summands. For the next step we fix some $i \in \Delta$.

**Step 2:** $a_j^u$ for $j \in \Delta_i$ are linearly independent.

It suffices to show that the matrix $A' := p^{k-d}(a_l^u, a_m^u)_{l,m\in\Delta_i}$ is invertible. We already know that the diagonal entries of $A'$ equal 2. Now write $m = l + jp^{k-1}$ for some $j \neq 0$. We consider the summands in (5.6). Assume that there is some $\tau \in \mathscr{A}$ such that $l \equiv m\tau \equiv (l + jp^{k-1})\tau \pmod{p^k}$. Then we have $\tau \equiv 1 \pmod{p^{k-1}}$ which implies $\tau = 1$. However, this contradicts $j \neq 0$. On the other hand we have $l' \equiv m'\tau \equiv l'\tau \pmod{p^k}$ for $\tau = 1 \in \mathscr{A}$. Now assume $-l \equiv m'\tau \pmod{p^k}$. Then the argument above implies $\tau = 1$ and $l + l' \equiv 0 \pmod{p^k}$ which is false. Similarly the last summand in (5.6) equals 0. Thus, we have shown that $A' = (1 + \delta_{lm})_{l,m\in\Delta_i}$ is invertible.

Therefore we have constructed a basis for the row space of $\tilde{A}$. Hence, there exists an integral matrix $U \in \mathrm{GL}(p^{k-2}(p-1)^2, \mathbb{Q})$ such that the only non-zero rows of $U\tilde{A}$ are $a_i^u$ for $i \in \bigcup_{j\in\Delta}\Delta_j$. Then we can leave out the zero rows and obtain a matrix (still denoted by $\tilde{A}$) of dimension $p^{k-2}(p-1)^2/r$. Moreover, the two steps above show that $p^{k-d}\tilde{A}\tilde{A}^{\mathrm{T}}$ consists of $p^{k-2}(p-1)/r$ blocks of the form $(1+\delta_{ij})_{1\le i,j\le p-1}$. Thus, an application of the quadratic form $q$ corresponding to the Dynkin diagram $A_{p^{k-2}(p-1)^2/r}$ in Eq. (5.5) gives

$$\Gamma(d,k,r) \le \frac{\Gamma(d,k-1,r)}{p} + \frac{p^{k-1}(p-1)}{p^k r}p^d = \frac{p^k + r^2 - 1}{p^k r}p^d.$$

The minimality of $\Gamma(d,k,r)$ is not so clear as in the proof of Proposition 5.10, since here we do not know if $\det U \in \{\pm 1\}$. However, it suffices to give an example where $k_0(B) = \Gamma(d,k,r)$. By Proposition 5.6 we already know that $\Gamma(d,k,r) = p^{d-k}\Gamma(k,k,r)$. Hence, we may assume $d = k$. Let $G = \langle u\rangle \rtimes C_r$ and $B$ be the principal block of $G$. Then it is easy to see that the hypothesis of the theorem is satisfied. Moreover,

$$k_0(B) = k(B) = \frac{|D|-1}{r} + r = \Gamma(d,k,r).$$

Hence, the proof is complete. □

We add some remarks. It is easy to see that the right hand side of (5.8) is always an integer. Moreover, if $\mathscr{A} = \mathscr{G}$ (i.e. $s = k-1$ and $r = p-1$) or $\mathscr{A}$ is a $p$-group (i.e. $r = 1$), we get the same bound as in Theorem 4.12 and Proposition 4.7. In all other cases Theorem 5.12 really improves Theorem 4.12 and Proposition 4.7. For $k \ge 2$ the case $s = 0$ and $r = p-1$ gives the best bound for $k_0(B)$. If $k$ tends to infinity, $\Gamma(d,k,p-1)$ goes to $p^d/(p-1)$.

Regarding Olsson's Conjecture, we have to say (in contrast to the case $p = 2$) that Olsson's Conjecture does not follow from Theorem 5.12 if it does not already follow from Theorem 4.12, since the right hand side of (5.8) is always larger than $p^{d-1}$.

In the proof we already saw that Inequality (5.8) is sharp for blocks with cyclic defect groups. Perhaps it is possible that this can provide a more elementary proof of Dade's Theorem 8.6. For this it would be sufficient to bound $l(B)$ from below, since the difference $k(B) - l(B)$ is locally determined.

As an application of Theorem 5.12 we give a concrete example. Let $B$ be an 11-block with defect group $D \cong C_{11} \times C_{11}$ and inertial index $e(B) = 5$ (for smaller primes results by Usami and Puig give more complete information, e.g. [227, 270]). Assume that $\mathrm{Aut}_{\mathscr{F}}(D)$ acts diagonally (and thus fixed point freely) on both factors $C_{11}$. Then we have $l(b_u) = 1$ for all non-trivial subsections $(u, b_u)$. Then Theorem 5.12 gives $k(B) \le 77$ while Theorem 4.2 only implies $k(B) \le 121$. Also Theorem 1.39 is useless here. However, for the principal block $B$ of $G = D \rtimes \mathrm{Aut}_{\mathscr{F}}(D)$ we have $k(B) = 29$.

As it was pointed out earlier, for odd primes $p$ and $l(b_u) > 1$ there is not always a stable character in $\mathrm{IBr}(b_u)$ under $\mathrm{N}_G(\langle u \rangle, b_u)$, even for $l(b_u) = 2$. However, the situation is better if we consider the principal block.

**Proposition 5.13** *Let $B$ be the principal $p$-block of $G$ for an odd prime $p$, and let $(u, b_u)$ be a $B$-subsection such that $l(b_u) = 2$, and $b_u$ has defect $d$ and Cartan matrix $C_u = (c_{ij})$. Then we may choose a basic set for $C_u$ such that $p^d c_{11} / \det C_u$ is divisible by $p$. Moreover, let $\mathscr{F}$ be the fusion system of $B$ and $|\mathrm{Aut}_{\mathscr{F}}(\langle u \rangle)| = p^s r$, where $p \nmid r$ and $s \ge 0$. Then we have*

$$k_0(B) \le \frac{|\langle u \rangle| + p^s(r^2 - 1)}{|\langle u \rangle| \cdot r} c_{11}.$$

*Proof* By Brauer's Third Main Theorem, $b_u$ is the principal block of $\mathrm{C}_G(u)$ and so $\mathrm{IBr}(b_u)$ contains the trivial Brauer character. Hence, both characters of $\mathrm{IBr}(b_u)$ are stable under $\mathrm{N}_G(\langle u \rangle)$. As in the proof of Theorem 5.3, $\frac{p^d}{\det C_u} C_u \pmod{p}$ has rank 1. Hence, we can choose a basic set for $C_u$ such that $p^d c_{11} / \det C_u$ and $p^d c_{12} / \det C_u$ are divisible by $p$. As in the proof of Theorem 5.3, the rows $d_i^u$ and $a_j^i$ become $\hat{d}_i^u$ and $\hat{a}_j^i$ for $i = 1, 2$ and $j = 0, \ldots, \varphi(|\langle u \rangle|) - 1$. Write $p^d C_u^{-1} = (\tilde{c}_{ij})$. For $\chi \in \mathrm{Irr}_0(B)$ we have

$$0 \not\equiv m_{\chi\chi}^u \equiv \tilde{c}_{11} \big(\hat{d}_{\chi\varphi_1}^u\big)^2 \pmod{\mathrm{Rad}\,\mathscr{O}}.$$

In particular, $\hat{a}_j^1(\chi) \ne 0$ for some $j \in \{0, \ldots, \varphi(p^k) - 1\}$. Now since

$$(\hat{d}_1^u, \gamma(\hat{d}_1^u)) = \begin{cases} c_{11} & \text{if } \gamma \in \mathscr{A}, \\ 0 & \text{if } \gamma \in \mathscr{G} \setminus \mathscr{A}, \end{cases}$$

the proof works as in case $l(b_u) = 1$. □

http://www.springer.com/978-3-319-12005-8

Blocks of Finite Groups and Their Invariants
Sambale, B.
2014, XIII, 243 p. 1 illus., Softcover
ISBN: 978-3-319-12005-8