Chapter 2
Preliminaries

This chapter provides the reader with a workable definition of an input–output system and the natural state. An input–output system is a causal mapping from an input space of time functions to an output space of time functions. We set up metrics on these spaces using seminorms referred to as fitted families. A wide class of fitted families of seminorms are given by the weighted L_p spaces. By causality, an input up to a time defines a mapping from future inputs to future outputs. This mapping is the natural state. The natural state set is seen to be minimal and a metric is defined on it.

The purpose of this chapter is to present the reader with a workable definition of an input–output system and natural state. An input–output system is herein denoted (Y, F, U) where F is a mapping from an input space U to an output space Y, and where U and Y are translation-invariant spaces of vector-valued time functions. The vector values of these time functions typically are $\Re^N$, $N = 1, 2, \cdots$, but only need to be Banach spaces. Other spaces and mappings related to (Y, F, U) will be introduced, but U and Y with whatever affixes they carry always refer to input and output spaces. Mappings from various input spaces to output spaces are denoted either F or ξ, again with qualifying affixes; ξ (or lowercase Greek) is reserved for state operators (i.e., for natural states).

The input and output space metrics are set up by seminorms referred to as fitted families (FFs) of seminorms. Roughly speaking, fitted families work like L_p norms on time functions with the additional feature that a time weighting can be incorporated so that the distant past of an input or output time function may be de-emphasized. The notation $\|u_{s,t}\|_{s,t}$ indicates the norm (e.g., weighted L_p norm) of the input u over the interval of time $(s, t]$. $U_{s,t}$ is the space of inputs over the same interval. FFs were initially described by Root [10].

Definition 1 ([10]). Let $\mathcal{L} = \mathcal{L}(\Re, E)$ be a linear space of time functions from $\Re$ into a Banach space E such that any translate of a function in $\mathcal{L}$ is also a function

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in $\mathcal{L}$. Let $\mathcal{N} = \{\|\cdot\|_{s,t}, -\infty < s < t < \infty\}$ be a family of seminorms on $\mathcal{L}$ satisfying the following conditions:

(1) For $f_1, f_2 \in \mathcal{L}$, if $f_1(\tau) = f_2(\tau)$ for $s < \tau \le t$, then $\|f_1 - f_2\|_{s,t} = 0$.
(2) Let L_τ denote shift to the left by τ. For all $f \in \mathcal{L}$, $\|\mathrm{L}_\tau f\|_{s-\tau,t-\tau} = \|f\|_{s,t}$.
(3) Let $r < s < t$. Then for all $f \in \mathcal{L}$, $\|f\|_{s,t} \le \|f\|_{r,t}$.
(4) Let $r < s < t$. Then for all $f \in \mathcal{L}$, $\|f\|_{r,t} \le \|f\|_{r,s} + \|f\|_{s,t}$.
(5) There exists $0 < \alpha \le \infty$ and $K \ge 1$ such that if $0 < t - r \le \alpha$ and $r < s < t$, then for all $f \in \mathcal{L}$, $\|f\|_{r,s} \le K\|f\|_{r,t}$.

The pair $(\mathcal{L}, \mathcal{N})$ is called an FF of seminorms on $\mathcal{L}$. The normed linear space formed from equivalence classes of functions in $\mathcal{L}$ with norm $\|\cdot\|_{s,t}$ is denoted $A_{s,t}$. The elements of $A_{s,t}$ are the equivalence classes determined by: $f \sim g$, $f, g \in \mathcal{L}$ if and only if $\|f - g\|_{s,t} = 0$. They are denoted $u_{s,t}$, $y_{s,t}$, etc. The set $\{A_{s,t}\}$, $-\infty < s < t < \infty$, is the *FF of normed linear spaces* given by $(\mathcal{L}, \mathcal{N})$.

A fairly wide class of examples of FFs are given by weighted L_p spaces. For $1 \le p < \infty$, let w be a fixed nonnegative, nonincreasing Lebesgue measurable real-valued function and let $\mathcal{L} = \mathcal{L}(\Re, \Re^N)$ be the set of N vector-valued functions on $\Re$ that are p-integrable Lebesgue on finite intervals. Then, for $f \in \mathcal{L}$ the seminorms

$$\|f\|_{s,t} = \left(\int_s^t \|f(\tau)\|^p \, w(t-\tau)d\tau\right)^{1/p} \tag{2.1}$$

satisfy Conditions (1),$\cdots$,(5) of Definition 1. For $p = \infty$, let $\mathcal{L}$ be the set of essentially bounded functions. When $\alpha = +\infty$ and $K = 1$, a uniform time weighting is given and $(\mathcal{L}, \mathcal{N})$ is referred to as a *standard* FF of seminorms. As time weighting was not essential to its purpose, standard FFs were used in [14].

We define another FF of seminorms. For $f \in \mathcal{L}$, put $\|f\|^{s,t} \triangleq \sup_{s<\tau\le t} \|f\|_{s,\tau}$. With $\mathcal{M}$ indicating this new set of seminorms, $(\mathcal{L}, \mathcal{M})$ is indeed an FF of seminorms, [10].

An FF $(\mathcal{L}, \mathcal{N})$ and $\{A_{s,t}\}$, $-\infty < s < t < \infty$, can be augmented to include $\|\cdot\|_{-\infty,t}$ by taking the limit $s \to -\infty$, since by (3) of Definition 1 $\|f\|_{s,t}$ is monotone nondecreasing as $s \to -\infty$ with t fixed. Let $\mathcal{L}_0 = \{f \in \mathcal{L} | \lim_{s\to-\infty} \|f\|_{s,t} < \infty, t \in \Re\}$. For $f \in \mathcal{L}_0$, define

$$\|f\|_t \triangleq \lim_{s\to-\infty} \|f\|_{s,t} = \|f\|_{-\infty,t} \,. \tag{2.2}$$

With the meaning of $(\mathcal{L}, \mathcal{N})$ thus extended, $\|\cdot\|_{s,t}$ is defined for $-\infty \le s < t < \infty$. The *left-expanded FF of seminorms* is thereby defined and is denoted $(\mathcal{L}_0, \mathcal{N})$. It still satisfies all the Conditions (1), $\cdots$, (5).

To discuss natural state we also need to define $\|\cdot\|_{s,\infty}$ and $A_{s,\infty}$ in a meaningful way. For an FF, this is done by taking the supremum. Let $\mathcal{L}_{00} = \{f \in \mathcal{L}_0 | \sup_t \|f\|_t < \infty\}$. For $f \in \mathcal{L}_{00}$ define

$$\|f\|_{s,\infty} \triangleq \sup_{t>s} \|f\|_{s,t} \;;\; -\infty \le s \,. \tag{2.3}$$

It may be readily verified that if $(\mathcal{L},\mathcal{N})$ is an FF for indices satisfying $-\infty < s < t < \infty$ then, with definitions given by (2.2) and (2.3), $(\mathcal{L}_{00},\mathcal{N})$ is an FF for indices satisfying $-\infty \le s < t < \infty$ and satisfies Conditions 1, 2, 3, and 5 of Definition 1 for indices $-\infty \le s < t \le \infty$. (For standard FFs Condition 4 holds for both cases.) $\{(\mathcal{L}_{00},\mathcal{N}),\ \|\cdot\|_{s,t}\,,\ -\infty \le s < t \le \infty\}$ is called the *expanded family of seminorms* determined by $(\mathcal{L},\mathcal{N})$. Note that $(\mathcal{L}_{00},\mathcal{M})$ similarly defined is an FF for $-\infty \le s < t \le \infty$.

For $f \in \mathcal{L}_{00}$ we put

$$\|f\| \triangleq \sup_{t\in\Re} \|f\|_t \ = \|f\|_{-\infty,\infty}\,. \tag{2.4}$$

The normed linear space consisting of equivalence classes of functions in $\mathcal{L}_{00}$ with the norm (2.4) is called the *bounding space* A for the family $\{A_{s,t}\}$.

The *extended space* A^e for the family $\{A_{s,t}\}$ is the set of all equivalence classes of functions f in $\mathcal{L}_0$ ($f \sim g$ iff $\|f-g\| = 0$) for which $\|f\|_s < \infty$ for all s. It does not have a norm and, indeed, is given no topology. This definition agrees with the notion of extended space commonly used in the control literature.

It is possible that an FF $(\mathcal{L},\mathcal{N})$ has a vacuous expansion in the sense that $\mathcal{L}_{00}$ is the empty set. An obvious example of this is given when $\mathcal{L}$ is the set of all constant real-valued functions on $\Re$ and $\mathcal{N}$ is the set of L_1-norms on finite intervals. To prevent this from happening and further to prevent the bounding space A from being too small (in a sense to be made explicit below) we can require that an FF be "full," as in the following definition.

Definition 2. The FF $(\mathcal{L},\mathcal{N})$ is *full* if each equivalence class $u_{s,t} \in A_{s,t}$, $-\infty < s < t < \infty$, has a representing function belonging to $\mathcal{L}_{00}$.

When this definition is satisfied, then for all pairs (s,t), $-\infty < s < t < \infty$, there is a 1:1 correspondence between the normed linear space $A_{s,t}$ determined by $(\mathcal{L},\mathcal{N})$ and the normed linear space $A'_{s,t}$ determined by $(\mathcal{L}_{00},\mathcal{N})$, which preserves the normed linear space structure. The correspondence is given by $u_{s,t} \leftrightarrow u'_{s,t}$, $u_{s,t} \in A_{s,t}$, $u'_{s,t} \in A'_{s,t}$ if and only if $u_{s,t}$ and $u'_{s,t}$ have a common representing function $f \in \mathcal{L}_{00}$. Thus, if $(\mathcal{L},\mathcal{N})$ is full, we need not distinguish between $A_{s,t}$ and $A'_{s,t}$. Henceforth, every FF mentioned is assumed to be full. The FFs formed with L_p spaces as described above are full. To emphasize the relations among the equivalence classes, suppose the function $f \in \mathcal{L}_{00}$ determines the equivalence classes $u \in A$, $u^t \in A^t$, $u_t \in A_t$, and $u_{s,t} \in A_{s,t}$. Since

$$\|f-g\|_{s,t} \le \|f-g\|_t \le \|f-g\|^t \le \|f-g\|$$

the equivalence class u considered as a set of functions is entirely contained in the equivalence class u^t considered as a set of functions, and similarly $u^t \subset u_t$ and $u_t \subset u_{s,t}$; also $u \subset u^{t,\infty}$. Thus, e.g., given t, u determines u_t and $u^{t,\infty}$. Therefore, if f determines u and $-\infty \le s < t \le \infty$, it is meaningful, for example, to write $\|f\|_{s,t}$, $\|u\|_{s,t}$, $\|u^t\|_{s,t}$, $\|u^{s,\infty}\|_{s,t}$, $\|u_t\|_{s,t}$, $\|u_{s,t}\|_{s,t}$, and they are all equal.

Let $-\infty \le r < s < t \le \infty$. Then, since $\|f\|_{s,t} \le \|f\|_{r,t}$ for $f \in \mathcal{L}_{00}$, the partitioning of $\mathcal{L}_{00}$ into equivalence classes by $\|\cdot\|_{r,t}$ results in a finer partition than that given by $\|\cdot\|_{s,t}$. That is, letting f determine $u \in U$, we have $u \subset u_{r,t} \subset u_{s,t}$.

In order to define the natural state, it is necessary to consider an arbitrary past input concatenated with an arbitrary future input; that is, to "splice" two inputs.

Definition 3. For $-\infty \le r < s < t < \infty$, and $h, g \in \mathcal{L}$, the *splice* of h and g over $(r, t]$ at s is defined and equals f if

$$f(\tau) = \begin{cases} h(\tau), & r < \tau \le s \\ g(\tau), & s < \tau \le t \end{cases}$$

belongs to $\mathcal{L}$. It is denoted $f_{r,t} = h_{r,s} \mapsto g_{s,t}$. For $t = \infty$, the splice of h and g equals f if

$$f(\tau) = \begin{cases} h(\tau), & r < \tau \le s \\ g(\tau), & s < \tau \end{cases}$$

belongs to $\mathcal{L}$. It is denoted $f_{r,\infty} = h^{r,s} \mapsto g_{s,\infty}$.

If $u_{r,s} \in A_{r,s}$, $v_{s,t} \in A_{s,t}$ are determined by functions h and g respectively, and $h_{r,s} \mapsto g_{s,t}$ exists, then for an FF, the splice of u and v (or $u_{r,s}$ and $v_{s,t}$) over $(r, t]$ at s is defined to be the element $w_{r,t} \in A_{r,t}$ determined by $h_{r,s} \mapsto g_{s,t}$; we write $w_{r,t} = u_{r,s} \mapsto v_{s,t}$. For $t = \infty$ we write $w_{r,\infty} = u^{r,s} \mapsto v_{s,\infty}$. These are not meaningful until it is proved that the splice is independent of the particular functions h and g representing the equivalence classes $u_{r,s}$ and $v_{s,t}$. However, this proof follows easily from Definition 1.

Any input space U is herein taken to be either the bounding space A of an FF $\{A_{s,t}\}$ that permits splicing or a translation-invariant subset of A. The extended space A^e can appear in an auxiliary role. Whether the normed linear space A is complete or not is irrelevant for the purposes of this book. We write $U_{s,t}$, $-\infty \le s < t \le \infty$, to denote the set ("space") of equivalence classes of functions belonging to U as determined by $\|\cdot\|_{s,t}$. If $U = A$, then $U_{s,t}$ is the normed linear space $A_{s,t}$; if U is a subset of A, $U_{s,t}$ is a space only in the sense that it is a subset of $A_{s,t}$. We call any $U_{s,t}$ a *truncated input space* and write U_t for $U_{-\infty,t}$.

The requirement that A permits splicing means that, if $U = A$, future inputs at any t can be arbitrary, independent of the past. Unfortunately, spaces of functions everywhere continuous on $\Re$ do not qualify, but this appears to be a minor drawback. It is sometimes desired that U be a translation-invariant bounded (or even totally bounded) subset of A; we always assume U contains the zero function. If U is a proper subset of A, a splice of two elements in U does not necessarily belong to U, of course. However, we require that for all $u \in U$, both $u^t \mapsto 0_{t,\infty}$ and $0^t \mapsto u_{t,\infty}$ belong to U.

The output space Y is taken to be the bounding space, here denoted B, of an FF of normed linear spaces $\{B_{s,t}\}$, or occasionally the corresponding extended space B^e.

In general, the families $\{B_{s,t}\}$ and $\{A_{s,t}\}$ need not be the same. The notations for output spaces are analogous to those for input spaces. The comments about equivalence classes are valid for the $y_{s,t} \in B_{s,t}$.

A mapping $F : U \to Y$ is called a *global input–output mapping* (or usually just input–output mapping).

Definition 4. Let (Y, F, U) be an input–output system. F is a *causal mapping* and (Y, F, U) is a *causal system* if and only if for all t and for all $u, v \in U$ such that $\|u - v\|_t = 0$ it follows that $\|F(u) - F(v)\|_t = 0$.

If F satisfies this definition, it determines a mapping from U_t into B_t, denoted $\tilde{F}_t$, that satisfies $\left\| \tilde{F}_t u_t - (Fu)_t \right\|_t = 0$. We call $\tilde{F}_t$ a *truncated input–output mapping* and define the *centered truncated input–output mapping* $F_t : U_0 \to Y_0$ by $F_t(u_0) \triangleq \mathrm{L}_t \tilde{F}_t \mathrm{R}_t(u_0)$, where $\mathrm{R}_t \triangleq \mathrm{L}_{-t}$ is the right-shift by t. We assume that all systems in this book are causal. If F satisfies this definition, it is causal in the usual sense. However, memory can affect causality.

Definition 5. Consider a left-expanded FF $\{A_{s,t}\}; -\infty \le s < t < \infty$. The family $\{A_{s,t}\}$ and the norms $\|\cdot\|_t$ are said to be *finite memory with memory length M* if there exists $0 < M < \infty$ such that $\|f\|_t = \|f\|_{t-M,t}$ for all $f \in L_0, t \in \Re$.

It is seen that if w in (2.1) has finite support, that is $w(t) = 0$ for $t \ge M$, the weighted L_p normed linear spaces are examples of finite memory spaces with memory length M. Finite memory may cause a system that is causal in the usual sense to not be causal according to Definition 4.

As in [14], the norms we use here for input–output mappings F, $\tilde{F}_t$, and F_t and for the natural states are the N-power norms, denoted $\|\!|\cdot|\!\|_{(N)}$. (We omit the subscript (N) when possible). Let Φ be a mapping from a normed linear space X into a normed linear space Z. For any nonnegative integer N, the N-power norm for Φ is given by

$$\|\!|\Phi|\!\|_{(N)} \triangleq \sup_{x \in X} \frac{\|\Phi(x)\|}{1 + \|x\|^N} \tag{2.5}$$

when the right side exists. We say Φ is *bounded (in N-power norm)* if $\|\!|\Phi|\!\|_{(N)} < \infty$. If Φ is bounded, it carries bounded sets into bounded sets by the inequality

$$\|\Phi(x)\| \le \|\!|\Phi|\!\|_{(N)} \cdot (1 + \|x\|^N) .$$

However, boundedness of Φ does not in general imply continuity nor vice versa. The space of bounded operators from U to Y is denoted by $\mathcal{F}_N(U, Y)$. The space of bounded and continuous operators from U to Y is denoted by $\mathcal{C}_N(U, Y)$. We have chosen to use N-power norms rather than Lipschitz norms because they are less restrictive and because they are not so much influenced by the fine structure of a mapping. This last property seems to be important when one is dealing with an approximate system representation (see Heitman [3] page 785). Other properties of these norms are given in the appendix of [14], which also gives comparisons with the Lipschitz norm. Although [14] uses the standard FFs, its appendix also applies to FFs in general. The N-power norms are special cases of a more general class of weighted supremum norms where $\|x\|^N$ in the denominator of the defining expression is replaced by an arbitrary, continuous, positive function of x; X need not be normed (see Appendix A in [15]).

Using (2.5) on the truncated system mapping and the system mapping, we have

$$\llbracket F_t \rrbracket_{(N)} = \sup_{u_0} \frac{\|F_t(u_0)\|_0}{1 + \|u_0\|_0^N} = \sup_{u_t} \frac{\left\|\tilde{F}_t(u_t)\right\|_t}{1 + \|u_t\|_t^N} = \llbracket \tilde{F}_t \rrbracket_{(N)} \tag{2.6}$$

and

$$\llbracket F \rrbracket_{(N)} = \sup_{u} \frac{\|F(u)\|}{1 + \|u\|^N} . \tag{2.7}$$

Hence, we have the useful relationship

$$\llbracket F \rrbracket_{(N)} \leq \sup_t \llbracket F_t \rrbracket_{(N)} = \sup_t \llbracket \tilde{F}_t \rrbracket_{(N)} . \tag{2.8}$$

Throughout this book, whenever there is reference to a system (Y, F, U), the following three hypotheses are in effect unless specifically noted otherwise:

(A) The input space U is either the bounding space A of an FF of normed linear spaces that permits splicing or a shift-invariant subset of such an A. If U is a proper subset of A, we require that it contain 0 , but also that $u \in U$ implies both $u^t \mapsto 0_{t,\infty}$ and $0^t \mapsto u_{t,\infty}$ belong to U.
(B) The output space Y is the bounding space B of an FF of normed linear spaces.
(C) The global system operator F satisfies Definition 4 (causality) with respect to the given A and B.

The noncentered natural states $\tilde{\xi}_t^u$ and the natural states ξ_t^u are to be defined as operators with domains $\tilde{\mathcal{D}}_t^u$ and $\mathcal{D}_t^u$, respectively, where

$$\tilde{\mathcal{D}}_t^u \triangleq \{v_{t,\infty} \in U_{t,\infty} | u^t \mapsto v_{t,\infty} \in U,\ \forall u^t \subset u_t\}$$

and

$$\mathcal{D}_t^u \triangleq \{v_{0,\infty} \in U_{0,\infty} | u^t \mapsto \mathrm{R}_t v_{0,\infty} \in U,\ \forall u^t \subset u_t\} .$$

Lemma 6. *Under the conditions just specified, given any $t \in \Re$ and any $u \in U$, there exists a mapping $\tilde{\xi}_t^u : \tilde{\mathcal{D}}_t^u \to Y_{t,\infty}$ such that*

$$\left\| \tilde{\xi}_t^u(v_{t,\infty}) - (F(u^t \mapsto v_{t,\infty}))_{t,\infty} \right\|_{t,\infty} = 0$$

for all $v_{t,\infty} \in \tilde{\mathcal{D}}_t^u$. Furthermore, if u and u' satisfy $\|u - u'\|_t = 0$, then $\tilde{\xi}_t^u = \tilde{\xi}_t^{u'}$. (Note that $\mathcal{D}_t^u = \mathcal{D}_t^{u'}$.)

The proof of Lemma 6 is similar to the proof of Lemma 1 in [14]. Denote the Nerode equivalence class of defining inputs for $\tilde{\xi}_t^u$ by $[u_t]_\xi$; i.e., $u_t' \in [u_t]_\xi$ if $\tilde{\xi}_t^{u'} = \tilde{\xi}_t^u$. Lemma 6 allows one to frame the following definition.

Definition 7 ([11,14,15]). The *natural state* for a system (Y, F, U) induced at time t by input u_t is defined to be the operator ξ_t^u from $\mathcal{D}_t^u$ to $Y_{0,\infty}$ given by

$$\xi_t^u(v_{0,\infty}) = \mathrm{L}_t \tilde{\xi}_t^u(\mathrm{R}_t v_{0,\infty}) \, . \tag{2.9}$$

The set of natural states is denoted Ξ; the set of natural states that can be achieved at time t is denoted $\Xi(t)$ so that $\Xi = \cup_{t \in \Re} \Xi(t)$. ($\Xi$ is referred to at times as the *natural state space* or the *natural state set*.)

Obviously, the state set Ξ is minimal since $\xi \in \Xi$ is defined as a mapping. Natural state may be defined for F not necessarily causal; however, we only consider causal F here.

The next hypothesis will often be needed but will not be in effect unless stated explicitly.

(D) The operators $F_t : U_0 \to Y_0$ are uniformly bounded in N-power norm for some fixed positive integer N by a constant $C < \infty$ for all $t \in \Re$, and are an equicontinuous family of uniformly continuous mappings.

Hypothesis (D) gives that the global system operator is $F : U \to Y \in \mathcal{C}_N(U, Y)$ bounded with bound C and is uniformly continuous, see Lemmas 4 and 5 of [14].[1] Let $\mathcal{F}_N(X_{0,\infty}, Y_{0,\infty})$ be the normed linear space of all mappings $\Phi : X_{0,\infty} \to Y_{0,\infty}$ with $\|\Phi\|_N < \infty$, and $\mathcal{C}_N(X_{0,\infty}, Y_{0,\infty})$ be the normed linear subspace of $\mathcal{F}_N(X_{0,\infty}, Y_{0,\infty})$ of all continuous Φ. Appendix A gives that when $F \in \mathcal{F}_N(U, Y)$, we have $\Xi \subset \mathcal{F}_N(X_{0,\infty}, Y_{0,\infty})$ and also $\Xi \subset \mathcal{C}_N(X_{0,\infty}, Y_{0,\infty})$ if $F \in \mathcal{C}_N(U, Y)$. In summary, with Hypothesis (D) in force, $\Xi \subset \mathcal{C}_N(U_{0,\infty}, Y_{0,\infty})$ and are also uniformly continuous. Other properties that result from Hypothesis (D) are given in [14] and [15] and are mentioned at the time they are invoked. As stated previously, properties of the natural state and its relationship to F are given in [11, 14], and [15].

[1]The proofs given in [14] were intended for a standard FF; however, such proofs usually hold for FFs in general if Condition 5 of Definition 1 is not used.

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