Existence and Uniqueness of Almost Automorphic Solutions to Cohen-Grossberg Neural Networks with Delays

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Abstract. The almost automorphic solution is a generalization of the almost periodic solution. In this paper, the almost automorphic solutions of Cohen-Grossberg neural networks with delays are considered. Using the semi-discretization method and the contraction mapping principle, some sufficient conditions are obtained to ensure the existence and the uniqueness of almost automorphic solutions to Cohen-Grossberg neural networks with delays.

Keywords: Cohen-Grossberg neural network · Almost automorphic solution · Contraction mapping principle

1 Introduction

In 1983, Cohen and Grossberg constructed an important kind of simplified neural networks model which is now called Cohen-Grossberg neural networks (CGNNS) [1]. This kind of neural networks is very general and includes Hopfield neural networks, cellular neural networks and BAM neural networks as its special cases. It has received increasing interest due to its applications in many fields such as pattern recognition, parallel computing, associative memory and combinatorial optimization. In recent years, the Cohen-Grossberg neural networks have been widely studied and many useful and interesting results have been obtained (see [6] and its references).

The concept of almost automorphy was introduced by Bochner [2] in 1964. It is a natural generalization of the classical almost periodicity. According to the properties of periodic functions, we know that periodic functions are all uniformly continuous. However, there exist some functions that have the similar properties to periodic functions, and meanwhile they are not uniformly continuous, such as $f(k) = sign(cos 2\pi k\theta)$. This kind of function is almost automorphic.

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At present, the almost automorphic functions have been used in many different kind of fields [3–5], [10,15,16], such as ordinary differential equation, partial differential equation, integral equation and dynamic system and so on. In [5], the authors studied a kind of partial differential equation based on biology, and the natural function classes of the solutions about this kind differential equation are almost automorphic functions. However, there is no paper discussed the almost automorphic solution to Cohen-Grossberg neural networks, so it is meaningful to discuss it and ours is the first one.

Generally speaking, the Cohen-Grossberg neural networks with delays can be described as following:

$$\dot{x_i}(t) = -a_i(x_i(t))\big[b_i(x_i(t)) - \textstyle\sum_{j=1}^{n} d_{ij}(t)g_j(x_j(t-\tau_j)) - I_i(t)\big] \tag{1}$$

where $i = 1, 2, \cdots, n$, $x_i(t)$ is the state variable associated with the i_{th} neuron, $a_i(\cdot)$ is an amplification function and $b_i(\cdot)$ represents a behaved function, $d_{ij}(t)$ presents the strength of connectivity between cells i and j at time t, the activation function $g_i(\cdot)$ tells how the i_{th} neuron reacts to the input, τ_i corresponds to the time delay. The initial condition of (1) is $x_i(t) = \varphi_i(t)$, $t \in [-\tau_i, 0]$.

In reality for the applications of neural networks to some practical problems, such as experiment, image processing, computational purposes and so on, it is essential to formulate a discrete-time system which is a version of the continuous-time system. The discrete-time system is desired to preserve the dynamical characteristics of the continuous-time system. There are many numerical schemes such as Euler scheme and Runge-Kutta scheme that can be utilized to obtain the discrete-time version of the continuous-time system. In this paper, we will use the semi-discretization scheme to obtain the discrete-time analogues of the continuous-time (1). The semi-discretization idea was originally used in the partial differential equations and then introduced to the ordinary differential equations. It has been proved that such kind of method can preserve the dynamical characteristics of the continuous-time systems to some extent, we can find examples in [7]-[9].

Using the semi-discretization method, the model (1) can be written as:

$$\dot{x_i}(t) = -a_i(x_i(t))\big[b_i(x_i(t)) - \textstyle\sum_{j=1}^{n} d_{ij}(t)g_j(x_j([\frac{t}{h}]h - [\frac{\tau_j}{h}]h)) - I_i(t)\big] \tag{2}$$

$t \in [nh, (n+1)h)$, $[\frac{t}{h}] = n$, h is the discretization step-size, it is a fixed positive real number.

In this paper, we consider the existence and the uniqueness of almost automorphic solutions of (2).

The remainder of this paper is organized as following: some definitions and assumptions are given in Section 2, and in Section 3, some sufficient conditions are given to ensure the existence of the almost automorphic solutions of (2). In the last section, Section 4, some conclusions about this paper are presented.

2 Preliminaries

For the readers' convenience, we first give some definitions (for details, see [11]-[14]).

Definition 1. *A continuous function $f : R \times X \to R$ is called almost automorphic for x in compact subsets of X, if for every compact subset K of x and every real sequence s_n, there exists a subsequence s_{n_k}, such that*

$$\lim_{n\to+\infty} f(t+s_{n_k},x) = g(t,x) \quad and \quad \lim_{n\to+\infty} g(t-s_{n_k},x) = f(t,x), \quad t\in R,\ x\in K$$

Definition 2. *A continuous $f : Z \times X \to X$ is a called almost automorphic sequence for $x \in X$ if for every sequence of integer $\{n\}$, there exists a subsequence $\{n_l\}_{l\in N}$, such that*

$$f(n+hn_l,x) \to g(n,x) \quad and \quad g(n-hn_l,x) \to f(n,x), \quad n\in Z \quad and \quad x\in X$$

The set of all such functions are denoted by $AAS(Z\times X \to X)$, AAS for short.

The following are some assumptions which will be used later.

$\mathbf{A_1}$: $a_i(\cdot)$, $d_{ij}(\cdot)$, and $I_i(\cdot)$ are almost automorphic to the variable t, and $0 < \underline{a}_i \le a_i(\cdot) \le \overline{a}_i$.

$\mathbf{A_2}$: for any $x, y \in R$, there exist some constants A_i, L_j, and G_j, such that $|a_i(x) - a_i(y)| < A_i|x-y|$, $|g_j(x)-g_j(y)| \le L_j|x-y|$, $|g_j(x)| \le G_j$.

$\mathbf{A_3}$: There exist positive almost automorphic functions $\underline{\beta}_i(t)$, $\overline{\beta}_i(t)$, such that $\forall x_i, y_i \in R$, $i = 1, 2, \cdots, n$, the following inequality holds:

$$0 < \underline{\beta}_i(t) \le \frac{a_i(x_i(t))b_i(x_i(t)) - a_i(y_i(t))b_i(y_i(t))}{x_i(t)-y_i(t)} \le \overline{\beta}_i(t)$$

and $\underline{\beta}_i = \inf_{t\ge0} |\underline{\beta}_i(t)|$, $\overline{\beta}_i = \sup_{t\ge0} |\overline{\beta}_i(t)|$, $b_i(0) \equiv 0$.

3 Main Results

According to A_3, the model (2) can be written as following:

$$\dot{x}_i(t) = -r_i(t)x_i(t) + a_i(x_i(t))\big[\textstyle\sum_{j=1}^n d_{ij}(t)g_j(x_j([\frac{t}{h}]h - [\frac{\tau_j}{h}]h)) + I_i(t)\big] \tag{3}$$

From (3) we can obtain:

$$\begin{aligned} x_i^h(n+1) = x_i^h(n)e^{-\int_{nh}^{(n+1)h} r_i(u)du} + a_i(x_i^h(n)) \\ \times\Big\{\textstyle\int_{nh}^{(n+1)h}\big[\sum_{j=1}^n d_{ij}(s)g_j(x_j(n-\tau_j^*)) + I_i(s)\big]e^{-\int_s^{(n+1)h} r_i(u)du}ds\Big\} \end{aligned} \tag{4}$$

where $x_i^h(n) = x_i(nh)$, and $\tau_j^* = [\frac{\tau_j}{h}]$.

Let

$$\begin{aligned} R_i(n) &= e^{-\int_{nh}^{(n+1)h} r_i(u)du}, \\ D_{ij}(n) &= \textstyle\int_{nh}^{(n+1)h} d_{ij}(s)e^{-\int_s^{(n+1)h} r_i(u)du}ds, \\ E_i(n) &= \textstyle\int_{nh}^{(n+1)h} I_i(s)e^{-\int_s^{(n+1)h} r_i(u)du}ds \end{aligned}$$

then (4) is reformulated as:

$$x_i^h(n+1) = R_i(n)x_i^h(n) + \textstyle\sum_{j=1}^n a_i(x_i^h(n))D_{ij}(n)g_j(x_j(n-\tau_j^*)) + a_i(x_i^h(n))E_i(n) \tag{5}$$

Denote $\overline{R}_i = \sup_{n\in Z}\{R_i(n)\}$, $\overline{D}_{ij} = \sup_{n\in Z}\{D_{ij}(n)\}$, $\overline{E}_i = \sup_{n\in Z}\{E_i(n)\}$.

Theorem 1. *Suppose that the assumptions $A_1 - A_3$ hold, then there exists a unique almost automorphic solution of (5) if*

$$\max_{1\leq i,j\leq n}\{\overline{R}_i+\sum_{j=1}^{n}\overline{a}_i\overline{D}_{ij}L_j+\sum_{j=1}^{n}\overline{A}_i\overline{D}_{ij}G_j+\overline{A}_i\overline{I}_i\}<1.$$

Proof. There are three steps to complete the proof.

Step1. To start the proof, we show that $R_i(n)$, $D_{ij}(n)$, $E_i(n)$ are almost automorphic for $i,j=1,2,\cdots,n$, firstly.

For $r_i(t)$ is almost automorphic, then for any sequence t_n, there exists a subsequence t_{n_l} such that $r_i(t+t_{n_l})\to\overline{r}_i(t)$ and $\overline{r}_i(t-t_{n_l})\to r_i(t)$ for $n_l\to\infty$. so

$$\begin{aligned}|R_i(n+t_{n_l})-\overline{R}_i(n)|&=|e^{-\int_{(n+t_{n_l})h}^{(n+1+t_{n_l})h}r_i(u)du}-e^{-\int_{nh}^{(n+1)h}\overline{r}_i(u)du}|\\&=|e^{-\int_{nh}^{(n+1)h}r_i(u+t_{n_l})du}-e^{-\int_{nh}^{(n+1)h}\overline{r}_i(u)du}|\to 0\end{aligned}$$

Thus $R_i(n+t_{n_l})\to\overline{R}_i(n)$. Likewise, $\overline{R}_i(n-t_{n_l})\to R_i(n)$.

Under assumption A_1, $d_{ij}(t)$ is almost automorphic and $d_{ij}(t+t_{n_l})\to\overline{d}_{ij}(t)$.

Let $\overline{D}_{ij}(n)=\int_{nh}^{(n+1)h}\overline{d}_{ij}(s)e^{-\int_s^{(n+1)h}\overline{r}_i(u)du}ds$, then for $\{t_{n_l}\}\in Z$,

$$\begin{aligned}&|D_{ij}(n+t_{n_l})-\overline{D}_{ij}(n)|\\=&\ |\int_{(n+t_{n_l})h}^{(n+1+t_{n_l})h}d_{ij}(s)e^{-\int_s^{(n+1+t_{n_l})h}r_i(u)du}ds-\int_{nh}^{(n+1)h}\overline{d}_{ij}(s)e^{-\int_s^{(n+1)h}\overline{r}_i(u)du}ds|\\\leq&\ |\int_{nh}^{(n+1)h}d_{ij}(s+t_{n_l})[e^{-\int_s^{(n+1)h}r_i(u+t_{n_l})du}-e^{-\int_s^{(n+1)h}r_i(u)du}]ds|\\&+|\int_{nh}^{(n+1)h}[d_{ij}(s+t_{n_l})-\overline{d}_{ij}(s)]e^{-\int_s^{(n+1)h}\overline{r}_i(u)du}ds|\\\to&\ 0.\end{aligned}$$

Likewise, $\overline{D}_{ij}(n-t_{n_l})\to D_{ij}(n)$. Then by the similar analysis, $E_i(n+t_{n_l})\to\overline{E}_i(n)$, $\overline{E}_i(n-t_{n_l})\to E_i(n)$. That is to say, $A_i(n)$, $D_{ij}(n)$, $E_i(n)\in AAS$.

Step2. Consider the following equation:

$$x_i^h(n+1)=R_i(n)x_i^h(n)+a_i(x_i^h(n))E_i(n) \tag{6}$$

Next, we will show that (6) has a unique almost automorphic sequence solution. Using the method of induction, according to (6), we can obtain

$$x_i^h(n+1)=\prod_{l=0}^{n}R_i(l)x_i^h(0)+\sum_{q=0}^{n}a_i(x_i^h(n-q))\int_{(n-q)h}^{(n+1-q)h}I_i(s)e^{-\int_s^{(n+1)h}r_i(u)du}ds$$

Let

$$\widetilde{x}_i^h(n)=\sum_{q=0}^{n-1}a_i(\widetilde{x}_i^h(n-1-q))\int_{(n-1-q)h}^{(n-q)h}I_i(s)e^{-\int_s^{nh}r_i(u)du}ds,$$

then

$$|\widetilde{x}_i^h(n)|\leq|\sum_{q=0}^{n-1}\overline{a}_i\frac{\overline{I}_i}{\underline{\beta}_i}(e^{-qh\underline{\beta}_i}-e^{-(q+1)h\underline{\beta}_i})|<|\frac{\overline{a}_i\overline{I}_i}{\underline{\beta}_i}(1-e^{-nh\underline{\beta}_i})|<|\frac{\overline{a}_i\overline{I}_i}{\underline{\beta}_i}|.$$

We can easily verify that
$\widetilde{x}_i^h(n+1) = R_i(n)\widetilde{x}_i^h(n) + a_i(\widetilde{x}_i^h(n))E_i(n)$.

Let $\widetilde{x}_{i_*}^h(n) = \sum_{q=0}^{n-1} \overline{a}_i(\widetilde{x}_{i_*}^h(n-1-q)) \int_{(n-1-q)h}^{(n-q)h} \overline{I}_i(s)e^{-\int_s^{(n-q)h} \overline{r}_i(u)du}ds$, where $a_i(\widetilde{x}_i^h(n+t_{n_l})) \to \overline{a}_i(\widetilde{x}_{i_*}^h(n))$, and $\overline{a}_i(\widetilde{x}_{i_*}^h(n-t_{n_l})) \to a_i(\widetilde{x}_i^h(n))$.

Then for any given sequence $\{t_{n_l}\} \in Z$,

$$\begin{aligned}
&|\widetilde{x}_i^h(n+t_{n_l}) - \widetilde{x}_{i_*}^h(n)|\\
\leq &\textstyle\sum_{q=0}^{n-1} \big|\big[a_i(\widetilde{x}_i^h(n-1-q)) - \overline{a}_i(\widetilde{x}_{i_*}^h(n-1-q))\big]\\
&\textstyle\int_{(n-1-q)h}^{(n-q)h} I_i(s+t_{n_l})e^{-\int_s^{(n-q)h} r_i(u+t_{n_l})du}ds\big|\\
&+\textstyle\sum_{q=0}^{n-1} \big|\overline{a}_i(\widetilde{x}_{i_*}^h(n-1-q)) \int_{(n-1-q)h}^{(n-q)h}[I_i(s+t_{n_l}) - \overline{I}_i(s)]e^{-\int_s^{(n-q)h} r_i(u+t_{n_l})du}ds\big|\\
&+\textstyle\sum_{q=0}^{n-1} \big|\overline{a}_i(\widetilde{x}_{i_*}^h(n-1-q)) \int_{(n-1-q)h}^{(n-q)h} \overline{I}_i(s)\\
&[e^{-\int_s^{(n-q)h} r_i(u+t_{n_l})du} - e^{-\int_s^{(n-q)h} r_i(u)du}]ds\big| \to 0
\end{aligned}$$

So $\widetilde{x}_i^h(n+t_{n_l}) \to \widetilde{x}_{i_*}^h(n)$. Likewise, $\widetilde{x}_{i_*}^h(n-t_{n_l}) - \widetilde{x}_i^h(n)$. Thus, $\widetilde{x}_i^h(n)$ is almost automorphic. In addition, $\widetilde{x}_i^h(n+1) = R_i(n)\widetilde{x}_i^h(n) + a_i(\widetilde{x}_i^h(n))E_i(n)$, then $\widetilde{x}_i^h(n)$ is the almost automorphic solution of (6).

Step3. Assume that

$$\theta = \max_{1\leq i\leq n} \frac{\overline{a}_i\overline{I_i}}{\underline{\beta_i}}, \quad \omega = \max_{1\leq i\leq n}\{\overline{R}_i + \overline{A}_i\overline{E}_i\}, \qquad \gamma = \max_{1\leq i\leq n}\{\overline{R}_i + \sum_{j=1}^n \overline{a}_i\overline{D}_{ij}L_j\}$$

Define a mapping F: $AAS \to AAS$, $x \to Fx$, $Fx = ((Fx)_2, \cdots, (Fx)_n)^T$,

$$(Fx)_i(n+1) = R_i(n)x_i^h(n) + \sum_{j=1}^n a_i(x_i^h(n))D_{ij}(n)g_j(x_j(n-\tau_j^*)) + a_i(x_i^h(n))E_i(n).$$

Denote $\|x\| = \sup_{n\in Z}\max_{1\leq i\leq n}|x_i(n)|$, let $\Omega = \{x : x$ is almost automorphic, $\|x-\widetilde{x}\| \leq \frac{\omega+\gamma}{1-\gamma}\theta\}$, then $\|x\| \leq \|x-\widetilde{x}\| + \|\widetilde{x}\| = \frac{\omega+1}{1-\gamma}\theta$.

$\forall x, y \in \Omega$, we have:

$$\begin{aligned}
\|Fx - \widetilde{x}\| &= \sup_{n\in Z}\max_{1\leq i\leq n}|R_i(n)(x_i^h(n) - \widetilde{x}_i^h(n))\\
&\quad+ \sum_{j=1}^n a_i(x_i^h(n))D_{ij}(n)g_j(x_j(n-\tau_j^*)) + [a_i(x_i^h(n)) - a_i(\widetilde{x}_i^h(n))]E_i(n)|\\
&\leq \overline{R}_i\|x\| + \overline{R}_i\|\widetilde{x}\| + \sum_{j=1}^n \overline{a}_i\overline{D}_{ij}L_j\|x\| + \overline{A}_i\overline{E}_i\|\widetilde{x}\|\\
&\leq (\overline{R}_i + \sum_{j=1}^n \overline{a}_i\overline{D}_{ij}L_j)\|x\| + \omega\|\widetilde{x}\|\\
&\leq \frac{\omega+\gamma}{1-\gamma}\theta
\end{aligned}$$

$$\begin{aligned}\|Fx - Fy\| = \sup_{n \in Z} \max_{1 \le i \le n} \Big\{ &|R_i(n)(x_i^h(n) - y_i^h(n)) \\ &+ \sum_{j=1}^{n} [a_i(x_i^h(n)) g_j(x_j(n - \tau_j^*)) - \sum_{j=1}^{n} a_i(y_i^h(n)) g_j(y_j(n - \tau_j^*))] D_{ij}(n) \\ &+ [a_i(x_i^h(n)) - a_i(y_i^h(n))] I_i(n)| \Big\} \\ \le \max_{1 \le i \le n} \Big\{ (\overline{R}_i + &\sum_{j=1}^{n} \overline{a}_i \overline{D}_{ij} L_j + \sum_{j=1}^{n} \overline{A}_i \overline{D}_{ij} G_j + \overline{A}_i \overline{I}_i) \|x - y\| \Big\} < \|x - y\|\end{aligned}$$

Then F is a construction mapping, thus (5) has a unique almost automorphic solution which satisfies that $\|x - \widetilde{x}\| < \frac{\omega+\gamma}{1-\gamma}\theta$. This completes the proof.

4 Conclusions

In this paper, the almost automorphic solutions of delayed Cohen-Grossberg neural networks are investigated. The almost automorphic solution is a generalization of the almost periodic solution, and it has been used in ordinary differential equation, partial differential equation, integral equation and dynamic system and so on. Our paper is the first one to discuss such solutions on Cohen-Grossberg neural networks. By the contraction mapping principle, the existence and the uniqueness of almost automorphic solutions are discussed, and some new results are obtained.

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