

The Non-confusing Travel Groupoids on a Finite Connected Graph

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Abstract. The notion of travel groupoids was introduced by L. Nebeský in 2006 in connection with a study on geodetic graphs. A travel groupoid is a pair of a set V and a binary operation $*$ on V satisfying two axioms. For a travel groupoid, we can associate a graph. We say that a graph G has a travel groupoid if the graph associated with the travel groupoid is equal to G . A travel groupoid is said to be non-confusing if it has no confusing pairs. Nebeský showed that every finite connected graph has at least one non-confusing travel groupoid.

In this note, we study non-confusing travel groupoids on a given finite connected graph and we give a one-to-one correspondence between the set of all non-confusing travel groupoids on a finite connected graph and a combinatorial structure in terms of the given graph.

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1 Introduction

A *groupoid* is the pair $(V, *)$ of a nonempty set V and a binary operation $*$ on V . The notion of travel groupoids was introduced by L. Nebeský [5] in 2006 in connection with his study on geodetic graphs [1–3] and signpost systems [4]. First, let us recall the definition of travel groupoids.

A *travel groupoid* is a groupoid $(V, *)$ satisfying the following axioms (t1) and (t2):

(t1) $(u * v) * u = u$ (for all $u, v \in V$),

(t2) if $(u * v) * v = u$, then $u = v$ (for all $u, v \in V$).

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A *geodetic graph* is a connected graph in which there exists a unique shortest path between any two vertices. Let G be a geodetic graph, and let $V := V(G)$. For two vertices u and v of G , let $A_G(u, v)$ denote the vertex adjacent to u which is on the unique shortest path from u to v in G . Define a binary operation $*$ on V as follows: For all $u, v \in V$, let $u * v := A_G(u, v)$ if $u \neq v$ and $u * v := u$ if $u = v$. This groupoid $(V, *)$ is called the *proper groupoid* of the geodetic graph G . Remark that the proper groupoid of any geodetic graph is a travel groupoid.

Let $(V, *)$ be a travel groupoid, and let G be a graph. We say that $(V, *)$ is *on* G or that G *has* $(V, *)$ if $V(G) = V$ and $E(G) = \{\{u, v\} \mid u, v \in V, u \neq v, \text{ and } u * v = v\}$. Note that every travel groupoid is on exactly one graph.

Let $(V, *)$ be a travel groupoid. For $u, v \in V$, we define $u *^0 v := u$ and $u *^{i+1} v := (u *^i v) * v$ for every nonnegative integer i . It is clear that $(u *^j v) *^k v = u *^{j+k} v$ holds for any nonnegative integers j and k . Note that, for any two distinct elements u and v in V , it holds that $u * v \neq u$ (see [5, Proposition 2 (2)]) and that $u *^2 v \neq u$ by (t2). An ordered pair (u, v) of two distinct elements of V is called a *confusing pair* in $(V, *)$ if there exists an integer $i \geq 3$ such that $u *^i v = u$. A travel groupoid $(V, *)$ is said to be *non-confusing* if there is no confusing pair in $(V, *)$.

Nebeský gave a characterization of non-confusing travel groupoids on a finite graph.

Theorem 1 ([5, Theorem 3]). *Let $(V, *)$ be a travel groupoid on a finite graph G . Then, $(V, *)$ is non-confusing if and only if, for all distinct elements u and v in V , there exists a positive integer k such that the sequence $(u *^0 v, \dots, u *^{k-1} v, u *^k v)$ is a path from u to v in G .* $\square$

Nebeský also showed a result on the existence of non-confusing travel groupoids. Note that a travel groupoid $(V, *)$ is said to be *simple* if it satisfies the following axiom (t3) if $v * u \neq u$, then $u * (v * u) = u * v$ (for all $u, v \in V$).

Theorem 2 ([5, Theorem 4]). *For every finite connected graph G , there exists a simple non-confusing travel groupoid on G .* $\square$

By Theorem 2, we know that every finite connected graph has always at least one non-confusing travel groupoid.

In this note, we study non-confusing travel groupoids on a given finite connected graph and we give a one-to-one correspondence between the set of all non-confusing travel groupoids on a finite connected graph and a combinatorial structure in terms of the given graph.

2 Main Result

To describe the structure of non-confusing travel groupoids on a graph, we define the following.

Definition. For a vertex v of a graph G , a v -tree is a spanning tree of G which contains all the edges incident to v . We denote by $\mathcal{S}_G(v)$ the set of all v -trees in G . $\square$

Lemma 3. Let $(V, *)$ be a non-confusing travel groupoid on a finite connected graph G . For an element v of V , let T_v be the graph defined by

$$V(T_v) := V \quad \text{and} \quad E(T_v) := \{\{u, u * v\} \mid u \in V \setminus \{v\}\}.$$

Then, T_v is a v -tree of G .

Proof. Since $(V, *)$ is on G , for any two distinct elements u and v in V , $\{u, u * v\}$ is an edge of G . Therefore, we have $E(T_v) \subseteq E(G)$. Moreover, $V(T_v) = V = V(G)$. So T_v is a spanning subgraph of G . By the definition of the graph T_v , we have $|E(T_v)| \leq |V| - 1$. Since $(V, *)$ is non-confusing, it follows from Theorem 1 that any vertex u of T_v distinct from v is connected to the vertex v by a path P_{uv} in G , where P_{uv} is of the form $(u *^0 v, \dots, u *^k v)$ for some positive integer k . Since $u *^{i+1} v = (u *^i v) * v$, the edge $\{u *^i v, u *^{i+1} v\}$ of the path P_{uv} is also an edge of T_v for any $i \in \{0, 1, \dots, k-1\}$. Therefore, the path P_{uv} from u to v in G is also a path in T_v . So we can conclude that T_v is connected. Since T_v is a connected graph with $|E(T_v)| \leq |V(T_v)| - 1$, T_v is a tree. Thus, T_v is a spanning tree of G . For each vertex x which is adjacent to the vertex v in G , we have $x * v = v$ and therefore $\{x, v\} \in E(T_v)$. So, all the edges incident to the vertex v in G are contained in T_v . Hence, the graph T_v is a v -tree of G . $\square$

The following theorem is our main result.

Theorem 4. Let G be a finite connected graph. Then, there exists a one-to-one correspondence between the set $\Pi_{v \in V(G)} \mathcal{S}_G(v)$ and the set of all non-confusing travel groupoids on G .

Proof. Let $V := V(G)$, and let $\mathcal{N}_G$ be the set of all non-confusing travel groupoids on G . Note that, since G is finite, both the sets $\Pi_{v \in V} \mathcal{S}_G(v)$ and $\mathcal{N}_G$ are finite.

For any $(T_v)_{v \in V} \in \Pi_{v \in V(G)} \mathcal{S}_G(v)$, we define a groupoid on V as follows: For two distinct elements u and v in V , $u * v$ is defined to be the vertex adjacent to u which is on the unique path from u to v in the tree T_v . If $u = v$, then let $u * v = u$. We show that this groupoid $(V, *)$ is a non-confusing travel groupoid. First, we check that $(V, *)$ satisfies the axioms (t1) and (t2). Take any two distinct elements u and v in V . Let $w := u * v$. Then $\{u, w\} \in E(G)$. Therefore $\{w, u\} \in E(T_u)$ and we have $(u * v) * u = w * u = u$. Moreover, if $u = v$, then $(u * v) * u = (u * u) * u = u * u = u$. Thus the axiom (t1) holds. Again, take any two distinct elements u and v in V . If u and v are adjacent (i.e. $\{u, v\} \in E(G)$), then $\{u, v\} \in E(T_v)$ and therefore $(u * v) * v = v * v = v \neq u$. If u and v are not adjacent, then $u * v \neq v$ and the element $(u * v) * v$ is the third vertex of the path from u to v in the tree T_v and therefore $(u * v) * v$ is not the element u . Thus the axiom (t2) holds. So, $(V, *)$ is a travel groupoid. Second, we check that $(V, *)$ is non-confusing. Take any two distinct elements u and v in V . Let k be the length of the path from u to v in T_v . Then, it follows from the definition of

$(V, *)$ that $u *^k v = v$ and that $(u *^0 v, \dots, u *^k v)$ is the path from u to v in T_v . Therefore, $(u *^0 v, \dots, u *^k v)$ is a path from u to v in G . By Theorem 1, the travel groupoid $(V, *)$ is non-confusing. Now, we define a map $\Phi : \Pi_{v \in V} \mathcal{S}_G(v) \rightarrow \mathcal{N}_G$ by $\Phi((T_v)_{v \in V}) = (V, *)$, where $(V, *)$ is the non-confusing travel groupoid defined as above for $(T_v)_{v \in V}$.

Next, let $(V, *)$ be a non-confusing travel groupoid on a finite connected graph G . For each $v \in V(G)$, we define a graph T_v by $V(T_v) := V$ and $E(T_v) := \{\{u, u * v\} \mid u \in V \setminus \{v\}\}$. By Lemma 3, T_v is a v -tree of G , i.e., $T_v \in \mathcal{S}_G(v)$. Therefore, $(T_v)_{v \in V} \in \Pi_{v \in V} \mathcal{S}_G(v)$. Now, we define a map $\Psi : \mathcal{N}_G \rightarrow \Pi_{v \in V} \mathcal{S}_G(v)$ by $\Psi((V, *)) = (T_v)_{v \in V}$, where $(T_v)_{v \in V}$ are v -trees defined as above for $(V, *)$.

Then, we can check that $\Psi \circ \Phi((T_v)_{v \in V}) = (T_v)_{v \in V}$ holds for any $(T_v)_{v \in V} \in \Pi_{v \in V} \mathcal{S}_G(v)$ and that $\Phi \circ \Psi((V, *)) = (V, *)$ holds for any $(V, *) \in \mathcal{N}_G$. Hence the map Φ is a one-to-one correspondence between the sets $\Pi_{v \in V} \mathcal{S}_G(v)$ and $\mathcal{N}_G$. $\square$

Corollary 5. *Let G be a finite connected graph. Then, the number of non-confusing travel groupoids on G is equal to $\Pi_{v \in V(G)} |\mathcal{S}_G(v)|$.*

Proof. It follows from Theorem 4. $\square$

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