Chapter 2
The Model and Preliminaries

2.1 The Goal of This Chapter and the System with Persistent Memory

In this book, we study certain results on the controllability of distributed systems with persistent memory (in the final section of this chapter, we show the derivation of the heat equation with memory and the equation of viscoelasticity). In this chapter, we define and give formulas for the solutions of the system under study and we derive their properties, using an operator approach. We define the notion of controllability and prove the key results relevant to the study of control problems. In particular, we prove that signal propagates with finite velocity, as in the case of the (memoryless) wave equation. A special and important case of the equation with persistent memory is the telegrapher's equation. In Sect. 2.6.3, we use this important example to contrast the properties of the systems with memory and those of the (memoryless) wave and heat equations.

We strive for simplicity of presentation and study the simplest significant case[1]:

$$w''(x,t) = 2cw'(x,t) + c_0^2 \Delta w(x,t) + \int_0^t M(t-s)\Delta w(x,s)\mathrm{d}s + F(x,t) \quad (2.1)$$

(here $c_0^2 > 0$ and Δ is the laplacian in the space variable x). The initial conditions are

$$w(x,0) = u_0(x)\,, \quad w'(x,0) = v_0(x). \quad (2.2)$$

We assume $x \subseteq \Omega \in \mathbb{R}^d$ ($d \leq 3$ is the case of physical interest).

System (2.1) can be written in the following equivalent form

[1] We introduced the velocity term $2cw'$, which has a role in the application of moment methods. A term $c_1 w$ does not make any difference and we ignore it.

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$$w'(x,t) = 2cw(x,t) + \int_0^t N(t-s)\Delta w(x,s)\mathrm{d}s + H(x,t)\,, \qquad w(x,0) = u_0(x) \tag{2.3}$$

where

$$H(x,t) = \int_0^t F(x,s)\mathrm{d}s + [v_0(x) - 2cu_0(x)]\,, \quad N(t) = c_0^2 + \int_0^t M(s)\mathrm{d}s. \tag{2.4}$$

We can pass from one system to the other and study these systems in a unified way, but they have different physical interpretations, see Sect. 2.6.

The "initial" time $t_0 = 0$ is the time after which a control f is applied to the system. Up to now, controllability has been mostly studied when the control acts in the Dirichlet boundary condition (the important case that the control is a traction on the boundary seems not yet sufficiently studied):

$$w(x,t) = f(x,t)\,, \qquad x \in \Gamma \subseteq \partial\Omega\,, \qquad w(x,t) = 0\,, \qquad x \in \partial\Omega \setminus \Gamma. \tag{2.5}$$

We call Γ the ACTIVE PART of the boundary, and we do not exclude $\Gamma = \partial\Omega$. We study whether it is possible to force $(w(x,t), w_t(x,t))$ to hit prescribed targets at some time $T > 0$ (see the precise definition in Sect. 2.3).

These general statements are now sufficient for the introduction of the assumptions and suitable shorthand notations.

The following assumptions are always used, and not explicitly repeated:

- the region Ω (on one side of its boundary) is bounded with C^2 boundary.
- Γ is relatively open in $\partial\Omega$.
- the kernel $M(t)$ is of class $C^2(0,T)$ for every $T > 0$ and $c_0^2 > 0$. So, the kernel $N(t)$ is of class $C^3(0,T)$ for every $T > 0$ and $N(0) > 0$.

We shall see that $c_0^2 > 0$ implies that the signals propagate with finite velocity. The second principle of thermodynamics imposes further restrictions to the kernels $M(t)$ and $N(t)$ (see [33]), which are of no use in the study of controllability.

Notations

We recall that γ_0 denotes the trace on $\partial\Omega$ and γ_1 denotes the exterior normal derivative on $\partial\Omega$.

As stated in Sect. 1.3, the convolution is denoted $*$:

$$(f * g)(t) = \int_0^t f(t-s)g(s)\mathrm{d}s\,.$$

Unless needed for clarity, dependence on the time or space variables is not indicated and the apex denotes time derivative. So, Eqs. (2.1) and (2.3) can be written respectively as

$$w'' = 2cw' + c_0^2 \Delta w + M * \Delta w + F\,, \qquad w' = 2cw + N * \Delta w + H\,.$$

The function f is called a *control* while the functions F and H are the (*distributed*) *affine terms*. As explained in Sect. 2.6, the functions F and H depend on the history of the system for $t < 0$.

Dependence of the solutions on the initial conditions and on the affine term is not indicated. We use w^f to denote dependence on the control f, when needed for clarity.

We have to consider both controlled systems, i.e., systems with f any (square integrable) function, and *uncontrolled systems,* i.e., systems with the Dirichlet boundary conditions put equal 0 on the whole of $\partial\Omega$. Solutions of controlled systems will be denoted u or w, while solutions of uncontrolled systems will be denoted with a Greek letter. For example, $\phi'' = \Delta\phi$, $\phi = 0$ on $\partial\Omega$.

Controllability is studied in *real* Hilbert spaces but, when needed the spaces are complexified without any specific observation (for example when defining the operator $\mathscr{A}$ and the sine and cosine operators).

The Semigroup and the Cosine Operator Generated by A

We consider the operator $A\colon L^2(\Omega) \mapsto L^2(\Omega)$ defined in (1.31):

$$\operatorname{dom} A = H^2(\Omega) \cap H_0^1(\Omega)\,, \qquad A\phi = \Delta\phi. \tag{2.6}$$

Some of its properties have been described in Sect. 1.3. In particular, we recall that $\{\phi_n\}$ denotes a sequence of eigenvectors of A, which is an orthormal basis of $L^2(\Omega)$.

We need further properties. First, we recall that a function $K(t)$ from $[a,b]$ to $\mathscr{L}(H)$ (H a Banach or Hilbert space) is a STRONGLY CONTINUOUS FUNCTION when the H-valued functions $K(t)h$ are continuous for every $h \in H$.

The operator valued function $t \mapsto \mathrm{e}^{At}$ defined by

$$\mathrm{e}^{A0} = I\,, \qquad \mathrm{e}^{At}\left(\sum_{n=1}^{+\infty} \alpha_n\phi_n\right) = \sum_{n=1}^{+\infty} \mathrm{e}^{-\lambda_n^2 t}\alpha_n\phi_n\,, \qquad t > 0 \tag{2.7}$$

is the STRONGLY CONTINUOUS SEMIGROUP generated by A. Its properties are:

- the operator e^{At} is defined for every $t \geq 0$ and $\mathrm{e}^{At} \in \mathscr{L}(L^2(\Omega))$ is selfadjoint.
- for $t \geq 0$, $\tau \geq 0$ we have: $\mathrm{e}^{A(t+\tau)} = \mathrm{e}^{At}\mathrm{e}^{A\tau}$ and $\mathrm{e}^{A0} = I$.
- the transformation $t \mapsto \mathrm{e}^{At}$ is strongly continuous.
- a strongly continuous semigroup is a STRONGLY CONTINUOUS GROUP when it is defined also for $t < 0$ and the properties stated above hold for t and τ in $\mathbb{R}$. It is known that the semigroup (2.7) *is not a group.*

Let $\mathscr{A} = \mathrm{i}(-A)^{1/2}$. It is known (see [10, 27]) that $\mathrm{e}^{\mathscr{A}t}$ is a C_0-*group* of operators on $L^2(\Omega)$. In terms of the Fourier expansion in (1.32) and (1.33) we have:

$$\text{if} \quad y = \sum_{n=1}^{+\infty} \alpha_n \phi_n \in L^2(\Omega) \quad \text{then} \quad \begin{cases} \mathscr{A} y = \mathrm{i}\left(\sum_{n=1}^{+\infty} \lambda_n \alpha_n \phi_n\right), \\ \mathrm{e}^{\mathscr{A}t} y = \sum_{n=1}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda_n t} \alpha_n \phi_n . \end{cases} \tag{2.8}$$

The $\mathscr{L}(L^2(\Omega))$-operator valued function

$$R_+(t) = \frac{1}{2}\left[\mathrm{e}^{\mathscr{A}t} + \mathrm{e}^{-\mathscr{A}t}\right] \qquad t \in \mathbb{R}$$

is the COSINE OPERATOR generated by A. Its key property is the COSINE FORMULA

$$R_+(t)R_+(\tau) = \frac{1}{2}\left(R_+(t) + R_+(\tau)\right) \qquad \forall t\,, \tau \in \mathbb{R}. \tag{2.9}$$

It is convenient to introduce the operators

$$R_-(t) = \frac{1}{2}\left[\mathrm{e}^{\mathscr{A}t} - \mathrm{e}^{-\mathscr{A}t}\right], \qquad S(t) = \mathscr{A}^{-1} R_-(t)\,, \qquad t \in \mathbb{R}\,.$$

The operator $S(t)$ is the SINE OPERATOR (generated by A).

The following properties are known (see [27, 60, 91]):

- $R_+(t)$, $R_-(t)$ and $S(t)$ are selfadjoint continuous operators for every $t \in \mathbb{R}$ and they are strongly continuous functions of $t \in \mathbb{R}$.
- $S(t)$ takes values in dom $\mathscr{A}$.
- for all $z \in L^2(\Omega)$ we have $S(t)z = \int_0^t R_+(r)z\,\mathrm{d}r$.
- for every $z \in$ dom $\mathscr{A}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} R_+(t)z = \mathscr{A} R_-(t)z = AS(t)z\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t} R_-(t)z = \mathscr{A} R_+(t)z\,.$$

- if $z \in \mathscr{D}(\Omega)$ then $R_+(t)z$, $R_-(t)z$, $\mathrm{e}^{\mathscr{A}t}z$ and $\mathrm{e}^{At}z$ are of class C^∞ because $\mathscr{D}(\Omega) \subseteq$ dom $\mathscr{A}^k$ for every k.
- The operators $R_+(t)$, $R_-(t)$, $S(t)$ transform $H_0^1(\Omega)$ to itself and can be extended by continuity to $H^{-1}(\Omega)$.

Formulas (2.8) give

$$R_+(t)\left(\sum_{n=1}^{+\infty} \alpha_n \phi_n\right) = \sum_{n=1}^{+\infty} \left(\alpha_n \cos \lambda_n t\right) \phi_n(x)\,,$$

$$R_-(t)\left(\sum_{n=1}^{+\infty}\alpha_n\phi_n\right)=i\left(\sum_{n=1}^{+\infty}(\alpha_n\sin\lambda_n t)\,\phi_n(x)\right),\tag{2.10}$$
$$S(t)\left(\sum_{n=1}^{+\infty}\alpha_n\phi_n\right)=\left(\sum_{n=1}^{+\infty}\left(\alpha_n\frac{\sin\lambda_n t}{\lambda_n}\right)\phi_n(x)\right).$$

We shall use the following integration by parts formulas, which hold for $u(t)\in C^1(0,T;L^2(\Omega))$ (see [74]):

$$\int_0^t R_+(c_0(t-s))u'(s)\mathrm{d}s=u(t)-R_+(c_0t)u(0)+c_0\mathscr{A}\int_0^t R_-(c_0(t-s))u(s)\mathrm{d}s,\tag{2.11}$$

$$\int_0^t R_-(c_0(t-s))u'(s)\mathrm{d}s=-R_-(c_0t)u(0)+c_0\mathscr{A}\int_0^t R_+(c_0(t-s))u(s)\mathrm{d}s.\tag{2.12}$$

The Dirichlet Operator

We introduce the DIRICHLET OPERATOR D:

$$u=Df\iff\begin{cases}\Delta u(x)=0\text{ in }\Omega\\ u(x)\ \ =f(x)\text{ if }x\in\Gamma,\quad u(x)=0\text{ if }x\in\partial\Omega\setminus\Gamma.\end{cases}\tag{2.13}$$

The operator D, initially defined on "smooth" functions f, admits an extension $D\in\mathscr{L}\left(L^2(\Gamma),L^2(\Omega)\right)$. The function $u=Df\in L^2(\Omega)$ is the (unique) solution of the boundary value problem (2.13).

Let $\phi\in\operatorname{dom}A=H^2(\Omega)\cap H_0^1(\Omega)$. Then, we have (see [93, Proposition 10.6.1] and note that our operator A is $-A_0$ in [93]):

$$\int_\Omega A\phi Df\,\mathrm{d}x=\int_\Gamma(\gamma_1\phi)\,f\,\mathrm{d}\Gamma+\int_\Omega\phi\Delta Df=\int_\Gamma(\gamma_1\phi)\,f\,\mathrm{d}\Gamma,\tag{2.14}$$

$$A\phi_n=-\lambda_n^2\phi_n\implies\int_\Omega\phi_n Df\,\mathrm{d}x=-\frac{1}{\lambda_n^2}\int_\Gamma(\gamma_1\phi_n)\,f\,\mathrm{d}\Gamma.\tag{2.15}$$

2.1.1 The Wave Equation

We shall consider Eqs. (2.1) or (2.3) as a perturbation of a suitable wave type equation. With the goal of defining the solutions of the systems with persistent memory, we recall few crucial facts on the problem

$$u'' = c_0^2 \Delta u + G(x,t)\ , \quad \begin{cases} u(x,0) = u_0(x), & u'(x,0) = v_0(x), \\ u(x,t) = f(x,t)\ x \in \Gamma, & u(x,t) = 0\ x \in \partial\Omega \setminus \Gamma. \end{cases} \tag{2.16}$$

Thanks to the linearity of the problem, we can consider separately the dependence of the solution on the initial conditions (u_0, v_0), the distributed affine term G and the boundary control f. We have (see for example [60, 61, 66]):

Theorem 2.1 *Let $T > 0$. The following properties hold for problem (2.16):*

1. *the transformation*

$$(u_0, v_0) \mapsto \big(u(\cdot,t), u'(\cdot,t)\big)\ :\ H_0^1(\Omega) \times L^2(\Omega) \mapsto C([0,T]; H_0^1(\Omega) \times L^2(\Omega))$$

is (affine) linear and continuous. It is also (affine) linear and continuous as a transformation $L^2(\Omega) \times H^{-1}(\Omega) \mapsto C\left([0,T]; L^2(\Omega) \times H^{-1}(\Omega)\right)$. A short-hand notation is

$$u \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))\ .$$

2. *the transformation $G \mapsto \big(u(\cdot,t), u'(\cdot,t)\big)$ is (affine) linear and continuous from $L^1(0,T; L^2(\Omega))$ to $C\left([0,T]; H_0^1(\Omega) \times L^2(\Omega)\right)$, hence also from $L^1(0,T; L^2(\Omega))$ to $C\left([0,T]; L^2(\Omega) \times H^{-1}(\Omega)\right)$.*
3. *the transformation from the boundary control f to $\big(u(\cdot,t), u'(\cdot,t)\big)$ is (affine) linear and continuous from $L^2(0,T; L^2(\partial\Omega))$ to $C\left([0,T]; L^2(\Omega) \times H^{-1}(\Omega)\right)$.*

So, (when the control acts on the boundary, the case we shall study) controllability has to be considered in $L^2(\Omega) \times H^{-1}(\Omega)$.

The solution $u = u(x,t) \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ of Problem (2.16) is given by (see [60, 61] and recall $c_0 > 0$)

$$\begin{aligned} u(t) = R_+(c_0 t)u_0 + \frac{1}{c_0}\mathscr{A}^{-1} R_-(c_0 t)v_0 + \frac{1}{c_0}\mathscr{A}^{-1}\int_0^t R_-(c_0(t-s))G(s)\mathrm{d}s \\ - c_0 \mathscr{A} \int_0^t R_-(c_0(t-s))Df(s)\mathrm{d}s, \end{aligned} \tag{2.17}$$

$$u'(t) = c_0 \mathscr{A} R_-(c_0 t)u_0 + R_+(c_0 t)v_0 + \int_0^t R_+(c_0(t-s))G(s)\mathrm{d}s$$

$$- c_0^2 A \int_0^t R_+(c_0(t-s))Df(s)\mathrm{d}s. \tag{2.18}$$

The function u in (2.17) is not a differentiable function, and cannot be replaced in both the sides of the equation. So, we define:

Definition 2.1 The function u in (2.17) is the MILD SOLUTION of Eq. (2.16) (in the next chapters the term "solution" will always denote a mild solution). It is a REGULAR SOLUTION when

1. $u(t) - Df(t) \in C([0,T]; \mathrm{dom} A) \cap C^1([0,T]; \mathrm{dom}\mathscr{A}) \cap C^2([0,T]; L^2(\Omega))$.
2. $u(0) = u_0$ and $u'(0) = v_0$.

Note the sense in which the boundary condition is satisfied by the regular solutions:

$$u(t) - Df(t) \in \mathrm{dom} A = H^2(\Omega) \cap H_0^1(\Omega). \tag{2.19}$$

The next result will be used to justify the definition of mild solution:

Theorem 2.2 *Let $u_0 \in$ dom A, $v_0 \in$ dom $\mathscr{A}$, $G \in \mathscr{D}(\Omega \times (0,T))$ and $f \in \mathscr{D}(\Gamma \times (0,T))$. Then, the function $u(t)$ in (2.17) is a regular solution of Eq. (2.16) and the following equality holds for every t:*

$$u''(t) = c_0^2 A(u(t) - Df(t)) + G(t). \tag{2.20}$$

Proof We introduce the functions

$$u_1(t) = R_+(c_0 t)u_0 + \frac{1}{c_0}\mathscr{A}^{-1}R_-(c_0 t)v_0 \ ,$$

$$u_2(t) = \frac{1}{c_0}\mathscr{A}^{-1}\int_0^t R_-(c_0(t-s))G(s)\mathrm{d}s = \frac{1}{c_0}\mathscr{A}^{-1}\int_0^t R_-(c_0 s)G(t-s)\mathrm{d}s \ ,$$

$$u_3(t) = -c_0\mathscr{A}\int_0^t R_-(c_0(t-s))Df(s)\mathrm{d}s = -c_0\mathscr{A}\int_0^t R_-(c_0 s)Df(t-s)\mathrm{d}s \ .$$

Thanks to the linearity we examine separately these functions.

The definitions of $R_+(t)$ and $R_-(t)$ clearly imply that $u_1(t)$ is a regular solution of (2.20) with $f = 0$, $G = 0$.

We consider the function $u_3(t)$ and we leave the similar analysis of $u_2(t)$ to the reader. Let $y(t) = u_3(t) - Df(t)$. First we note that

$$y(t) = -c_0\mathscr{A}\int_0^t R_-(c_0 s)Df(t-s)\mathrm{d}s - Df(t)\ ,$$

$$y'(t) = -c_0\mathscr{A}\int_0^t R_-(c_0 s)Df'(t-s)\mathrm{d}s - Df'(t)\ ,$$

$$y''(t) = -c_0\mathscr{A}\int_0^t R_-(c_0 s)Df''(t-s)\mathrm{d}s - Df''(t)\ .$$

An integration by parts, using formula (2.11), proves $y''(t) \in C([0,T]; L^2(\Omega))$. In fact,

$$y''(t) = -\int_0^t R_+(c_0 s)Df'''(t-s)\mathrm{d}s\ . \tag{2.21}$$

Integrating by parts twice we get $y' \in C([0,T]; \mathrm{dom}\mathscr{A})$. In fact:

$$y'(t) = -\frac{1}{c_0}\mathscr{A}^{-1}\int_0^t R_-(c_0 s)Df'''(t-s)\mathrm{d}s.$$

We give the details of the proof that $y(t) \in C([0,T]; \mathrm{dom}A)$. We integrate by parts three times, using both the formulas (2.11) and (2.12):

$$\begin{aligned} y(t) &= -\int_0^t R_+(c_0 s)Df'(t-s)\mathrm{d}s = -\frac{1}{c_0}\mathscr{A}^{-1}\int_0^t R_-(c_0 s)Df''(t-s)\mathrm{d}s \\ &= \frac{1}{c_0^2}A^{-1}\left[Df''(t) - \int_0^t R_+(c_0 s)Df'''(t-s)\mathrm{d}s\right]\ . \end{aligned}$$

We compare this last equality and (2.21), using $y = u_3 - Df$, and we see that u_3 solves (2.20) (with $G = 0$):

$$c_0^2 Ay(t) = Df''(t) - \int_0^t R_+(c_0 s)Df'''(t-s)\mathrm{d}s = Df''(t) + y''(t) = u_3''(t)\ .$$

The linearity of the problem shows that $u(t) = u_1(t)+u_2(t)+u_3(t)$ solves (2.20), hence also (2.16) since when $\phi \in \mathrm{dom}A$, then $A\phi = \Delta\phi$.

Remark 2.1 Stronger properties give more regular solutions. Note that $\mathscr{D}(\Omega) \subseteq \operatorname{dom} A^k$ for every k. So, if $f = 0$, $G = 0$ and u_0, v_0 belong to $\mathscr{D}(\Omega)$, then $u(t) \in C^\infty([0,T]; H^r(\Omega))$ for every r and so $u(t) \in C^\infty(\Omega \times (0,T))$ and $\gamma_0 u(t)$, $\gamma_1 u(t)$ belong to $C^\infty([0,T]; L^2(\partial\Omega))$. In particular, if $D_i = \partial/\partial x_i$ then $D_i u(x,t)$ solves (2.16) with initial conditions $D_i u_0$ and $D_i v_0$ (but in general $\gamma_0 D_i u \neq 0$). If furthermore $\partial\Omega \in C^\infty$, then the derivatives of u of every order have smooth traces on $\partial\Omega$.

Different methods can be used to justify the definition of mild solutions, see [27, 84]. We adopt the following one: let u_n be given by (2.17) with data $\{f_n\}$, $\{G_n\}$, $\{u_{0,n}\}$, $\{v_{0,n}\}$ each one of class C^∞ with compact support and such that

$$\begin{gathered} u_{0,n} \to u_0 \text{ in } L^2(\Omega)\,, \quad v_{0,n} \to v_0 \text{ in } H^{-1}(\Omega)\,, \\ f_n \to f \text{ in } L^2(0,T; L^2(\partial\Omega))\,, \quad G_n \to G \text{ in } L^2(0,T; L^2(\Omega))\,. \end{gathered}$$

Then, u_n is a regular solution and Theorem 2.1 show that

$$u_n \to u \text{ in } C([0,T]; L^2(\Omega))\,, \quad u_n' \to u' \text{ in } C([0,T]; H^{-1}(\Omega))$$

for every $T > 0$: *a mild solution is the limit of a sequence of regular solutions.* This observation justifies the definition of mild solution.

Remark 2.2 The property in Item 3 of Theorem 2.1 is called ADMISSIBILITY of the operator $\begin{bmatrix} 0 \\ c_0^2 AD \end{bmatrix}$.

2.2 The Solutions of the System with Memory

The properties of the wave equation just outlined can be used to derive definitions and formulas for the solutions of the systems with memory (2.1) and (2.3). We must find a suitable formula, which can be used to *define* the solution, and then of course we must justify our choice. With this goal in mind, we perform formal manipulations as follows. First a transformation, which is known as MACCAMY TRICK. We rewrite the Eq. (2.1) in the form

$$\Delta w + \frac{1}{c_0^2} M * \Delta w = \frac{1}{c_0^2} w'' - \frac{2c}{c_0^2} w' - \frac{1}{c_0^2} F \tag{2.22}$$

and we denote $R(t)$ the resolvent kernel of $M(t)/c_0^2$ so that (see Sect. 1.3)

$$R = \frac{1}{c_0^2} M - \frac{1}{c_0^2} M * R. \tag{2.23}$$

Then we have

$$c_0^2 \Delta w = w'' - 2cw' - F - R * w'' + 2cR * w' + R * F \ .$$

We integrate by parts the integrals $R * w''$ and $R * w'$ and we get:

$$w''(t) = c_0^2 \Delta w(t) + aw' + bw + \int_0^t K(t-s)w(s)\mathrm{d}s + F_1(t), \tag{2.24}$$

$$F_1(t) = F(t) - \int_0^t R(t-s)F(s)\mathrm{d}s - R(t)v_0 - (R'(t) - 2cR(t))u_0 \ ,$$

$$a = 2c + R(0) \ , \ b = R'(0) - 2cR(0) \ , \quad K(t) = \left[R''(t) - 2cR'(t)\right].$$

The kernel $K(t)$ is continuous.

Remark 2.3 The noticeable fact of the MacCamy trick is that *the laplacian does not appear in the memory term of* (2.24). Furthermore, note that when $a = 0$, $b = 0$ we get a system a special case of which has been studied in Chap. 1 (when the space variable is in $(0, +\infty)$).

Equation (2.24) is the same as the wave equation (2.16) with $G(t) = F_1(t) + aw' + bw + K * w$. So, we can combine formulas (2.17) and (2.24) to obtain a Volterra integral equation for $w(t)$:

$$\begin{aligned} w(t) = & -c_0 \mathscr{A} \int_0^t R_-(c_0(t-s))Df(s)\mathrm{d}s + \frac{1}{c_0}\mathscr{A}^{-1} \int_0^t R_-(c_0(t-s))F_1(s)\mathrm{d}s \\ & + R_+(c_0 t)u_0 + \frac{1}{c_0}\mathscr{A}^{-1} R_-(c_0 t)v_0 + \frac{a}{c_0}\mathscr{A}^{-1} \int_0^t R_-(c_0(t-s))w'(s)\mathrm{d}s \\ & + \frac{1}{c_0}\mathscr{A}^{-1} \int_0^t R_-(c_0(t-s)) \left[bw(s) + \int_0^s K(s-r)w(r)\mathrm{d}r \right] \mathrm{d}s \\ = & \ H(t) + \int_0^t L(s)w(t-s)\mathrm{d}s. \end{aligned} \tag{2.25}$$

where (use (2.12) to integrate by parts the integral which contains w')

$$H(t) = -c_0\mathscr{A}\int_0^t R_-(c_0(t-s))Df(s)\mathrm{d}s + \frac{1}{c_0}\mathscr{A}^{-1}\int_0^t R_-(c_0(t-s))F_1(s)\mathrm{d}s$$
$$+ R_+(c_0t)u_0 + \frac{1}{c_0}\mathscr{A}^{-1}R_-(c_0t)\left[v_0 - au_0\right] \ ,$$
$$L(t)w = \left[aR_+(c_0t) + \frac{b}{c_0}\mathscr{A}^{-1}R_-(c_0t)\right]w + \frac{1}{c_0}\mathscr{A}^{-1}\int_0^t K(r)R_-(c_0(t-r))w\mathrm{d}r \ .$$

The affine term $H(t)$ appears in the solutions of the wave equation (2.17) and so its properties are known (see Theorem 2.1 and recall that the function $F_1(t)$ here depends also on u_0 and v_0): $H(t) \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$. The properties of $L(t)$ are:

- $L(t) \in \mathscr{L}(L^2(\Omega))$ for every $t \geq 0$ and $t \mapsto L(t)$ is strongly continuous.
- $L(t)L(\tau) = L(\tau)L(t)$ for every $t \geq 0$, $\tau \geq 0$.

So, using known properties of the Volterra integral equations (see Sect. 1.3):

Theorem 2.3 *Let $T > 0$. There exists a unique solution $w \in C([0,T]; L^2(\Omega))$ with $w' \in C([0,T]; H^{-1}(\Omega))$ of the Volterra integral equation (2.25) which depends continuously on $u_0 \in L^2(\Omega)$, $v_0 \in H^{-1}(\Omega)$, $F \in L^1(0,T; L^2(\Omega))$ and $f \in L^2(0,T; L^2(\Gamma))$.*

We use the Volterra integral Eq. (2.25) to define the solutions of Eq. (2.1):

Definition 2.2 The MILD SOLUTION of (2.1) (or of (2.3)) with conditions (2.2) and (2.5) is the function $w(t)$ which solves (2.25). The mild solution is a REGULAR SOLUTION when

$$w(t) - Df(t) \in C([0,T]; \mathrm{dom}A) \cap C^1([0,T]; \mathrm{dom}\mathscr{A}) \cap C^2([0,T]; L^2(\Omega)) \ .$$

In the next chapters, the term "solution" will always denote a mild solution.

Also in the case of systems with memory, regular solutions satisfy the boundary conditions in the sense that $w(t) - Df(t) \in \mathrm{dom}A = H^2(\Omega) \cap H_0^1(\Omega)$.

Remark 2.4 Note that $w(t)$ takes real values when the initial conditions, the affine term and the control are real.

The following result justifies the definition of the regular and mild solutions. The first statement is a reformulation of Theorem 2.3.

Theorem 2.4 *Let $T > 0$ be fixed.*

1. *The transformation $(u_0, u_1, F, f) \mapsto w$ is continuous from $L^2(\Omega) \times H^{-1}(\Omega) \times L^1(0,T; L^2(\Omega)) \times L^2(0,T; L^2(\Gamma))$ to $C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$.*

2. *Let* $u_0 \in \operatorname{dom} A$*,* $v_0 \in \operatorname{dom} \mathscr{A}$*,* $F \in \mathscr{D}(\Omega \times (0,T))$ *and* $f \in \mathscr{D}(\Gamma \times (0,T))$*. Then, the mild solution defined by the Volterra integral equation* (2.25) *is a regular solution and the following equalities hold for every* t *(here* $y(t) = w(t) - Df(t)$*):*

$$w''(t) = c_0^2 Ay(t) + aw'(t) + bw(t) + \int_0^t K(t-s)w(s)\mathrm{d}s + F_1(t), \tag{2.26}$$

$$w''(t) = 2cw' + c_0^2 Ay(t) + \int_0^t M(t-s)Ay(s)\mathrm{d}s + F(t). \tag{2.27}$$

Proof We prove the second statement. Thanks to the linearity of the problem, as in the case of the memoryless wave equation, we can study separately the effect of the initial conditions, of the affine term and of the control. We confine ourselves to study the effect of the control. So, $u_0 = 0$, $v_0 = 0$ and $F = 0$. We use the function u_3 introduced in the proof of Theorem 2.2 but now $y(t) = w(t) - Df(t)$. With these notations, Eq. (2.25) takes the form

$$y(t) = (u_3(t) - Df(t)) + \int_0^t L(s)Df(t-s)\mathrm{d}s + \int_0^t L(s)y(t-s)\mathrm{d}s. \tag{2.28}$$

Using $f \in \mathscr{D}(\Gamma \times (0,T))$ and Theorem 2.2 we know that

$$A\,(u_3 - Df)\,,\;\; \mathscr{A}\,(u_3 - Df)'\,,\;\; (u_3 - Df)'' \text{ belong to } C([0,T]; L^2(\Omega)). \tag{2.29}$$

We prove the analogous properties of $y = w - Df$.

Using the definition of $L(t)$ and Theorem 2.2 we can prove:

$$\begin{aligned} &A\int_0^t L(t-s)Df(s)\mathrm{d}s \in C([0,T]; L^2(\Omega)),\\ &\mathscr{A}\int_0^t L(t-s)Df'(s)\mathrm{d}s \in C([0,T]; L^2(\Omega)). \end{aligned} \tag{2.30}$$

We sketch the proof of the first property (the proof of the second one is similar). The definition of $L(t)$ shows that $A\int_0^t L(t-s)Df(s)\mathrm{d}s$ is the linear combination of three terms. It is easy to see that they belong to $C([0,T]; L^2(\Omega))$. In fact, the first one is (integrate by parts twice)

$$A\int_0^t R_+(c_0(t-s))Df(s)\mathrm{d}s = -\frac{1}{c_0^2}\left[Df'(t) - \int_0^t R_+(c_0(t-s))Df''(s)\mathrm{d}s\right],$$

an $L^2(\Omega)$-valued continuous function. The second and third terms are treated analogously.

Now, we proceed to prove that $w(t)$ is a regular solution. We first prove that $y = w - Df \in C([0,T]; \mathrm{dom}A)$. This follows because (2.28) shows that $Ay(t)$ solves

$$Ay(t) = A\left(u_3(t) - Df(t) + \int_0^t L(t-s)Df(s)\mathrm{d}s\right) + \int_0^t L(t-s)Ay(s)\mathrm{d}s$$

so that $Ay \in C([0,T]; L^2(\Omega))$ i.e., $y \in C([0,T]; \mathrm{dom}A)$ thanks to the first properties in (2.29) and in (2.30).

Now, we study the derivatives of y. The equations of $\xi = y'$ and $\zeta = y''$ are respectively

$$\xi = (u_3 - Df)' + \int_0^t L(s)Df'(t-s)\mathrm{d}s + \int_0^t L(s)\xi(t-s)\mathrm{d}s, \tag{2.31}$$

$$\zeta = (u_3 - Df)'' + \int_0^t L(s)Df''(t-s)\mathrm{d}s + \int_0^t L(s)\zeta(t-s)\mathrm{d}s \tag{2.32}$$

(note that $y(0) = 0$, $y'(0) = 0$). From (2.32) and continuity of $(u_3 - Df)''$ we get $\zeta \in C([0,T]; L^2(\Omega))$. We consider Eq. (2.31): the second property in (2.30) and continuity of $\mathscr{A}(u_3 - Df)'$ prove $\mathscr{A}\xi \in C([0,T]; L^2(\Omega))$, as wanted.

Now, we prove the equalities (2.26) and (2.27). We present the proof in the case $u_0 = 0$, $v_0 = 0$ and $F = 0$. We proved that w is twice differentiable (when $f \in \mathscr{D}(\Gamma \times (0,T))$) so that we can integrate by parts back in (2.25). Then, (2.25) takes the form

$$\begin{aligned} w(t) = &-c_0\mathscr{A}\int_0^t R_-(c_0(t-s))Df(s)\mathrm{d}s \\ &+ \frac{1}{c_0}\mathscr{A}^{-1}\int_0^t R_-(c_0(t-s))\left[aw'(s) + bw(s) + \int_0^s K(s-r)w(r)\mathrm{d}r\right]\mathrm{d}s\,. \end{aligned}$$

We compare with the functions $u_2(t)$ and $u_3(t)$ in Theorem 2.2 and with (2.20) and we see that (2.26) holds.

Using again differentiability of $w(t)$ and the definition of $K(t)$, we integrate by parts the last integral in (2.26). We find:

$$(w'' - 2cw') - R * (w'' - 2cw') = c_0^2 Ay. \tag{2.33}$$

The definition (2.23) of $R(t)$ gives

$$M * Ay = R * (w'' - 2cw') \ .$$

Using (2.33) to replace the convolution on the right side, we find (2.27).

We sum up: *the definition of the mild solutions is justified since mild solutions are limits of regular solutions.*

Remark 2.5 It is easy to extend Remark 2.1 to Eq. (2.26): if $f = 0$, $F = 0$ and both the initial data u_0 and v_0 belong to $\mathscr{D}(\Omega)$ then $w(t) \in C^\infty([0, T]; H^r(\Omega))$ for every r and so it has continuous derivatives of every order. In particular, $D_i w(x, t)$ solves Eq. (2.26) with initial conditions $D_i u_0$ and $D_i v_0$ ($D_i w(x, t)$ might not be zero on the boundary). If furthermore $\partial\Omega \in C^\infty$ then the derivatives of w of every order have smooth traces on $\partial\Omega$.

2.3 Description of the Control Problems

Now, we describe the control problem. In the study of linear systems, exact or approximate controllability does not depend on the choice of initial conditions and affine terms (which are kept fixed). For this reason *from now on, when studying control problems, we assume*

$$u_0 = 0 \ , \quad v_0 = 0 \ , \quad F = 0 \ .$$

Definition 2.3 A "target" $(\xi, \eta) \in L^2(\Omega) \times H^{-1}(\Omega)$ is *reachable at time T* when there exists a STEERING CONTROL $f \in L^2(0, T; L^2(\Gamma))$ such that

$$w^f(T) = \xi \ , \quad (w^f)'(T) = \eta \ .$$

The REACHABLE SET at time T is the set of the reachable targets (at time T) and system (2.1) is CONTROLLABLE (at time T) when *every* target in $L^2(\Omega) \times H^{-1}(\Omega)$ is reachable (APPROXIMATE CONTROLLABILITY when the reachable set is dense in $L^2(\Omega) \times H^{-1}(\Omega)$. In order to contrast approximate controllability and controllability, the last property is also called EXACT CONTROLLABILITY).

Note that:

- targets are taken to be real so that we use *real control functions.*
- if a target is reachable at time T_0 it is also reachable at any later time $T > T_0$, using the control $f(t-(T-T_0))$ (equal to zero if $t < T - T_0$).
- we might wish to study controllability of the sole component w. This has an interest in some applications, see the identification problem in Sect. 5.4.

It might seem that we can relax the property that the control time is "universal", and that we might require that every target be reached in a certain time T, not the same for all of them. In fact, this would not give a more general definition of controllability, since we can prove:

Theorem 2.5 *Assume that for every target $\xi \in L^2(\Omega)$ there exists a time $T = T_\xi$ and a control f such that $w(T_\xi) = \xi$. Then, there exists a time T_0 such that the system is controllable at time T_0.*

A similar statement holds for the pair (w, w')

Proof In this proof we use the following facts (see [11, Chap. 3]. The statements are adapted to Hilbert spaces): (1) a *convex* set is closed in the norm topology if and only if it is weakly closed, i.e., if every weakly convergent sequence converges to a point of the set; (2) a *bounded* set which is *convex* and *closed* is also weakly compact, i.e., every sequence in the set admits a weakly convergent subsequence, which converges to a point of the set; (3) any linear and continuous transformation is weakly continuous, and so it transforms weakly compact sets into weakly compact sets. Furthermore, we shall use Baire Theorem, see below.

Now we prove the theorem, studying controllability of the sole component w. The proof of the second assertion, concerning the pair (w, w'), is similar. Let

$$R(T) = \left\{ w^f(T)\,, \quad f \in L^2(0,T;L^2(\Gamma)) \right\}\,,$$
$$R_N(T) = \left\{ w^f(T)\,, \quad |f|_{L^2(0,T;L^2(\Gamma))} \le N \right\}\,.$$

By assumption, every target can be reached. So,

$$L^2(\Omega) = \bigcup_{T>0\,,\ N\in\mathbb{N}} R_N(T) = \bigcup_{T\in\mathbb{N}\,,\ N\in\mathbb{N}} R_N(T)$$

(the first equality is the assumption of the theorem and the second follows since $T \mapsto R_N(T)$ and $N \mapsto R_N(T)$ are increasing).

The set

$$\left\{ f, \quad |f|_{L^2(0,T;L^2(\Gamma))} \le N \right\}$$

is convex and weakly compact in $L^2(0,T;L^2(\Gamma))$ and $R_N(T) \subseteq L^2(\Omega)$ is the image of such convex weakly compact set under a linear continuous transformation.

Hence $R_N(T)$ is convex and weakly compact, hence also weakly closed. So, it is closed in the norm topology of $L^2(\Omega)$.

We sum up: the space $L^2(\Omega)$ is union of the (double) sequence $\{R_N(T)\}_{N,T\in\mathbb{N}}$ of *closed sets*. Baire Theorem (see [72, 120]) implies the existence of T_0 and N_0 such that $R_{N_0}(T_0)$ has a nonempty interior. Hence, also

$$R(T_0) = \bigcup_{N\in\mathbb{N}} R_N(T_0) \subseteq L^2(\Omega)$$

is a subspace with a nonempty interior. Hence, it must be the whole space $L^2(\Omega)$ and *every* $\xi \in L^2(\Omega)$ is reachable in time T_0.

Remark 2.6 In the memoryless case and for distributed control the analogous of Theorem 2.5 has been first proved in [30] and then independently reproved or extended by several authors, essentially with the same proof, see for example [73, 83, 86].

The content of Theorem 2.5 combined with the fact that a system, which is controllable at a certain time T is also controllable at later times justifies the following definition:

Definition 2.4 The infimum of those times T at which the system is controllable is called the SHARP CONTROL TIME.

2.4 Useful Transformations

In this section, we show different representations of the control systems (2.1) or (2.3), or of their solutions, *which do not change the controllability property.* Different approaches to controllability can profit of one or the other representation. Accepting these facts, the proofs in this section can be skipped. However, the reader should keep in mind that the transformation used to achieve the condition $c_0 = 1$, i.e., $N(0) = 1$, changes the control time.

1. Let us represent

$$w(x,t) = \sum_{n=1}^{+\infty} \phi_n(x) w_n(t)\,, \qquad w_n(t) = \int_\Omega \phi_n(x) w_n(x,t)\mathrm{d}x \tag{2.34}$$

($\{\phi_n\}$ orthonormal basis of real eigenvectors of A in $L^2(\Omega)$). It turns out that $w_n(t)$ satisfies

$$w_n''(t) = 2cw_n'(t) - \lambda_n^2 \left(c_0^2 w_n(t) + \int_0^t M(t-s) w_n(s)\mathrm{d}s \right)$$

$$
-\int_{\Gamma}\left(c_0^2 f(t)+\int_0^t M(t-s)f(s)\mathrm{d}s\right)(\gamma_1\phi_n)\mathrm{d}\Gamma + F_n(t) \tag{2.35}
$$

$$
w_n(0)=u_{0,n}=\int_{\Omega}u_0(x)\phi_n(x)\mathrm{d}x,\quad w_n'(0)=v_{0,n}=\int_{\Omega}v_0(x)\phi_n(x)\mathrm{d}x,
$$

$$
F_n(t)=\int_{\Omega}F(x,t)\phi_n(x)\mathrm{d}x.
$$

We integrate both the sides and we get:

$$
\begin{aligned}
w_n'(t)&=2cw_n(t)-\lambda_n^2\int_0^t N(t-s)w_n(s)\mathrm{d}s\\
&\quad -\int_0^t N(t-s)\left(\int_{\Gamma}(\gamma_1\phi_n)\,f(x,s)\mathrm{d}\Gamma\right)\mathrm{d}s+H_n(t),
\end{aligned} \tag{2.36}
$$

$$
w_n(0)=u_{0,n}\ ,\qquad H_n(t)=\int_{\Omega}H(x,t)\phi_n(x)\mathrm{d}x
$$

(H defined in (2.4)) and so

$$
\begin{aligned}
w_n(t)&=e^{2ct}u_{0,n}-\int_0^t e^{2c(t-\tau)}\left\{\lambda_n^2\int_0^{\tau}N(\tau-s)w_n(s)\mathrm{d}s\right.\\
&\quad\left.+\int_0^{\tau}N(\tau-s)\left(\int_{\Gamma}(\gamma_1\phi_n)\,f(x,s)\mathrm{d}\Gamma\right)\mathrm{d}s+H_n(\tau)\right\}\mathrm{d}\tau.
\end{aligned} \tag{2.37}
$$

2. It is possible to remove the velocity term from formula (2.24). Let

$$
w_{\alpha}(t)=\mathrm{e}^{-\alpha t}w(t)\,,\qquad \alpha=\frac{a}{2}\,.
$$

Then, $w_{\alpha}(t)$ solves

$$
\left\{\begin{array}{ll}
w_{\alpha}''=c_0^2\Delta w_{\alpha}+hw_{\alpha}+\displaystyle\int_0^t K_{\alpha}(t-s)w_{\alpha}(s)\mathrm{d}s+F_{\alpha}(t), & \\
w_{\alpha}(0)=u_0, & w_{\alpha}'(0)=v_0-\alpha u_0,\\
w_{\alpha}(t)=\mathrm{e}^{-\alpha t}f(t)\text{ on }\Gamma, & w_{\alpha}(t)=0\text{ on }\partial\Omega\setminus\Gamma,\\
h=\left(b-\alpha^2\right)\,, & F_{\alpha}(t)=\mathrm{e}^{-\alpha t}F(t)\,,\quad K_{\alpha}(t)=\mathrm{e}^{-\alpha t}K(t).
\end{array}\right. \tag{2.38}
$$

3. When we study the control system represented in the form (2.3), it is not restrictive to assume

$$c_0^2 = N(0) = 1 \,, \quad N'(0) = 0. \tag{2.39}$$

Remark 2.7 It is possible to study the convergence of the series (2.34) using the formulas for $w_n(t)$ (see [77]). This is a different way to define the solutions of system (2.1) and to study their properties.

Justification of the Formulas

Formal computations can easily justify the previous formulas. For example, recalling that the eigenfunctions $\phi_n(x)$ are real, equality (2.35) is formally obtained computing the inner product of both the sides of (2.1) with $\phi_n(x)$ and using formula (2.15). The rigorous justifications are as follows:

1. The easiest way to prove formulas (2.35) and (2.36) is to show correctness for regular solutions, and then to extend by continuity. So, let u_0 and v_0 belong to $\mathscr{D}(\Omega)$, $F \in \mathscr{D}(\Omega \times (0, T))$ and $f \in \mathscr{D}(\Gamma \times (0, T))$. We integrate the product of both the sides of (2.27) with ϕ_n and we use formula (2.15). Formulas (2.35), i.e., (2.36) and (2.37), are easily derived under these special assumptions. We can pass to the limit in (2.37), thanks to the first assertion of Theorem 2.4 and continuity of the inner product, and so we conclude that formula (2.37) holds in general. Hence, also the formulas (2.35) and (2.36) hold.

2. The series exapansion of $\mathrm{e}^{-\alpha t} w(x, t)$ has coefficients $w_{\alpha,n}(t)$ solutions of an equation similar to (2.35), but with $M(t)$ and $F(t)$ replaced with $F_\alpha(t)$ and $M_\alpha(t)$; the initial conditions replaced by the Fourier coefficients of u_0 and $v_0 - \alpha u_0$ and $\mathrm{e}^{-\alpha t} f(t)$ in the place of $f(t)$. These are the coefficients of the Fourier expansion of the solutions of (2.38).

3. Finally, we show how the conditions (2.39) can be achieved.

We use the representation (2.34) of $w(t)$ and the Eq. (2.36) for $w_n(t)$. The condition $c_0^2 = N(0) = 1$ is achieved using the transformation

$$w \mapsto \theta = \sum_{n=1}^{+\infty} \phi_n(x)\theta_n(t) \,, \qquad \theta_n(x, t) = w_n(t/c_0) \,, \quad c_0 = \sqrt{N(0)} > 0 \,.$$

In fact,

$$\begin{aligned} \theta_n'(t) &= 2\frac{c}{c_0}\theta_n - \lambda_n^2 \int_0^t \frac{1}{c_0^2} N((t-s)/c_0) \left[\theta_n(s) + \int_\Gamma (\gamma_1\phi_n) f(s/c_0) \mathrm{d}\Gamma \right] \mathrm{d}s \\ &\quad + \frac{1}{c_0} H_n(t/c_0) \,. \end{aligned}$$

Clearly, controllability of w is equivalent to controllability of θ while controllability of (w, w') is equivalent to controllability of (θ, θ'), but the control time is divided by c_0: *if θ is controllable at the time S then w is controllable at the time*

$T = S/c_0 = S/\sqrt{N(0)}$ *(and conversely).* The bonus of this transformation is that

$$\text{the kernel is } N_1(t) = \frac{1}{c_0^2} N\left(t/c_0\right) \text{ so that} N_1(0) = 1\ .$$

A second transformation is as follows: we introduce

$$\tilde{\theta}(x,t) = \mathrm{e}^{2\gamma t}\theta(x,t)\ ,\ \text{i.e.,}\quad \tilde{\theta}_n(x,t) = \mathrm{e}^{2\gamma t}\theta_n(x,t)\ ,\quad \gamma = -\frac{1}{2}N_1'(0)\ .$$

Then, we have, with $N_2(t) = \mathrm{e}^{2\gamma t}N_1(t)$ and $\alpha = (\gamma + c/c_0)$

$$\begin{aligned}\tilde{\theta}_n'(t) &= 2\alpha\tilde{\theta}_n(t) - \lambda_n^2 \int_0^t N_2(t-s)\left[\tilde{\theta}_n(s) + \int_\Gamma (\gamma_1\phi_n)\left(\mathrm{e}^{2\gamma s} f(s/c_0)\right)\mathrm{d}\Gamma\right]\mathrm{d}s \\ &\quad + \frac{1}{c_0}\mathrm{e}^{2\gamma t} H_n(t/c_0).\end{aligned} \tag{2.40}$$

Neither controllability nor the control time are affected and

$$N_2(0) = 1\ ,\quad N_2'(0) = 2\gamma N_1(0) + N_1'(0) = 0\ .$$

2.4.1 *Finite Propagation Speed*

FINITE PROPAGATION SPEED of the associated wave equation (2.16) is the following property (see for example [66, Remarque 1.2]): let $G = 0$ and let u_0 and v_0 have support in a compact set $K \subseteq \Omega$. Let $d > 0$ be *smaller then* the distance of x_0 from $K \cup \Gamma$. The known property is that[2] if $c_0 t < d$ then $u(x,t) = 0$ in $B(x_0, d - c_0 t)$. This can be expressed as follows: let $S(t) \subseteq \Omega$ be the union of the supports of the three functions (of the space variable x, while the time t is fixed)

$$R_+(c_0 t)u_0\ ,\quad \mathscr{A}^{-1}R_-(c_0 t)v_0\ ,\quad u(t) = -\mathscr{A}\int_0^t R_-(c_0(t-s))Df(s)\mathrm{d}s. \tag{2.41}$$

Then,

$$c_0 t < d \implies S(t) \cap B(x_0, d - c_0 t) = \emptyset. \tag{2.42}$$

We prove that this property is retained by Eq. (2.1). We are interested in the propagation of signals produced by the control so we confine ourselves to the case

---

[2] $B(x_0, r)$ denotes the ball of center x_0 and radius r.

$u_0 = 0$, $v_0 = 0$ and $F = 0$. Furthermore, we can assume that f is "smooth", $f \in \mathscr{D}(\Gamma \times (0, +\infty))$.

It is easier if we represent Eq. (2.1) in the form (2.38) (index α is not indicated) so that (the definition of $u(t)$ is in (2.41))

$$w(t) = u(t) + \frac{1}{c_0}\mathscr{A}^{-1}\int_0^t R_-(c_0(t-s))\left[hw(s) + \int_0^s K(s-r)w(r)\mathrm{d}r\right]\mathrm{d}s. \quad (2.43)$$

The solution of this Volterra integral equation is represented using the Picard series (1.20).

Our goal is the proof that the support of $w(x,t)$ does not intersect $B(x_0, d - c_0t)$ if $c_0t < d$. It is sufficient to prove this property for every term of the Picard series. In fact, it is enough to see this fact for the first term of the series, since it is clear that the method can be applied to every term.

The first term of the series is

$$\frac{h}{c_0}\int_0^t \mathscr{A}^{-1}R_-(c_0(t-s))u(s)\mathrm{d}s \quad (2.44)$$

$$+\frac{1}{c_0}\int_0^t \mathscr{A}^{-1}R_-(c_0(t-s))\int_0^s K(s-r)u(r)\mathrm{d}r\,\mathrm{d}s. \quad (2.45)$$

We consider the addendum (2.44). We already noted that the support of $u(t)$ does not intersect $B(x_0, d - c_0t)$. Then, with s in the place of t, the support of $u(s)$ does not intersect $B(x_0, d - c_0s)$. Hence,

$$\text{the support of } \mathscr{A}^{-1}R_-(c_0(t-s))u(s) \text{ does not intersect } B(x_0, d - c_0t)\ .$$

This property is retained by the integral on $[0, t]$.

The integral (2.45) is treated analogously and so the support of the first term in the Picard series does not intersect $B(x_0, d - c_0t)$. It is clear that this property holds for every term of the series. Hence, we have:

Theorem 2.6 *If the distance of x_0 from Γ is larger than d and if $c_0t < d$, then the support of $w(t)$ does not intersect $B(x_0, d - c_0t)$.*

The interpretation is that signals (the "waves" in the body) propagate in a viscoelastic material with finite velocity not larger then c_0 (strict positivity of c_0^2 is crucial for this result). This observation suggests the following result:

Theorem 2.7 *The sharp control time of the system with memory is not shorter than that of the associated wave equation.*

We refer to [82] for the proof.

Remark 2.8 We mention that the velocity of the signals in the viscoelastic body is precisely c_0, the same as for the memoryless wave equation, hence *independent on* c and $M(t)$, see [22, 28, 46].

2.5 Final Comments

The solutions of the Eqs. (2.1) and (2.3) have been defined in this chapter using an "operator" approach as in [10, 74]. Fourier expansions can be used, see Remark 2.7. A different approach, the definition "by transposition" is related to the arguments in Chap. 6. The solutions can also be defined using the semigroup approach introduced in [18, 19]. This approach has a particular interest in the study of stability, and we cite the book [67].

A formal integration of both the sides of Eq. (2.3) gives

$$w(t) = u_0 + 2c\int_0^t w(s)\mathrm{d}s + \int_0^t \left[\int_0^{t-r} N(s)\mathrm{d}s\right] \Delta w(r)\mathrm{d}r + \int_0^t F(s)\mathrm{d}s \ .$$

In this form, the equation is deeply studied in [85], using Laplace transform techniques. See [40] for control problems.

An important case in applications is when the kernel is a linear combination of exponentials,

$$N(t) = \sum_{k=1}^{K} a_k \mathrm{e}^{-b_k t} \ .$$

In applications, it must be $a_k > 0$ and $b_k \geq 0$. The special case in which Ω is a segment, $\Omega = (0, 1)$ has been studied in [94], using an interesting idea which we describe when $K = 2$. We introduce the auxiliary functions

$$r_1(t) = \int_0^t \mathrm{e}^{-b_1(t-s)} w_x(s)\mathrm{d}s \ , \qquad r_2(t) = \int_0^t \mathrm{e}^{-b_2(t-s)} w_x(s)\mathrm{d}s$$

and we note that

$$\frac{\partial}{\partial x} r_i = \int_0^t \mathrm{e}^{-b_i(t-s)} w_{xx}(s)\mathrm{d}s \ , \qquad r_i' = w_x - b_i r_i \ .$$

Hence, the equation

$$w' = \int_0^t \left(a_1 e^{-b_1(t-s)} + a_2 e^{b_2(t-s)}\right) w_{xx}(s)\mathrm{d}s$$

is equivalent to the system

$$w' = a_1 \frac{\partial}{\partial x} r_1 + a_2 \frac{\partial}{\partial x} r_2 \,, \quad r_1' = w_x - b_1 r_1 \,, \quad r_2' = w_x - b_2 r_2 \,.$$

We do not insist on this approach (used to study stability and spectrum in [38, 94]) which has not yet been exploited in the study of control problems (controllability when $N(t) = 1 + a\mathrm{e}^{bt}$ is studied in [68] using moment methods, as in Chap. 5).

2.6 The Derivation of the Models

Among the several applications of systems with persistent memory, we keep in mind applications to thermodynamics and (nonfickian) diffusion, and to viscoelasticity. We give a short account of the derivation of the models (2.1) and (2.3) in these cases. See [20] for a detailed analysis.

2.6.1 Thermodynamics with Memory and Nonfickian Diffusion

We first recall the derivation of the memoryless heat equation in a bar, which follows from two fundamental physical facts: conservation of energy and the fact that the temperature is a measure of energy, i.e.,

$$e'(x,t) = -q_x(x,t) \,, \qquad \theta'(x,t) = \gamma e'(x,t) \,, \quad \gamma > 0 \,.$$

Here, q is the flux of heat, e is the density of energy and θ is the temperature.[3] So, we have

$$\theta' = -\gamma q_x(x,t) \,.$$

This equality is combined with a "constitutive law". Fourier law assumes that the flux responds immediately to changes in temperature:

$$q(x,t) = -k\theta_x(x,t) \tag{2.46}$$

[3] the minus sign because the total internal energy $\int_a^b e(x,t)\mathrm{d}x$ of a segment (a,b) decreases when the flux of heat is directed to the exterior of the segment.

($k > 0$ and the minus sign because the flux is toward parts which have lower temperature). Combining these equalities we get the memoryless heat equation

$$\theta' = (k\gamma)\theta_{xx}. \tag{2.47}$$

The heat equation with memory is obtained when (as done in [39], following the special case examined in [13]) we take into account the fact that the transmission of heat is not immediate, and Fourier law is replaced with

$$q(x,t) = -k \int_{-\infty}^{t} N(t-s)\theta_x(x,s)\mathrm{d}s, \tag{2.48}$$

which gives (the product $k\gamma N(t)$ is renamed $N(t)$)

$$\theta' = \int_{-\infty}^{t} N(t-s)\theta_{xx}(x,s)\mathrm{d}s. \tag{2.49}$$

If the system is subject to the action of an external control f, for example if we impose

$$\theta(0,t) = f(t)\ , \qquad \theta(\pi,t) = 0\ ,$$

then the control acts after a certain initial time t_0 and we can assume $t_0 = 0$. So, we get the control problem

$$\theta' = \int_{0}^{t} N(t-s)\theta_{xx}(s)\mathrm{d}s + H(t), \quad \theta(0,t) = f(t), \quad \theta(\pi,t) = 0. \tag{2.50}$$

The affine term $H(t) = H(x,t)$ takes into account the previous history of the system,

$$H(x,t) = \int_{-\infty}^{0} N(t-s)\theta_{xx}(x,s)\mathrm{d}s$$

(we noted that in the study of control problems we can assume $H = 0$).

So, we get system (2.3).

A similar equation is obtained when θ represents the concentration of a solute in a solvent, and q represents the flux across the position x at time t. Then, the constitutive law (2.46) is the Fick law, which leads to (2.47) as the law for the variation in time and space of the concentration (denoted θ) but it is clear that the assumption that the flux of matter reacts immediately to the variation of concentration is even less acceptable and, in particular in the presence of complex molecular structures, the law (2.48) is preferred, leading to (2.49) as the equation for the concentration. In fact, in Sect. 2.6.3 we shall see a further reason for this choice.

2.6.2 Viscoelasticity

We consider the simple case of a finite string (of constant density $\rho > 0$), which in the undeformed configuration is on the segment $[a, b]$ of the horizontal axis. After a deformation (which we assume "small" in the sense specified below) the point in position $(x, 0)$ will be found in position $(x, w(x))$ (so, we assume negligible horizontal motion, a first "smallness" constraint).

We call $w(x, t)$ the "deformation" of the string at time t and position x.

A deformation by itself does not produce an elastic traction: for example, a translation $w(x) = h$, the same for every x, does not produce any traction in the string. An elastic traction appears when neighboring points undergo different deformations.

Let us fix a point x_0. A small segment $(x_0, x_0 + \boldsymbol{\delta})$ on the right of x_0 exerts a traction on $[a, x_0]$ if $w(x_0 + \boldsymbol{\delta}) \neq w(x_0)$, i.e., $w(x_0 + \boldsymbol{\delta}) - w(x_0) \neq 0$, and elasticity assumes that this traction is $k\,(w(x_0 + \boldsymbol{\delta}) - w(x_0) + \mathrm{o}\,(w(x_0 + \boldsymbol{\delta}) - w(x_0)))$.

It is an experimental fact that $k = k(\boldsymbol{\delta})$ and, with an acceptable error,

$$k = \frac{k_0}{\boldsymbol{\delta}}$$

(in order to produce the same deformation in a longer string a smaller force is required). The number k_0 is positive since if $\boldsymbol{\delta} > 0$ and $w(x_0 + \boldsymbol{\delta}) - w(x_0) > 0$, then the traction on $[a, x_0]$ exerted by the part of the segment $x > x_0$ points upwards.

We proceed assuming the conditions of linear elasticity, which are new "smallness" condition. The first one is that the effect of $\mathrm{o}\,(w(x_0 + \boldsymbol{\delta}) - w(x_0))$ is negligibly small, and we ignore it.

Now, we balance the momentum on a segment (x_0, x_1) and the exterior forces acting on this segment, which are its weight, $p = -\rho g(x_1 - x_0)$ (g is the acceleration of gravity) and the difference of the tractions in x_1 and in x_0. Hence, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_{x_0}^{x_1} w'(x, t)\mathrm{d}x\right) &= -(x_1 - x_0)\rho g \\ &\quad + \frac{k_0}{\boldsymbol{\delta}}\left[(w(x_1 + \boldsymbol{\delta}, t) - w(x_1, t)) - (w(x_0 + \boldsymbol{\delta}, t) - w(x_0, t))\right] .\end{aligned}$$

We approximate the second line with $k_0\,(w_x(x_1, t) - w_x(x_0, t))$ (the last "smallness" condition):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_{x_0}^{x_1} w'(x,t)\mathrm{d}x\right) = -(x_1-x_0)\rho g + k_0\left[w_x(x_1,t)-w_x(x_0,t)\right]. \tag{2.51}$$

Now, we divide both the sides with $x_1 - x_0$ and we pass to the limit for $x_1 \to x_0$. As x_0 is a generic abscissa of points of the string, we rename it x and we get the string equation

$$\rho w''(x,t) = -\rho g + k_0 w_{xx}(x,t). \tag{2.52}$$

A hidden assumption in this derivation is that the elastic traction appears at the same time as the deformations $[w(x_1+\boldsymbol{\delta},t)-w(x_1,t)]$, $[w(x_0+\boldsymbol{\delta},t)-w(x_0,t)]$, and that it adjusts itself instantaneously to the variation of the deformation. In a sense, this is a common experience: a rubber cord shortens abruptly when released. But, if the cord has been in the freezer for some time, it will shorten slowly: the effect of the traction fades slowly with time. This is taken into account as follows: at every time t the traction depends also on the configuration of the string on every previous time s. Hence, instead of $k_0\left[w_x(x_1,t)-w_x(x_0,t)\right]$, the traction exerted on $[x_0,x_1]$ at time t *and due to the deformation at time* s is approximated with

$$\begin{aligned}\boldsymbol{\sigma}(x_0,x_1,t,s) &= k_0\left[(w_x(x_1,s)-w_x(x_1,s-\boldsymbol{\delta}))-((w_x(x_0,s)-w_x(x_0,s-\boldsymbol{\delta}))\right]\\ &= k_0\left[w_x'(x_1,s)-w_x'(x_0,s)\right]\boldsymbol{\delta}+\mathrm{o}(\boldsymbol{\delta})\ .\end{aligned}$$

We approximate this expression with:

$$\boldsymbol{\sigma}(x_0,x_1,t,s) = k_0(t,s)\left[w_x'(x_1,s)-w_x'(x_0,s)\right]\boldsymbol{\delta}\ .$$

Most often, $k(t,s)=k(t-s)$ and $k(t)$ is a decreasing function (the effect of previous deformations fades with time).

All the tractions originating at every time $s<t$ "sum" to give the traction acting on the segment $[x_0,x_1]$ at time t. So, Eq. (2.51) is replaced with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_{x_0}^{x_1} w'(x,t)\mathrm{d}x\right) &= -(x_1-x_0)\rho g\\ &\quad+\int_{-\infty}^{t} k(t-s)\left[w_x'(x_1,s)-w_x'(x_0,s)\right]\mathrm{d}s = -\rho g(x_1-x_0)\\ &\quad+k(0)\left[w_x(x_1,t)-w_x(x_0,t)\right]+\int_{-\infty}^{t} k'(t-s)\big[w_x(x_1,s)\\ &\quad-w_x(x_0,s)\big]\mathrm{d}\tau\ .\end{aligned}$$

We divide both the sides with $x_1 - x_0$ and we pass to the limit for $x_1 - x_0 \to 0$, as above. This gives Eq. (2.1).

Note that in this model the stress at position x is

$$\boldsymbol{\sigma}(x,t) = k(0)w_x(x,s) + \int_{-\infty}^{t} k'(t-s)w_x(x,s)\mathrm{d}s \ .$$

Remark 2.9 An implicit assumption in the previous derivation: the traction acting on the extremum x_1 of the segment $[x_0, x_1]$ is approximated with $k_0 w'_x(x_1, s)$: we disregard the traction, which may "diffuse" from distant points. And so, in Eq. (2.1) $w''(x,t)$ on the left side depends on the history of $w(x,s)$, at the *same* position x.

Finally, when studying control problems we can integrate on $(0, t)$, since we can assume that the system is at rest for $t < 0$.

2.6.3 The Special Case of the Telegraphers' Equation

The simplest example of a system with persistent memory is the system

$$u'(x,t) = \int_0^t \mathrm{e}^{-\gamma(t-s)} u_{xx}(x,s)\mathrm{d}s \qquad u(x,0) = u_0(x)$$

(physical considerations imply $\gamma \geq 0$). We compute the derivative (in time) of both the sides. If $\gamma = 0$ we get the string equation (2.52) (with $\rho = k_0 = 0$ and without the affine term) while if $\gamma > 0$ we get the the TELEGRAPHER'S EQUATION

$$u''(x,t) = u_{xx}(x,t) - u'(x,t) \qquad u(x,0) = u_0(x) \ , \quad u'(x,0) = 0 \ .$$

So, it has an interest to contrast the properties of the wave equation, the telegrapher's equation and, in view of Sect. 2.6.1, also of the heat equation. The telegraphers' equation has been studied in details and, when $x \in \mathbb{R}$, formulas for the solutions are known, see [89, Ch. VII-2]. We are not going to discuss these formulas. We use them to represent the solutions in the case described below (Figs. 2.1 and 2.2 (right)).

We consider the case $x \in \mathbb{R}$ (no boundary, i.e., we assume that $x \in [a, b]$ and that $b - a$ is "very large" when compared with the interval of time during which the system is studied). We assign a discontinuous initial condition[4] to the wave, telegrapher's and heat equations, an we compare the corresponding solutions. The

[4] this is not realistic for an elastic or viscoelastic body. We should consider continuous initial conditions, which are not everywhere differentiable, but the qualitative facts described below are the same. For completeness, Fig. 2.2 (right) shows the case that the initial deformation is zero, with nonzero and discontinuous initial velocity.

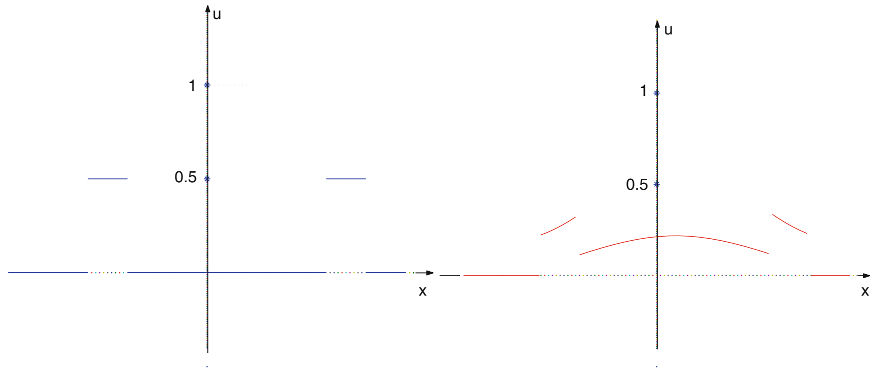

Fig. 2.1 *left* the graph of the initial condition (*dots*). When $t = 3$: the deformation computed from the (memoryless) wave equation (*left*) and from the telegrapher's equation (*right*)

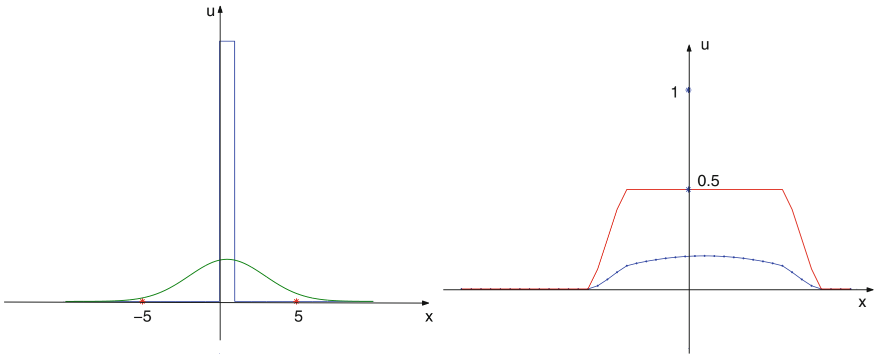

Fig. 2.2 Solution of the heat equation at the time $t = 3$ and the graph of the initial condition (*left*); wave and telegraph equations (at the time $t = 3$) with $u_0(x) = 0$ and $v_0(x) = 1$ for $x \in (0, 1)$ and $v_0(x) = 0$ otherwise (*right*)

equations are, respectively,

$$u'' = u_{xx} \ , \quad u'' = u_{xx} - u' \ , \quad u' = u_{xx}$$

and the initial conditions are

$$u(x, 0) = u_0(x) = \begin{cases} 1 & \text{if } t \in (0, 1)\,, \\ 0 & \text{otherwise} \end{cases} \qquad u'(x, 0) = v_0(x) = 0$$

(of course, in the case of the heat equation, we disregard the condition on $u'(x, 0)$).

We compare the graph of the function $x \mapsto u(x, 0) = u_0(x)$ and the graphs of the solutions at time $t = 3$, i.e., the maps $x \mapsto u(x, 3)$.

The noticeable facts for the *wave equation* are as follows:

- the solution at time $t = 3$ reached the intervals $[-3, -2]$ traveling to the left and $[3, 4]$ traveling to the right. During the times $t \in [0, 3)$ the solutions encountered every point in $(-2, 3)$, but the signal did not leave any memory of itself: once passed, its effect is abruptly forgotten.
- the discontinuity of the initial condition is preserved. There is a sharp *wavefront* both "in front" and "behind" the "traveling wave".

Instead, for the solution of the *telegraphers' equation:*

- the solution is a wave traveling with the same speed 1 as the solution of the wave equation: at time $T = 3$ it reaches the points $x = 4$ and $x = -3$. In particular, there is a discontinuity, i.e., a sharp forward "wavefront", at these points.
- the solution is discontinuous also at the points -2 and 3; i.e., it has the same jump points as the solution of the wave equation, but once the signal "encounters" a point x its memory *persists forever in that position* (in fact, its effect is attenuated with time, but never becomes identically zero).

We contrast also with the solution of the *heat equation,* whose key property is:

- the solution of the heat equation is of class C^∞ and *it is not zero for every x*. It soon becomes "very small" and it cannot be detected in the graph, but it is nonzero, since it is given by

$$\frac{1}{\sqrt{4\pi t}} \int_0^1 e^{-(x-s)^2/4t} \, ds \, .$$

  Consequence of this: signals travels with infinite speed in the case of the (memoryless) heat equation.

If we interpret w as the concentration of a "liquid" which diffuses in a polymer then we have:

- if the diffusion is according to Fick law, concentration varies smoothly and there is no visible separation between wet and dry regions.
- if the law of the diffusion is the telegraph equation then there is a sharp separation between wet and dry parts of the body, a fact experimentally verified in presence of complex molecular structure.

This last observation suggested to replace the heat equation with Eq. (2.3) to model diffusion in polymers, see [21].

2.7 Problems[5] to Chap. 2

2.1. Consider the following system of two ordinary differential equations:

$$u'' = -3u + w\,, \quad w'' = -3w + u\,.$$

Let the initial conditions be $u(0) = u_0$, $u'(0) = 0$, $w(0) = w_0$, $w'(0) = 0$. Represent the solution as

$$\begin{pmatrix} u(t) \\ w(t) \end{pmatrix} = R(t) \begin{pmatrix} u_0 \\ w_0 \end{pmatrix}\,.$$

Prove that $R(t)$ is a cosine operator in the sense that it verifies equality (2.9).

2.2. Show that the family of the operators $R_+(t)$ defined by

$$(R_+(t)\phi)(x) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) \tag{2.53}$$

is a cosine operator in $L^2(\mathbb{R})$, in the sense that equality (2.9) holds for this family of operators. Decide whether it is possible to represent this cosine operator with a Fourier type expansion, as in (2.10).

2.3. Show that the series representation (2.10) for the cosine operator can be obtained by separation of variables. Write it explicitly in the case $\Omega = (0, \pi)$.

2.4. Write the series (2.10), which represents the cosine operator of the problem

$$u'' = u_{xx} + u_{yy} \qquad (x, y) \in Q = (0, \pi) \times (0, \pi)\,, \quad u = 0 \text{ on } \partial Q$$

(∂Q is not of class C^2, but the results we have seen extend to this case).

2.5. Prove the integration by parts formulas (2.11) and (2.12).

2.6. Prove Theorem 2.2 when $f = 0$, $G = 0$, but the initial conditions are not zero; and do the same when control and initial conditions are zero, but $G \neq 0$.

2.7. Prove that under the condition of Theorem 2.2, and if furthermore u_0 and v_0 belong to $\mathscr{D}(\Omega)$ then the function $u(t)$ is of class $C^\infty([0, T]; L^2(\Omega))$.

2.8. Let $x \in (0, \pi)$ and $u = u(x, t)$ while $w = w(t)$ depends only on the time. Assuming zero initial conditions, discuss the time at which the effect of the input f affects the output y in the case of the systems

$$\begin{cases} u'' &= u_{xx}\,, \quad u(0, t) = f(t)\,, \quad u(\pi, t) = 0 \\ w'' &= -w + \int_{1/3}^{1/2} u(x, s)\mathrm{d}x \\ y(t) &= w(t), \end{cases} \qquad \begin{cases} w'' &= -w + f(t) \\ u_{tt} &= u_{xx} + \mathbf{1}_{(1/3,1/2)}(x) w(t) \\ u(0, t) &= u(\pi, t) = 0 \\ y(t) &= \int_{3/4}^{1} u(x, t)\mathrm{d}t \end{cases}$$

[5] Solutions at the address http://calvino.polito.it/~lucipan/materiale_html/P-S-CH-2.

The function $\mathbf{1}_{(1/3,1/2)}(x)$ is the characteristic function of $(1/3, 1/2)$.

2.9. The notations u and w are as in Problem 2.8 while $x \in (0, 7)$. Let $y(t) = w(t) \in \mathbb{R}$ be the output of the following system ($\alpha > 1$ is a real parameter):

$$u'' = u_{xx} + \mathbf{1}_{(1,\alpha)}(x) f(t) \ , \quad w'' = -w + \int_3^6 u(x,t)\mathrm{d}x$$

($u = u(x, t)$ with $x \in (0, 7)$) and conditions

$$\begin{cases} u(0,t) = 0 \ , \ u(7,t) = f(t) \ , \\ u(x,0) = 0 \ , \ u'(x,0) = 0 \ , \ w(0) = 0 \ . \end{cases}$$

Study at what time the effect of the external input $f(t)$ will influence the observation $y(t)$ and specify whether the time depends on α.

2.10. On a region Ω, consider the problem

$$w' = \Delta w + \int_0^t \Delta w(s)\mathrm{d}s, \qquad w(0) = u_0 \in L^2(\Omega), \quad w = 0 \text{ on } \partial\Omega. \qquad (2.54)$$

This equation is not of the same type as those studied in this book and has different control properties, see [37, 41–43]. Prove that Eq. (2.54) can be reduced to a Volterra integral equation using the semigroup e^{At}.

2.11. Let $\Omega = (0, 1)$ and consider the heat equation with memory (2.50) with $H = 0$ and $N(t) \equiv 1$ (hence, an integrated version of the string equation). Assume zero initial condition and $\theta(0, t) = f(t)$, $\theta(1, t) = 0$. Use (2.48) with initial time $t_0 = 0$ and $k = 1$. Compute the flux $q(t)$ on $t \in (0, 2)$ and show that the function $t \mapsto q(t)$ belongs to $C([0, 2]; L^2(0, 1))$.

Let either $T = 1$ or $T = 2$. Study whether the pair $(\theta(T), q(T))$ can be controlled to hit any target $(\xi, \eta) \in L^2(0, 1) \times L^2(0, 1)$ using a control $f \in L^2(0, T)$.

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