Introduction to Modern Physics

1

Chapter 1 consists of **59 problems** that are spread over 31 sections providing an introduction to modern physics by addressing the basic elements of atomic, nuclear, relativistic, and quantum physics as well as electromagnetic theory. Medical physics has its origin in Wilhelm Röntgen's discovery of x rays in 1895, Antoine-Henri Becquerel's discovery of natural radioactivity in 1896 and discovery of radium by Marie Skłodowska-Curie and Pierre Curie in 1898. Just as knowledge of basic physics was important to physicists working with physicians on the early uses of ionizing radiation in medicine, so is knowledge of basic physics and modern physics of great importance to contemporary medical physicists.

After introducing the basic physical constants and the derived physical constants of importance in modern physics and medical physics, this chapter deals with rules that govern physical quantities and units and the classification of natural forces, fundamental particles, and ionizing radiation. It also addresses the basic definitions for atomic and nuclear structure, concepts of physics of small dimensions (quantum physics) and concepts of physics of large velocities (relativistic physics). Problems at the end of Chap. 1 deal with wave–particle duality, basic wave mechanics, Maxwell equations, and normal probability distribution.

Medical physics is a perfect and long-standing example of translational research where basic experimental and theoretical discoveries are rapidly implemented into benefiting humanity through improved diagnostic and therapeutic procedures based on ionizing radiation. This chapter provides the background knowledge that is required for a study of radiation physics as well as for working as medical physicist on a medical team dealing with patients in diagnostic radiology, nuclear medicine, and radiotherapy.

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1.1 Fundamental Physical Constants

1.1.Q1 **(1)**

(a) The use of prefixes (decimal based multipliers) in conjunction with units of physical quantities is a common method for avoiding very large and very small numbers when describing the magnitude of physical quantities. The prefix precedes a fundamental unit of measure to indicate a decimal multiple or decimal fraction of the physical unit. Give the names and symbols for the following two groups of prefixes:

(1) Factors > 1: 10^1, 10^2, 10^3, 10^6, 10^9, 10^{12}, 10^{15}, 10^{18}, 10^{21}, 10^{24}.
(2) Factors < 1: 10^{-1}, 10^{-2}, 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}, 10^{-18}, 10^{-21}, 10^{-24}.

(b) Complete Table 1.1A on fundamental physical constants. The entries should be rounded off to four significant figures and based on the 2006 CODATA set of values available from the website supported by the National Institute of Science and Technology (NIST) in Washington, D.C., USA (http://physics.nist.gov/cuu/Constants/).

Table 1.1A Selected physical constants of importance to modern physics and radiation physics

	Physical constant	Value
1	Speed of light in vacuum	$c =$
2	Avogadro constant	$N_A =$
3	Electron charge	$e =$
4	Electron rest mass	$m_e =$
5	Proton rest mass	$m_p =$
6	Neutron rest mass	$m_n =$
7	Electric constant	$\varepsilon_0 =$
8	Magnetic constant	$\mu_0 =$
9	Ratio proton to electron rest mass	$m_p/m_e =$

SOLUTION:

(a) Standard prefixes for physical quantities are used to form decimal multiples of fundamental and derived units with special names. Currently 20 agreed upon prefixes are in use; 10 for multipliers larger than 1 and 10 for multipliers smaller than 1. Most of the prefixes are based on Greek and Latin language; however, a few can also be traced to Dutch, Norwegian and Italian languages.

(1) For multipliers exceeding 1 the 10 agreed upon prefixes are as follows: 10^2 deca (da); 10^2 hecto (h); 10^3 kilo (k); 10^6 mega (M); 10^9 giga (G); 10^{12} tera (T); 10^{15} peta (P); 10^{18} exa (E); 10^{21} zetta (Z); and 10^{24} yotta (Y).
(2) For multipliers smaller than 1 the 10 agreed upon prefixes are as follows: 10^{-1} deci (d); 10^{-2} centi (c); 10^{-3} milli (m); 10^{-6} micro (μ); 10^{-9} nano (n); 10^{-12} pico (p); 10^{-15} femto (f); 10^{-18} atto (a); 10^{-21} zepto (z); 10^{-24} yocto (y).

(b) Values of selected physical constants of importance to modern physics and medical physics are given in Table 1.1B.

Table 1.1B Selected physical constants of importance to modern physics and radiation physics

	Physical constant	Value	
1	Speed of light in vacuum	$c = 2.998 \times 10^8$ m/s $\approx 3 \times 10^8$ m/s	(1.1)
2	Avogadro constant	$N_A = 6.022 \times 10^{23}$ mol^{-1}	(1.2)
3	Electron charge	$e = 1.602 \times 10^{-19}$ C	(1.3)
4	Electron rest mass	$m_e = 0.5110$ MeV	(1.4)
5	Proton rest mass	$m_p = 938.3$ MeV	(1.5)
6	Neutron rest mass	$m_n = 939.6$ MeV	(1.6)
7	Electric constant	$\varepsilon_0 = 8.854 \times 10^{-12}$ C/(V · m)	(1.7)
8	Magnetic constant	$\mu_0 = 4\pi \times 10^{-7}$ (V · s)/(A · m)	(1.8)
9	Ratio of proton to electron rest mass	$m_p/m_e = 1836$	(1.9)

1.2 Derived Physical Constants and Relationships

1.2.Q1 **(2)**

Complete Table 1.2A of important derived physical constants and provide the value of the derived constants to four significant figures as well as appropriate units.

Table 1.2A Selected derived physical constants

(a) Speed of light in vacuum	$c =$
(b) Reduced Planck constant × speed of light in vacuum	$\hbar c =$
(c) Bohr radius constant	$a_0 =$
(d) Fine structure constant	$\alpha =$
(e) Rydberg energy	$E_\mathrm{R} =$
(f) Rydberg constant	$R_\infty =$
(g) Classical electron radius	$r_\mathrm{e} =$
(h) Compton wavelength of the electron	$\lambda_\mathrm{C} =$
(i) Thomson classical cross section	$\sigma_\mathrm{Th} =$

SOLUTION:

Table 1.2B Selected derived physical constants

(a) Speed of light in vacuum

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.998 \times 10^8 \text{ m/s} \approx \mathbf{3 \times 10^8 \text{ m/s}} \tag{1.10}$$

(b) Reduced Planck constant × speed of light in vacuum

$$\hbar c = \frac{h}{2\pi} c = 197.3 \text{ MeV} \cdot \text{fm} = \mathbf{197.3 \text{ eV} \cdot \text{nm}} \approx 200 \text{ MeV} \cdot \text{fm} \tag{1.11}$$

(c) Bohr radius constant

$$a_0 = \frac{4\pi \varepsilon_0}{e^2} \frac{(\hbar c)^2}{m_e c^2} = \mathbf{0.5292 \text{ Å}} \tag{1.12}$$

(d) Fine structure constant

$$\alpha = \frac{e^2}{4\pi \varepsilon_0} \frac{1}{\hbar c} = \frac{\hbar c}{a_0 m_e c^2} = 7.297 \times 10^{-3} \approx \mathbf{\frac{1}{137}} \tag{1.13}$$

(e) Rydberg energy

$$E_R = \frac{1}{2} m_e c^2 \alpha^2 = \frac{1}{2} \left[\frac{e^2}{4\pi \varepsilon^2} \right] \frac{m_e c^2}{(\hbar c)^2} = \mathbf{13.61 \text{ eV}} \tag{1.14}$$

(f) Rydberg constant

$$R_\infty = \frac{E_R}{2\pi \hbar c} = \frac{m_e c^2 \alpha^2}{4\pi \hbar c} = \frac{1}{4\pi} \left[\frac{e^2}{4\pi \varepsilon^2} \right] \frac{m_e c^2}{(\hbar c)^2} = \mathbf{109\,737 \text{ cm}^{-1}} \tag{1.15}$$

(g) Classical electron radius

$$r_e = \frac{e^2}{4\pi \varepsilon_0} \frac{1}{m_e c^2} = \mathbf{2.818 \text{ fm}} \tag{1.16}$$

(h) Compton wavelength of the electron

$$\lambda_C = \frac{h}{m_e c} = \frac{2\pi \hbar c}{m_e c^2} = \mathbf{0.02426 \text{ Å}} \tag{1.17}$$

(i) Thomson classical cross section

$$\sigma_{Th} = \frac{8\pi}{3} r_e^2 = \mathbf{0.6653 \text{ b}} = 0.6653 \times 10^{-24} \text{ cm}^2 \tag{1.18}$$

1.3 Milestones in Modern Physics and Medical Physics

1.3.Q1 **(3)**

Complete Table 1.3A related to major discoveries in modern physics and medical physics. Provide the name of the discoverer and the year of discovery.

Table 1.3A Major discoveries of importance to modern physics and medical physics

	Discovery	Discoverer	Year
1	X rays		
2	Natural radioactivity		
3	Electron		
4	Radium-226		
5	Special theory of relativity		
6	Photoelectric effect		
7	Thermionic emission		
8	Model of hydrogen atom		
9	Coolidge x-ray tube		
10	Proton		
11	Incoherent (Compton) scattering		
12	Cyclotron		
13	Neutron		
14	Positron		
15	Artificial radioactivity		
16	Uranium fission		
17	Cobalt-60 teletherapy machine		
18	GammaKnife		
19	CT scanner		
20	Magnetic resonance scanner		
21	Positron emission tomography		

SOLUTION:

Table 1.3B Major discoveries of importance to modern physics and radiation physics

	Discovery	Discoverer	Year
1	X rays	Wilhelm Konrad **Röntgen**	1895
2	Natural radioactivity	Antoine-Henri **Becquerel**	1896
3	Electron	Joseph John **Thomson**	1897
4	Radium-226	Pierre **Curie**, Marie **Curie-Skłodowska**	1898
5	Special theory of relativity	Albert **Einstein**	1905
6	Photoelectric effect	Albert **Einstein**	1905
7	Thermionic emission	Owen W. **Richardson**	1911
8	Model of hydrogen atom	Niels **Bohr**	1913
9	Coolidge x-ray tube	William D. **Coolidge**	1914
10	Proton	Ernest **Rutherford**	1919
11	Incoherent (Compton) scattering	Arthur H. **Compton**	1922
12	Cyclotron	Ernest O. **Lawrence**	1931
13	Neutron	James **Chadwick**	1932
14	Positron	Carl D. **Anderson**	1932
15	Artificial radioactivity	Irène **Joliot-Curie**, Frédéric **Joliot**	1934
16	Uranium fission	Lise **Meitner**, Otto **Frisch**, Otto **Hahn**, Friedrich **Strassmann**	1939
17	Cobalt-60 teletherapy machine	Harold E. **Johns**	1951
18	GammaKnife	Lars **Leksell**	1968
19	CT scanner	Godfrey **Hounsfield**, Alan **Cormack**	1971
20	Magnetic resonance scanner	Paul C. **Lauterbur**, Peter **Mansfield**	1973
21	Positron emission tomography	Michael **Phelps**	1973

1.4 Physical Quantities and Units

1.4.Q1 **(4)**

The text presented below with five bullets appears like a standard scientific text but contains several errors in style commonly found in scientific texts and scientific presentations. Correct the text following the common rules used in scientific publishing. The corrected text is presented on next page, with footnotes identifying the mistakes and providing an explanation for each error.

- Exposure X is related to the ability of photons to ionize air. Its unit Röntgen (R) is defined as charge of $2.58 \cdot 10^{-4}$C produced per kilogram of air.
- Kerma K is defined for indirectly ionizing radiations (photons and neutrons) as the energy transferred to charged particles per unit mass of the absorber. Its SI unit gray (Gy) is defined as 1 J of energy absorbed per kilogram of medium, i.e., 1 Gy = 1 Jkg^{-1}.
- Dose, D, is defined as the energy absorbed per unit mass of medium. Its SI unit gray (Gy) is defined as 1 J of energy absorbed per kilogram of medium, i.e., 1 $Gy = 1$ J/kg. The old unit of dose is rad where 1 Gy = 100 cGy = 100 rad = 100,000 mrad.
- Equivalent dose H is defined as the dose multiplied by a radiation-weighting factor w_R. The SI unit of equivalent dose is sievert (Sv), the old unit is rem where 1 Sv = 100 rem.
- Activity $\mathcal{A}$ of a radioactive substance is defined as the number of nuclear decays per unit time. Its SI unit is becquerel (Bq) corresponding to one decay per second or $1\mathrm{Bq} = 1\mathrm{s}^{-1}$.

SOLUTION:

The correct text should read as follows:

- Exposure X is related to the ability of photons to ionize air. Its unit röntgen (R) is defined as charge of 2.58×10^{-4} C produced per kilogram of air.
- Kerma K is defined for indirectly ionizing radiations (photons and neutrons) as the energy transferred to charged particles per unit mass of the absorber. Its SI unit gray (Gy) is defined as 1 J of energy absorbed per kilogram of medium, i.e., $1\ \mathrm{Gy} = 1\ \mathrm{J} \cdot \mathrm{kg}^{-1}$.
- Dose D is defined as the energy absorbed per unit mass of medium. Its SI unit gray (Gy) is defined as 1 J of energy absorbed per kilogram of medium, i.e., 1 Gy = 1 J/kg. The old unit of dose is rad where 1 Gy = 100 cGy = 100 rad = 100000 mrad.
- Equivalent dose H is defined as the dose multiplied by a radiation-weighting factor w_{R}. The SI unit of equivalent dose is sievert (Sv), the old unit is rem where 1 Sv = 100 rem.
- Activity $\mathcal{A}$ of a radioactive substance is defined as the number of nuclear decays per unit time. Its SI unit is becquerel (Bq) corresponding to one decay per second or $1\ \mathrm{Bq} = 1\ \mathrm{s}^{-1}$.

The errors are highlighted as follows and explained with footnotes (a) through (i) below:

- Exposure X[(a)] is related to the ability of photons to ionize air. Its unit Roentgen[(b)] (R) is defined as charge of $2.58 \cdot^{(c)} 10^{-4}$ C produced per kilogram of air.

- Kerma K is defined for indirectly ionizing radiations (photons and neutrons) as the energy transferred to charged particles per unit mass of the absorber. Its SI unit gray (Gy) is defined as 1 J of energy absorbed per kilogram of medium, i.e., 1 Gy = 1 Jkg^{-1} [d].
- Dose, D,[e] is defined as the energy absorbed per unit mass of medium. Its SI unit gray (Gy) is defined as $1J$[f] of energy absorbed per kilogram of medium, i.e., $1\ Gy = 1\ J/kg$[f]. The old unit of dose is rad where 1 Gy = 100,[g] 000 mrad = 100 rad = 100 cGy.
- Equivalent dose H is defined as the dose multiplied by a radiation-weighting factor w_R[h]. The SI unit of equivalent dose is sievert (Sv), the old unit is rem where 1 Sv = 100 rem.
- Activity $\mathcal{A}$ of a radioactive substance is defined as the number of nuclear decays per time. Its SI unit is becquerel (Bq) corresponding to one decay per second or 1[i]Bq = 1[i]s^{-1}.

Footnotes

(a) Exposure is a physical quantity and its symbol should be written in italic font. Therefore, "X" should read: "X".

(b) Full names of physical units, even when they originate from a surname of a person, are customarily spelled out with initial letter in low case. Thus, "Röntgen" designating the unit of exposure should read: "röntgen". However, abbreviations for physical units linked to surnames are commonly spelled out with initial capital letter. Thus, the symbol for the unit "röntgen" is "R", for "volt" it is "V", for "gray" it is "Gy", and so on.

(c) Multiplication of numbers with powers of 10 is usually indicated with the multiplication sign "×" rather than with a period "." or a half-high dot "·". Thus the corrected version of "$2.58 \cdot 10^{-4}$" should read: "2.58×10^{-4}".

(d) Multiplication of physical units, on the other hand, is usually designated with blank space between the units or, preferably, with a half-high dot "·" separating the units. Thus, "Jkg^{-1}" should read: "J kg^{-1}" or, preferably, "J · kg^{-1}". Another example: "1 N = 1 kg · m · s^{-2}" and NOT "1 N = 1 kgms^{-2}".
Note: "ms" stands for millisecond and NOT "meter × second".

(e) In many texts symbols for physical units are placed between commas. This unnecessary use of commas is generally not used in current scientific texts. Thus, we say: "Dose D is defined…" rather than: "Dose, D, is defined…"

(f) While physical quantities are designated with italic symbols, units of physical quantities are designated with roman type font. Thus, we have "1 J" and NOT "$1J$", and "$1\ Gy = 1\ J/kg$" is corrected to "1 Gy = 1 J/kg".

(g) While banks and economic texts often use commas to identify thousands in large numbers, this practice is not acceptable in science. In many countries comma is actually used to designate the decimal point; thus to avoid confusion between the "decimal" comma and a comma designating groups of thousands, the latter is not allowed in scientific literature. Thus, "100,000 mrad" in the text has been corrected to read: "100000 mrad". If thousands must be grouped together, the use of blank space is allowed in scientific literature. Thus, "100000 mrad" can also be written as: "100 000 mrad" for better clarity.

(h) Subscripts and superscripts used with physical quantities are in italic type if they represent variables, physical quantities, or running numbers; they are in roman type if they are descriptive. Thus, "w_R" is corrected to read: "w_{R}", since "R" is descriptive representing radiation. However, the exposure calibration coefficient is written as "N_X" with italic "X" and NOT "N_{X}" with roman "X", since X represents the quantity exposure.

(i) Physical quantities have a numerical value and physical unit. The two MUST be separated with one blank space. We thus have: "$E = 6$ MeV" and NOT "E = 6MeV". The text reading: "1Bq = 1s^{-1} " has therefore been corrected to read: "1 Bq = 1 s^{-1}".

1.4.Q2 **(5)**

In Table 1.4A list the seven basic physical quantities and their units in the international system (SI) of units. Also list a few non-SI units for the basic physical quantities that are used in radiation physics and medical physics.

Table 1.4A The seven basic physical quantities and their units in the SI system of units

	Basic physical quantity	SI unit	Other units
1			
2			
3			
4			
5			
6			
7			

SOLUTION:

Table 1.4B The seven basic physical quantities and their units in the SI system of units

	Basic physical quantity	SI unit	Other units
1	Length ℓ	meter (m)	nm, Å, fm
2	Mass m	kilogram (kg)	g, mg, μg, eV/c^2, keV/c^2, MeV/c^2
3	Time t	second (s)	a, h, ms, μs, ns, ps ("a" stands for "year")
4	Electric current I	ampere (A)	mA, μA, nA, pA
5	Temperature T	kelvin (K)	°C
6	Amount of substance	mole (mol)	mmol, μmol
7	Luminous intensity	candela (cd)	

1.5 Classification of Forces in Nature

1.5.Q1 **(6)**

In Table 1.5A list the four distinct forces acting between various particles in decreasing order of magnitude. For each force also list its source, transmitted particle, and relative strength.

Table 1.5A The four fundamental forces in nature, their source, their transmitted particle, and their relative strength normalized to 1 for the strong force

	Natural force	Source of force	Transmitted particle	Relative strength
1				
2				
3				
4				

SOLUTION:

Four distinct forces, listed in Table 1.5B are observed in the interaction between various types of particles. These forces, in decreasing order of strength, are the strong force, electromagnetic (EM) force, weak force, and gravitational force, with relative strengths of 1, $1/137$, 10^{-6}, and 10^{-39}, respectively. As far as the range of the four fundamental forces is concerned, the forces are divided into two groups: two are infinite range force (electromagnetic force and gravitational force) and two are very short-range force (strong force and weak force).

Each force results from a particular intrinsic property of the particles, such as strong charge for the strong force, electric charge for the EM force, weak charge for the weak force, and energy for the gravitational force:

Table 1.5B The four fundamental forces in nature, their source, their transmitted particle, and their relative strength normalized to 1 for the strong force

	Natural force	Source of force	Transmitted particle	Relative strength
1	Strong	Strong charge	Gluon	1
2	Electromagnetic	Electric charge	Photon	$1/137$
3	Weak	Weak charge	W^+, W^-, Z^0	10^{-6}
4	Gravitational	Energy	Graviton	10^{-39}

1.5.Q2 **(7)**

For proton-electron system determine:

(a) Gravitational force constant Gm_pm_e, where G is the gravitational constant $G = 6.67 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$.
(b) Electromagnetic (EM) force constant $e^2/(4\pi\varepsilon_0)$.
(c) Ratio between the gravitational force constant of (**a**) and the EM force constant of (**b**).

SOLUTION:

(a) Gravitational force constant for proton-electron system:
Mass of proton:

$$m_p = 938.3 \times 10^6\ \mathrm{eV}/c^2 = \frac{938.3 \times 10^6\ \mathrm{eV}}{(3 \times 10^8\ \mathrm{m/s})^2} \times (1.602 \times 10^{-19}\ \mathrm{J/eV})$$
$$= 1.67 \times 10^{-27}\ \mathrm{kg}. \tag{1.19}$$

Mass of electron:

$$m_e = 0.511 \times 10^6\ \mathrm{eV}/c^2 = \frac{0.511 \times 10^6\ \mathrm{eV}}{(3 \times 10^8\ \mathrm{m/s})^2} \times (1.602 \times 10^{-19}\ \mathrm{J/eV})$$
$$= 9.11 \times 10^{-31}\ \mathrm{kg}. \tag{1.20}$$

Charge of proton and electron: $e = 1.602 \times 10^{-19}\ \mathrm{C}$

$$Gm_pm_e = (6.67 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}) \times \frac{938.3 \times 10^6\ \mathrm{eV}}{(3 \times 10^8\ \mathrm{m} \cdot \mathrm{s}^{-1})^2} \frac{0.511 \times 10^6\ \mathrm{eV}}{(3 \times 10^8\ \mathrm{m} \cdot \mathrm{s}^{-1})^2}$$
$$\times (1.602 \times 10^{-19}\ \mathrm{J/eV})^2$$
$$= 1.01 \times 10^{-67}\ \mathrm{J} \cdot \mathrm{m} = 6.32 \times 10^{-39}\ \mathrm{eV} \cdot \text{Å}. \tag{1.21}$$

(b) Electromagnetic force constant for proton-electron system:
Electric constant of vacuum: $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C/(V \cdot m)}$
Charge of proton and electron: $e = 1.602 \times 10^{-19}\ \mathrm{C}$

$$\frac{e^2}{4\pi\varepsilon_0} = \frac{(1.6 \times 10^{-19}\ \mathrm{C})^2}{4\pi \times [8.85 \times 10^{-12}\ \mathrm{C/(V \cdot m)}]} = 2.3 \times 10^{-28}\ \mathrm{J} \cdot \mathrm{m} = 14.4\ \mathrm{eV} \cdot \text{Å}. \tag{1.22}$$

(c) Ratio of gravitational force constant Gm_pm_e for proton-electron system to electromagnetic force constant $e^2/(4\pi\varepsilon_0)$ for proton-electron system

$$\frac{Gm_\text{p}m_\text{e}}{e^2/(4\pi\varepsilon_0)} = \frac{1.01\times10^{-67}\ \text{J}\cdot\text{m}}{2.3\times10^{-28}\ \text{J}\cdot\text{m}} = \frac{6.32\times10^{-39}\ \text{eV}\cdot\text{Å}}{14.4\ \text{eV}\cdot\text{Å}} = 4.39\times10^{-40}. \quad (1.23)$$

1.6 Classification of Fundamental Particles

1.6.Q1 **(8)**

Complete the block diagram of Fig. 1.1A dealing with the basic classification of fundamental particles.

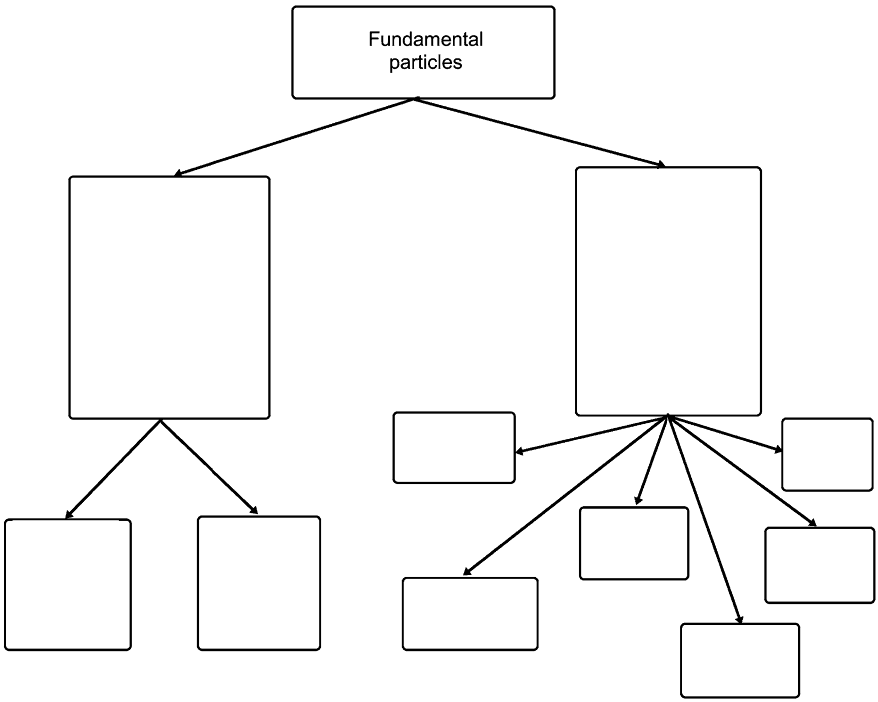


Fig. 1.1A Classification of fundamental particles

SOLUTION:

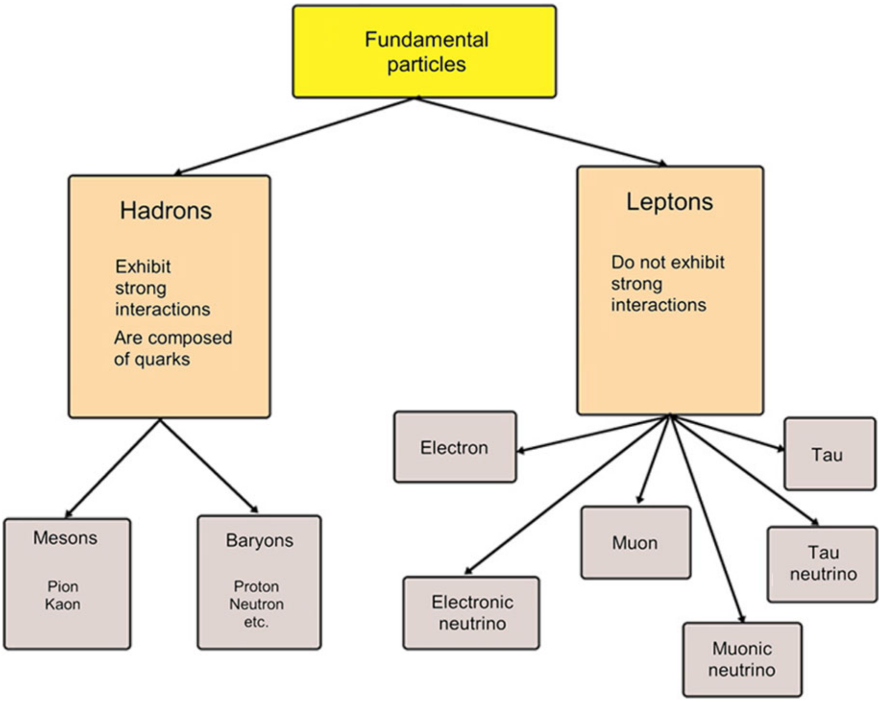


Fig. 1.1B Classification of fundamental particles

1.7 Classification of Radiation

1.7.Q1 **(9)**

Radiation is classified into two main categories: non-ionizing and ionizing, depending on its ability to ionize matter. The ionization potential of atoms, i.e., the minimum energy required for ionizing an atom, ranges from a few electron volts for alkali elements to 24.6 eV for helium (noble gas). Ionization potentials of all other atoms are between these two extremes.

Complete the block diagram of Fig. 1.2A dealing with classification of radiation.

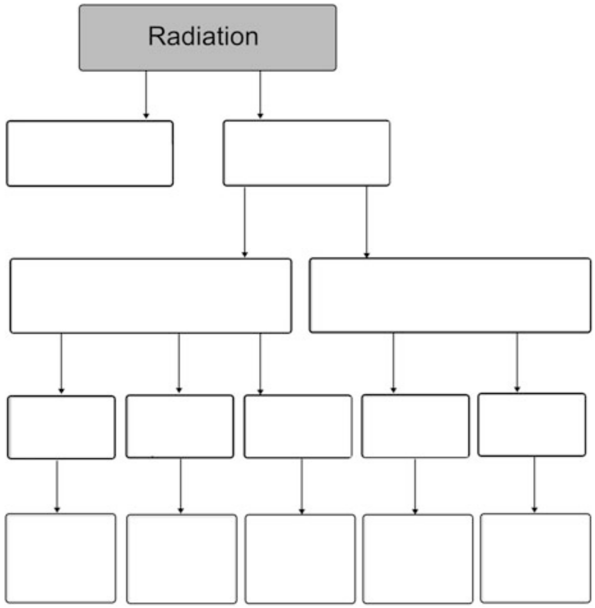


Fig. 1.2A Classification of radiation

SOLUTION:

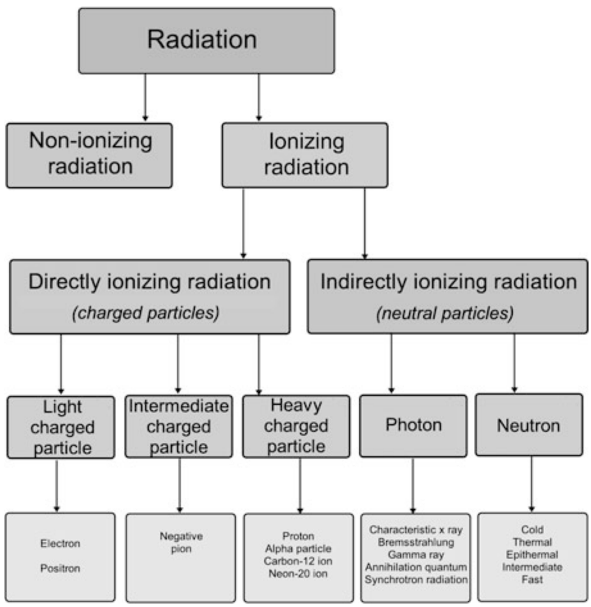


Fig. 1.2B Classification of radiation

1.8 Classification of Ionizing Radiation

1.8.Q1 **(10)**

Complete Table 1.6A listing areas of modern life in which ionizing radiation is used.

Table 1.6A Use of ionizing radiation in science and industry

	Use of ionizing radiation	Brief description of the particular use
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

SOLUTION:

Table 1.6B Use of ionizing radiation in science and industry

	Use of ionizing radiation	Brief description of the particular use
1	Medicine	Diagnostic radiology. Nuclear medicine. Radiotherapy.
2	Nuclear reactor	Basic research. Production of radionuclides. Electric power.
3	Industrial radiography	Nondestructive inspection of welds in airplanes and pipelines.
4	Well logging	Inspection of geologic and recoverable hydrocarbon zones.
5	Insect pest control	For pest sterilization in insect pest eradication.
6	Security services	Screening of cargo and luggage. Sanitation of mail (antrax).
7	Food production	Killing of bacteria and viruses in food. Slowing ripening process.
8	Waste management	Killing of pathogenic microorganisms and harmful bacteria.
9	Chemical industry	Production of polymers and vulcanized tires.
10	Production of weapons	Military production of weapons of mass destruction.

1.9 Classification of Directly Ionizing Radiation

1.9.Q1 **(11)**

Directly ionizing radiation consists of charged particles and for use in medical physics falls into two categories: light charged particles (electrons and positrons) and heavy charged particles such as protons, etc. In the table below list the most common sources of electrons as well as the specific names of the electrons that the sources emit.

Table 1.7A Common sources of electrons and nomenclature for electrons produced

	Source of electrons	Name of electron that the source produces
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		

SOLUTION:

Table 1.7B Common sources of electrons and nomenclature for electrons produced

	Source of electrons	Name of electron that the source produces
1	Photoelectric effect	Photoelectron
2	Compton effect	Compton recoil electron
3	Nuclear pair production	Pair production electron
4	Triplet production	Pair production electron
5	Beta minus nuclear decay	Beta particle (electron)
6	Internal conversion	Internal conversion electron
7	Linac, betatron, microtron	Megavoltage electron
8	Auger effect	Auger electron, Coster-Kronig electron
9	Charged particle collision	Delta ray electron
10	Thermionic emission	Thermion
11	Exoelectron emission	Exoelectron

1.9.Q2 **(12)**

In Table 1.8A list the first five lowest-mass heavy charged particles. List their designation, such as "hydrogen-1", their symbol, as well as the name for the nucleus and the associated atom.

Table 1.8A Basic properties of common heavy charged particles used in nuclear physics and medicine

	Designation	Symbol	Name of nucleus	Protons	Neutrons	Nuclear stability	Name of atom
1							
2							
3							
4							
5							

SOLUTION:

According to the mode of ionization, ionizing radiation is classified into two distinct categories: *directly ionizing radiation* and *indirectly ionizing radiation*. Directly ionizing radiation comprises charged particles that deposit energy in the absorber through a direct one-step process involving Coulomb interactions between the directly ionizing charged particle and orbital electrons of the atoms of the absorber.

For use in radiotherapy heavy charged particles are defined as particles such a proton and heavier ions with mass exceeding that of the electron. The first five lowest-mass heavy charged particles of importance in nuclear physics and also potentially useful in medicine for treatment of disease are listed in Table 1.8B.

Table 1.8B Basic properties of common heavy charged particles used in nuclear physics and medicine

	Designation	Symbol	Name of nucleus	Protons	Neutrons	Nuclear stability	Name of atom
1	Hydrogen-1	$^{1}_{1}H$	Proton	1	0	Stable	Protium
2	Hydrogen-2	$^{2}_{1}H$	Deuteron	1	1	Stable	Deuterium
3	Hydrogen-3	$^{3}_{1}H$	Triton	1	2	Radioactive	Tritium
4	Helium-3	$^{3}_{2}He$	Helion	2	1	Stable	Helium-3
5	Helium-4	$^{4}_{2}He$	Alpha particle	2	2	Stable	Helium-4

1.10 Classification of Indirectly Ionizing Photon Radiation

1.10.Q1 **(13)**

In addition to ultraviolet radiation there are five other types of indirectly ionizing photon radiation. In Table 1.9A provide a list of the five types of indirectly ionizing photon radiation used in medical imaging and radiotherapy. Also provide the source of each radiation type.

Table 1.9A Classification of indirectly ionizing photon radiation

	Indirectly ionizing photon radiation	Origin of radiation
1		
2		
3		
4		
5		

SOLUTION:

Indirectly ionizing radiation comprises neutral particles (photons and neutrons) that deposit energy in the absorber through a two-step process with the first step releasing charged particles (photons release either electrons or electron/positron pairs; neutrons release protons or heavier ions) and the second step the released charged particles deposit a portion of their energy in the absorber through direct Coulomb interactions with orbital electrons of the atoms of the absorber.

Indirectly ionizing photon radiation consists of three categories of photon: ultraviolet (uv), x ray, and gamma ray. While ultraviolet photons are of some limited use in medicine, imaging and treatment of disease with radiation are carried out with photons of higher energy than that of uv photons. With regard to their origin, these higher energy photons fall into five categories as listed in Table 1.9B:

Table 1.9B Classification of indirectly ionizing photon radiation

	Indirectly ionizing photon radiation	Origin of radiation
1	Gamma rays	Result from nuclear transitions in excited radionuclides referred to as gamma decay.
2	Annihilation quanta	Result from positron-electron annihilation, be it with stationary positron or an energetic positron.
3	Characteristic (fluorescence) x rays	Result from electron transitions between atomic shells in an excited atom.
4	Bremsstrahlung x rays	Result from interactions between an energetic electron and a nucleus of absorber.
5	Synchrotron radiation	Results from electrons moving in circular orbits in a magnetic field (storage ring).

1.11 Radiation Quantities and Units

1.11.Q1 **(14)**

Several quantities and units are used for quantifying radiation. In Table 1.10A list at least five of these quantities, their definition, their unit in the SI system, their traditional old unit, and the relationship between their SI unit and the old unit.

Table 1.10A Basic physical quantities and their units used in radiation measurement

	Quantity	Definition	SI unit	Old unit	Conversion
1					
2					
3					
4					
5					

SOLUTION:

Table 1.10B Basic physical quantities and their units used in radiation measurement

	Quantity	Definition	SI unit	Old unit	Conversion
1	Exposure X	$X = \frac{\Delta Q}{\Delta m_{\text{air}}}$	$2.58 \times \frac{10^{-4}\,\text{C}}{\text{kg air}}$	$1\ \text{R} = \frac{1\ \text{esu}}{\text{cm}^3\ \text{air}_{\text{STP}}}$	$1\ \text{R} = 2.58 \times \frac{10^{-4}\,\text{C}}{\text{kg air}}$
2	Kerma K	$K = \frac{\Delta E_{\text{tr}}}{\Delta m}$	$1\ \text{Gy} = 1\ \frac{\text{J}}{\text{kg}}$	–	–
3	Dose D	$D = \frac{\Delta E_{\text{ab}}}{\Delta m}$	$1\ \text{Gy} = 1\ \frac{\text{J}}{\text{kg}}$	$1\ \text{rad} = 100\ \frac{\text{erg}}{\text{g}}$	$1\ \text{Gy} = 100\ \text{rad}$
4	Equivalent dose H	$H = D w_{\text{R}}$	1 Sv	1 rem	$1\ \text{Sv} = 100\ \text{rem}$
5	Activity $\mathcal{A}$	$\mathcal{A} = \lambda N$	$1\ \text{Bq} = 1\ \text{s}^{-1}$	$1\ \text{Ci} = 3.7 \times 10^{10}\ \text{s}^{-1}$	$1\ \text{Bq} = \frac{1\ \text{Ci}}{3.7 \times 10^{10}}$

1.12 Dose Distribution in Water for Various Radiation Beams

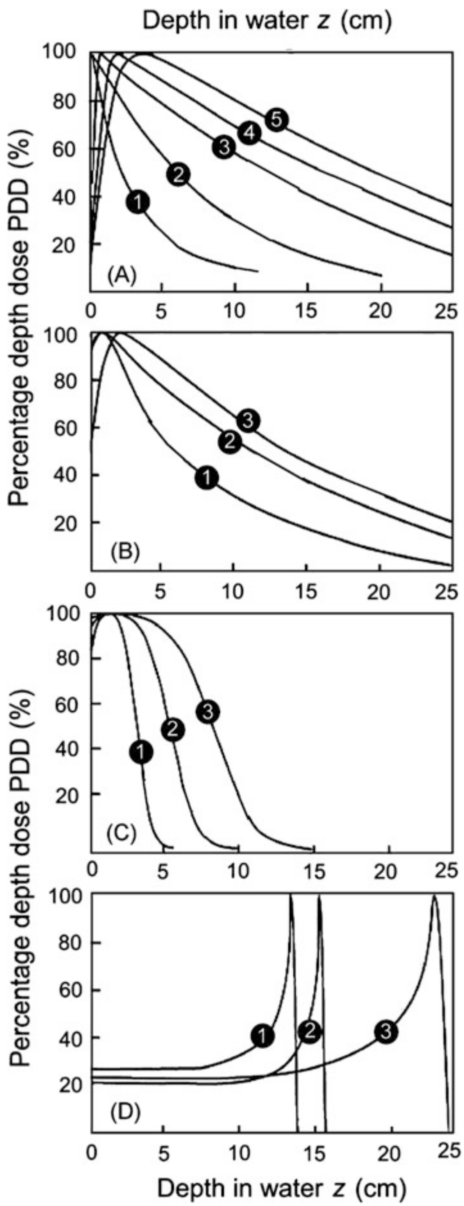
1.12.Q1 **(15)**

Dose deposition in water is one of the most important characteristics of the interaction of radiation beams with matter. This is true in general radiation physics and even more so in medical physics, where the dose deposition properties in tissue govern both the diagnosis of disease with radiation (*imaging physics*) as well as treatment of disease with radiation (*radiotherapy physics*).

Imaging with ionizing radiation is limited to the use of x-ray beams in *diagnostic radiology* and gamma ray beams in *nuclear medicine*, while in *radiotherapy* the use of radiation is broader and covers essentially all ionizing radiation types ranging from x rays and gamma rays through electrons to neutrons, protons and heavier charged particles. For a given radiation beam, its dose deposition in water is usually depicted in the form of percentage depth dose (PDD) curve that plots the radiation dose (normalized to 100 % at the depth of dose maximum) against depth z in water.

Diagrams in Fig. 1.3 depict percentage depth doses (PDD) against depth in water z for various directly and indirectly ionizing radiation beams used in radiotherapy. For PDD curves in **(A)**, **(B)**, **(C)**, and **(D)** identify:

(a) Mode of radiation (directly or indirectly ionizing).
(b) Type of radiation (photon, electron, etc.).
(c) Beam energy (80 kVp, 18 MV, 10 MeV, etc.)



(A) Radiation mode: ____________

Radiation type: ____________

Beam energy:
(1) __________
(2) __________
(3) __________
(4) __________
(5) __________

(B) Radiation mode: ____________

Radiation type: ____________

Beam energy:
(1) __________
(2) __________
(3) __________

(C) Radiation mode: ____________

Radiation type: ____________

Beam energy:
(1) __________
(2) __________
(3) __________

(D) Radiation mode: ____________

Radiation type: ____________

Beam energy:
(1) __________
(2) __________
(3) __________

Fig. 1.3 Percentage depth dose (PDD) against depth z in water for radiation beams of various modes, types and energies

SOLUTION:

Figure 1.4 depicts percentage depth doses (PDD) against depth z in water for radiation beams of various modes, types and energies.

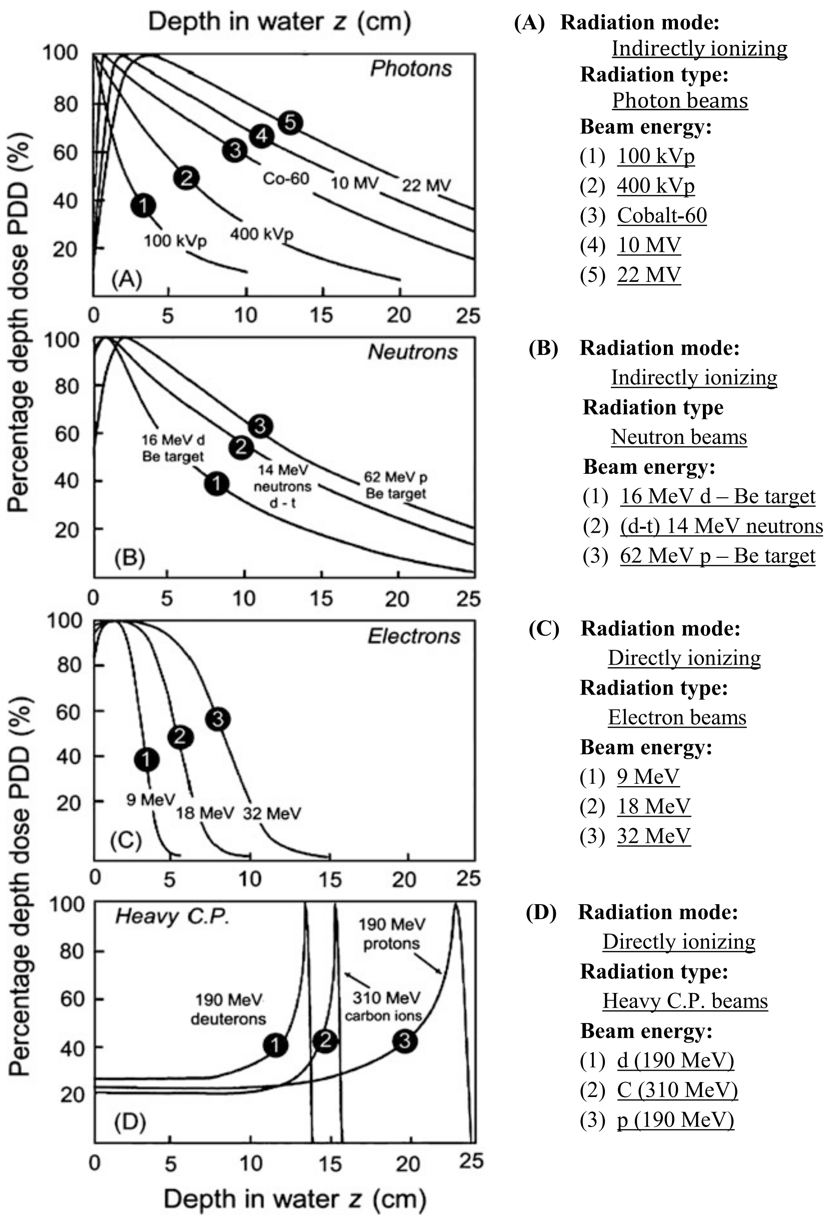


Fig. 1.4 Percentage depth dose (PDD) against depth z in water for radiation beams of various modes, types and energies. **(A)** and **(B)** are for indirectly ionizing radiation (photons and neutrons, respectively); **(C)** and **(D)** are for directly ionizing radiation (electrons and heavy charged particles, respectively). C.P. = charged particle, d = deuteron, t = triton, C = carbon ion, p = proton

Of the four beam categories of Fig. 1.4, photon beams in the indirectly ionizing radiation category and electron beams in the directly ionizing radiation category are considered conventional beams, well understood, and readily available for radiotherapy in all major medical institutions around the world. On the other hand, neutron beams in the indirectly ionizing radiation category and heavy ions including protons in the directly ionizing radiation category remain in the category of special beams, available in only a limited number of institutions around the world, despite having been in use for the past 5 decades. These beams offer some advantages in treatment of certain malignant diseases; however, in comparison with conventional beams, they are significantly more complicated to use as well as to maintain and their infrastructure and operating costs are also significantly higher, currently precluding a widespread clinical use.

Special beams (neutrons and protons) provide certain advantages when used in treatment of selected tumor types; however, their choice and prescribed dose must account not only for the physical beam characteristics but also for the biological effects associated with radiation beams: the relative biological effectiveness (RBE) and the oxygen enhancement ratio (OER). The RBE is defined by the ratio between dose of test radiation to dose of standard radiation (250 kVp x rays) to produce the same biological effect. The OER is defined by the ratio of doses without and with molecular oxygen (hypoxic versus well oxygenated cells) to produce the same biological effect.

1.13 Basic Definitions for Atomic Structure

1.13.Q1 **(16)**

Complete Table 1.11A dealing with basic constituents of nuclides of importance to medical physics.

Table 1.11A Basic characteristics of selected nuclides of importance to medical physics

	Physical quantity	Symbol	${}^{1}_{1}$H	${}^{12}_{6}$C	${}^{60}_{27}$Co	${}^{137}_{55}$Cs	${}^{192}_{77}$Ir	${}^{207}_{82}$Pb	${}^{226}_{88}$Ra	${}^{235}_{92}$U
1	Atomic number									
2	Number of protons									
3	Number of electrons									
4	Atomic mass number									
5	Number of nucleons									
6	Number of neutrons									

SOLUTION:

Table 1.11B Basic characteristics of selected nuclides of importance to medical physics

	Physical quantity	Symbol	${}^{1}_{1}\mathrm{H}$	${}^{12}_{6}\mathrm{C}$	${}^{60}_{27}\mathrm{Co}$	${}^{137}_{55}\mathrm{Cs}$	${}^{192}_{77}\mathrm{Ir}$	${}^{207}_{82}\mathrm{Pb}$	${}^{226}_{88}\mathrm{Ra}$	${}^{235}_{92}\mathrm{U}$
1	Atomic number	Z	1	6	27	55	77	82	88	235
2	Number of protons	Z	1	6	27	55	77	82	88	235
3	Number of electrons	Z	1	6	27	55	77	82	88	235
4	Atomic mass number	A	1	12	60	137	192	207	226	92
5	Number of nucleons	A	1	12	60	137	192	207	226	92
6	Number of neutrons	$A–Z$	0	6	33	82	115	125	138	143

1.13.Q2 **(17)**

Lithium borate with chemical formula $Li_2B_4O_7$ is used as ingredient in production of glass and ceramics. It is also used as a sensitive phosphor in thermoluminescence dosimetry (TLD). Determine the mean molecular mass of lithium borate, having the following atomic constituents: lithium Li, boron B, and oxygen O. Also determine the mean rest energy of a lithium borate molecule.

SOLUTION:

From the NIST (http://physics.nist.gov/PhysRefData/Compositions/index.html) we obtain the isotopic composition and atomic masses for the stable isotopes of the constituent nuclides: Li, B, and O. The relevant data are shown in the table below:

Table 1.12 Basic atomic data for constituents of lithium borate ($Li_2B_4O_7$) according to the NIST

Nuclide	Atomic number	Stable isotopes	Abundance (%)	Isotopic atomic mass
Li	3	Li-6	7.59	6.0151223
		Li-7	92.41	7.0160041
B	5	B-10	20	10.012937
		B-11	80	11.009306
O	8	O-16	99.76	15.994915
		O-17	0.04	16.999132
		O-18	0.20	17.999161

Using the data given in Table 1.12 we now calculate the mean atomic masses for the three atoms comprising lithium borate as follows:

$$\bar{\mathcal{M}}(\mathrm{Li}) = 0.0759\times 6.0151223\ \mathrm{u} + 0.9241\times 7.0160041\ \mathrm{u} = 6.941\ \mathrm{u}, \quad (1.24)$$

$$\bar{\mathcal{M}}(\mathrm{B}) = 0.20\times 10.012937\ \mathrm{u} + 0.80\times 11.0093055\ \mathrm{u} = 10.811\ \mathrm{u}, \quad (1.25)$$

$$\begin{aligned}\bar{\mathcal{M}}(\mathrm{O}) &= 0.9976\times 15.994915\ \mathrm{u} + 0.0004\times 16.9991315\ \mathrm{u} \\ &\quad + 0.0020\times 17.999161\ \mathrm{u} \\ &= 15.9994\ \mathrm{u}. \end{aligned} \quad (1.26)$$

Mean molecular mass of $Li_2B_4O_7$ is determined by adding the weighted mean atomic masses of the individual constituents to get

$$\begin{aligned}\bar{\mathcal{M}}(\mathrm{Li_2B_4O_7}) &= 2\bar{\mathcal{M}}(\mathrm{Li}) + 4\bar{\mathcal{M}}(\mathrm{B}) + 7\bar{\mathcal{M}}(\mathrm{O}) \\ &= 2\times 6.941\ \mathrm{u} + 4\times 10.811\ \mathrm{u} + 7\times 15.9994\ \mathrm{u} = 169.12\ \mathrm{u}. \end{aligned} \quad (1.27)$$

Finally, the mean rest energy of a lithium borate molecule is

$$\begin{aligned}\bar{\mathcal{M}}(\mathrm{Li_2B_4O_7})c^2 &= \bar{\mathcal{M}}(\mathrm{Li_2B_4O_7},\mathrm{u})\times 931.494028\ \mathrm{MeV/u} \\ &= (169.12\ \mathrm{u})\times 931.494028\ \mathrm{MeV/u} = 157\,534.27\ \mathrm{MeV}. \end{aligned} \quad (1.28)$$

1.13.Q3 **(18)**

Cells in human body use sugar (glucose: $C_6H_{12}O_6$) as their major source of energy. Blood circulating in the body delivers glucose to cells and the blood sugar concentration (also referred to as blood glucose level) is controlled through various negative feedback mechanisms and kept within a relatively narrow range.

The blood sugar concentration is specified either in mmol/ℓ representing molar concentration of sugar per liter of blood or in mg/dℓ representing mass concentration of sugar per deciliter of blood.

Determine the relationship between mmol/ℓ and mg/dℓ in measurement of blood glucose.

SOLUTION:

To find the relationship between mmol/ℓ and mg/dℓ we first determine the mean molecular mass (standard molecular weight) $\mathcal{M}$ of glucose $C_6H_{12}O_6$ using the following standard molecular weight of carbon C, hydrogen H, and oxygen O

(see T1.22 and T1.23)

$$\mathcal{M}(\mathrm{C}) = 12.0107 \text{ u},$$
$$\mathcal{M}(\mathrm{H}) = 1.00794 \text{ u},$$
$$\mathcal{M}(\mathrm{O}) = 15.9994 \text{ u}$$

to get the following result for the mean molecular mass (standard molecular weight) of glucose

$$\mathcal{M}(\mathrm{C_6H_{12}O_6}) = 6\times 12.0107 \text{ u} + 12\times 1.00794 \text{ u} + 6\times 15.9994\text{u} = 180.12 \text{ u}. \quad (1.29)$$

Thus,

1 mole of glucose corresponds to 180.12 g of glucose,

1 mmol/ℓ corresponds to 180.12 mg/ℓ or 18.012 mg/dℓ, usually approximated to 18 mg/dℓ.

The normal fasting glucose level in human blood is about 4.5 mmol/ℓ to 5.6 mmol/ℓ or 80 mg/dℓ to 100 mg/ℓ.

1.14 Basic Definitions for Nuclear Structure

1.14.Q1 **(19)**

For all stable nuclides listed in Table A.1 of Appendix A prepare a Cartesian diagram plotting their atomic number Z on the ordinate (y) axis and atomic mass number A on the abscissa (x) axis. On the same diagram also plot the function

$$Z(A) = \frac{A}{2 + 0.0155A^{2/3}}, \quad (1.30)$$

which has been proposed empirically as a reasonable approximation linking Z and A for all stable nuclides.

SOLUTION:

A plot of atomic number Z against atomic mass number A for the stable isotopes listed in Table A.1 is given in Fig. 1.5.

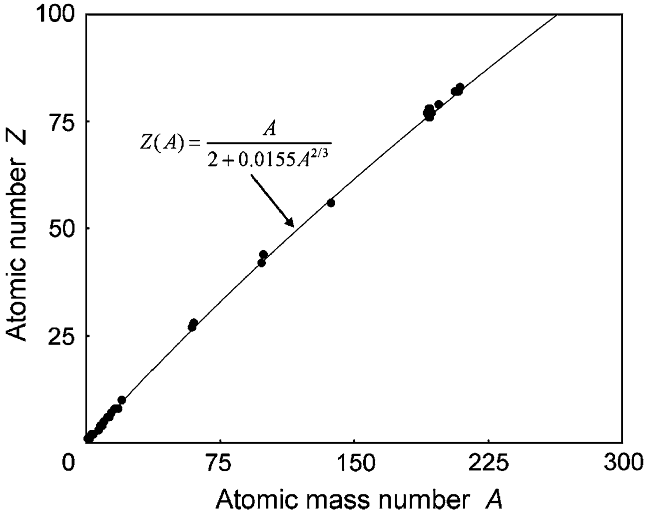


Fig. 1.5 Plot of (1.30) shown with *solid line* and data from Table A.1 of Appendix A for all stable nuclides shown with *data points*. Atomic mass number A is plotted on the abscissa (x) axis and atomic number Z on the ordinate (y) axis. The agreement between the data points and (1.30) is excellent

1.15 Nuclear Binding Energies

1.15.Q1 **(20)**

The sum of masses of the individual components of a nucleus that contains Z protons and $(A–Z)$ neutrons is larger than the actual mass of the nucleus. This difference in mass is called the mass defect (mass deficit) Δm and its energy equivalent Δmc^2 is called the total binding energy E_B of the nucleus. The binding energy per nucleon (E_B/A) in a nucleus (i.e., the total binding energy of a nucleus divided by the number of nucleons) varies with the number of nucleons A and is of the order of $\sim$8 MeV/nucleon.

(a) Determine the binding energy per nucleon $E_B/A = \Delta mc^2/A$, where A is the atomic mass number, for the following nuclei: (1) deuteron ($^2_1$H), (2) alpha particle ($^4_2$He), (3) boron ($^{10}_5$B), (4) oxygen-16 ($^{16}_8$O), (5) cobalt-60 ($^{60}_{27}$Co), (6) cesium-137 ($^{137}_{55}$Cs), (7) lead-208 ($^{208}_{82}$Pb), and (8) uranium-235 ($^{235}_{92}$U).

(b) Plot E_B/A against A for nuclides listed in **(a)**.

(c) The peculiar shape of the E_B/A curve against A suggests two methods for converting mass into energy: fusion of nuclei at low A and fission of nuclei at large A. Briefly discuss the principles of: (1) Fusion and (2) Fission.

SOLUTION:

(a) Binding energy per nucleon E_B/A is given as follows

$$\frac{E_B}{A} = \frac{\Delta mc^2}{A} = \frac{Zm_p c^2 + (A-Z)m_n c^2 - Mc^2}{A}, \tag{1.31}$$

where

Z and A	are the nuclide atomic number and atomic mass number, respectively.
$m_p c^2$	is the proton rest energy (938.272013 MeV).
$m_n c^2$	is the neutron rest energy (939.565346 MeV).
Mc^2	is the nuclear rest energy that may be obtained directly from nuclear data tables or from atomic mass $\mathcal{M}(\mathrm{u})$ available from the NIST with the following expression

$$\begin{aligned} Mc^2 &= \mathcal{M}(\mathrm{u})c^2 - Zm_e c^2 \\ &= \mathcal{M}(\mathrm{u}) \times 931.494\,028\ \mathrm{MeV/u} - Z \times 0.510\,999\ \mathrm{MeV}, \end{aligned} \tag{1.32}$$

with $m_e c^2$ the electron rest mass (0.510999 MeV).

By way of example, we calculate E_B/A for the $^4_2$**He nucleus** using the following steps:

(1) According to the NIST (http://physics.nist.gov/PhysRefData/Compositions/index.html) the atomic mass of $^4_2$He is 4.002603 u.
(2) According to (1.32) the nuclear rest energy of $^4_2$He is

$$\begin{aligned} Mc^2 &= (4.002603\ \mathrm{u}) \times (931.494028\ \mathrm{MeV/u}) - 2 \times 0.510999\ \mathrm{MeV} \\ &= 3728.4009\ \mathrm{MeV} - 1.0220\ \mathrm{MeV} = 3727.3791\ \mathrm{MeV}. \end{aligned} \tag{1.33}$$

(3) Using (1.31) we now calculate E_B/A for $^4_2$He and get

$$\begin{aligned} \frac{E_B}{A} &= \frac{2 \times (938.272013\ \mathrm{MeV}) + 2 \times (939.565346\ \mathrm{MeV}) - (3727.3791\ \mathrm{MeV})}{4} \\ &= 7.0739\ \mathrm{MeV}. \end{aligned} \tag{1.34}$$

Binding energies per nucleon for the other nuclei of part (**a**) were calculated with the same technique as the one used for the helium-4 nucleus and the final results are tabulated in Table 1.13 and plotted in Fig. 1.6.

Table 1.13 Atomic mass $\mathcal{M}$(u), nuclear rest energy Mc^2, nuclear binding energy E_B, and binding energy per nucleon E_B/A for selected nuclides. Data are available in Table A.1 of Appendix A

	Nucleus	Atomic mass $\mathcal{M}$(u)	Nuclear rest energy Mc^2 (MeV)	Binding energy E_B (MeV)	Binding energy per nucleon E_B/A (MeV)
1	$^{1}_{2}$H	2.014102	1875.6128	2.22458	1.1123
2	$^{2}_{4}$He	4.002603	3727.3791	28.29569	7.0739
3	$^{10}_{5}$B	10.012937	9324.4362	64.75071	6.4751
4	$^{16}_{8}$O	15.994915	14895.0796	127.61927	7.9762
5	$^{60}_{27}$Co	59.933822	55814.2014	524.80028	8.7467
6	$^{137}_{55}$Cs	136.907084	127500.0283	1149.29287	8.3890
7	$^{208}_{82}$Pb	207.976636	193687.0956	1636.44573	7.8675
8	$^{235}_{92}$U	235.043923	218895.0023	1783.87084	7.5909

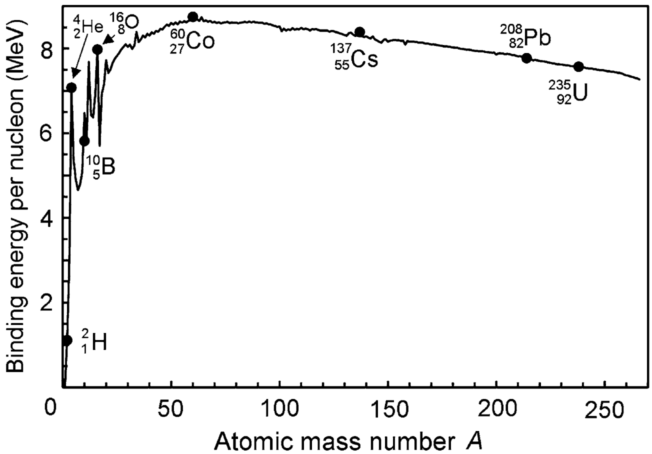


Fig. 1.6 Binding energy per nucleon E_B/A in MeV/nucleon against atomic mass number A for all known nuclides shown with solid curve and for the nuclides of Table 1.13 shown with data points

(b) Plot of E_B/A data of Table 1.13 against the atomic mass number A is shown in Fig. 1.6.

(c) Nuclear fusion and fission are practical examples of converting mass into energy. Nuclear energy is contained within the atomic nucleus that consists of protons and neutrons held together by a strong force. The larger the binding energy per nucleon of an atom, the more stable is the atom. As shown in Fig. 1.6, atoms with atomic mass number $A \approx 60$ are the most stable in nature. Therefore, fusing light nuclei into a heavier nucleus or splitting a heavy nucleus into lighter fragments both result in nuclei with stronger binding energy per nucleon and, consequently, conversion of a fraction of nuclear mass into energy.

(1) *Fusion* of two nuclei of very small mass, e.g., ${}^{2}_{1}\mathrm{H} + {}^{3}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + \mathrm{n}$, creates a more massive nucleus and releases a certain amount of energy. Experiments using controlled nuclear fusion for production of energy have so far not been successful; however, steady progress in fusion research is being made in various laboratories around the world, so it is reasonable to expect that in the future controlled fusion will become possible and will result in a relatively clean and abundant means for sustainable power generation.

(2) *Fission* of elements of large mass, e.g., ${}^{235}_{92}\mathrm{U} + \mathrm{n}$, creates two smaller mass and more stable nuclei with release of energy in the form of heat and radiation. Nuclear fission was observed first in 1934 by Enrico Fermi and described correctly by Otto Hahn, Friedrich Strassmann, Lise Meitner, and Otto Frisch in 1939. In 1942 at the University of Chicago Enrico Fermi and colleagues carried out the first controlled chain reaction based on nuclear fission.

1.16 Nuclear Models

1.16.Q1 **(21)**

Use the Weizsäcker binding energy formula to determine the binding energy of the following three nuclei: **(a)** Boron-10 (B-10); **(b)** Cobalt-60 (Co-60); and **(c)** Uranium-235 (U-235).

SOLUTION:

The liquid drop nuclear model assumes that nuclei resemble a very dense incompressible spherical liquid drop. The Weizsäcker empirical binding energy formula

$$E_{\mathrm{B}}\left({}^{A}_{Z}\mathrm{X}\right) \approx C_1 A - C_2 A^{2/3} - C_3 \frac{Z^2}{A^{1/3}} - C_4 \frac{(A-2Z)^2}{A}, \tag{1.35}$$

where Z and A are the atomic number and atomic mass number, respectively, of nucleus X, accounts for the nuclear volume effect (C_1), nuclear surface effect (C_2), Coulomb repulsion of protons (C_3), and excess of neutrons over protons in the nucleus (C_4). Constants C_1, C_2, C_3, and C_4 were determined empirically and equal to 15.75 MeV, 17.8 MeV, 0.711 MeV, and 23.7 MeV, respectively.

Using Weiszäcker equation we get the following results for the binding energies of ${}^{10}_{5}\mathrm{B}$, ${}^{60}_{27}\mathrm{Co}$, and ${}^{235}_{92}\mathrm{U}$:

(a)

$$
\begin{aligned}
E_{\mathrm{B}}\left({}^{10}_{5}\mathrm{B}\right) &= 15.75\times 10\ \mathrm{MeV} - 17.8\times 10^{2/3}\ \mathrm{MeV} \\
&\quad - 0.711\times\frac{5^2}{10^{1/3}}\ \mathrm{MeV} - 23.7\times\frac{(10-2\times 5)^2}{10}\ \mathrm{MeV} \\
&= 66.6\ \mathrm{MeV},
\end{aligned}
\tag{1.36}
$$

deviating from the actual value of 64.8 MeV determined in Prob. 20 by 2.8 %.

(b)

$$
\begin{aligned}
E_{\mathrm{B}}\left({}^{60}_{27}\mathrm{Co}\right) &= 15.75\times 60\ \mathrm{MeV} - 17.8\times 60^{2/3}\ \mathrm{MeV} \\
&\quad - 0.711\times\frac{27^2}{60^{1/3}}\ \mathrm{MeV} - 23.7\times\frac{(60-2\times 27)^2}{60}\ \mathrm{MeV} \\
&= 525\ \mathrm{MeV},
\end{aligned}
\tag{1.37}
$$

deviating from the actual value of 524.8 MeV determined in Prob. 20 by 0.04 %.

(c)

$$
\begin{aligned}
E_{\mathrm{B}}\left({}^{235}_{92}\mathrm{U}\right) &= 15.75\times 235\ \mathrm{MeV} - 17.8\times 235^{2/3}\ \mathrm{MeV} \\
&\quad - 0.711\times\frac{92^2}{235^{1/3}}\ \mathrm{MeV} - 23.7\times\frac{(235-2\times 92)^2}{235}\ \mathrm{MeV} \\
&= 1782.9\ \mathrm{MeV},
\end{aligned}
\tag{1.38}
$$

deviating from the actual value of 1783.9 MeV determined in Prob. 20 by −0.06 %.

1.17 Physics of Small Dimensions and Large Velocities

1.17.Q1 **(22)**

At the end of the 19-th century physics was considered a completed discipline within which most of the natural physical phenomena were satisfactorily explained. However, as physicists broadened their interests and refined their experimental techniques, it became apparent that classical physics suffered severe limitations in two areas: (i) dealing with dimensions comparable to small atomic dimensions and (ii) dealing with velocities comparable to speed of light. Modern physics handles these limitations in two distinct, yet related, subspecialties: quantum physics and relativistic physics, respectively.

State or calculate typical dimensions of:

(a) Rutherford-Bohr atomic model (radii of nucleus and atom) for hydrogen-1 (H-1 protium) atom and for uranium-235 atom.
(b) Copernican heliocentric planetary celestial model (radii of Sun and Earth's orbit) for the Earth and Sun planetary system.
(c) Normalize both models to the same scale and determine which model exhibits larger radius and by what factor?

SOLUTION:

(a) Rutherford-Bohr atomic model: Electrons are in planetary motion about the stationary nucleus. Most of the atomic mass is concentrated in the nucleus that is some five orders of magnitude smaller than the atomic radius.

Nuclear radius R is estimated with the following expression

$$R = R_0 \sqrt[3]{A}, \tag{1.39}$$

where R_0 is the nuclear radius constant (1.25 fm) and A is the atomic mass number of a given nuclide.

Using (1.39) we get the following nuclear radii for the hydrogen-1 (protium) atom and for the uranium-235 atom, respectively

$$R\left({}^{1}_{1}\mathrm{H}\right) = R_0 \approx 1.25\ \mathrm{fm} \quad \text{and} \quad R\left({}^{235}_{92}\mathrm{U}\right) = (1.25\ \mathrm{fm}) \times \sqrt[3]{235} \approx 7.7\ \mathrm{fm}. \tag{1.40}$$

Atomic radius of the Rutherford-Bohr atomic model is defined well for one-electron structures such as the hydrogen atom for which one can determine the atomic Bohr radius a_0 from first principles as

$$a_0 = a_\mathrm{H} = \frac{4\pi\varepsilon_0}{e^2}\frac{(\hbar c)^2}{m_\mathrm{e}c^2} \approx 0.53\ \text{Å}. \tag{1.41}$$

Contrary to the impression that a high atomic number Z atom is much larger than the hydrogen atom, measurements have shown that the outer shell radius of a high Z atom such as uranium ($Z = 92$) exceeds the hydrogen atomic radius by only a factor of ~ 2. We thus estimate the radius of the uranium atom to be about 1 Å.

(b) Heliocentric Copernican planetary system: Planets revolve about the stationary Sun in or close to the ecliptic plane.

Solar radius r_S is used as unit of length suitable for expressing the size of stars and is estimated to be $1\ r_\mathrm{S} = 6.955 \times 10^8\ \mathrm{m} \approx 7 \times 10^8\ \mathrm{m}$.

Mean distance between the Earth and the Sun (mean radius of Earth's orbit around the Sun) is used as unit of length suitable for expressing the distance between a planet and the Sun in the Solar system. The unit is called the astronomical unit (au) and has the following magnitude 1 au $= 149.6\times10^{11}$ m $= 215\ r_S \approx 150\times10^6$ km.

(c) Ratios:

$$\frac{\text{Earth's orbit}}{\text{Solar radius}} \approx \frac{150\times10^9 \text{ m}}{7\times10^8 \text{ m}} = 215 \tag{1.42}$$

$$\frac{\text{Hydrogen radius}}{\text{Proton radius}} \approx \frac{0.53\times10^{-10} \text{ m}}{1.25\times10^{-15} \text{ m}} = 42400 \tag{1.43}$$

$$\frac{\text{Uranium radius}}{\text{Radius of U-235 nucleus}} \approx \frac{1.5\times10^{-10} \text{ m}}{7.7\times10^{-15} \text{ m}} = 19500 \tag{1.44}$$

Thus, when the Sun and the atomic nucleus are normalized to the same size, the Rutherford-Bohr atom has a radius at least two orders of magnitude larger (range: factor of $\sim$90 for the uranium-235 atom to $\sim$200 for the hydrogen-1 atom) than the Sun-Earth planetary system. The amount of empty space in an atom is truly staggering and can be explained only by the enormous mass density of the nucleus which amounts to $\sim 1.5\times10^{15}$ g/cm^3.

1.18 Planck Energy Quantization

1.18.Q1 **(23)**

Quantum physics was born in 1900 when Max Planck presented his revolutionary idea of energy quantization of physical systems that undergo simple harmonic oscillations. Planck energy ε quantization is expressed as $\varepsilon = nh\nu$, where n is the quantum number ($n = 0, 1, 2, 3, \ldots$), h is a universal constant referred to as the Planck constant, and ν is the frequency of oscillation.

(a) Define the process of quantization.
(b) Give at least five examples of quantization in daily life.
(c) Provide at least five examples of quantization in modern physics.
(d) Describe Planck postulate and briefly discuss Planck's pioneering use of the quantization idea in 1900.

SOLUTION:

(a) Quantization is a process of constraining a quantity from a set of continuous values to a set of discrete values. The process is applied in various domains resulting in audio quantization, video and image quantization, and color quantization. It is also applied in mathematics and modern physics where it is used to develop quantum field theory from the classical field theory.

(b) Examples of quantization in daily life are:
Height quantization using steps in stairwells or rungs on ladders.
Quantization of currency and prices of goods and services.
Quantization of time.
Quantization of person's age.

(c) Examples of quantization in modern physics are:
Planck's quantization of oscillators in emission of blackbody radiation in 1900.
Einstein's quantization of light quanta in photoelectric effect in 1905.
Einstein's quantization of atomic vibrations in theory of specific heat in 1907.
Millikan's elementary charge quantization in 1910.
Bohr's quantization of angular momentum and energy in 1913.
Hydrogen emission spectrum and derivation of Rydberg constant in 1913.

(d) Classical physics predicts that the relationship between $\mathrm{d}\rho(T)/\mathrm{d}\nu$, the spectral energy density (energy per volume per frequency), and frequency ν of the emitted radiation is given by the Rayleigh-Jeans law as follows

$$\frac{\mathrm{d}\rho(T)}{\mathrm{d}\nu} = \frac{8\pi kT}{c^3}\nu^2, \tag{1.45}$$

with

T absolute temperature of the blackbody.
k Boltzmann constant.
c speed of light in vacuum.

Rayleigh-Jeans law predicts that energy density $\rho_\nu(T)$ increases as the square of frequency ν, approaching ∞ as $\nu \to \infty$, as shown with dashed curves in Fig. 1.7. This phenomenon, termed the ultraviolet catastrophe, is not borne out by experiments which show that blackbody emitters have clear maxima in their emission spectra with $\rho_\nu(T) \to 0$ as $\nu \to \infty$, as shown schematically with the solid curves in Fig. 1.7.

To solve the discrepancy between the classical theory and experiment Planck modeled a blackbody as a collection of oscillators that can only take on discrete, quantized energies described as

$$\varepsilon_n = nh\nu, \tag{1.46}$$

where n is an integer $(0, 1, 2, \ldots)$; ν is the frequency of emitted radiation; and h is a constant, now referred to as the Planck constant.

Based on oscillator energy quantization, Planck's alternative to Rayleigh-Jeans classical law for spectral energy density $\rho_\nu(T)$ is called the Planck law of blackbody

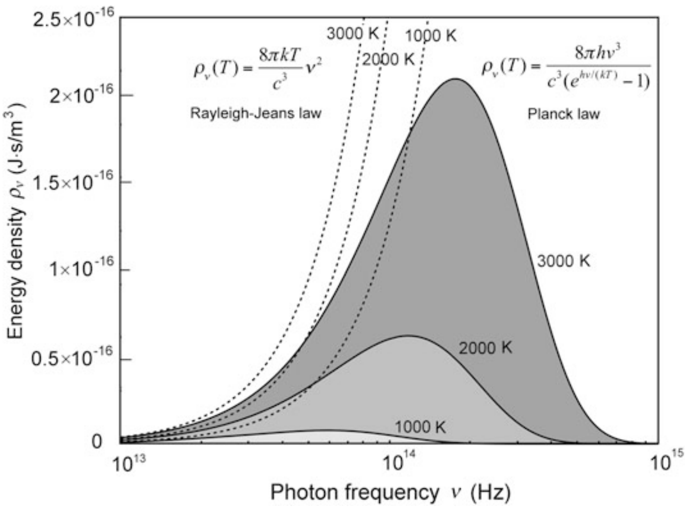


Fig. 1.7 Spectral energy density $\mathrm{d}\rho(T)/\mathrm{d}\nu$ against photon frequency ν in black body radiation for three temperatures: 1000 K, 2000 K, and 3000 K. *Dashed lines* are for Rayleigh-Jeans theory, *solid lines* for Planck theory. Note the shift of the peak in $\mathrm{d}\rho(T)/\mathrm{d}\nu$ toward higher frequencies with an increase in temperature T (see Prob. 24)

radiation and reads as follows

$$\frac{\mathrm{d}\rho(T)}{\mathrm{d}\nu} = \frac{8\pi h\nu^3}{c^3(e^{\frac{h\nu}{kT}} - 1)}. \tag{1.47}$$

The Planck law accurately predicts experimental data and, as shown in Fig. 1.7, exhibits a maximum; approaches 0 at low and high frequencies υ and, furthermore, at low frequencies transforms into Rayleigh-Jeans law, making the energy spectrum of each oscillator effectively continuous. However, at high frequencies the use of energy quantization is required in order to reach agreement between theory and experiment avoiding the ultraviolet catastrophe.

At low frequencies ν we can expand the exponential function of (1.47) as follows

$$\left\{e^{\frac{h\nu}{kT}} - 1\right\} \approx 1 + \frac{h\nu}{kT} + \cdots - 1 \tag{1.48}$$

and (1.47) transforms into Rayleigh-Jeans equation given in (1.45).

Planck's use of the simple oscillator energy quantization $\varepsilon_n = nh\nu$ in theory of blackbody radiation ushered in the quantum physics which subsequently made use of several other quantization processes, listed in (**c**). For example, in 1905 Einstein took the quantization idea a step further and explained the surface photoelectric effect by introducing the "photon quantum hypothesis" whereby an electron is ejected from metallic surface by the impact of a particle of light (photon). The relationship between the photon frequency υ and the kinetic energy of the emitted electron E_K

is given as

$$h\nu = E_K + e\phi, \tag{1.49}$$

where $e\phi$ is the work function, characteristic of the particular metal.

1.18.Q2 **(24)**

In Prob. 23 the Planck law of blackbody radiation expressing the spectral density of a blackbody emitter was written in the frequency ν domain as

$$\frac{\mathrm{d}\rho(T)}{\mathrm{d}\nu} = \frac{8\pi h\nu^3}{c^3(e^{\frac{h\nu}{kT}} - 1)}. \tag{1.50}$$

The spectral energy density can also be written in the wavelength λ domain and is usually designated as $\mathrm{d}\rho(T)/\mathrm{d}\lambda$. A plot of $\mathrm{d}\rho(T)/\mathrm{d}\lambda$ against λ, similarly to a plot of $\mathrm{d}\rho(T)/d\nu$ against ν (see Fig. 1.7), exhibits a peak $\lambda_{\max}$ which is proportional to temperature T; however, its position shifts toward shorter λ as the temperature increases, in contrast to the behavior of the $\mathrm{d}\rho(T)/\mathrm{d}\nu$ against ν graphs where $\nu_{\max}$ shifts toward higher frequencies ν as the temperature increases.

(a) Derive an expression for $\mathrm{d}\rho(T)/\mathrm{d}\lambda$ from $\mathrm{d}\rho(T)/\mathrm{d}\nu$ given in (1.50).

(b) Show that the maximum in $\mathrm{d}\rho(T)/\mathrm{d}\nu$ which occurs at frequency $\nu_{\max}$ is proportional to temperature T (Wien displacement law in frequency domain).

(c) Show that the maximum in $\mathrm{d}\rho(T)/\mathrm{d}\lambda$ which occurs at wavelength $\lambda_{\max}$ is inversely proportional to temperature T (Wien displacement law in wavelength domain).

SOLUTION:

(a) The spectral energy density in the wavelength domain $\mathrm{d}\rho(T)/\mathrm{d}\lambda$ is given as

$$\frac{\mathrm{d}\rho(T)}{\mathrm{d}\lambda} = \frac{\mathrm{d}\rho(T)}{\mathrm{d}\nu}\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{8\pi h\nu^3}{c^3(e^{\frac{h\nu}{kT}} - 1)}\frac{c}{\lambda^2} = \frac{8\pi hc}{\lambda^5(e^{\frac{h\nu}{kT}} - 1)}. \tag{1.51}$$

Equation (1.51) is plotted in Fig. 1.8 for three temperatures (2000 K, 2500 K, and 3000 K), while Fig. 1.9 displays (1.50) for the same three temperatures. The shifts in $\nu_{\max}$ and $\lambda_{\max}$ with temperature T are clearly noticeable with $\nu_{\max}$ proportional to T and $\lambda_{\max}$ inversely proportional to T.

(b) The Wien displacement law in the frequency domain states that $\nu_{\max}$ is proportional to temperature T of the blackbody emitter, i.e., $V_{\max} = C_\nu T$ (see Fig. 1.9). The constant C_ν is obtained from the Planck law by setting the second derivative

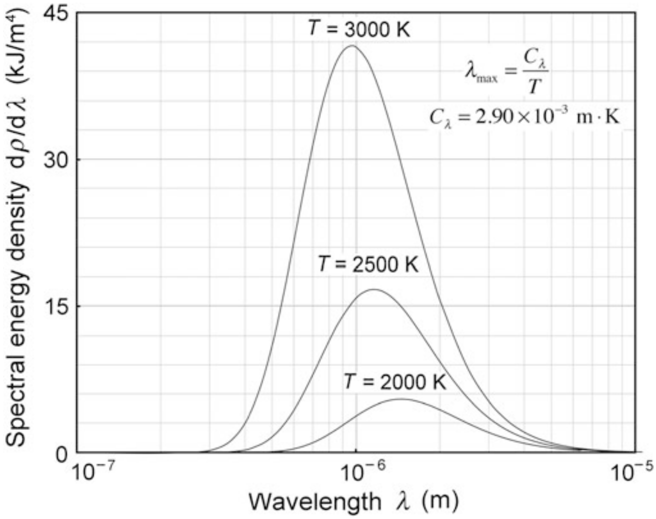


Fig. 1.8 Spectral energy density in the wavelength domain, $\mathrm{d}\rho(T)/\mathrm{d}\lambda$, against wavelength λ. Note the shift of $\lambda_{\max}$ to lower wavelengths with increasing temperature T of the blackbody emitter

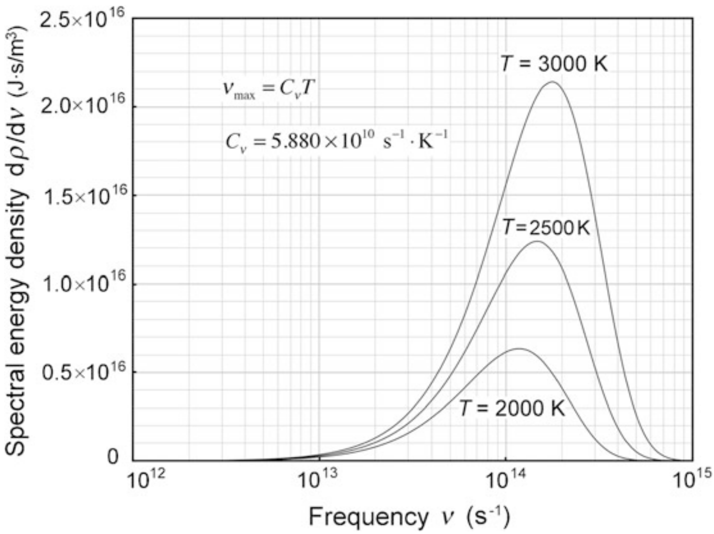


Fig. 1.9 Spectral energy density in the frequency domain, $\mathrm{d}\rho(T)/\mathrm{d}\nu$, against frequency ν. Note the shift of $\nu_{\max}$ to higher frequencies with increasing temperature T of the blackbody emitter

$\mathrm{d}^2\rho(T)/\mathrm{d}\nu^2|_{\nu=\nu_{\max}}$ equal to zero to find the relationship between $\nu_{\max}$ and T as follows

$$\left.\frac{\mathrm{d}^2\rho(T)}{\mathrm{d}\nu^2}\right|_{\nu=\nu_{\max}} = \frac{24\pi h\nu^2c^3(e^{\frac{h\nu_{\max}}{kT}}-1) - 8\pi h\nu^3c^3\frac{h}{kT}e^{\frac{h\nu_{\max}}{kT}}}{c^6(e^{\frac{h\nu_{\max}}{kT}}-1)^2} = 0 \tag{1.52}$$

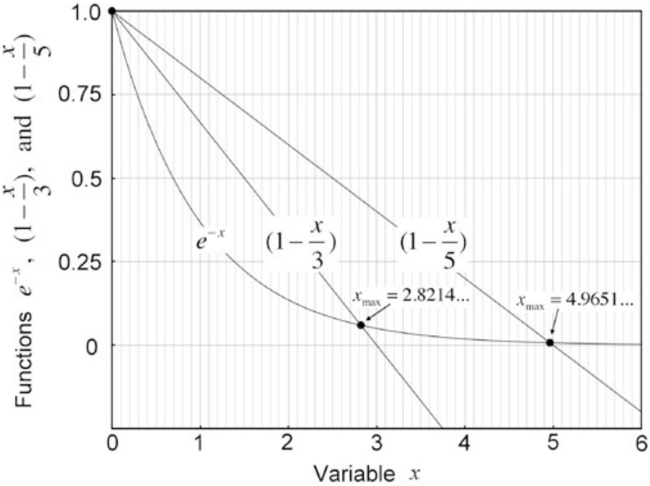


Fig. 1.10 Functions e^{-x}, $1-\frac{1}{3}x$, and $1-\frac{1}{5}x$ plotted against variable x and used in graphical solution of transcendental equations (1.54) and (1.58)

or

$$3\left(e^{\frac{h\nu_{max}}{kT}}-1\right)-\frac{h\nu}{kT}e^{\frac{h\nu_{max}}{kT}}=0. \tag{1.53}$$

Introducing a new variable $x_{max}=h\nu_{max}/(kT)$ into (1.53) we get a transcendental equation

$$3\left(e^{x_{max}}-1\right)-x_{max}e^{x_{max}}=0 \quad \text{or} \quad 1-\frac{x_{max}}{3}=e^{-x_{max}} \tag{1.54}$$

that cannot be solved in a closed form; however, we can solve it numerically or graphically by plotting its two functions: e^{-x} and $1-\frac{1}{3}x$ and finding solutions to (1.54) through determining the intercepts between the two functions occurring at $x=x_{max}$. As shown in Fig. 1.10, the two functions have two intercepts that provide two solutions: a trivial solution at $x_{max}=0$ and a physical solution at $x_{max}=2.8214$. The frequency ν_{max} where $\mathrm{d}\rho(T)/\mathrm{d}\nu$ attains its maximum value at a given T is now determined as follows

$$\begin{aligned}\nu_{max}&=x_{max}\frac{k}{h}T=\frac{2.8214\times1.38\times10^{-23}\ \mathrm{J\cdot K^{-1}}}{6.626\times10^{-34}\ \mathrm{J\cdot s}}T=C_\nu T\\&=\left(5.880\times10^{10}\ \mathrm{s^{-1}/K}\right)\times T.\end{aligned} \tag{1.55}$$

The Wien displacement constant in the frequency domain is given as $C_\nu = 5.880\times10^{10}\ \mathrm{s^{-1}/K}$.

(c) The Wien displacement law in the wavelength domain states that λ_{max} is inversely proportional to the temperature T of the blackbody emitter, i.e., $\lambda_{max} = C_\lambda/T$ (see Fig. 1.8). The constant C_λ is obtained by setting the second derivative

$\mathrm{d}^2\rho_\lambda(T)/d\lambda^2|_{\lambda=\lambda_{\max}}$ equal to zero to find the relationship between $\lambda_{\max}$ and T as follows

$$\left.\frac{\mathrm{d}^2\rho_\lambda(T)}{\mathrm{d}\lambda^2}\right|_{\lambda=\lambda_{\max}} = \frac{-8\pi hc[5\lambda^4(e^{\frac{hc}{kT\lambda_{\max}}}-1)-\lambda_{\max}^5\frac{hc}{kT\lambda_{\max}^2}e^{\frac{hc}{kT\lambda_{\max}}}]}{\lambda^{10}(e^{\frac{hc}{kT\lambda_{\max}}}-1)^2} = 0 \quad (1.56)$$

or

$$5\left(e^{\frac{hc}{kT\lambda_{\max}}}-1\right) - \frac{hc}{kT\lambda_{\max}}e^{\frac{hc}{kT\lambda_{\max}}} = 0. \quad (1.57)$$

Introducing a new variable $x_{\max} = hc/(kT\lambda_{\max})$ into (1.57) we get a transcendental equation

$$5\left(e^{x_{\max}}-1\right) - x_{\max}e^{x_{\max}} = 0 \quad \text{or} \quad 1-\frac{x}{5} = e^{-x_{\max}} \quad (1.58)$$

that like (1.54) cannot be solved in a closed form; however, we can solve it numerically or graphically by plotting its two functions: e^{-x} and $(1-\frac{1}{5}x)$ and finding solutions to (1.58) through determining the intercepts between the two functions occurring at $x = x_{\max}$. As shown in Fig. 1.10, the two functions have two intercepts that provide two solutions: a trivial solution at $x_{\max} = 0$ and a physical solution at $x_{\max} = 4.9651$. The wavelength $\lambda_{\max}$ where $\mathrm{d}\rho(T)/\mathrm{d}\lambda$ attains its maximum value at a given T is now determined as follows

$$\begin{aligned}\lambda_{\max} &= \frac{hc}{x_{\max}kT} = \frac{(6.626\times10^{-34}\ \mathrm{J\cdot s})\times(3\times10^8\ \mathrm{m\cdot s^{-1}})}{4.9651\times1.381\times10^{-23}\ \mathrm{J\cdot K^{-1}}}\frac{1}{T} \\ &= \frac{C_\lambda}{T} = \frac{2.90\times10^{-3}\ \mathrm{m\cdot K}}{T}. \end{aligned} \quad (1.59)$$

The Wien displacement constant in the wavelength domain is given as $C_\lambda = 2.90\times10^{-3}\ \mathrm{m\cdot K}$.

1.19 Quantization of Electromagnetic Radiation

1.19.Q1 **(25)**

(a) Show that photon energy E_ν is related to photon wavelength λ through the following relationship

$$E_\nu = \frac{12.4\times10^3\ \mathrm{eV\cdot Å}}{\lambda} = \frac{1.24\times10^{-6}\ \mathrm{eV\cdot m}}{\lambda}. \quad (1.60)$$

(b) Show that photon momentum p_ν is related to photon wavelength λ through the following relationship

$$p_\nu = \frac{0.6613 \times 10^{-33}\ \mathrm{kg \cdot m^2 \cdot s^{-1}}}{\lambda}. \tag{1.61}$$

(c) In Table 1.14A list the major components of the electromagnetic spectrum and for each component indicate the range in wavelength λ, frequency ν, and energy E_ν.

Table 1.14A Main characteristics of the electromagnetic (EM) spectrum

	Component of EM spectrum	Wavelength λ (m)	Frequency ν (Hz) $\nu =$	Energy E_ν (eV) $E_\nu =$
1		To $\Uparrow$ From	From $\Downarrow$ To	From $\Downarrow$ To
2		To $\Uparrow$ From	From $\Downarrow$ To	From $\Downarrow$ To
3		To $\Uparrow$ From	From $\Downarrow$ To	From $\Downarrow$ To
4	Visible	To $\Uparrow$ From	From $\Downarrow$ To	From $\Downarrow$ To
5		To $\Uparrow$ From	From $\Downarrow$ To	From $\Downarrow$ To
6		To $\Uparrow$ From	From $\Downarrow$ To	From $\Downarrow$ To

SOLUTION:

The oscillator energy quantization that Max Planck proposed for solving the blackbody emission spectrum problem introduced the notion that energy of electromagnetic radiation can only be released in packets of energy called quanta. These quanta have subsequently been named photons. Photon is characterized with its wavelength λ, frequency ν, energy E_ν, and momentum p_ν. Furthermore, the photon has no mass, possesses no charge, does not decay in empty space, and moves in vacuum with speed $c = 3\times 10^8$ m/s that is a universal constant independent of the motion of the source. The following basic relationships apply for photons:

(a)

$$c = \lambda\nu \tag{1.62}$$

$$E_\nu = h\nu = h\frac{c}{\lambda} = \frac{2\pi\hbar c}{\lambda} = \frac{2\pi\times 197.3\ \mathrm{MeV\cdot fm}}{\lambda} = \frac{1.24\times 10^{-6}\ \mathrm{eV\cdot m}}{\lambda} \tag{1.63}$$

(b)

$$\begin{aligned} p_\nu &= \frac{E_\nu}{c} = \frac{h}{\lambda} = \frac{1.24\times 10^{-6}\ \mathrm{eV\cdot m}}{(3\times 10^8\ \mathrm{m\cdot s^{-1}})\times\lambda} \\ &= \frac{(1.24\times 10^{-6}\ \mathrm{eV\cdot m})\times(1.6\times 10^{-19}\ \mathrm{N\cdot m})}{(3\times 10^8\ \mathrm{m\cdot s^{-1}})\times\lambda} \\ &= \frac{0.6613\times 10^{-33}\ \mathrm{kg\cdot m^2\cdot s^{-1}}}{\lambda} \end{aligned} \tag{1.64}$$

(c) Major components of the electromagnetic spectrum are listed in Table 1.14B.

Table 1.14B Main characteristics of the electromagnetic (EM) spectrum

	Component of EM spectrum	Wavelength λ (m)		Frequency ν (Hz) $\nu = c/\lambda$		Energy E_ν (eV) $E_\nu = 1.24\times 10^{-6}$ eV · m	
		To	10^3	From	3×10^5	From	1.24×10^{-9}
1	Radio waves	$\Uparrow$		$\Downarrow$		$\Downarrow$	
		From	10^{-1}	To	3×10^9	To	1.24×10^{-5}
		To	10^{-1}	From	3×10^9	From	1.24×10^{-5}
2	Microwaves	$\Uparrow$		$\Downarrow$		$\Downarrow$	
		From	10^{-3}	To	3×10^{11}	To	1.24×10^{-3}
		To	10^{-3}	From	3×10^{11}	From	1.24×10^{-3}
3	Infrared radiation	$\Uparrow$		$\Downarrow$		$\Downarrow$	
		From	7×10^{-7}	To	0.43×10^{15}	To	1.77
		To	7×10^{-7}	From	0.43×10^{15}	From	1.77
4	Visible light	$\Uparrow$		$\Downarrow$		$\Downarrow$	
		From	4×10^{-7}	To	0.75×10^{15}	To	3.10
		To	4×10^{-7}	From	0.75×10^{15}	From	3.10
5	Ultraviolet radiation	$\Uparrow$		$\Downarrow$		$\Downarrow$	
		From	10^{-8}	To	3×10^{16}	To	124
		To	10^{-8}	From	3×10^{16}	From	124
6	X rays and γ rays	$\Uparrow$		$\Downarrow$		$\Downarrow$	
		From	10^{-15}	To	3×10^{23}	To	1.24×10^9

1.20 Special Theory of Relativity

1.20.Q1 **(26)**

Lorentzian transformations relate the spatial and temporal coordinates x, y, z, and t in a stationary frame F to spatial and temporal coordinates x', y', z', and t' in a reference frame F' moving with uniform velocity υ in the direction of the abscissa (x) axis. The two frames are parallel to one another, i.e., the x' axis is parallel to the x axis, the y' axis is parallel to the y axis, and the z' axis is parallel to the z axis. Equations for the forward Lorentzian transformation are as follows: $x' = \gamma(x - \upsilon t)$; $y' = y$; $z' = z$; and $t' = \gamma[t - \upsilon x/(c^2)]$, where γ is the standard Lorentz factor $\gamma = (1 - \upsilon^2/c^2)^{-1/2}$.

Determine the relationships that govern the inverse Lorentzian transformation.

SOLUTION:

Expressions for $y' = y$ and $z' = z$ remain the same in the inverse transformation; expressions for x and t we determine by solving expressions for x' and t' in the forward transformation. We first derive the expression for x using expressions for x' and t' as follows

$$x' = \gamma(x - \upsilon t) \quad \text{or} \quad x = \frac{1}{\gamma}x' + \upsilon t, \tag{1.65}$$

$$t' = \gamma\left(t - \frac{\upsilon}{c^2}x\right) \quad \text{or} \quad t = \frac{1}{\gamma}t' + \frac{\upsilon}{c^2}x. \tag{1.66}$$

Inserting (1.66) into (1.65) we get

$$x = \frac{1}{\gamma}x' + \frac{\upsilon}{\gamma}t' + \frac{\upsilon^2}{c^2}x \quad \text{or} \quad x\left(1 - \frac{\upsilon^2}{c^2}\right) = \frac{1}{\gamma}\left(x' + \upsilon t'\right). \tag{1.67}$$

Since $(1 - \upsilon^2/c^2) = 1/\gamma^2$, we get the following expression for x

$$x = \gamma\left(x' + \upsilon t'\right). \tag{1.68}$$

We now derive the expression for t using expressions for x' and t' as follows

Inserting (1.65) into (1.66) we get

$$t = \frac{1}{\gamma}t' + \frac{1}{\gamma}\frac{\upsilon}{c^2}x' + \frac{\upsilon^2}{c^2}t \quad \text{or} \quad t\left(1 - \frac{\upsilon^2}{c^2}\right) = \frac{1}{\gamma}\left(t' + \frac{\upsilon}{c^2}x'\right). \tag{1.69}$$

Since $(1 - \upsilon^2/c^2) = 1/\gamma^2$, we get the following expression for t

$$t = \gamma\left(t' + \frac{\upsilon}{c^2}x'\right). \tag{1.70}$$

Equations for the inverse Lorentzian transformation

$$x = \gamma\left(x' + \upsilon t'\right); \qquad y = y'; \qquad z = z'; \quad \text{and} \quad t = \gamma\left[t' + \upsilon x'/\left(c^2\right)\right] \quad (1.71)$$

are equivalent to equations governing the forward Lorentzian transformation, except that the velocity of the moving frame is $-\upsilon$ in the inverse transformation as opposed to $+\upsilon$ in the forward transformation.

1.20.Q2 **(27)**

(a) Show that the wave equation

$$\frac{\partial \phi}{\partial x^2} + \frac{\partial \phi}{\partial y^2} + \frac{\partial \phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial \phi}{\partial t^2} \quad (1.72)$$

is invariant under Lorentzian transformation.

(b) What is the effect of this invariance on Maxwell equations?

SOLUTION:

(a) The Lorentzian transformation for two inertial frames, F and F', with parallel corresponding axes, with x and x' axes being common, and with frame F' moving along the x axis with velocity υ relative to frame F, is expressed as follows

$$x' = \gamma(x - \upsilon t); \qquad y' = y; \qquad z' = z; \quad \text{and} \quad t' = \gamma\left(t - \frac{\upsilon}{c^2}c\right) \quad (1.73)$$

where γ is the standard Lorentz factor $\gamma = (1 - \upsilon^2/c^2)^{-1/2}$ and the following derivatives should be noted

$$\frac{\partial x'}{\partial x} = \gamma, \qquad \frac{\partial x'}{\partial t} = -\gamma\upsilon, \qquad \frac{\partial t'}{\partial t} = \gamma, \quad \text{and} \quad \frac{\partial t'}{\partial x} = -\gamma\frac{\upsilon}{c^2}. \quad (1.74)$$

The space derivatives are expressed as follows

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \phi}{\partial t'}\frac{\partial t'}{\partial x} = \frac{\partial \phi}{\partial x'}\gamma - \frac{\partial \phi}{\partial t'}\gamma\frac{\upsilon}{c^2}, \quad (1.75)$$

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x^2} &= \frac{\partial}{\partial x'}\left[\frac{\partial \phi}{\partial x'}\right]\frac{\partial x'}{\partial x}\gamma + \frac{\partial}{\partial t'}\left[\frac{\partial \phi}{\partial x'}\right]\frac{\partial t'}{\partial x}\gamma - \frac{\partial}{\partial x'}\left[\frac{\partial \phi}{\partial t'}\right]\frac{\partial x'}{\partial x}\gamma\frac{\upsilon}{c^2} \\
&\quad - \frac{\partial}{\partial t'}\left[\frac{\partial \phi}{\partial t'}\right]\frac{\partial t'}{\partial x}\gamma\frac{\upsilon}{c^2} \\
&= \gamma^2\frac{\partial^2 \phi}{\partial x'^2} - \gamma^2\frac{\upsilon}{c^2}\frac{\partial^2 \phi}{\partial x'\partial t'} - \gamma^2\frac{\upsilon}{c^2}\frac{\partial^2 \phi}{\partial x'\partial t'} + \gamma^2\frac{\upsilon^2}{c^4}\frac{\partial^2 \phi}{\partial t'^2} \\
&= \gamma^2\frac{\partial^2 \phi}{\partial x'^2} + \gamma^2\frac{\upsilon^2}{c^4}\frac{\partial^2 \phi}{\partial t'^2} - 2\gamma^2\frac{\upsilon}{c^2}\frac{\partial^2 \phi}{\partial x'\partial t'},
\end{aligned} \quad (1.76)$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y'^2}, \tag{1.77}$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial z'^2}. \tag{1.78}$$

The time derivatives are

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \phi}{\partial x'}\frac{\partial x'}{\partial t} = \frac{\partial \phi}{\partial t'}\gamma - \frac{\partial \phi}{\partial x'}\gamma \upsilon, \tag{1.79}$$

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial t^2} &= \frac{\partial}{\partial t'}\left[\frac{\partial \phi}{\partial t'}\right]\frac{\partial t'}{\partial t}\gamma + \frac{\partial}{\partial x'}\left[\frac{\partial \phi}{\partial t'}\right]\frac{\partial x'}{\partial t}\gamma - \frac{\partial}{\partial t'}\left[\frac{\partial \phi}{\partial x'}\right]\frac{\partial t'}{\partial t}\gamma \upsilon - \frac{\partial}{\partial x'}\left[\frac{\partial \phi}{\partial x'}\right]\frac{\partial x'}{\partial t}\gamma \upsilon \\
&= \gamma^2 \frac{\partial^2 \phi}{\partial t'^2} - \gamma^2 \upsilon \frac{\partial^2 \phi}{\partial x' \partial t'} - \gamma^2 \upsilon \frac{\partial^2 \phi}{\partial x' \partial t'} - \gamma^2 \upsilon^2 \frac{\partial^2 \phi}{\partial x'^2} \\
&= \gamma^2 \frac{\partial^2 \phi}{\partial t'^2} + \gamma^2 \upsilon^2 \frac{\partial^2 \phi}{\partial x'^2} - 2\gamma^2 \upsilon \frac{\partial^2 \phi}{\partial x' \partial t'}.
\end{aligned} \tag{1.80}$$

The components of the wave equation can now be expressed as follows

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \gamma^2 \frac{\partial^2 \phi}{\partial x'^2} + \gamma^2 \frac{\upsilon^2}{c^4}\frac{\partial^2 \phi}{\partial t'^2} - 2\gamma^2 \frac{\upsilon}{c^2}\frac{\partial^2 \phi}{\partial x' \partial t'} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} \tag{1.81}$$

and

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\gamma^2}{c^2}\frac{\partial^2 \phi}{\partial t'^2} + \gamma^2 \frac{\upsilon^2}{c^2}\frac{\partial^2 \phi}{\partial x'^2} - 2\gamma^2 \frac{\upsilon}{c^2}\frac{\partial^2 \phi}{\partial x' \partial t'}, \tag{1.82}$$

resulting in the following expression for the wave equation

$$\gamma^2\left(1 - \frac{\upsilon^2}{c^2}\right)\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} = \gamma^2 \frac{1}{c^2}\left(1 - \frac{\upsilon^2}{c^2}\right)\frac{\partial^2 \phi}{\partial t'^2}. \tag{1.83}$$

Recognizing that $\gamma^2(1 - \upsilon^2/c^2) = \gamma^2(1 - \beta^2) = 1$ we finally get the following expression for the wave equation in the inertial frame F'

$$\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t'^2}, \tag{1.84}$$

showing that equation $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = (1/c^2)\partial^2\phi/\partial t^2$ is invariant under Lorentzian transformation.

(b) The form of the wave equation in a frame F' moving with uniform velocity υ is identical to the form of the wave equation in fixed inertial frame F. A result of this invariance is the invariance of Maxwell equations governing the electric and magnetic fields produced by a charge moving with uniform velocity υ.

1.20.Q3 **(28)**

Theory of relativity has several consequences, some of them quite dramatic, counter-intuitive, and irreconcilable with classical physics. Three important phenomena in this category are: (i) length contraction, (ii) time dilation, and (iii) mass increase with velocity, all three phenomena governed by the Lorentz factor $\gamma = [1 - \beta^2]^{-1/2} = [1 - (\upsilon/c)^2]^{-1/2}$ which becomes significant when the speed υ of an object or particle is an appreciable fraction of the speed of light c in vacuum.

Use the Lorentzian transformation between two inertial frames, one stationary and the other moving with velocity υ with respect to the stationary frame, to show that

(a) Length contraction is inversely proportional to γ, i.e., $L = \frac{L_0}{\gamma}$.

(b) Time dilation is proportional to γ as $\Delta t = \gamma \Delta t_0$. In equations for L and Δt, L_0 and Δt_0 are the proper length and the proper time interval, respectively, measured by an observer moving with the object, while L and Δt are the length and time interval, respectively, measured by an observer in the stationary frame.

(c) The increase in relativistic mass $m(\upsilon)$ of a particle as a function of its velocity υ makes the speed of light c in vacuum the upper limit of speed in the universe. Calculate the velocity relative to speed of light c of a particle at which the relativistic mass of the particle exceeds its rest mass m_0 by 2 %, 10 %, 50 %, a factor of 10, and a factor of 100.

SOLUTION:

(a) **Length contraction**. The measured length of an object depends on the relative velocity of the object and observer. The largest length of an object is measured in a frame in which the object is at rest. This length is referred to as the proper or rest length L_0.

Assume that we have a rod placed along the x' axis in the moving frame F'. The length L_0 measured by an observer at rest with respect to the moving frame F' is given as $L_0 = x_2' - x_1'$. The length of the rod L measured in the fixed reference frame F is $L = x_2 - x_1$ with the coordinates x_1 and x_2 related to coordinates x_1' and x_2', respectively, through the Lorentzian transformation. We can thus express L_0 as follows

$$L_0 = x_2' - x_1' = \gamma(x_2 - \upsilon t_2) - \gamma(x_1 - \upsilon t_1). \tag{1.85}$$

Since the two measurements made in the fixed frame F were made simultaneously, we have $t_1 = t_2$ and (1.85) is simplified to read

$$L_0 = \gamma(x_2 - x_1) = \gamma L \quad \text{and} \quad L = \frac{1}{\gamma} L_0. \tag{1.86}$$

Length L of an object that is moving with velocity υ relative to the observer is inversely proportional to Lorentz factor γ and is smaller than or equal to the proper length L_0, since $\gamma \geq 1$ This phenomenon is referred to as the length contraction.

(b) Time dilation. The shortest time interval is measured in a frame in which the clock is at rest. The time interval so measured is referred to as the proper or rest time interval Δt_0. We now assume that we are measuring a time interval Δt_0 in the moving frame F' where $\Delta t_0 = t_2' - t_1'$. The time interval Δt measured for the same event in the fixed frame F is given as

$$\Delta t = t_2 - t_1 = \gamma\left(t_2' + \frac{\upsilon}{c^2}x_2'\right) - \gamma\left(t_1' + \frac{\upsilon}{c^2}x_1'\right). \tag{1.87}$$

Since the two measurements made in F' are made at the same location, we have $x_1' = x_2'$ and (1.87) simplifies to read $\Delta t = \gamma(t_2' - t_1') = \gamma\Delta t_0$. This phenomenon is referred to as time dilation.

(c) Relativistic mass. Particle mass m depends on particle velocity υ through the following relationship

$$\begin{aligned} m(\upsilon) &= \gamma m_0 = \frac{m_0}{\sqrt{1-\beta^2}} = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \quad \text{or} \\ \frac{m(\upsilon)}{m_0} &= \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \end{aligned} \tag{1.88}$$

where m_0 is the particle mass at rest at $\upsilon = 0$ referred to as the particle rest mass or invariant mass and γ is the Lorentz factor. The relativistic mass of a particle becomes infinite as the velocity of the particle approaches the speed of light c.

Solving (1.88) for $\beta = \upsilon/c$ against $\gamma = m(\upsilon)/m_0$ results in the following relationship

$$\beta = \frac{\upsilon}{c} = \frac{\sqrt{\gamma^2 - 1}}{\gamma} = \sqrt{1 - \frac{1}{\gamma^2}}. \tag{1.89}$$

Results for β with various values of $\gamma = m(\upsilon)/m_0$ are shown in Table 1.15.

Table 1.15 Speed υ of particle normalized to speed of light c against Lorentz factor $\gamma = m(\upsilon)/m_0$

$\gamma = m(\upsilon)/m_0$	1.01	1.02	1.10	1.50	10	100
$\beta = \upsilon/c$	0.140	0.197	0.417	0.745	0.995	0.99995

1.20.Q4 **(29)**

Pions π also called π mesons belong to a group of short-lived subatomic particles called mesons. They are either neutral (π^0) or come with positive (π^+) or negative (π^-) electron charge and their rest mass is about $273m_e$ where $m_e = 0.511$ MeV is the rest mass of the electron. Pions do not exist in free state in nature; they reside inside the nuclei of atoms and, based on their mass, were identified as the quanta of the strong interaction. They can be ejected from the nucleus in nuclear reactions by bombarding target nuclei with energetic electrons or protons. Mean lifetime of a free negative pion (π^-) and positive pion π^+ in its own reference frame (proper or rest mean lifetime) is 2.6×10^{-8} s and they decay through weak interaction.

Of the three pion types negative pions have been used for radiotherapy, since by virtue of their negative charge, they produce the so-called "pion stars" in irradiated nuclei. They showed great promise for use in radiotherapy; however, during recent years pions, because of their complexity and cost, were largely abandoned in favor of heavy charged particles such as protons and heavier ions.

If the pion travels with velocity of $0.99c$ where c is the speed of light in vacuum, determine:

(a) Mean lifetime of the pion when measured by a stationary observer on earth.

(b) Mean distance the pion travels before it decays, as measured by a stationary observer on earth.

SOLUTION:

First we determine the Lorentz factor γ for $\beta = 0.99c$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.99^2}} = 7.09. \tag{1.90}$$

(a) Mean lifetime $\bar{\tau}$ of the pion as measured on earth is longer than the proper (rest) mean lifetime $\bar{\tau}_0$ because of the relativistic time dilation effect. The following relationship between $\bar{\tau}$ and $\bar{\tau}_0$ applies (see Prob. 28)

$$\bar{\tau} = \gamma\bar{\tau}_0 = 7.09\times 2.6\times 10^{-8}\text{ s} = 18.4\times 10^{-8}\text{ s} = 0.184\text{ ms}. \tag{1.91}$$

(b) Mean distance $\bar{\ell}$ that the pion with velocity of 0.99c travels is calculated by multiplying the pion's mean lifetime $\bar{\tau}$ with its velocity to get

$$\bar{\ell} = \bar{\tau}\upsilon = \left(0.184\times 10^{-6}\text{ s}\right)\times\left(0.99\times 3\times 10^{8}\text{ m/s}\right) = 54.6\text{ m}. \tag{1.92}$$

1.20.Q5 **(30)**

(a) A meter stick moving in the direction parallel to its long dimension appears to be only 80 cm long to a stationary observer. Calculate the speed of the stick.

(b) Estimate the length L of an acceleration waveguide of a linear accelerator (linac) as seen by an accelerated electron, if the length L_0 of the accelerator waveguide is 1.5 m, the accelerator gun voltage is 100 kV, and the linac nominal energy is 25 MV.

SOLUTION:

(a) The proper (rest) length of the stick is $L_0 = 100$ cm. Since, as a result of the relativistic length contraction, the stick appears shorter ($L = 80$ cm), it must be moving with significant speed that can be determined from the length contraction expression, given in Prob. 28 as

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-\beta^2} = L_0\sqrt{1-\frac{\upsilon^2}{c^2}}, \tag{1.93}$$

where γ is the Lorentz factor.

Solving (1.93) for the normalized velocity $\beta = \upsilon/c$ we get

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} = \sqrt{1-\frac{L^2}{L_0^2}} = 0.6 \quad \text{or} \quad \upsilon = 0.6c = 1.8\times10^8 \text{ m/s}. \tag{1.94}$$

(b) The proper (rest) length of a linac waveguide measured in the reference frame of the linac is L_0 and the length of the waveguide as it appears to an observer traveling with the accelerated electron is L. The two lengths L and L_0 are related through relativistic length contraction (see Prob. 28) as follows

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-\beta^2} \quad \text{or} \quad \mathrm{d}z' = \frac{\mathrm{d}z}{\gamma} = \mathrm{d}z\sqrt{1-\beta^2}, \tag{1.95}$$

where $\mathrm{d}z$ is an element of accelerated electron's path in the linac reference frame and $\mathrm{d}z'$ is an element of path in the rest frame.

Since the total energy E of the electron is related to the rest energy E_0 of the electron through the Lorentz factor γ as $E = \gamma E_0$ we note that $\gamma = E/E_0$ and (1.95) becomes

$$\mathrm{d}z' = \frac{\mathrm{d}z}{\gamma} = \mathrm{d}z\sqrt{1-\beta^2} = \frac{E_0}{E}\mathrm{d}z = \frac{E_0}{E_i + e\mathcal{E}_z z}\mathrm{d}z = \frac{E_0}{E_0 + (E_\mathrm{K})_\mathrm{i} + e\mathcal{E}_z z}\mathrm{d}z, \tag{1.96}$$

where the total energy E as a function of z along the waveguide axis is given as the sum of the total energy E_i of the electrons injected from the electron gun into the accelerator waveguide and energy $e\mathcal{E}_z z$ gained by the electron in the electric field $\mathcal{E}_z$ used for electron acceleration in the waveguide. The total energy E_i of the injected electron is the sum of electron rest energy E_0 and its kinetic energy $(E_\mathrm{K})_\mathrm{i}$ at the time of injection into the waveguide. We thus have

$$E = E_\mathrm{i} + e\mathcal{E}_z z = E_0 + (E_\mathrm{K})_\mathrm{i}. \tag{1.97}$$

Next, we integrate (1.96) over z from 0 to L_0 and over z' from 0 to L to get

$$\int_0^L \mathrm{d}z' = \int_0^{L_0} \frac{E_0}{E_0 + (E_\mathrm{K})_\mathrm{i} + e\mathcal{E}_z z}\,\mathrm{d}z = \frac{E_0}{e\mathcal{E}_z}\int_0^{L_0} \frac{\mathrm{d}(E_0 + (E_\mathrm{K})_\mathrm{i} + e\mathcal{E}_z z)}{E_0 + (E_\mathrm{K})_\mathrm{i} + e\mathcal{E}_z z} \tag{1.98}$$

leading to

$$L = \frac{E_0}{e\mathcal{E}_z}\left\{\ln\left[E_0 + (E_\mathrm{K})_\mathrm{i} + e\mathcal{E}_z z\right]\right\}_0^{L_0} = \frac{E_0}{e\mathcal{E}_z}\ln\frac{E_\mathrm{f}}{E_0 + (E_\mathrm{K})_\mathrm{i}}, \tag{1.99}$$

where we recognize that $E_0 + (E_\mathrm{K})_\mathrm{i} + e\mathcal{E}_z L_0$ is the final total energy E_f of the electron as it exits the acceleration waveguide.

Equation (1.99) is the general equation for estimating the waveguide contraction of an acceleration waveguide and we now use this equation to solve our specific problem given with the following parameters: $L_0 = 1.5$ m, $E_0 = 0.511$ MeV, $E_\mathrm{f} = 25$ MeV, and $(E_\mathrm{K})_\mathrm{i} = 100$ keV. The electric field $\mathcal{E}_z$ is estimated from the capture condition for a linac waveguide (T13.110) to be $\sim$**8 MV/m** for electron gun voltage of 100 kV.

The apparent length L of the waveguide is from (1.99) given as follows

$$L = \frac{E_0}{e\mathcal{E}_z}\ln\frac{E_\mathrm{f}}{E_0 + (E_\mathrm{K})_\mathrm{i}} = \frac{0.511\ \mathrm{MeV}}{e\times(8\ \mathrm{MV/m})}\ln\frac{25\ \mathrm{MeV}}{(0.511+0.100)\ \mathrm{MeV}} = 0.24\ \mathrm{m} \tag{1.100}$$

compared to a waveguide length of 1.5 m in the linear accelerator frame.

1.21 Important Relativistic Relations

1.21.Q1 **(31)**

Determine the speed of a particle (as fraction of the speed of light in vacuum c) at which the particle mass $m(\upsilon)$ becomes:

(a) Twice its rest mass m_0,
(b) Three times its rest mass m_0,
(c) Four times its rest mass m_0.

SOLUTION:

(a) Using the basic Einstein expression that states that the ratio between particle's relativistic mass $m(\upsilon)$ and rest mass m_0 equals to particle's Lorentz factor γ, i.e., $m(\upsilon)/m_0 = \gamma$ we set

$$\frac{m(\upsilon)}{m_0} = 2 = \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} \tag{1.101}$$

and solve for υ/c to get

$$\frac{\upsilon}{c} = \sqrt{1 - \frac{1}{[\frac{m(\upsilon)}{m_0}]^2}} = \sqrt{\frac{3}{4}} = 0.866 \tag{1.102}$$

corresponding to $\upsilon = 0.866c = 2.60\times10^8$ m/s.

(b) For $m(\upsilon) = 3m_0$ we get

$$\frac{\upsilon}{c} = \sqrt{1 - \frac{1}{[\frac{m(\upsilon)}{m_0}]^2}} = \sqrt{\frac{8}{9}} = 0.943 \tag{1.103}$$

corresponding to $\upsilon = 0.943c = 2.83\times10^8$ m/s.

(c) For $m(\upsilon) = 4m_0$ we get

$$\frac{\upsilon}{c} = \sqrt{1 - \frac{1}{[\frac{m(\upsilon)}{m_0}]^2}} = \sqrt{\frac{15}{16}} = 0.968 \tag{1.104}$$

corresponding to $\upsilon = 0.968c = 2.90\times10^8$ m/s. Figure 1.11 is a plot of γ against β for a particle of rest mass m_0, where

β is the particle velocity υ normalized to speed of light c in vacuum, i.e., $\beta = \upsilon/c$.
γ is the so called Lorentz factor defined as:

(i) $\gamma = 1/\sqrt{1-\upsilon^2/c^2}$.
(ii) $\gamma = m(\upsilon)/m_0$ with $m(\upsilon)$ particle mass m at velocity υ and m_0 particle rest mass.
(iii) $\gamma = E/E_0 = (E_K + E_0)/E_0 = 1 + E_K/E_0$ with E particle total energy, E_K particle kinetic energy, and E_0 particle rest energy.

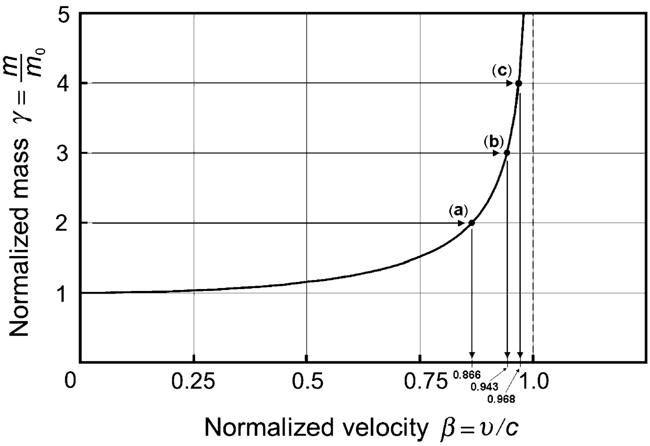


Fig. 1.11 Particle normalized mass $m(\upsilon)/m_0$ against its normalized velocity $\beta = \upsilon/c$. Data points represent results of **(a)**, **(b)**, and **(c)**

1.21.Q2 **(32)**

The standard expression of special relativity

$$E^2 = p^2c^2 + E_0^2 \tag{1.105}$$

linking particle's total energy E with its momentum p and rest energy E_0 is universally valid for all particles ($m_0 \neq 0$) as well as for photons ($m_0 = 0$).

(a) Derive (1.105) from the basic Einstein expression for relativistic mass m as a function of velocity υ, i.e., from $m(\upsilon) = \gamma m_0$ where m is the particle's relativistic mass depending on particle's velocity υ and m_0 is the particle's rest mass.

(b) Show that the expression $E^2 = p^2c^2 + E_0^2$ can also be derived directly from the two basic relativistic expressions: $E = \gamma E_0$ and $p = m\upsilon$ with γ the Lorentz factor given as $\gamma = [1 - \upsilon^2/c^2]^{-1/2}$ and $\beta = \upsilon/c$ the velocity υ normalized to speed of light c.

(c) Show that a massless particle ($m_0 = 0$) always travels at the speed of light c.

(d) Show that a particle that travels with speed of light c possesses no rest energy ($E_0 = 0$) and no rest mass ($m_0 = 0$).

SOLUTION:

(a) The derivation of (1.105) from the basic Einstein relationship for relativistic mass proceeds as

(1) Start with the basic equation

$$m = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}. \tag{1.106}$$

(2) Square (1.106), multiply the result by c^4, and rearrange the terms to obtain

$$m^2c^4 - m^2c^2\upsilon^2 = m_0^2c^4. \tag{1.107}$$

(3) Equation (1.107) can be written as

$$E^2 - p^2c^2 = E_0^2 \quad \text{or} \quad E = \sqrt{E_0^2 + p^2c^2}, \tag{1.108}$$

incorporating into (1.107) the common relativistic relationships for total energy $E = mc^2$, rest energy $E_0 = m_0c^2$, and momentum $p = m\upsilon$.

(b) Particle momentum $p = m\upsilon$ may be expressed as $p = m\upsilon = \gamma E_0\beta/c$ using the standard relationships for γ and E_0. Multiplying the expression for p with c, squaring the result, and recognizing that $\beta^2 = 1 - \gamma^{-2}$ gives

$$p^2c^2 = \gamma^2E_0^2\beta^2 = E^2\left[1 - \frac{1}{\gamma^2}\right] = E^2 - \frac{E^2}{\gamma^2} = E^2 - E_0^2 \tag{1.109}$$

or

$$E^2 = p^2c^2 + E_0^2. \tag{1.110}$$

(c) From (1.110) we get

$$E^2 = E_0^2 + p^2c^2 = 0 + p^2c^2 \quad \text{or} \quad E = pc. \tag{1.111}$$

Equation (1.111) gives the following relationship for particle speed υ, total energy E, and momentum p

$$\frac{\upsilon}{c} = \beta = \frac{pc}{E} \quad \text{or} \quad E = \frac{pc}{\beta}. \tag{1.112}$$

Equations (1.109) and (1.110) can hold simultaneously only for $\beta = 1$, i.e., $\upsilon = c$.

(d) From (1.111) we have $\upsilon/c = pc/E = 1$ or $E = pc$. From (1.110) we get the following expression for the rest energy E_0

$$E^2 = E_0^2 + p^2c^2 \quad \text{or} \quad E_0 = \sqrt{E^2 - p^2c^2} = 0. \tag{1.113}$$

Thus, when the total energy E equals pc, the rest energy E_0 is zero and the particle's rest mass m_0 is zero.

1.21.Q3 **(33)**

An electron is accelerated in a 10 MV linear accelerator (linac) and strikes an x-ray target. For the electron determine:

(a) Kinetic energy E_K.
(b) Total energy E.
(c) Lorentz factor γ.
(d) Velocity υ.
(e) Mass m.

SOLUTION:

(a) By definition a 10 MV linac produces a 10 MV x-ray beam whereby electrons of kinetic energy of 10 MeV strike the x-ray target and produce a 10 MV spectral distribution. Kinetic energy of the electron when it strikes the target is thus $E_K =$ **10 MeV**. It is customary to describe an electron with its kinetic energy, so that a label "10 MeV electron" implies that kinetic energy of the electron is 10 MeV.

(b) Total energy E of an electron with kinetic energy $E_K = 10$ MeV is given as the sum of electron's kinetic energy E_K and its rest energy $E_0 = 0.511$ MeV.

$$E = E_K + E_0 = 10\text{ MeV} + 0.511\text{ MeV} = 10.511\text{ MeV}. \tag{1.114}$$

(c) Lorentz factor γ is given as $\gamma = (1 - \upsilon^2/c^2)^{-1/2}$ and depends, in addition to velocity υ, also indirectly on electron kinetic energy E_K

$$E_K = (\gamma - 1)E_0. \tag{1.115}$$

Solving (1.115) for γ results in the following expression

$$\gamma = 1 + \frac{E_K}{E_0} = 1 + \frac{10\text{ MeV}}{0.511\text{ MeV}} = 20.57. \tag{1.116}$$

The Lorentz factor γ can also be calculated from the basic definitions that implicitly state the following relationships

$$E = mc^2 = \gamma E_0 = \gamma m_0 c^2 \quad \text{or} \quad \gamma = \frac{E}{E_0} = \frac{10.511\text{ MeV}}{0.511\text{ MeV}} = 20.57. \tag{1.117}$$

A plot of the Lorentz factor γ against velocity υ normalized to speed of light c in vacuum of Fig. 1.12 shows that an electron with $\gamma = 20.57$ is highly relativistic and travels with a velocity close to speed of light c in vacuum.

(d) Now that we have the Lorentz factor γ for a 10 MeV electron, we can calculate the normalized electron velocity υ/c. Solving $\gamma = (1 - \upsilon^2/c^2)^{-1/2}$ for υ/c results

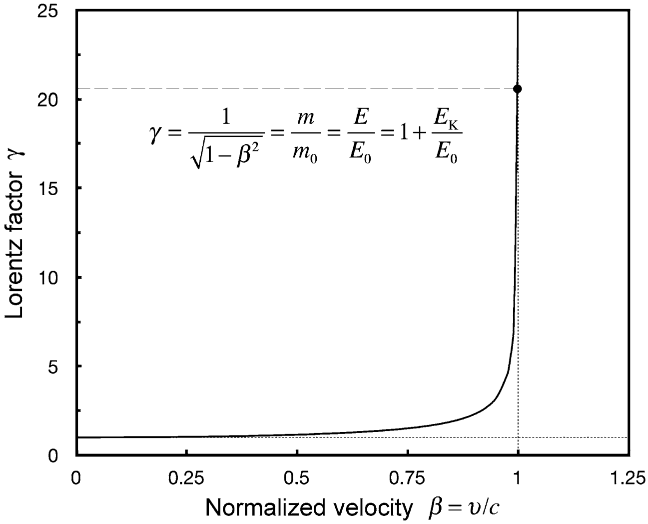


Fig. 1.12 Lorentz factor γ against normalized velocity β for electron. The *solid dot* on the curve indicates the Lorentz factor γ of 20.57 for a 10 MeV electron ($E_K = 10$ MeV) with normalized velocity $\beta = 0.999$

in the following expression

$$\frac{\upsilon}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{20.57^2}} = 0.999. \tag{1.118}$$

We can also get υ/c of a 10 MeV electron using the general expression for kinetic energy E_K given in (T1.58) as

$$E_K = (\gamma - 1)E_0 = \left(\frac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} - 1\right)E_0. \tag{1.119}$$

Solving (1.119) for υ/c we get the following result of the normalized electron velocity

$$\frac{\upsilon}{c} = \sqrt{1 - \frac{1}{(1 + \frac{E_K}{E_0})^2}} = \sqrt{1 - \frac{1}{(1 + \frac{10\ \text{MeV}}{0.511\ \text{MeV}})^2}} = 0.999. \tag{1.120}$$

(e) Mass m of the electron is calculated from the basic Einstein relationship as follows

$$m = \gamma m_0 = 20.57 \times 0.511\ \text{MeV}/c^2 = 10.511\ \text{MeV}/c^2. \tag{1.121}$$

1.21.Q4 **(34)**

(a) Present at least three methods for calculation of momentum p of a relativistic particle.
(b) Show that for particle velocity υ much less than speed of light c all expressions presented in **(a)** transform into classical relationship for momentum given as $p = m_0\upsilon$.
(c) Use methods presented in **(a)** to determine momentum p of a 10 MeV electron. Express momentum in units of MeV/c^2.
(d) Use methods presented in **(a)** to determine momentum p of a 10 MeV proton. Express momentum in units of MeV/c^2.
(e) Express the common unit of momentum MeV/c^2 in SI units.

SOLUTION:

(a) Several expressions are available for calculation of momentum p in relativistic physics. The methods are, of course, all related and the choice of which one to use at a given time depends on the available input data. Four related methods are presented below:

(1) The most general relationship for momentum p is the product of relativistic mass m and particle velocity υ

$$p = m\upsilon = mc\frac{\upsilon}{c} = \frac{mc^2}{c}\beta = \frac{E\beta}{c} = mc\beta = \gamma m_0\upsilon = \frac{m_0\upsilon}{\sqrt{1-\beta^2}} = \frac{E_0\beta}{c\sqrt{1-\beta^2}} \tag{1.122}$$

where m is the particle's relativistic mass, E is its total energy, m_0 is its rest mass, E_0 is its rest energy, υ is its velocity, and γ is the Lorentz factor given as $\sqrt{1-\beta^2}$ with $\beta = \upsilon/c$.

(2) Momentum can also be determined from the basic expression for relativistic total energy E of a particle

$$E^2 = p^2c^2 + E_0^2. \tag{1.123}$$

Solving (1.123) for p we get the general expression for relativistic momentum p

$$\begin{aligned} p &= \frac{1}{c}\sqrt{E^2 - E_0^2} = \frac{1}{c}\sqrt{(E_\text{K} + E_0)^2 - E_0^2} = \frac{1}{c}\sqrt{E_\text{K}^2 + 2E_\text{K}E_0} \\ &= \frac{E_\text{K}^2}{c}\sqrt{1 + \frac{2E_0}{E_\text{K}}}, \end{aligned} \tag{1.124}$$

where E_K is its kinetic energy of the particle.

(3) From the basic expression (1.122) for momentum p we get the following expression

$$p = m\upsilon = \gamma m_0 \upsilon = \gamma m_0 c^2 \frac{\upsilon}{c^2} = \frac{E\beta}{c}. \tag{1.125}$$

(4) Momentum p can also be calculated expanding (1.125) to read

$$p = m\upsilon = \gamma m_0 \upsilon = \gamma m_0 c^2 \frac{\beta}{c} = \frac{E_0 \gamma \beta}{c} = \frac{E_0}{c}\sqrt{\gamma^2 - 1}, \tag{1.126}$$

since, as can easily be shown, the product $\gamma\beta$ is given as $\gamma\beta = \beta/\sqrt{1-\beta^2} = \sqrt{\gamma^2 - 1}$.

(b) All four relativistic expressions for momentum p introduced in **(a)** transform into classical expression for momentum that reads $p = m_0\upsilon$ for $\upsilon/c = \beta \to 0$ as follows:

(1) From (1.122) given as $p = m_0\upsilon/\sqrt{1-\beta^2}$ we get the following classical limit

$$\lim_{\beta\to 0} p = \lim_{\beta\to 0} \frac{m_0\upsilon}{\sqrt{1-\beta^2}} = \lim_{\beta\to 0} m_0 c\beta \left(1-\beta^2\right)^{-1/2} \approx \lim_{\beta\to 0} m_0 c\beta \left(1 + \frac{1}{2}\beta^2\right)$$
$$\approx m_0 c\beta = m_0 \upsilon. \tag{1.127}$$

(2) From (1.124) given as $p = (E_K^2/c)\sqrt{1 + 2E_0/E_K}$ we get the following classical limit for $E_K \ll E_0$

$$\lim_{\beta\to 0} p = \lim_{\beta\to 0} \frac{E_K^2}{c}\sqrt{1 + \frac{2E_0}{E_K}} \approx \lim_{E_K \ll E_0} \frac{E_K^2}{c}\sqrt{\frac{2E_0}{E_K}} = \frac{1}{c}\sqrt{2E_K m_0 c^2}$$
$$= \sqrt{2\frac{m_0\upsilon^2}{2} m_0} = m_0\upsilon. \tag{1.128}$$

(3) From (1.125) given as $p = E\beta/c$ we get the following classical limit for $E_K \ll E_0$

$$\lim_{\beta\to 0} p = \lim_{\beta\to 0} \frac{E\beta}{c} = \lim_{E_K \ll E_0} \frac{(E_K + E_0)\beta}{c} \approx \frac{E_0\beta}{c} = \frac{m_0 c^2 \upsilon}{c^2} = m_0\upsilon. \tag{1.129}$$

(4) From (1.126) given as $p = (E_0/c)\sqrt{\gamma^2 - 1}$ we get the following classical limit as $E_K \ll E_0$

$$\lim_{\beta\to 0} p = \lim_{\beta\to 0} \frac{E_0}{c}\sqrt{\gamma^2 - 1} = \lim_{\beta\to 0} \frac{E_0}{c}\beta\left(1-\beta^2\right)^{-\frac{1}{2}} \approx \lim_{\beta\to 0} m_0 c\beta\left(1 + \frac{1}{2}\beta^2\right)$$
$$\approx m_0 c\beta = m_0\upsilon. \tag{1.130}$$

(c) By way of example, the expressions for relativistic momentum p that were presented in **(a)** will be used to calculate momentum p of a 10 MeV electron. By definition a label "10 MeV electron" designates an electron with kinetic energy of 10 MeV. Characteristics of a 10 MeV electron are: $E_K = 10$ MeV; $E_0 = 0.511$ MeV; and $E = E_K + E_0 = 10\text{ MeV} + 0.511\text{ MeV} = 10.511$ MeV.

Lorentz factor γ of a 10 MeV electron is determined as

$$\gamma = \frac{E}{E_0} = \frac{E_0 + E_K}{E_0} = 1 + \frac{E_K}{E_0} = 1 + \frac{10}{0.511} = 20.569. \tag{1.131}$$

Normalized velocity $\beta = \upsilon/c$ of a 10 MeV electron is calculated either from kinetic energy E_K as

$$E_K = E - E_0 = (\gamma - 1)E_0 = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)E_0 \tag{1.132}$$

leading to

$$\beta = \sqrt{1 - \frac{1}{(1 + \frac{E_K}{E_0})^2}} = \sqrt{1 - \frac{1}{(1 + \frac{10}{0.511})^2}} = \sqrt{1 - \frac{1}{20.569^2}} = 0.9988 \tag{1.133}$$

or from the Lorentz factor γ of 20.569 given in (1.131)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \rightarrow \quad \beta = \frac{\sqrt{\gamma^2 - 1}}{\gamma} = \frac{\sqrt{20.569^2 - 1}}{20.569} = 0.9988. \tag{1.134}$$

Based on the information above we now determine the momentum of a 10 MeV electron:

(1)

$$p = \frac{E_0 \beta}{c\sqrt{1-\beta^2}} = \frac{(0.511\text{ MeV}) \times 0.9988}{c\sqrt{1 - 0.9988^2}} = 10.42\text{ MeV}/c^2. \tag{1.135}$$

(2)

$$p = \frac{E_K}{c}\sqrt{1 + \frac{2E_0}{E_K}} = \frac{(10\text{ MeV})}{c}\sqrt{1 + \frac{2 \times 0.511}{10}} = 10.5\text{ MeV}/c. \tag{1.136}$$

(3)

$$p = \frac{E\beta}{c} = \frac{E_K + E_0}{c}\beta = \frac{10\text{ MeV} + 0.511\text{ MeV}}{c} \times 0.999 = 10.50\text{ MeV}/c. \tag{1.137}$$

(4)

$$p = \frac{E_0}{c}\sqrt{\gamma^2 - 1} = \frac{0.511\text{ MeV}}{c}\sqrt{20.569^2 - 1} = 10.5\text{ MeV}/c. \tag{1.138}$$

As shown by (1.135) through (1.138), all methods of momentum calculation for a 10 MeV electron give essentially identical results with some minor discrepancies arising from rounding errors.

(d) Momentum of a 10 MeV proton. By definition a label “10 MeV proton” designates a proton with kinetic energy of 10 MeV. Characteristics of a 10 MeV proton are as follows: $E_K = 10$ MeV; $E_0 = 938.3$ MeV; and $E = E_K + E_0 = 10\text{ MeV} + 938.3\text{ MeV} = 948.3$ MeV.

Lorentz factor γ of a 10 MeV proton is determined as follows

$$\gamma = \frac{E}{E_0} = \frac{E_0 + E_K}{E_0} = 1 + \frac{E_K}{E_0} = 1 + \frac{10}{938.3} = 1.0107, \tag{1.139}$$

making the proton almost classical but not quite.

Normalized velocity $\beta = \upsilon/c$ of a 10 MeV proton is calculated (i) either from kinetic energy E_K as

$$E_K = E - E_0 = (\gamma - 1)E_0 = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)E_0 \tag{1.140}$$

leading to

$$\beta = \sqrt{1 - \frac{1}{(1 + \frac{E_K}{E_0})^2}} = \sqrt{1 - \frac{1}{(1 + \frac{10}{938.3})^2}} = \sqrt{1 - \frac{1}{1.0107^2}} = 0.145 \tag{1.141}$$

or (ii) from the Lorentz factor γ of 1.10107 given in (1.139)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \rightarrow \quad \beta = \frac{\sqrt{\gamma^2 - 1}}{\gamma} = \frac{\sqrt{1.0107^2 - 1}}{1.0107} = 0.145. \tag{1.142}$$

Based on the information above we now determine the momentum of a 10 MeV proton:

(1)

$$p = \frac{E_0\beta}{c\sqrt{1-\beta^2}} = \frac{(938.3\text{ MeV})\times 0.145}{c\sqrt{1-0.145^2}} = 137.5\text{ MeV}/c^2. \tag{1.143}$$

(2)

$$p = \frac{E_K}{c}\sqrt{1 + \frac{2E_0}{E_K}} = \frac{(10\text{ MeV})}{c}\sqrt{1 + \frac{2\times 938.3}{10}} = 137.4\text{ MeV}/c. \tag{1.144}$$

(3)

$$p = \frac{E\beta}{c} = \frac{E_K + E_0}{c}\beta = \frac{10\text{ MeV} + 938.3\text{ MeV}}{c}\times 0.145 = 137.5\text{ MeV}/c. \tag{1.145}$$

(4)

$$p = \frac{E_0}{c}\sqrt{\gamma^2 - 1} = \frac{938.3\ \text{MeV}}{c}\sqrt{1.0107^2 - 1} = 137.6\ \text{MeV}/c. \qquad (1.146)$$

As shown by (1.142) through (1.146), all methods of momentum calculation for a 10 MeV proton give essentially identical results with some minor discrepancies arising from rounding errors.

(e) The units of momentum in (1.135) through (1.138) for 10 MeV electron and from (1.143) through (1.146) for 10 MeV proton are given in MeV/c. This is a common and convenient unit of momentum used in nuclear, medical, and relativistic physics, especially since momentum, despite being a very important physical quantity, does not have an assigned special unit in contrast to other physical quantities, such as force with newton N, energy with joule J, and power with watt W, etc., that do.

The SI derived unit of momentum is $\text{kg} \cdot \text{m} \cdot \text{s}^{-1}$ or $\text{N} \cdot \text{s}$ and the relationship between 1 MeV/c and $\text{kg} \cdot \text{m} \cdot \text{s}^{-1}$ is given as follows

$$\begin{aligned} \frac{1\ \text{MeV}}{c} &= \frac{(10^6\ \text{eV}) \times (1.602 \times 10^{-19}\ \text{J/eV})}{3 \times 10^8\ \text{m} \cdot \text{s}^{-1}} = 5.34 \times 10^{-22}\ \text{N} \cdot \text{s} \\ &= 5.34 \times 10^{-22}\ \text{kg} \cdot \text{m} \cdot \text{s}^{-1} \end{aligned} \qquad (1.147)$$

or for 1 eV/c

$$\begin{aligned} \frac{1\ \text{eV}}{c} &= \frac{(1\ \text{eV}) \times (1.602 \times 10^{-19}\ \text{J/eV})}{3 \times 10^8\ \text{m} \cdot \text{s}^{-1}} = 5.34 \times 10^{-16}\ \text{N} \cdot \text{s} \\ &= 5.34 \times 10^{-16}\ \text{kg} \cdot \text{m} \cdot \text{s}^{-1}. \end{aligned} \qquad (1.148)$$

Momentum of a 10 MeV electron, expressed as 10.5 MeV/c in **(c)**, is in the SI system of units given as

$$p = 10.5\ \text{MeV}/c = 10.5 \times \left(5.34 \times 10^{-22}\ \text{kg} \cdot \text{m} \cdot \text{s}^{-1}\right) = 5.61 \times 10^{-21}\ \text{kg} \cdot \text{m} \cdot \text{s}^{-1}. \qquad (1.149)$$

Momentum of a 10 MeV proton, expressed as 137.5 MeV/c in **(c)**, is in the SI system of units given as

$$p = 137.5\ \text{MeV}/c = 137.5 \times \left(5.34 \times 10^{-22}\ \text{kg} \cdot \text{m} \cdot \text{s}^{-1}\right) = 7.343 \times 10^{-20}\ \text{kg} \cdot \text{m} \cdot \text{s}^{-1}. \qquad (1.150)$$

1.21.Q5 **(35)**

The standard Newton second law $\mathbf{F} = m\mathbf{a}$ stating that force $\mathbf{F}$ is proportional to acceleration $\mathbf{a}$ with mass m the proportionality constant does not hold in relativistic mechanics where mass m of an object moving with velocity υ is not constant and depends on velocity υ.

(a) Express the Newton second law in relativistic form accounting for the variation in mass m with velocity $\boldsymbol{\upsilon}$.

(b) Show that for $\boldsymbol{\upsilon} \to 0$ the relativistic form of Newton second law derived in **(a)** transforms into the standard classical Newton second law.

(c) A particle with charge q moves along a straight path in a uniform electric field $\boldsymbol{\mathcal{E}}$ with velocity υ. The two vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\upsilon}$ are parallel. Calculate the relativistic acceleration $\mathbf{a}$ of the charged particle and show that for relativistic particles the acceleration depends on velocity $\boldsymbol{\upsilon}$.

(d) If particle in **(c)** starts its motion at rest at position $x = 0$ and time $t = 0$, calculate its speed $\upsilon(t)$ and its position $x(t)$ at time t. Also, calculate the classical limits of $\upsilon(t)$ and $x(t)$.

(e) Lorentz force $\mathbf{F}_\mathrm{L}$ on a charged particle q of rest mass m_0 moving with velocity $\boldsymbol{\upsilon}$ in magnetic field $\boldsymbol{\mathcal{B}}$ is given as $\mathbf{F}_\mathrm{L} = q\boldsymbol{\upsilon}\times\boldsymbol{\mathcal{B}}$. Use Newton second law in relativistic form derived in **(a)** to show that the angular "cyclotron frequency" ω_cyc of the relativistic particle is given as $\omega_\mathrm{cyc} = q\mathcal{B}/(\gamma m_0)$.

SOLUTION:

(a) Relativistic force $\mathbf{F}$ with mass m a function of particle velocity $|\boldsymbol{\upsilon}| = \upsilon$ is in general given as follows

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}m\boldsymbol{\upsilon}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} + \boldsymbol{\upsilon}\frac{\mathrm{d}m}{\mathrm{d}t} = \gamma m_0\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} + m_0\boldsymbol{\upsilon}\frac{\mathrm{d}\gamma}{\mathrm{d}t} \tag{1.151}$$

with $\mathbf{p}$ the particle momentum, m_0 the particle rest mass, and γ the Lorentz factor of the particle: $\gamma = (1 - \upsilon^2/c^2)^{-1/2}$. Force $\mathbf{F}$ as expressed in (1.151) depends on $\mathrm{d}\gamma/\mathrm{d}t$ that can be expanded to read as follows

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}\gamma}{\mathrm{d}\boldsymbol{\upsilon}}\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\upsilon}}\frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}\cdot\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \left[1-\frac{\upsilon^2}{c^2}\right]^{-\frac{3}{2}}\frac{\boldsymbol{\upsilon}}{c^2}\cdot\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \gamma^3\frac{\boldsymbol{\upsilon}}{c^2}\cdot\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t}. \tag{1.152}$$

After inserting (1.152) into (1.151) we get the following expression for $\mathbf{F}$

$$\mathbf{F} = \gamma m_0\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} + m_0\boldsymbol{\upsilon}\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma m_0\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} + \gamma^3 m_0\left[\frac{\boldsymbol{\upsilon}}{c}\left(\frac{\boldsymbol{\upsilon}}{c}\cdot\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t}\right)\right]. \tag{1.153}$$

A scalar product of $\mathbf{F}$ and $\boldsymbol{\upsilon}$ will be of help in expanding (1.153) further

$$\mathbf{F}\cdot\boldsymbol{\upsilon} = \gamma m_0 \boldsymbol{\upsilon}\cdot\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} + \gamma^3\beta^2 m_0 \boldsymbol{\upsilon}\cdot\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \gamma^3 m_0 \boldsymbol{\upsilon}\cdot\frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t}, \tag{1.154}$$

recalling that $\frac{\boldsymbol{\upsilon}}{c}\cdot\frac{\boldsymbol{\upsilon}}{c} = \beta^2$ and $\beta^2\gamma^2 = \gamma^2 - 1$.

Force (1.153) incorporating (1.154) is now expressed as follows

$$\mathbf{F} = \gamma m_0 \frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} + (\mathbf{F}\cdot\boldsymbol{\beta})\boldsymbol{\beta}, \tag{1.155}$$

and leads directly to a relativistic expression relating force $\mathbf{F}$ and acceleration $\mathbf{a} = \mathrm{d}\boldsymbol{\upsilon}/\mathrm{d}t$, allowing us to express acceleration $\mathbf{a}$ as

$$\mathbf{a} = \frac{\mathrm{d}\boldsymbol{\upsilon}}{\mathrm{d}t} = \frac{1}{\gamma m_0}\big[\mathbf{F} - (\mathbf{F}\cdot\boldsymbol{\beta})\boldsymbol{\beta}\big]. \tag{1.156}$$

(b) As shown in (1.156), acceleration $\mathbf{a}$ is not parallel to force $\mathbf{F}$ at large velocities because a particle velocity cannot exceed the speed of light c in vacuum. However, for velocities $\upsilon \ll c$, where $\beta \to 0$ and $\gamma \to 1$, the relativistic expression for acceleration $\mathbf{a}$ given in (1.156) transforms into Newton's classical result $\mathbf{a} = \mathrm{d}\boldsymbol{\upsilon}/\mathrm{d}t = \mathbf{F}/m_0$, with $\mathbf{a}$ parallel to $\mathbf{F}$.

(c) Since force $\mathbf{F}$ is given as $\mathbf{F} = q\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}$ is parallel to $\boldsymbol{\upsilon}$, we can express (1.156) in scalar form to get

$$a = \frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \frac{F - F\beta^2}{\gamma m_0} = \frac{F}{\gamma m_0}\big(1-\beta^2\big) = \frac{q\mathcal{E}}{m_0}\big(1-\beta^2\big)^{\frac{3}{2}} = \frac{q\mathcal{E}}{m_0}\frac{1}{\gamma^3}. \tag{1.157}$$

The relativistic expression for acceleration a shows that a depends on velocity υ through the Lorentz factor γ. The following conditions apply in the limits: (1) as $\upsilon \to c$ and (2) as $\upsilon \to 0$.

(1) As υ increases into the highly relativistic region ($\upsilon \approx c$), acceleration a decreases reaching 0 as the velocity approaches c, irrespective of the magnitude of force F.
(2) For $\upsilon \ll c$ that results in $\beta \to 0$ and $\gamma \to 1$ the relativistic expression (1.157) transforms into the classical relationship $a = q\mathcal{E}/m_0$ that is independent of velocity of the particle υ.

(d) The speed $\upsilon(t)$ of a particle with charge q and rest mass m_0 moving in a uniform electric field parallel to the particle's velocity can be obtained using (1.157). We rearrange (1.157) to read

$$\gamma^3\,\mathrm{d}\upsilon = \frac{\mathrm{d}\upsilon}{(1-\frac{\upsilon^2}{c^2})^{\frac{3}{2}}} = \frac{q\mathcal{E}}{m_0}\,\mathrm{d}t. \tag{1.158}$$

For a particle at rest at time $t=0$, the speed of the particle υ at time t is obtained by integration of (1.158) in velocity υ from 0 to $\upsilon(t)$ and in time t from 0 to t

$$\int_0^{\upsilon(t)} \frac{\mathrm{d}\upsilon}{(1-\frac{\upsilon^2}{c^2})^{3/2}} = \frac{q\mathcal{E}}{m_0}\int_0^t \mathrm{d}t = \frac{q\mathcal{E}}{m_0}t = a_{\mathrm{E}}t, \tag{1.159}$$

where $a_{\mathrm{E}} = q\mathcal{E}/m_0$ is the classical acceleration limit under the influence of electric field.

We solve the first integral of (1.159) with the help of the substitution $\upsilon = c\sin u$ and $\mathrm{d}\upsilon = c\cos u\,\mathrm{d}u$ to get

$$\begin{aligned}\int_0^{\upsilon(t)} \frac{\mathrm{d}\upsilon}{(1-\frac{\upsilon^2}{c^2})^{3/2}} &= c\int \frac{\cos u\,\mathrm{d}u}{(1-\sin^2 u)^{\frac{3}{2}}} = c\int \frac{\mathrm{d}u}{\cos^2 u} = c\tan u\\ &= c\frac{\sin u}{\sqrt{1-\sin^2 u}} = \left[\frac{\upsilon}{\sqrt{1-\frac{\upsilon^2}{c^2}}}\right]_0^{\upsilon(t)}.\end{aligned} \tag{1.160}$$

Merging (1.159) and (1.160) we get a simple expression relating velocity υ and acceleration a_{E} as follows

$$\frac{\upsilon}{\sqrt{1-\frac{\upsilon^2}{c^2}}} = \gamma\upsilon = a_{\mathrm{E}}t. \tag{1.161}$$

Finally, solving (1.161) for $\upsilon(t)$ yields the relativistic relationship between velocity $\upsilon(t)$ and acceleration a_{E}

$$\upsilon(t) = \frac{a_{\mathrm{E}}t}{\sqrt{1+(c^{-1}a_{\mathrm{E}}t)^2}}. \tag{1.162}$$

The classical limit of velocity $\upsilon(t)$ from the general expression given in (1.162) is obtained for small time t such that $a_{\mathrm{E}}t \ll c$ or $a_{\mathrm{E}}t \ll 1$. For such conditions, velocity υ is expressed as

$$\upsilon(t) \approx a_{\mathrm{E}}t, \tag{1.163}$$

the well-known non-relativistic (classical) velocity of a particle undergoing constant acceleration.

If the initial position of the particle is $x(t=0)=0$, the position $x(t)$ of the particle at time t is

$$\begin{aligned}x(t) &= \int_0^t \upsilon(t)\,\mathrm{d}t = \int_0^t \frac{a_{\mathrm{E}}t}{\sqrt{1+(a_{\mathrm{E}}t/c)^2}}\,\mathrm{d}t\\ &= \frac{c^2}{2a_{\mathrm{E}}}\int_0^t \left[1+(a_{\mathrm{E}}t/c)^2\right]^{-\frac{1}{2}}\mathrm{d}\left[1+(a_{\mathrm{E}}t/c)^2\right]\\ &= \frac{c^2}{a_{\mathrm{E}}}\left[\sqrt{1+(a_{\mathrm{E}}t/c)^2}\right]_0^t = \frac{[\sqrt{1+(a_{\mathrm{E}}t/c)^2}-1]c^2}{a_{\mathrm{E}}}.\end{aligned} \tag{1.164}$$

To calculate the classical limit of (1.164), we express (1.164) using Taylor's expansion to get

$$x(t) = \frac{c^2}{a_E}\left[\sqrt{1+\frac{a_E^2 t^2}{c^2}} - 1\right] \approx \frac{c^2}{a_E}\left[1 + \frac{1}{2}\frac{a_E^2 t^2}{c^2} + \cdots - 1\right]. \tag{1.165}$$

For $a_E t \ll c$ or $a_E t/c \ll 1$, (1.165) becomes

$$x(t) \approx \frac{1}{2} a_E t^2 \tag{1.166}$$

that is the non-relativistic position of a particle undergoing a constant acceleration.

(e) Equation (1.156) gives the general expression for relativistic acceleration **a**. In our case of charged particle q motion in magnetic field $\boldsymbol{\mathcal{B}}$ under the influence of the Lorentz force $\mathbf{F}_L$, the particle velocity $\boldsymbol{\upsilon}$ is perpendicular to magnetic field $\boldsymbol{\mathcal{B}}$ as well as to Lorentz force $\mathbf{F}_L$, so we write the Lorentz force $F_L = q\upsilon\mathcal{B}$ and acceleration from (1.156) as $a = F/(\gamma m_0)$ using $|\boldsymbol{\upsilon}\times\boldsymbol{\mathcal{B}}| = \upsilon\mathcal{B}$ and $\mathbf{F}\cdot\boldsymbol{\beta} = 0$. Since $a = \upsilon^2/r = \omega_{cyc}^2 r$ we get the following expression for the two forces that are in equilibrium

$$F = \gamma m_0 a = \gamma m_0 \omega_{cyc}^2 r = q\upsilon\mathcal{B} = q\omega_{cyc} r\mathcal{B}. \tag{1.167}$$

From (1.167) the cyclotron angular frequency is expressed as follows

$$\omega_{cyc} = \frac{q\mathcal{B}}{\gamma m_0} = \frac{q\mathcal{B}}{m_0}\sqrt{1-\beta^2} = \frac{q\mathcal{B}}{m_0}\sqrt{1-\frac{\upsilon^2}{c^2}}. \tag{1.168}$$

For $\upsilon \ll c$ that results in $\beta \to 0$ and $\gamma \to 1$ the relativistic expression (1.168) for the angular cyclotron frequency transforms into the classical cyclotron angular frequency given as

$$\omega_{cyc} = \frac{q\mathcal{B}}{m_0}. \tag{1.169}$$

1.21.Q6 **(36)**

An electron has velocity υ_e of $0.95c$. Calculate velocity of a proton that has:

(a) Same kinetic energy $(E_K)_p$ as the electron; i.e., $(E_K)_p = (E_K)_e$.
(b) Same momentum p_p as the electron; i.e., $p_p = p_e$.

SOLUTION:

We first calculate the kinetic energy $(E_K)_e$ and momentum p_e of an electron with velocity $\upsilon_e = 0.95c$ corresponding to $\beta_e = 0.95$ and $\gamma_e = [1-\beta_e^2]^{-1/2} = 3.2$. Electron kinetic energy $(E_K)_e$ is calculated using (1.115) as follows

$$(E_K)_e = (\gamma_e - 1)(E_0)_e = \left[\frac{1}{\sqrt{1-\beta_e^2}} - 1\right](E_0)_e$$
$$= (3.2-1)\times 0.511 \text{ MeV} = 1.124 \text{ MeV}. \tag{1.170}$$

Electron momentum p_e is calculated using (T1.67) as follows

$$p_e = \frac{(E_0)_e}{c}\sqrt{\gamma_e^2 - 1} = 0.511 \text{ (MeV}/c)\times\sqrt{3.2^2-1} = 1.55 \text{ MeV}/c. \tag{1.171}$$

Now we can address the two questions dealing with protons, first question **(a)** with protons of kinetic energy $(E_K)_p = 1.124$ MeV and then question **(b)** with protons of momentum $p_p = 1.55$ MeV$/c$.

(a) Kinetic energy of the proton $(E_K)_p$ equals kinetic energy of the electron $(E_K)_e$; i.e., $(E_K)_p = (E_K)_e = 1.124$ MeV.

We again use (1.115) and express $(E_K)_p$ as follows

$$(E_K)_p = (\gamma_p - 1)(E_0)_p = 1.124 \text{ MeV}, \tag{1.172}$$

then solve this relationship for γ_p to get

$$\gamma_p = 1 + \frac{(E_K)_p}{(E_0)_p} = 1 + \frac{1.124 \text{ MeV}}{938.3 \text{ MeV}} = 1.0012. \tag{1.173}$$

Next we calculate β_p using the standard definition of Lorentz factor $\gamma_p = [1-\beta_p^2]^{-1/2}$ to get

$$\beta_p = \sqrt{1 - \frac{1}{\gamma_p^2}} = \sqrt{1 - \frac{1}{1.0012^2}} = 0.049 \tag{1.174}$$

or

$$\upsilon_P = \beta_P c = 0.049\times\left(3\times10^8 \text{ m/s}\right) = 1.47\times10^7 \text{ m/s}. \tag{1.175}$$

Thus, a proton with kinetic energy $(E_K)_p$ of 1.124 MeV has a velocity υ_p equal to about 5 % of the speed of light in vacuum c, while an electron with same kinetic energy $(E_K)_e = 1.124$ MeV travels at 95 % of c. A proton with kinetic energy of the order of 1 MeV can be treated as a classical particle, an electron of same kinetic energy is highly relativistic as result of the significant difference in the rest masses of the two particles with $m_p/m_e = 1836$.

(b) Proton momentum p_p equals electron momentum p_e, i.e., $p_p = p_e = 1.55$ MeV$/c$. We now again use (1.171), write the proton momentum p_p as follows

$$p_p = \frac{(E_0)_p}{c}\sqrt{\gamma_p^2 - 1} = 1.55 \text{ MeV}/c, \tag{1.176}$$

and then solve this expression for γ_{p}^2 to get

$$\gamma_{\mathrm{p}}^2 = 1 + \frac{p_{\mathrm{p}}^2}{(E_0)_{\mathrm{p}}^2/c^2} = \frac{1}{1-\beta_{\mathrm{p}}^2} = 1.00000273. \tag{1.177}$$

Next, we solve (1.177) for β_{p} and get

$$\beta_{\mathrm{p}} = \sqrt{1 - \frac{1}{\gamma_{\mathrm{p}}^2}} = \sqrt{1 - \frac{1}{1.00000273^2}} = 2.34 \times 10^{-3} \tag{1.178}$$

or

$$\upsilon_{\mathrm{P}} = \beta_{\mathrm{P}} c = 7.02 \times 10^5 \text{ m/s}. \tag{1.179}$$

Thus, a proton with momentum p_{p} of 1.55 MeV$/c$ has a velocity of about 0.23 % of the speed of light in vacuum c, while an electron of this momentum has a velocity of 95 % of c. Proton of momentum $p_{\mathrm{p}} = 1.55$ MeV$/c$ is a classical particle, while an electron of same momentum is clearly a relativistic particle.

1.21.Q7 **(37)**

Determine the energy required to accelerate an electron:

(a) From a velocity of $0.25c$ to velocity of $0.75c$.
(b) From a velocity of $0.95c$ to velocity of $0.99c$.

SOLUTION:

We determine the rise in kinetic energy E_{K} of the electron using the following standard relationship for kinetic energy

$$\begin{aligned}
\Delta E_{\mathrm{K}} &= c^2 \int_{m(\upsilon_1)}^{m(\upsilon_2)} \mathrm{d}m = m(\upsilon_2) - m(\upsilon_1) \\
&= m_{\mathrm{e}} c^2 \left[\frac{1}{\sqrt{1 - \frac{\upsilon_2^2}{c^2}}} - \frac{1}{\sqrt{1 - \frac{\upsilon_1^2}{c^2}}} \right] = E_0 \left[\frac{1}{\sqrt{1-\beta_2^2}} - \frac{1}{\sqrt{1-\beta_1^2}} \right] \\
&= E_0(\gamma_2 - \gamma_1),
\end{aligned} \tag{1.180}$$

where γ is the Lorentz factor of an electron with velocity υ: $\gamma = (1-\beta^2)^{-\frac{1}{2}} = (1 - \upsilon^2/c^2)^{-\frac{1}{2}}$.

(a) We now calculate ΔE_{K} for a velocity increase from $\upsilon_1 = 0.25c$ to $\upsilon_2 = 0.75c$ corresponding to an increase in γ from $\gamma_1 = 1.033$ to $\gamma_2 = 1.512$ as follows

$$\Delta E_{\mathrm{K}} = (\gamma_2 - \gamma_1) E_0 = (1.512 - 1.033) \times 0.511 \text{ MeV} = 0.245 \text{ MeV}. \tag{1.181}$$

It takes energy of 0.245 MeV to accelerate an electron from velocity of $0.25c$ to velocity of $0.75c$.

(b) Energy ΔE_K for velocity increase from $\upsilon_1 = 0.95c$ to $\upsilon_1 = 0.99c$ corresponding to an increase in γ from $\gamma_1 = 3.203$ to $\gamma_2 = 31.623$ is calculated as follows

$$\Delta E_K = (\gamma_2 - \gamma_1) E_0 = (31.623 - 3.203) \times 0.511 \text{ MeV} = 14.52 \text{ MeV}. \quad (1.182)$$

It takes energy of 14.52 MeV to accelerate an electron from velocity of $0.95c$ to velocity of $0.99c$.

A conclusion can be made from **(a)** and **(b)** that in the vicinity of the speed of light c much more energy is required to increase the velocity of a particle than at relatively low particle velocities.

1.21.Q8 **(38)**

The relativistic red and blue Doppler shifts play an important role in astrophysics, as they allow the determination of the motion of distant galaxies relative to Earth. A blue shift suggests that the source is moving toward the observer, a red shift that the source is moving away from the observer.

(a) Derive the basic expression for the relativistic Doppler shift using Lorentz transformations on energy and momentum.
(b) Plot the ratio $\lambda_{\text{observed}}/\lambda_{\text{source}}$ for the red and blue shift against β, the normalized velocity υ of the photon source.

SOLUTION:

The simplest and fastest method for deriving the relativistic Doppler equation is through the use of Lorentz transformations relating total energy E and momentum p of a particle in two inertial reference frames F and F'. Assume that frame F' moves with uniform velocity υ along the abscissa (x) axis of frame F and that a particle of interest moves in frame F' also along the abscissa axis.

The general Lorentz transformations for E and p of the particle are given as follows

$$E' = \gamma(E - \beta c p) \quad \text{and} \quad p' = \gamma\left(p - \frac{\upsilon}{c^2}E\right), \quad (1.183)$$

where γ is the Lorentz factor $\gamma = (1 - \beta^2)^{-1/2}$ and β is the relative velocity υ normalized to speed of light c in vacuum. Note that energy E transform takes the form of the x coordinate Lorentz transformation and momentum p transform takes the form of the transformation for time t coordinate, as given in Prob. 26.

For a particle with zero rest mass, such as photon, energy $E = h\nu$ is directly proportional to momentum $p = E/c = h\nu/c = h/\lambda$ and the Lorentz transformation of (1.183) for a photon source moving away from the observer simplifies to read

$$E' = h\nu' = \gamma(E - \beta E) = \gamma E(1-\beta) = \gamma h\nu(1-\beta). \tag{1.184}$$

We can obtain the same result using the momentum transformation

$$\frac{h\nu'}{c} = \gamma\left(\frac{h\nu}{c} - \frac{\upsilon}{c^2}h\nu\right) = \gamma\frac{h\nu}{c}(1-\beta). \tag{1.185}$$

The ratio of photon energies $h\nu'/h\nu$ coincides with the ratio of photon frequencies ν'/ν found in the relativistic Doppler shift usually expressed as

$$\frac{\nu'}{\nu} = \frac{\nu_{\text{observed}}}{\nu_{\text{source}}} = \gamma(1-\beta) = \frac{\sqrt{1-\beta}\sqrt{1-\beta}}{\sqrt{1-\beta^2}} = \sqrt{\frac{1-\beta}{1+\beta}}. \tag{1.186}$$

This result for $\nu_{\text{observed}}/\nu_{\text{source}}$ translates into the following expression for the ratio of wavelengths $\lambda_{\text{observed}}/\lambda_{\text{source}}$

$$\frac{\lambda'}{\lambda} = \frac{\lambda_{\text{observed}}}{\lambda_{\text{source}}} = \sqrt{\frac{1+\beta}{1-\beta}} > 1. \tag{1.187}$$

For the case of photon source moving away from the observer, $\lambda_{\text{observed}}/\lambda_{\text{source}} > 1$ and the increase in wavelength of the observed photon emission is referred to as the *red shift*.

If the photon wavelength is measured in a frame that moves toward the photon source, the velocity υ is negative and the measured energy E' is expressed as

$$E' = h\nu' = \gamma(E + \beta E) = \gamma E(1+\beta) = \gamma h\nu(1+\beta). \tag{1.188}$$

The ratio of photon energies E'/E corresponds to the ratio of photon frequencies

$$\frac{E'}{E} = \frac{\nu'}{\nu} = \frac{\nu_{\text{observed}}}{\nu_{\text{source}}} = \frac{\sqrt{1+\beta}\sqrt{1+\beta}}{\sqrt{1-\beta^2}} = \sqrt{\frac{1+\beta}{1-\beta}} \tag{1.189}$$

and translates into the following wavelength ratio

$$\frac{\lambda'}{\lambda} = \frac{\lambda_{\text{observed}}}{\lambda_{\text{source}}} = \sqrt{\frac{1-\beta}{1+\beta}} < 1. \tag{1.190}$$

Since $\lambda_{\text{observed}}/\lambda_{\text{source}} < 1$, the decrease in wavelength λ with the source approaching the observer is referred to as the *blue shift*.

The speed of light c is the same in all reference frames; however, photon energy $h\nu$, frequency ν, and wavelength λ all depend on the relative velocity υ between the source and observer. The following conclusions apply:

(1) With the source moving away from the observer, the frequency ratio and photon energy ratio decrease, the ratio of wavelengths increases, the velocity υ in the Lorentz transformation is positive, and the effect is referred to as the red Doppler shift.

(2) With the source moving toward the observer, the frequency ratio and photon energy ratio increase, the ratio of wavelengths decreases, the velocity υ in the Lorentz transformation is negative, and the effect is referred to as the blue Doppler shift.

(b) A plot of $\lambda_{\text{observed}}/\lambda_{\text{source}}$ is shown in Fig. 1.13.

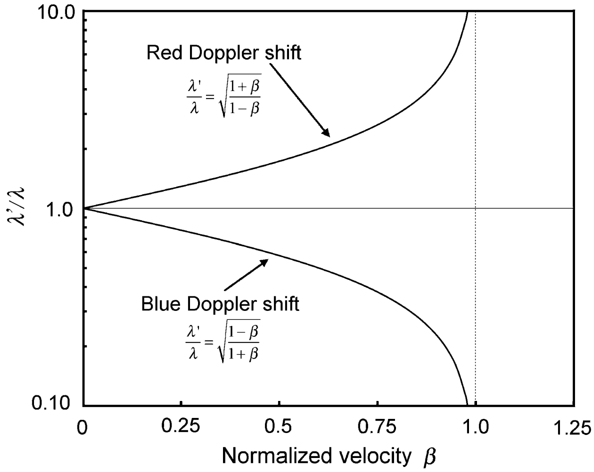


Fig. 1.13 Red and blue Doppler shifts ($\lambda/\lambda' = \lambda_{\text{observed}}/\lambda_{\text{source}}$) against normalized velocity $\beta = \upsilon/c$. For the red shift the source is approaching the observed and $\lambda/\lambda' > 1$, while for the blue shift the source is moving away from the observer and $\lambda/\lambda' < 1$

1.22 Particle-Wave Duality

1.22.Q1 **(39)**

Calculate the de Broglie wavelength λ of the following:

(a) Electron with velocity of $0.01c$.
(b) Electron with kinetic energy of 100 eV.
(c) Electron with kinetic energy of 200 MeV.
(d) Proton with kinetic energy of 10 MeV.
(e) Marble of mass 100 g moving with velocity of 50 m/s.

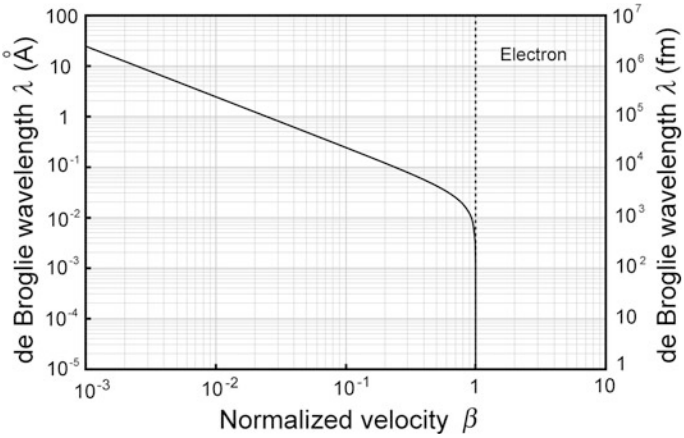


Fig. 1.14 De Broglie wavelength λ against normalized velocity β for electrons in the range of β from 10^{-3} to 1

Figure 1.14 plots the de Broglie wavelength λ against normalized velocity β (T1.79) of electrons in the range $10^{-3} \le \beta \le 10$, while Fig. 1.15 plots the de Broglie wavelength λ against kinetic energy E_K for both electrons as well as protons in the kinetic energy range 10^{-5} MeV $\le E_K \le 10^3$ MeV.

(f) Verify your data calculated in **(a)**, **(b)**, and **(c)** for electrons by superimposing them as data points on Fig. 1.14 and Fig. 1.15.

(g) Verify the de Broglie wavelength calculated in **(d)** for protons by superimposing the point on the diagram of Fig. 1.15.

(h) Energetic electrons and protons can serve as excellent probes in studies of atomic and nuclear structure. Can the particles of **(a)** through **(d)** be of any use in atomic and nuclear physics?

SOLUTION:

De Broglie wavelength λ of an object or particle is defined as $\lambda = h/p$, where h is the Planck constant and p is the momentum of the object or particle. As shown in Prob. 34, momentum is in general expressed as

$$p = m\upsilon = \gamma m_0 \beta c \equiv \frac{m_0 c^2}{c} \frac{\beta}{\sqrt{1-\beta^2}} \equiv \frac{E_0}{c} \frac{\beta}{\sqrt{1-\beta^2}} = \frac{E_K}{c} \sqrt{1 + \frac{2E_0}{E_K}}, \quad (1.191)$$

where γ is the Lorentz factor, m_0 is the rest mass of the object or particle, and E_0 is the rest energy of the object or particle. It is easy to show that for $\upsilon \ll c$

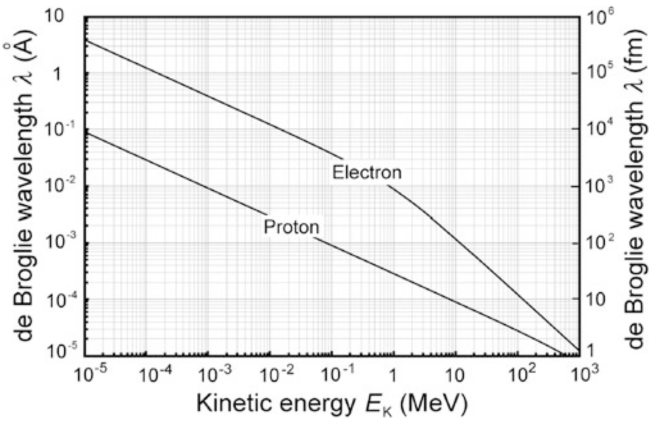


Fig. 1.15 De Broglie wavelength λ against kinetic energy E_K for electrons and protons in kinetic energy range from 10 eV to 1000 MeV

where $E_K \ll E_0$ the general relationship for the momentum reduces to the classical expression $p = m_0 \upsilon = \sqrt{2m_0 E_K}$, while in the extreme relativistic region with $\upsilon \approx c$ where $E_K \gg E_0$ the general relationship for the momentum p simplifies to read $p \approx E_K/c$.

(a) **Electron** travelling with velocity $\upsilon = 0.01c$ equivalent to $\beta = 0.01$.

According to (1.115) an electron with β of 0.01 possesses kinetic energy E_K of 25.6 eV. Since $\beta \ll 1$, the classical expression $E_K = \frac{1}{2} m_0 \upsilon^2 = \frac{1}{2} m_0 c^2 \beta^2$ gives the same result (25.6 eV). The de Broglie wavelength λ is calculated as follows

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m_0 \upsilon} = \frac{2\pi \hbar c \sqrt{1-\beta^2}}{\beta m_0 c^2} = \frac{2\pi (197.3\ \text{MeV} \cdot \text{fm}) \times \sqrt{1-0.01^2}}{0.01 \times 0.511\ \text{MeV}} = 2.42\ \text{Å}. \tag{1.192}$$

(b) **Electron** with kinetic energy $E_K = 100$ eV, i.e., 100 eV electron.

According to classical and relativistic expressions an electron with kinetic energy E_K of 100 eV travels with velocity of $0.02c$. Since $\beta \ll 1$, the classical expression $\upsilon = \sqrt{2E_K/m_e}$ gives the same result as the relativistic one. The de Broglie wavelength λ is calculated as follows

$$\lambda = \frac{h}{p} = \frac{2\pi \hbar c}{E_K \sqrt{1 + \frac{2E_0}{E_K}}} \approx \frac{2\pi \hbar c}{\sqrt{2 m_e c^2 E_K}} = \frac{2\pi \times (197.3 \times 10^6\ \text{eV} \cdot \text{fm})}{\sqrt{2 \times (0.511 \times 10^6\ \text{eV}) \times (10^2\ \text{eV})}}$$

$$= 1.2 \times 10^5\ \text{fm} = 1.2\ \text{Å}, \tag{1.193}$$

where we used (1.128) to calculate momentum p of the 100 eV electron in (1.193).

(c) **Electron** with kinetic energy $E_K = 200$ MeV, i.e., 200 MeV electron.

According to (1.115) an electron with kinetic energy E_K of 200 MeV travels with velocity of $0.99999675c$ and is thus highly relativistic. In the case of β for highly relativistic particles we have no choice but to use the value of β with many significant figures, since we are in a region of extremely rapid change of E_K with β. The de Broglie wavelength λ is calculated with the standard de Broglie expression $\lambda = h/p$ expressing p with (1.124) or (1.126) (both expressions should give the same result)

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{E_K\sqrt{1+\frac{2E_0}{E_K}}} = \frac{2\pi \times (197.3 \text{ MeV} \cdot \text{fm})}{(200 \text{ MeV}) \times \sqrt{1+\frac{2\times 0.511 \text{ MeV}}{200 \text{ MeV}}}}$$
$$= 6.17 \text{ fm} = 6.17 \times 10^{-15} \text{ m} \tag{1.194}$$

or

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{E_0}\frac{\sqrt{1-\beta^2}}{\beta} = \frac{2\pi \times (197.3 \text{ MeV} \cdot \text{fm})}{0.511 \text{ MeV}} \times \frac{\sqrt{1-0.99999675^2}}{0.99999675}$$
$$= 6.17 \text{ fm} = 6.17 \times 10^{-15} \text{ m}. \tag{1.195}$$

(d) **Proton** with kinetic energy $E_K = 10$ MeV, i.e., 10 MeV proton. A proton with kinetic energy E_K of 10 MeV travels with classical velocity of $0.146c$. Since $\beta \ll 1$, the classical expression $\upsilon = \sqrt{2E_K/m_P}$ can be used in calculation of υ_P. The de Broglie wavelength λ is calculated as follows

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{E_K\sqrt{1+\frac{2E_0}{E_K}}} = \frac{2\pi \times (197.3 \text{ MeV} \cdot \text{fm})}{(10 \text{ MeV}) \times \sqrt{1+\frac{2\times (938.3 \text{ MeV})}{10 \text{ MeV}}}}$$
$$= 9.02 \text{ fm} = 9.02 \times 10^{-15} \text{ m}. \tag{1.196}$$

(e) **Marble** of mass $m = 100$ g travelling with velocity $\upsilon = 50$ m/s. The de Broglie wavelength λ is calculated as follows

$$\lambda = \frac{h}{p} = \frac{h}{m_0\upsilon} = \frac{2\pi\hbar c}{m_0\upsilon c}$$
$$= \frac{2\pi \times (197.3\times 10^6 \text{ eV} \cdot \text{fm}) \times (1.6\times 10^{-19} \text{ J/eV}) \times (10^{-15} \text{ m/fm})}{(10^{-1} \text{ kg}) \times (3\times 10^8 \text{ m/s}) \times (50\text{m/s})}$$
$$= 1.32\times 10^{-34} \text{ m} = 1.32\times 10^{-19} \text{ fm}. \tag{1.197}$$

(f) Figure 1.16 plots the de Broglie wavelength λ for electrons against the normalized velocity β. Data points calculated in **(a)**, **(b)**, and **(c)** for electrons are shown superposed onto the de Broglie wavelength λ curve.

(g) Figure 1.17 plots the de Broglie wavelength λ for electrons and protons against kinetic energy E_K. Data points calculated in **(a)**, **(b)**, and **(c)** for electrons and in **(d)** for protons are shown superposed onto appropriate de Broglie wavelength λ curves.

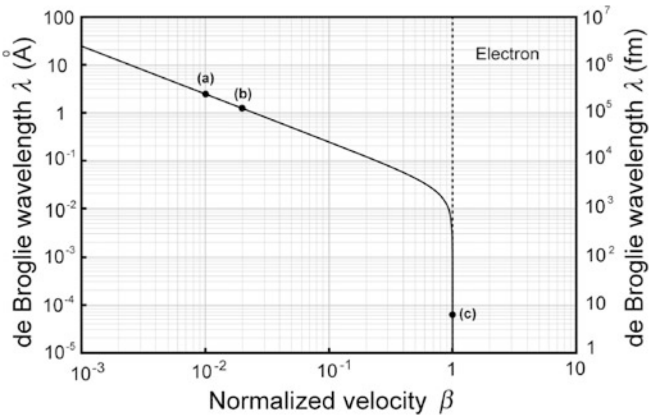


Fig. 1.16 De Broglie wavelength λ against normalized velocity $\beta = \upsilon/c$ for electron. The data points **(a)**, **(b)**, and **(c)** correspond to results calculated in (**a**), (**b**), and **(c)**, respectively

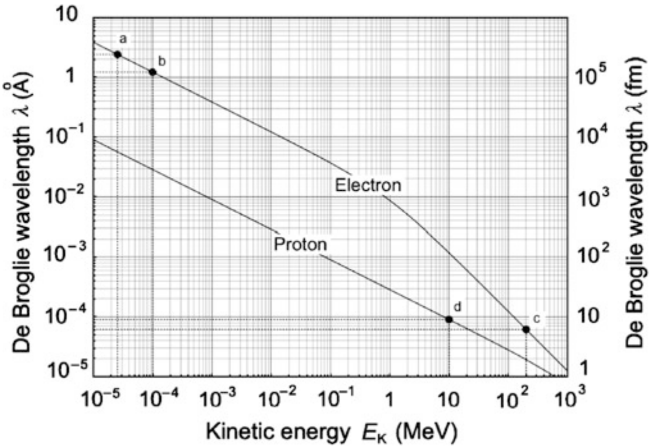


Fig. 1.17 De Broglie wavelength λ against kinetic energy E_K for electron and proton. Data points **(a)**, **(b)**, **(c)**, and **(d)** correspond to results calculated in (**a**), (**b**), (**c**), and (**d**), respectively

(h) To be useful as a probe in scattering experiments a particle must have the de Broglie wavelength λ of the order of the dimension of the investigated target: few angstroms for atoms and few fermis (femtometers) for nuclei.

Electrons in (**a**) and (**b**) with kinetic energy of 25.6 eV and 100 eV, respectively, and de Broglie wavelengths λ of 2.4 Å and 1.2 Å, respectively, would be suitable for scattering experiments on atoms, while electrons of **(c)** with kinetic energy of 200 MeV corresponding to λ of 6.2 fm and protons of **(d)** with kinetic energy of 10 MeV corresponding to a wavelength λ of 9 fm would be suitable as nuclear probes.

1.22.Q2 **(40)**

In 1927 Clinton J. Davisson and Lester H. Germer confirmed experimentally the wave nature of electrons by bombarding a nickel target with electrons and measuring the intensity of electrons scattered from the target. The target was in the form of a regular crystalline alloy that was formed through a special annealing process. The beam of electrons was produced by thermionic emission from a heated tungsten filament. The electrons were accelerated through a relatively low variable potential difference U that enabled the selection of the electron kinetic energy E_K incident onto the nickel crystal. Electrons scattered from the crystal were collected with a Faraday cup and their intensity was measured with a galvanometer.

In a Davisson-Germer experiment electrons (rest energy $m_e = 0.511$ MeV and charge $e = 1.602\times10^{-19}$ C) are accelerated through a potential difference U of the order of 100 V and scattered on nickel crystals with crystalline plane separation of the order of 2 Å. Show that the non-relativistic de Broglie wavelength λ_{clas} of electrons in Davisson-Germer experiment has the following characteristics:

(a) Is inversely proportional to $\sqrt{U}$.
(b) Can be expressed as $\lambda = (12.26\ \text{Å}/\sqrt{U})$ when U is given in volts.
(c) Is equal to 1.73 Å for $U = 50\ V$.
(d) Requires a correction factor $C = \frac{1}{\sqrt{1+eU/(2m_ec^2)}}$ for relativistic electrons.

SOLUTION:

(a) Momentum p for non-relativistic electron.

Recalling that kinetic energy E_K of charge q accelerated through potential difference U is given as a product qU, we can use the classical relationship $E_K = \frac{1}{2}p^2/m_0$ to express momentum p as $p = \sqrt{2m_eE_K} = \sqrt{2m_eeU}$ and get the following classical expression for de Broglie wavelength λ

$$\lambda_{class} = \frac{h}{p} = \frac{2\pi\hbar c}{\sqrt{2m_ec^2E_K}} = \frac{2\pi\hbar c}{\sqrt{2m_ec^2eU}} \propto \frac{1}{\sqrt{U}}. \tag{1.198}$$

(b) Equation (1.198) shows that the non-relativistic relationship for the de Broglie wavelength λ is inversely proportional to $\sqrt{U}$ and the proportionality constant when U is given in volts is

$$\lambda_{class} = \frac{h}{p} = \frac{2\pi\hbar c}{\sqrt{2m_ec^2eU}} = \frac{2\pi\ 1973\ \text{eV}\cdot\text{Å}}{\sqrt{2\times0.511\times10^6\ e^2\text{V}^2\ U(\text{in V})}} = \frac{12.26\ \text{Å}}{\sqrt{U(\text{in V})}}. \tag{1.199}$$

(c) We use (1.199) to determine λ_{clas} for $U = 50$ V and get

$$\lambda_{clas} = \frac{h}{p} = \frac{12.26\ \text{Å}}{\sqrt{50}} = 1.73\ \text{Å}. \tag{1.200}$$

(d) Momentum p for relativistic electron is calculated from the basic expression for the relativistic total energy $E = \sqrt{p^2c^2 + E_0^2}$ where E_0 is the particle rest mass

$$p = \frac{1}{c}\sqrt{E^2 - E_0^2} = \frac{1}{c}\sqrt{E_K^2 - 2E_0E_K} = \frac{1}{c}\sqrt{(eU)^2 + 2m_0c^2eU}$$
$$= \sqrt{2m_e eU\left(1 + \frac{eU}{2m_e c^2}\right)}. \tag{1.201}$$

Equation (1.201) gives the following de Broglie wavelength λ_{rel} for a relativistic electron

$$\lambda_{rel} = \frac{h}{p} = \frac{2\pi\hbar c}{\sqrt{2m_e c^2 eU}\sqrt{1 + \frac{eU}{2m_e c^2}}} = \frac{\lambda_{clas}}{\sqrt{1 + \frac{eU}{2m_e c^2}}} = C\lambda_{clas}, \tag{1.202}$$

where λ_{clas} is the classical de Broglie wavelength of (1.198) and the constant C is given as

$$C = \frac{1}{\sqrt{1 + \frac{eU}{2m_e c^2}}}. \tag{1.203}$$

1.22.Q3 **(41)**

Neutron diffraction is a powerful tool for studying the structure of crystals, especially organic hydrogen-rich crystals. In a Davisson-Germer type diffraction experiment with monochromatic neutrons on an organic sample with plane separation $d = 1.85$ Å, the resulting diffraction pattern exhibited a maximum at an angle $\varphi = 50°$. Calculate the kinetic energy E_K of the monochromatic neutrons that were used in the experiment.

SOLUTION:

The diffraction pattern with its specific intensity maximum results from the wave nature of neutrons. Similarly to the behavior of light and sound, we assume that neutrons also exhibit wavelike behavior that is governed by: (1) the Bragg diffraction law expressed as

$$m\lambda = 2d \sin\varphi, \tag{1.204}$$

with m an integer, φ the Bragg angle measured between the incident beam direction and the crystal surface, and d atomic lattice spacing as well as (2) the de Broglie particle-wave duality hypothesis expressed with the de Broglie wavelength λ as

$$\lambda = \frac{h}{m_\mathrm{n}\upsilon}, \tag{1.205}$$

where υ is the velocity of the neutron and m_n is the neutron rest mass.

We first calculate the neutron de Broglie wavelength λ from Bragg law for $m = 1$ and get

$$\lambda = 2d\sin\varphi = 2\times(1.85\ \text{Å})\times\sin 50^\circ = 2.8\ \text{Å} \tag{1.206}$$

and next we calculate the kinetic energy E_K of neutrons with de Broglie wavelength λ of 2.8 Å from the following basic de Broglie expression

$$\lambda = \frac{h}{m_\mathrm{n}\upsilon} = \frac{h}{\sqrt{2m_\mathrm{n}E_\mathrm{K}}} \tag{1.207}$$

to get

$$E_\mathrm{K} = \frac{h^2}{2m_\mathrm{n}\lambda^2} = \frac{(2\pi)^2(\hbar c)^2}{2m_\mathrm{n}c^2\lambda^2} = \frac{(2\pi)^2\times(1973\ \text{eV}\cdot\text{Å})^2}{2\times(939.5\times10^6\ \text{eV})\times(2.8\ \text{Å})^2} = 0.01\ \text{eV}. \tag{1.208}$$

Neutrons with kinetic energy E_K of 0.01 eV fall into the upper level of so-called cold neutrons but they can also be classified as thermal neutrons that are defined as having energy of about 0.025 eV.

1.22.Q4 **(42)**

A beam of thermal neutrons with kinetic energy E_K of 0.04 eV is used in a scattering experiment on a crystal of sodium chloride with a lattice separation $d = 2.8$ Å. Calculate:

(a) The Bragg angle φ at which the first order ($m = 1$) Bragg reflection occurs.

(b) Energy $h\nu$ of monoenergetic x rays that would undergo first order Bragg reflection on sodium chloride crystal at the same Bragg angle φ.

SOLUTION:

(a) Equation $m\lambda = 2d\sin\varphi$ illustrates the Bragg condition for constructive interference between waves scattered from two planes with separation d, with m the order of the Bragg reflection, λ the wavelength of the incident radiation or matter wave, and ϕ the angle of incidence.

Thermal neutrons with $E_K = 0.04$ eV are characterized with the following de Broglie wavelength λ, as given by (1.193) or its classical approximation for $E_K \ll E_0$

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{E_K\sqrt{1+\frac{2E_0}{E_K}}} \approx \frac{2\pi\hbar c}{\sqrt{2E_0E_K}} = \frac{2\pi\times(197.3\text{ MeV}\cdot\text{fm})\times(10^{-5}\text{ Å/fm})}{\sqrt{2\times(939.6\text{ MeV})\times(4\times10^{-8}\text{ MeV})}}$$
$$= 1.43\text{ Å}, \tag{1.209}$$

where h is the Planck constant; $\hbar$ is the reduced Planck constant ($\hbar = h/(2\pi)$; p is momentum of the neutron; c is speed of light in vacuum; and E_0 is rest energy of the neutron (939.6 MeV).

According to (1.209) the de Broglie wavelength λ of a 0.04 MeV thermal neutron is 1.43 Å resulting in the following angle of incidence φ for the first order Bragg reflection

$$\sin\varphi = \frac{\lambda}{2d} = \frac{1.43\text{ Å}}{2\times2.8\text{ Å}} = 0.255 \quad \text{corresponding to } \varphi = 14.8°. \tag{1.210}$$

(b) A mono-energetic x-ray photon that would experience first order reflection at the same angle of incidence ($\varphi = 14.8°$) as the thermal neutron beam, would have a wavelength λ equal to the de Broglie wavelength λ of the thermal neutron beam. The energy of the mono-energetic x-ray photons is calculated using the Planck law

$$h\nu = h\frac{c}{\lambda} = \frac{2\pi\hbar c}{\lambda} = \frac{2\pi\times(1973\text{ eV}\cdot\text{Å})}{1.43\text{ Å}} = 8.67\text{ eV}. \tag{1.211}$$

1.22.Q5 **(43)**

Based on experimental nuclear data:

(a) Evaluate the feasibility of a nuclear model in which atomic electrons are confined within the atomic nucleus.

(b) Compare the results with those for the model in which protons are confined within the atomic nucleus.

SOLUTION:

(a) Nuclear radius is given as $R(A) = R_0\sqrt[3]{A}$ where R_0 is the nuclear radius constant (1.25 fm) and A is the atomic mass number or number of nucleons in the nucleus. This defines the range of nuclear diameters between 2 fm for low-A nuclei and 20 fm for high-A nuclei, and stipulates that an electron residing in the nucleus would have a de Broglie wavelength λ below ~20 fm.

Equation (1.193) gives the following relationship between de Broglie wavelength λ and kinetic energy E_K of a particle

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{E_K\sqrt{1+\frac{2E_0}{E_K}}} = \frac{2\pi\hbar c}{\sqrt{E_K^2 + 2E_0 E_K}}, \tag{1.212}$$

where E_0 is the rest energy of the particle.

Squaring (1.212) results in a quadratic equation for E_K with the following general solution

$$E_K = \frac{-2E_0 \pm \sqrt{4E_0^2 + 4\frac{(2\pi\hbar c)^2}{\lambda^2}}}{2} = E_0\left[\sqrt{1+\left(\frac{2\pi\hbar c}{\lambda E_0}\right)^2} - 1\right], \tag{1.213}$$

with the $+$ sign in the $\pm$ option providing a physically relevant solution.

In the non-relativistic (classical) region where $2\pi\hbar c \ll \lambda E_0$ we simplify (1.213) to read

$$E_K = E_0\left[1 + \frac{1}{2}\left(\frac{2\pi\hbar c}{\lambda E_0}\right)^2 + \cdots - 1\right] \approx \frac{2\pi^2}{E_0}\left(\frac{\hbar c}{\lambda}\right)^2 \tag{1.214}$$

and in the extreme relativistic region where $2\pi\hbar c \gg \lambda E_0$ we simplify (1.213) to

$$E_K \approx \frac{2\pi\hbar c}{\lambda}. \tag{1.215}$$

Inserting $\lambda = 20$ fm and $E_0 = 0.511$ MeV into (1.213) or just $\lambda = 20$ fm into (1.215) we obtain kinetic energy $E_K = 62$ MeV for the electron and establish that λ of less than 20 fm corresponds to electron kinetic energy E_K of more than 62 MeV.

Even if we take the reduced de Broglie wavelength $\lambdabar$ that is defined as $\lambda/(2\pi)$ and compare it to a nuclear diameter of 20 fm, we note that $\lambdabar$ of 20 fm corresponds to electron kinetic energy E_K of ~ 9.8 MeV. Since kinetic energies of 10 MeV have never been observed experimentally for electrons emitted from the nucleus in any of the nuclear processes such as beta decay, we conclude that a nuclear model housing electrons in addition to protons is not feasible.

Actually, electrons emitted from nuclei have maximum kinetic energies at least an order of magnitude smaller than 10 MeV and amount to only about 1 MeV corresponding to de Broglie wavelengths λ of about 900 fm or reduced de Broglie wavelengths $\lambdabar$ of ~ 150 fm, significantly larger than the nuclear diameter that is of the order of 10 fm.

(b) A similar look at protons confined to the nucleus shows that this model of proton confinement in the nucleus is feasible because of the much larger mass of the proton m_p compared to that of the electron m_e ($m_p/m_e = 1836$). Inserting $\lambda = 20$ fm and $E_0 = 938.3$ MeV into (1.213) we get the following result for the proton

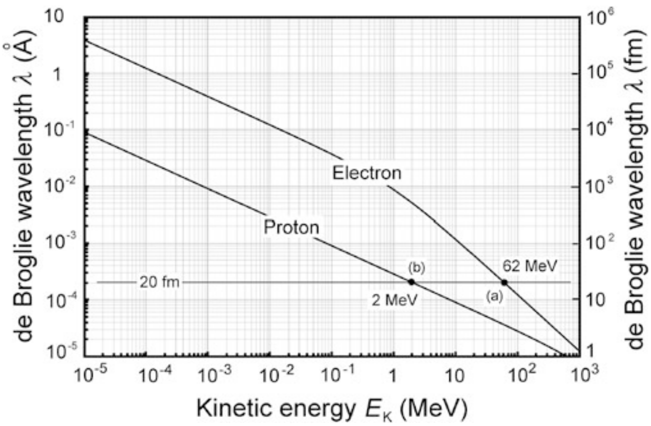


Fig. 1.18 De Broglie wavelength λ for electron and proton against kinetic energy E_K in the range from 10^{-5} MeV to 10^3 MeV. Points **(a)** and **(b)** correspond to kinetic energy of electron of 62 MeV and of proton of 2 MeV for de Broglie wavelength λ of 20 fm, determined in **(a)** and **(b)**, respectively, and estimated as the upper limit on the size of the nucleus for high atomic mass elements

kinetic energy E_K

$$E_\mathrm{K} = (938.3\ \mathrm{MeV}) \times \left[\sqrt{1 + \left(\frac{2\pi \times (197.3\ \mathrm{MeV \cdot fm})}{(20\ \mathrm{fm}) \times (938.3\ \mathrm{MeV})}\right)^2} - 1\right] \approx 2\ \mathrm{MeV}. \quad (1.216)$$

The estimated proton kinetic energy E_K of 2 MeV is well within the binding energy per nucleon E_B/A that is of the order of 8 MeV/nucleon and ranges from 1.2 MeV/nucleon for deuterium to the highest value at slightly less than 9 MeV/nucleon. The estimated proton kinetic energy is also of the same order of magnitude as the energy of protons emitted from the nuclei in proton emission decay.

Figure 1.18 plots the de Broglie wavelengths of proton and electron against kinetic energy E_K. In **(a)** we established 20 fm as the upper limit on the size of the nucleus for high atomic mass elements and this provides us with an order of magnitude for de Broglie wavelength λ of electron and proton residing in the nucleus. From the two curves in Fig. 1.18 we note that for $\lambda = 20$ fm kinetic energy of the electron would exceed 62 MeV, while it exceeds 2 MeV for a proton. Since electrons emitted from nuclei have kinetic energies of the order of 1 MeV, it is clear that a nuclear model incorporating electrons in the atomic nucleus would not be practical.

We thus conclude that, based upon the de Broglie hypothesis of matter waves, electrons cannot reside within the nucleus while the protons can.

1.23 Matter Waves

1.23.Q1 **(44)**

Write the one-dimensional time-dependent Schrödinger wave equation for a free electron and show that a plane wave function is a solution to the wave equation.

SOLUTION:

The 1-dimensional time-dependent Schrödinger wave equation is written as follows

$$-\frac{\hbar^2}{2m_0}\frac{\partial^2 \Psi(z,t)}{\partial z^2} + V(z,t)\Psi(z,t) = i\hbar\frac{\partial^2 \Psi(z,t)}{\partial t^2}, \tag{1.217}$$

where

$\Psi(z,t)$ is the wave function containing the information about the given particle.
$V(z,t)$ is the potential energy operator governing the motion of the particle.
m_0 is the particle rest mass.
$\hbar$ is the reduced Planck constant.

A free electron is an electron subject to no force ($V(z,t) = 0$) and the Schrödinger equation (1.217) for a free electron in one dimension simplifies to

$$-\frac{\hbar^2}{2m_0}\frac{\partial^2 \Psi(z,t)}{\partial z^2} = i\hbar\frac{\partial \Psi(z,t)}{\partial t}, \tag{1.218}$$

with $\Psi(z,t)$ a simple plane wave of the form

$$\Psi(z,t) = Ce^{i\varphi} = Ce^{i(kz-\omega t)} = Ce^{ikz}e^{-i\omega t}, \tag{1.219}$$

where

$\varphi = kz - \omega t$ is the phase of the plane wave and
k is the wave number ($k = 2\pi/\lambda$).
ω is the angular frequency ($\omega = 2\pi\nu$, with ν the frequency).
C is the normalization constant and here it is also the amplitude of the oscillation.

Using the Planck-Einstein quantum hypothesis $E = h\nu = \hbar\omega$ and the de Broglie particle-wave hypothesis $\lambda = h/p = 2\pi/k$ in conjunction with the classical relationship $E_K = p_e^2/(2m_e)$, we note that $\omega = E/\hbar$ and $\Psi(z,t)$ is a solution to the time-dependent Schrödinger equation (1.218) separable as a product of two functions: $\psi(z) = Ce^{ikz}$ that is a function of the spatial coordinate z only and $T(t) = e^{-i\omega t}$ that is a function of the temporal coordinate t only.

The equation for the space function $\psi(z)$ is the so-called time-independent Schrödinger equation, given for a free particle as follows

$$-\frac{\hbar^2}{2m_0}\frac{\mathrm{d}^2\psi(z)}{\mathrm{d}z^2} = E\psi(z), \tag{1.220}$$

with solution $\psi(z) = Ce^{ikz}$ and energy $E = p^2/(2m_0) = \hbar^2k^2/(2m_0)$. Thus, the total energy E equals the kinetic energy.

The general solution to the Schrödinger equation for a free electron (1.218) is thus given as

$$\Psi(z,t) = Ce^{ikz}e^{-i\frac{E}{\hbar}t}. \tag{1.221}$$

For a free particle the energy is not quantized; the wave number k can take any value and the same holds for energy E. Thus, the states of k and E form a continuum.

To verify that (1.221) is indeed a viable solution to (1.218) we evaluate the derivatives $\partial^2\Psi/\partial z^2$ and $\partial\Psi/\partial t$

$$\frac{\partial\Psi(z,t)}{\partial z} = ikCe^{ikz}e^{-i\omega t} = ik\Psi(z,t), \tag{1.222}$$

$$\frac{\partial^2\Psi(z,t)}{\partial z^2} = -k^2Ce^{ikz}e^{-i\omega t} = -k^2\Psi(z,t), \tag{1.223}$$

$$\frac{\partial\Psi(z,t)}{\partial t} = -i\omega Ce^{ikz}e^{-i\omega t} = -i\omega\Psi(z,t). \tag{1.224}$$

Insert the appropriate derivatives into (1.218) and get

$$-\frac{\hbar^2}{2m_0}\frac{\partial^2\Psi(z,t)}{\partial z^2} = -\frac{\hbar^2}{2m_0}k^2\Psi(z,t) \tag{1.225}$$

and

$$i\hbar\frac{\partial\Psi(z,t)}{\partial t} = \hbar\omega\Psi(z,t). \tag{1.226}$$

Wave function $\Psi(z,t) = C\exp[i(kz-\omega t)]$ (in quantum physics referred to as an eigenfunction) satisfies (1.218) provided that $\hbar^2k^2/(2m_0)$ from (1.223) equals to $\hbar\omega$ from (1.226). This condition holds since the following three conditions apply:

(i) $\hbar\omega = E$ according to the Planck-Einstein quantum hypothesis.
(ii) $k = p/\hbar$ according to the de Broglie particle–wave hypothesis.
(iii) $E_\mathrm{K} = p^2/(2m_0)$ according to the classical kinetic energy–momentum relationship.

Energy E is the total energy of the particle; energy E_K is its kinetic energy. For a free particle, the potential energy $V(z,t)$ is zero, so that the total energy E equals the kinetic energy E_K.

1.23.Q2 **(45)**

As shown in Prob. 32, in special relativity the relationship between total energy E and momentum p of a free particle of rest mass m_0 is given by

$$E^2 = p^2c^2 + m_0^2c^4. \tag{1.227}$$

(a) Use the quantum operators for total energy $E \to i\hbar\partial/\partial t$ and momentum $p \to -i\hbar\nabla$ to derive the relativistic Klein-Gordon equation for a free particle.
(b) Show that the Klein-Gordon equation for a free particle transforms into the common wave equation when the free particle is a photon. (Note: photon has zero rest mass).
(c) Show that inserting operators for E and p of **(a)** into the classical expression $E = p^2/2m_0$ results in the 3-dimensional time-dependent Schödinger equation for a free particle.

SOLUTION:

(a) Insert $E \to i\hbar\frac{\partial}{\partial t}$ and $p \to -i\hbar\nabla$ into $E^2 = p^2c^2 + m_0^2c^4$ and get

$$-\hbar^2\frac{\partial^2\Psi}{\partial t^2} = -\hbar^2c^2\nabla^2\Psi + m_0^2c^2\Psi \quad \text{or} \quad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = \frac{m_0^2c^4}{\hbar^2}\Psi. \tag{1.228}$$

Equation (1.228) is known as the Klein-Gordon wave equation and it correctly describes the propagation of relativistic particles of rest mass m_0.

(b) Insert $m_0 = 0$ into the Klein-Gordon equation (1.228) and the result will be the standard wave equation governing EM waves expressed as

$$\nabla^2\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}. \tag{1.229}$$

(c) Insert $E \to i\hbar\frac{\partial}{\partial t}$ and $p \to -i\hbar\nabla$ into the classical relationship for kinetic energy $E = p^2/2m_0$ to get

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_0}\nabla^2\Psi \quad \text{or} \quad -\frac{\hbar^2}{2m_0}\nabla^2\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \tag{1.230}$$

Equation (1.230) is the non-relativistic 3-dimensional time-dependent Schrödinger equation for a free particle.

1.23.Q3 **(46)**

For a free electron with a spatial wave function $\psi(z) = Ce^{i(7.5\ \text{Å}^{-1})z}$.

(a) Write the spatial component of the Schrödinger equation.
(b) Determine the wave number k of the electron.
(c) Calculate the de Broglie wavelength λ of the electron
(d) Calculate the momentum p_e of the electron
(e) Calculate the kinetic energy E_K^e of the electron.
(f) Calculate the velocity υ_e of the electron.

SOLUTION:

(a) The general wave equation for the free electron is given as

$$-\frac{\hbar^2}{2m_e}\nabla^2\psi(z) = E\psi(z), \tag{1.231}$$

where E is the total energy of the free electron and $\psi(z)$ is the time-independent wave function.

(b) Wave number of the free electron with a spatial wave function $\psi(z) = Ce^{i(7.5\ \text{Å})z}$ is obtained directly from the wave function as $k = 7.5\ \text{Å}^{-1} = 7.5\times 10^{-10}\ \text{m}^{-1}$.

(c) De Broglie wavelength λ of the free electron with a spatial wave function $\psi(z) = Ce^{i(7.5\ \text{Å})z}$ is calculated from the definition for the wave number expressed as $k = 2\pi/\lambda$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{7.5}\ \text{Å} = 0.84\ \text{Å} = 0.84\times 10^{-10}\ \text{m}. \tag{1.232}$$

(d) Momentum p of free electron with a spatial wave function $\psi(z) = Ce^{i(7.5\ \text{Å})z}$ is calculated as

$$\begin{aligned} p = \hbar k = (\hbar c)\frac{k}{c} &= (1973\ \text{eV}\cdot\text{Å})\times\frac{7.5\ \text{Å}^{-1}}{c} = 1.48\times 10^4\ \text{eV}/c = 14.8\ \text{keV}/c \\ &= \frac{(1.48\times 10^4\ \text{eV})\times(1.602\times 10^{-19}\ \text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}/\text{eV})}{3\times 10^8\ \text{m}\cdot\text{s}^{-1}} \\ &= 7.9\times 10^{-24}\ \text{kg}\cdot\text{m}\cdot\text{s}^{-1}. \end{aligned} \tag{1.233}$$

(e) Kinetic energy E_K of the free electron with a spatial wave function $\psi(z) = Ce^{i(7.5\ \text{Å})z}$ is calculated from: (1) classical relationship, (2) de Broglie wavelength λ, and (3) general relativistic expression. The three calculations are expected to give same results.

(1) From classical expression for kinetic energy E_K

$$E_K = \frac{1}{2}m_e\upsilon^2 = \frac{p^2}{2m_e} = \frac{(pc)^2}{2m_ec^2} = \frac{(1.48\times 10^4\ \text{eV})^2}{2\times 0.511\times 10^6\ \text{eV}} = 214.3\ \text{eV}. \quad (1.234)$$

(2) From de Broglie wavelength λ of 0.84 Å corresponding to wave number k of 7.5 Å

$$E_K = \frac{\hbar^2k^2}{2m_e} = \frac{(\hbar c)^2k^2}{2m_ec^2} = \frac{(1973\ \text{eV}\cdot\text{Å})^2\times(7.5\ \text{Å}^{-1})^2}{2\times 0.511\times 10^6\ \text{eV}} = 214.3\ \text{eV}. \quad (1.235)$$

(3) From the basic relativistic relationship for total energy $E = \sqrt{p^2c^2 + E_e^2} = E_K + E_e$

$$\begin{aligned} E_K &= E - E_e = \sqrt{E_e^2 + p^2c^2} - E_e = E_e\left(\sqrt{1 + \frac{p^2c^2}{E_e^2}} - 1\right) \\ &\approx E_e\left(1 + \frac{1}{2}\frac{p^2c^2}{E_e^2} + \cdots - 1\right) \\ &\approx \frac{p^2c^2}{2E_e} = \frac{(1.48\times 10^4\ \text{eV})^2}{1.022\times 10^6\ \text{eV}} = 214.3\ \text{eV}. \end{aligned} \quad (1.236)$$

(f) Velocity υ_e of the electron can be calculated: (1) from momentum p_e of the electron in **(d)** or (2) from kinetic energy E_K^e of the electron in **(e)**. We use here the relativistic expressions for momentum and kinetic energy to highlight the general relativistic case even though the electron in this problem can also be treated classically

(1) Using momentum $p_e = 14.8\ \text{keV}/c$ and the relativistic expression for momentum we get

$$p_e = \gamma m_e\upsilon_e = \gamma\beta m_ec = \frac{\beta}{\sqrt{1-\beta^2}}m_ec \quad (1.237)$$

where γ is the Lorentz factor and m_e is the rest mass of the electron.

Solving (1.237) for β we obtain the following result for the electron velocity υ_e

$$\beta = \frac{\upsilon_e}{c} = \frac{\frac{p_e}{m_ec}}{\sqrt{1 + [\frac{p_e}{m_ec}]^2}} = \frac{\frac{14.8\ \text{keV}/c}{0.511\times 10^3\ \text{keV}/c}}{\sqrt{1 + [\frac{14.8\ \text{keV}/c}{0.511\times 10^3\ \text{keV}/c}]^2}} = 0.029. \quad (1.238)$$

(2) As expected, using the standard relationship (see T2.7) between particle velocity υ and its kinetic energy E_K, we get the same result as in (1.238) for the

electron velocity υ_e

$$\beta = \frac{\upsilon_e}{c} = \sqrt{1 - \frac{1}{(1 + \frac{E_K^e}{m_e c^2})^2}} = \sqrt{1 - \frac{1}{(1 + \frac{214.3}{0.511 \times 10^6})^2}} = 0.029. \quad (1.239)$$

1.23.Q4 **(47)**

A particle has the following one-dimensional, time independent wave function $\psi(z) = Ce^{-\frac{1}{2}\alpha^2 z^2}$ Calculate:

(a) Normalization constant C of the particle wave function $\psi(z)$.
(b) Average or expectation value $\bar{z}$ of particle position z.
(c) Average or expectation value $\overline{z^2}$ of z^2 of the particle.
(d) Quantum uncertainty Δz in particle position z.

SOLUTION:

(a) The normalization condition for wave function $\psi(z)$ is given as

$$\int_{-\infty}^{\infty} |\psi(z)|^2 \, dz = \int_{-\infty}^{\infty} \psi^*(z)\psi(z) \, dz = C^2 \int_{-\infty}^{\infty} e^{-\alpha^2 z^2} \, dz = 1, \quad (1.240)$$

resulting in the following expression for the normalization constant C

$$C = \left\{ \int_{-\infty}^{\infty} e^{-\alpha^2 z^2} \, dz \right\}^{-\frac{1}{2}}. \quad (1.241)$$

In standard *Tables of Integrals* we find the following definite integral

$$\int_{-\infty}^{\infty} e^{-az^2} \, dz = \sqrt{\frac{\pi}{a}}. \quad (1.242)$$

Inserting $a = \alpha^2$ into (1.242) we obtain the following value for the constant C

$$C = \frac{1}{\sqrt{\int_{-\infty}^{\infty} e^{-\alpha^2 z^2} \, dz}} = \frac{\sqrt{\alpha}}{\sqrt[4]{\pi}}. \quad (1.243)$$

The wave function of our particle is thus given as

$$\psi(z) = \frac{\sqrt{\alpha}}{\sqrt[4]{\pi}} e^{-\frac{1}{2}\alpha^2 z^2}, \quad (1.244)$$

and, when inserted into the normalization condition (1.240), the result is 1, as expected.

(b) The mean (average) or expectation value $\bar{z}$ of particle position z is calculated as follows

$$\bar{z} = \int_{-\infty}^{\infty} z\left|\psi(z)\right|^2 \mathrm{d}z = C^2 \int_{-\infty}^{\infty} ze^{-\alpha^2 z^2}\,\mathrm{d}z = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} ze^{-\alpha^2 z^2}\,\mathrm{d}z. \qquad (1.245)$$

In standard *Tables of Integrals* we find the following definite integral

$$\int_{-\infty}^{\infty} ze^{-a(z-b)^2}\,\mathrm{d}z = b\sqrt{\frac{\pi}{a}}. \qquad (1.246)$$

Since in our example the constant b equals zero, the value of the definite integral (1.246) is zero, which is to be expected, since the wave function is essentially Gaussian as well as even and centered at $z = 0$. We thus conclude that the mean or expectation value of position z is equal to zero, i.e., $\bar{z} = 0$.

(c) The (mean) average or expectation value $\overline{z^2}$ is calculated as follows

$$\overline{z^2} = \int_{-\infty}^{\infty} z^2\left|\psi(z)\right|^2 \mathrm{d}z = C^2 \int_{-\infty}^{\infty} z^2 e^{-\alpha^2 z^2}\,\mathrm{d}z = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 e^{-\alpha^2 z^2}\,\mathrm{d}z. \qquad (1.247)$$

In standard *Tables of Integrals* we find the following definite integral

$$\int_{-\infty}^{\infty} z^2 e^{-az^2}\,\mathrm{d}z = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}. \qquad (1.248)$$

Inserting (1.248) with $a = \alpha^2$ into (1.247) we get the following result for $\overline{z^2}$

$$\overline{z^2} = \frac{1}{2\alpha^2}. \qquad (1.249)$$

(d) Δz, the quantum uncertainty in position z, is given as

$$\Delta z = \sqrt{\overline{z^2} - (\bar{z})^2}. \qquad (1.250)$$

Inserting $\bar{z} = 0$ from (1.246) and $\overline{z^2} = 1/(2\alpha^2)$ from (1.249) into (1.250) we obtain the following result for the quantum uncertainty Δz

$$\Delta z = \sqrt{\frac{1}{2\alpha^2} - 0} = \frac{1}{\alpha\sqrt{2}} = \frac{0.707}{\alpha}. \qquad (1.251)$$

1.24 Uncertainty Principle

1.24.Q1 **(48)**

Take the radius a and momentum p of the electron occupying the innermost orbit (shell) of the hydrogen atom and show that the two quantities satisfy the *Heisenberg uncertainty principle* $\Delta z \Delta p \geq \frac{1}{2}\hbar$. Assume that maximum uncertainties in position z and momentum p are equal to a and p, respectively, of the $n = 1$ hydrogen orbit.

SOLUTION:

The innermost electronic shell in a hydrogen atom is characterized with $n = 1$ and $Z = 1$, where n is the principal quantum number and Z is the atomic number of hydrogen.

The innermost electron orbit in hydrogen atom has the following radius: $r_1 = a_0 = 0.53\ \text{Å} = 0.53 \times 10^{-5}$ fm, where a_0 is a constant called the Bohr radius constant. The maximum uncertainty in position is then $\Delta z \approx a_0 = 0.53 \times 10^{-5}$ fm.

The velocity of the electron in the first orbit of the hydrogen atom is given as: $\upsilon_1 = \alpha c = (1/137)c$, where α is the fine structure constant and c is the speed of light in vacuum. Since $\upsilon_1 \ll c$, we can use the classical expression for the electron momentum p as follows

$$p = m_e \upsilon_1 = \frac{m_e c^2}{c} \frac{\upsilon_1}{c} = \frac{m_e c^2}{c} \alpha = 0.511\ \frac{\text{MeV}}{c} \times \frac{1}{137} = 3.73 \times 10^{-3}\ \frac{\text{MeV}}{c}, \tag{1.252}$$

resulting in maximum uncertainty in momentum p of $\Delta p \approx 3.73 \times 10^{-3}$ MeV/c.

Next we determine the product $\Delta z \Delta p$ and get

$$\Delta z \Delta p \approx a_0 m_e \upsilon_1 = \left(0.53 \times 10^{-5}\ \text{fm}\right) \times 3.73 \times 10^{-3}\ \frac{\text{MeV}}{c} = 197.7\ \frac{\text{MeV} \cdot \text{fm}}{c} \tag{1.253}$$

and evaluate $\frac{1}{2}\hbar$ to get

$$\frac{1}{2}\hbar = \frac{\hbar c}{2c} = \frac{197.3}{2c}\ \text{MeV} \cdot \text{fm} = 98.7\ \frac{\text{MeV} \cdot \text{fm}}{c}. \tag{1.254}$$

A comparison of (1.253) to (1.254) shows that the Heisenberg uncertainty principle stating that $\Delta z \Delta p \geq \frac{1}{2}\hbar$ is satisfied for the electron in the innermost shell of the hydrogen atom, since

$$\Delta z \Delta p = 197.7\ \frac{\text{MeV} \cdot \text{fm}}{c} \geq \frac{1}{2}\hbar = 98.7\ \frac{\text{MeV} \cdot \text{fm}}{c}. \tag{1.255}$$

1.25 Complementarity Principle

1.25.Q1 **(49)**

(a) Define the "*Principle of Complementarity*" in quantum physics.
(b) List a few examples for which the principle of complementarity applies.

SOLUTION:

(a) Introduced in 1927 by Niels Bohr, the principle of complementarity in quantum physics, also known as wave–particle duality, stipulates that the complete description of a phenomenon in physics of nano-dimensions relies on two contradictory, yet complementary, models. One or the other model alone cannot fully explain a particular phenomenon; complete understanding of a given phenomenon is only obtained if both models are combined in a complementary fashion.

In a narrower sense the principle of complementarity states that a quantum mechanical phenomenon behaves either as particle (corpuscle) or as wave but not as particle and wave simultaneously. A stronger emphasis on the wave nature diminishes the particle nature and vice-versa. This links the Bohr principle of complementarity, which stipulates that the particle and wave models complement rather than contradict one another, with the Heisenberg principle of uncertainty which sets a limit to the precision with which two conjugate physical quantities, such as particle's position and momentum or its energy and duration of energy measurement, can be determined.

(b) There are several phenomena in quantum physics where the particle-wave duality is apparent. In the broadest sense we can say that both radiation as well as matter exhibit the particle–wave duality to which the Bohr principle of complementarity applies. Results of a particular experiment with radiation or matter will be explained with only one of the two models and never with both; however, for a complete understanding of radiation and matter both models must be invoked. For example:

(1) Aspects of radiation explained by corpuscular nature: (i) photoelectric effect and (ii) Compton effect.
(2) Aspects of radiation explained by wave nature: (i) diffraction of x rays (Bragg law); (ii) diffraction of visible light (single slit, double slit (Young experiment), diffraction grating, circular aperture, (iii) wave equations for electric and magnetic fields for EM radiation (Maxwell equations).
(3) Aspects of matter explained by corpuscular nature: (i) electron ionization track in ionization chamber and bubble chamber, (ii) neutron track in neutron bubble detector, (iii) electron in Rutherford-Bohr atom (particle model).

(4) Aspects of matter explained by wave nature: (i) Bragg diffraction of electrons and neutrons on crystals with de Broglie wavelength λ much shorter than the separation d of crystallographic planes, (ii) electron in Rutherford-Bohr atom (electron wave functions).

1.26 Emission of Electrons from Material Surface: Work Function

1.26.Q1 **(50)**

In a surface photoelectric experiment the surface of sodium metal is exposed to incident monochromatic light of various wavelengths λ. The measured retarding potentials U_0^{ret} required to stop completely the emitted photoelectron current at a given λ are listed in Table 1.16.

Table 1.16 Retarding potential U_0^{ret} required for stopping photoelectron current against λ

Photon wavelength λ(Å)	5051	4475	4100	3591	3193	2723
Retarding potential U_0^{ret}(V)	0.15	0.37	0.71	1.23	1.61	2.17

From a graphical presentation of appropriate data determine:

(a) Ratio h/e where h is the Planck constant and e the electron charge.
(b) Planck constant assuming that we know the electron charge ($e = 1.602\times10^{-19}$ C).
(c) Work function $e\phi$ for sodium metal.

SOLUTION:

The surface photoelectric experiment consists of measuring the number of photoelectrons emitted and their kinetic energies as functions of intensity I and wavelength λ of monochromatic incident visible or ultraviolet light. Experimental apparatus allows application of retarding $U^{\text{ret}} < 0$ or accelerating $U^{\text{accel}} > 0$ electric field to the emitted photoelectrons. For $U^{\text{accel}} > 0$, all emitted electrons strike the collecting electrode and the current is essentially independent of the applied voltage. For $U^{\text{ret}} < 0$, the retarding potential prevents lower energy electrons from reaching the collecting electrode, so the current decreases with increasing negative potential until at U_0^{ret} no photoelectrons can overcome the retarding potential and the current drops to 0. For a given material of the emitting electrode, U_0^{ret} is constant and independent of light intensity I, a finding that cannot be explained with classical theory. This peculiar result caused significant difficulties for physicists at the beginning of the 20-th century and it was Albert Einstein who in 1905 explained the experimental data by introducing the idea of the photon quantum and the corpuscular nature of the photon.

According to Einstein, the kinetic energy E_{K} of the emitted photoelectron is given by

$$E_{\text{K}} = h\nu - e\phi, \tag{1.256}$$

where $h\nu$ is the energy of the incident photon quantum and $e\phi$ is the work function representing the minimum energy the photon must possess to be able to eject an electron from the surface of the electrode material.

The retarding potential U^{ret} decreases the measured electron current I at a given photon energy $h\nu = hc/\lambda$ and at U_0^{ret} completely stops the photoelectron current such that the kinetic energy E_{K} of the photoelectron equals to eU_0^{ret}. We now modify (1.256) to accommodate the measured data as follows

$$E_{\text{K}} = U_0^{\text{ret}} = \frac{h}{e}\nu - \frac{e\phi}{e}, \tag{1.257}$$

calculate frequencies ν from wavelengths λ using $\nu = c/\lambda$, and plot U_0^{ret} against the calculated frequencies ν based on the following frequency ν and U_0^{ret} data:

Table 1.17 Photon frequency for wavelength λ data of Table 1.16

	(1)	(2)	(3)	(4)	(5)	(6)
Photon wavelength λ(Å)	5051	4475	4100	3591	3193	2723
Photon frequency $\nu(\text{s}^{-1}) = c/\lambda$	5.94×10^{14}	6.70×10^{14}	7.32×10^{14}	8.35×10^{14}	10.6×10^{14}	11.0×10^{14}
Retarding potential U_0^{ret}(V)	0.15	0.37	0.71	1.23	1.61	2.17

(a) As shown in Fig. 1.19, the (ν, U_0^{ret}) plot is linear and allows us to determine the h/e ratio of (1.257) from the slope of the straight line obtained through the least squares fit to measured data. The slope is 4.12×10^{-15} V · s which means that $h/e = 4.12\times10^{-15}$ V · s.

(b) We can now determine the Planck constant h from the h/e ratio of **(a)** using the known value of the electron charge $e = 1.602\times10^{-19}$ C as follows

$$\begin{aligned} h &= \frac{h}{e}e = \left(4.12\times10^{-15}\ \text{V}\cdot\text{s}\right)\times e = 4.12\times10^{-15}\ \text{V}\cdot\text{s} \\ &= \left(4.12\times10^{-15}\ \text{eV}\cdot\text{s}\right)\times\left(1.602\times10^{-19}\ \text{J/eV}\right) = 6.60\times10^{-34}\ \text{J}\cdot\text{s}, \end{aligned} \tag{1.258}$$

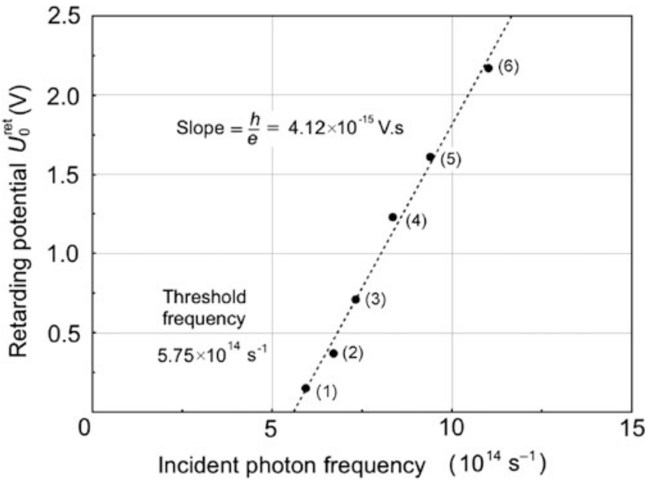


Fig. 1.19 Measured retarding potential U_0^{ret} against photon frequency ν. The slope of the linear plot obtained with the least squares fit to measured data is equal to h/e with h the Planck constant and e the electron charge. Data points (1) through (6) represent data presented in Table 1.17

that compares reasonably well with the currently accepted value of the Planck constant quoted as $h = 6.626\times 10^{-34}$ J · s.

(c) The work function $e\phi$ of sodium will be determined from the intercept of the linear (ν, U_0^{ret}) plot with the abscissa (ν) axis $(U_0^{\text{ret}} = 0)$ which yields $\nu_0 = 5.75\times 10^{14}\ \text{s}^{-1}$. We then get the following value for the work function $e\phi$ of sodium

$$e\phi = h\nu_0 = \frac{(6.60\times 10^{-34}\ \text{J}\cdot\text{s})\times(5.75\times 10^{14}\ \text{s}^{-1})}{1.602\times 10^{-19}\ \text{J/eV}} = 2.37\ \text{eV}, \tag{1.259}$$

in good agreement with currently used value of 2.36 eV for the photoelectric work function $e\phi$ of sodium.

1.26.Q2 **(51)**

In a surface photoelectric experiment photons with a wavelength λ are incident on the photo-cathode and eject photoelectrons with kinetic energy $E_K(\lambda)$ of 2.5 eV. When photons with a wavelength $\frac{1}{2}\lambda$ are incident on the photocathode, they eject photoelectrons with kinetic energy $E_K(\frac{1}{2}\lambda)$ of 9.25 eV. The threshold frequency ν_0 of the photocathode is $1.03\times 10^{15}\ \text{s}^{-1}$. Calculate:

(a) Photoelectric work function $e\phi$.
(b) Wavelength λ of the incident photons.

SOLUTION:

(a) The photoelectric work function $e\phi$ is calculated from the threshold frequency ν_0

$$e\phi = h\nu_0 = \frac{2\pi\hbar c}{c}\nu_0 = \frac{2\pi \times (1973\ \text{eV}\cdot\text{Å}) \times (1.03\times 10^{15}\ \text{s}^{-1})}{3\times 10^{18}\ \text{Å}\cdot\text{s}^{-1}} = 4.25\ \text{eV}. \tag{1.260}$$

(b) We now express mathematically the two sets of experimental data for photoelectron kinetic energies: $E_\text{K}(\lambda)$ and $E_\text{K}(\frac{1}{2}\lambda)$ using the Einstein photoelectric equation linking the photoelectron kinetic energy E_K with photon energy $h\nu$ and the photocathode photoelectric work function $e\phi$

$$E_\text{K} = h\nu - e\phi = \frac{2\pi\hbar c}{\lambda} - e\phi. \tag{1.261}$$

For the two wavelengths λ and $\frac{1}{2}\lambda$ we get the following expressions

$$E_\text{K}(\lambda) = \frac{2\pi\hbar c}{\lambda} - e\phi \tag{1.262}$$

and

$$E_\text{K}\left(\frac{1}{2}\lambda\right) = \frac{4\pi\hbar c}{\lambda} - e\phi. \tag{1.263}$$

Subtracting (1.262) from (1.263) one obtains

$$E_\text{K}\left(\frac{1}{2}\lambda\right) - E_\text{K}(\lambda) = \frac{4\pi\hbar c}{\lambda} - \frac{2\pi\hbar c}{\lambda} = \frac{2\pi\hbar c}{\lambda} = 9.25\ \text{eV} - 2.25\ \text{eV} = 6.75\ \text{eV} \tag{1.264}$$

or

$$\lambda = \frac{2\pi \times (1973\ \text{eV}\cdot\text{Å})}{6.75\ \text{eV}} = 1836\ \text{Å}. \tag{1.265}$$

Photoelectric work function $e\phi$ is 4.25 eV and wavelength λ of the incident photons is 1836 Å.

1.27 Thermionic Emission

1.27.Q1 **(52)**

The thermionic technique was used for measurement of the work function $e\phi$ of tungsten and the data listed in Table 1.18 were measured:

Table 1.18 Data for measurement of work function $e\phi$ with the thermionic technique

T (K)	1250	1500	1750	2000	2250	2500
j (A/m^2)	6.62×10^{-7}	1.03×10^{-3}	0.345	14.9	220	3.05×10^3

Plot the data in an Arrhenius-type graph using the Richardson-Dushman equation (T1.132) and from the graph determine:

(a) Richardson constant A_R for tungsten.
(b) Thermionic work function $e\phi$ for tungsten.

SOLUTION:

The Richardson-Dushman equation expresses the thermionic current density j in A/m^2 as a function of the temperature T in degree K of a heated metallic emitter as follows

$$j = A_R T^2 e^{-\frac{e\phi}{kT}}, \tag{1.266}$$

where k is the Boltzmann constant (8.617×10^{-5} eV$\cdot$K^{-1}) and A_R is the Richardson constant for a given emitter material. Equation (1.266) can be written in the form of Arrhenius-type equation as follows

$$\ln\frac{j}{T^2} = -\frac{e\phi}{k}\frac{1}{T} + \ln A_R, \tag{1.267}$$

to get a linear relationship of the form $y = Kx + b$ which, if borne out by experimental data, enables the determination of the work function $e\phi$ and constant A_R for a given thermionic emitter.

Thermionic data of Table 1.18 for tungsten were expanded to make them useful for Arrhenius diagram and are plotted in Fig. 1.20 with $\ln(j/kT)$ on the ordinate (y) axis against $10^4/T$ on the abscissa (x) axis.

The Arrhenius diagram of Fig. 1.20 for tungsten results in a straight line which:

(a) When extrapolated to $T \to \infty$, i.e., $1/T \to 0$, results in $\ln(j/kT) = \ln A_R = 13.4$ or $A_R = 0.6\times10^6$ A/(m$^2\cdot$K^2).

Table 1.19 Expanded Table 1.18 to make data measured with the thermionic technique suitable for determination of Richardson constant A_R and work function $e\phi$ of tungsten

	(1)	(2)	(3)	(4)	(5)	(6)
T (K)	1250	1500	1750	2000	2250	2500
$10^4/T$ (K^{-1})	8	6.67	5.71	5.00	4.44	4.00
j (A/m^2)	6.62×10^{-7}	1.03×10^{-3}	0.345	14.9	220	3.05×10^3
$\ln(j/T^2)$	−28.49	−21.51	−16.00	−12.50	−10.04	−7.63

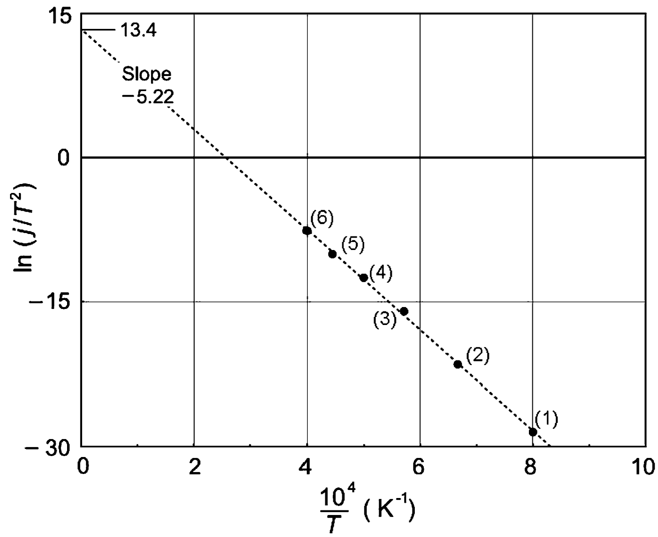


Fig. 1.20 Experimental data of Table 1.19 for tungsten plotted in the form $\ln(j/T^2)$ against $(10^4/T)$ to find solution to the Richardson-Dushman equation (1.266)

(b) Exhibits a slope of $e\phi/k = -5.22\times10^4$ K^{-1} yielding a work function for tungsten $e\phi = 4.5$ eV.

Thus, the Richardson constant for tungsten is $A_\mathrm{R} = 0.6\times10^6$ A/(m$^2 \cdot$ K^2) and its work function is $e\phi = 4.5$ eV. *Note*: the photoelectric work function and thermionic work function are assumed to be the same for the same emitter material.

1.27.Q2 **(53)**

Thermionic emission is a phenomenon in which an electronic current with density j evaporates from a metallic surface heated to temperature T in the absence of an external electric field. The current density j is given by the Richardson-Dushman equation

$$j = A_\mathrm{R} T^2 e^{-\frac{e\phi}{kT}}, \tag{1.268}$$

where $e\phi$ is the work function of the metal, A_R is the theoretical Richardson constant [$A_\mathrm{R} \approx 0.6\times10^6$ A/(m$^2 \cdot$ K^2)] and k is the Boltzmann constant ($k = 8.617\times10^{-5}$ eV $\cdot$ K^{-1}).

When the surface of a heated metallic cathode is immersed in an electric field, the field enhances the thermionic emission of the cathode by lowering the cathode's work function $e\phi$ by $e\Delta\phi$ and this results in a correction to the Richardson-Dushman equation.

The effect and the corrected equation are called the Schottky effect and the Schottky equation, respectively, in honor of German physicist Walter H. Schottky who discovered and explained the effect classically in 1914.

Based on the Richardson-Dushman equation (1.268), the current density j for the Schottky effect is written as

$$j = A_R T^2 e^{-[\frac{e\phi}{kT} - \frac{e\Delta\phi}{kT}]} = A_R T^2 e^{-\frac{e}{kT}[\phi - \Delta\phi]}, \tag{1.269}$$

where $e\Delta\phi$ represents the work function reduction because of the influence of the external electric field $\mathcal{E}$ on the work function $e\phi$.

(a) Sketch the work function $e\phi$ of a typical metal showing the potential energy E_P of an electron as a function of electron's distance x from the surface of the metal. On the same graph sketch the Schottky barrier that arises when external electric field $\mathcal{E}$ is applied to the metal. Indicate all parameters of interest in studies of the Schottky barrier.

(b) Show that $e\Delta\phi$, the decrease in work function $e\phi$ in Schottky effect is proportional to $\alpha\sqrt{\mathcal{E}}$, where $\mathcal{E}$ is the external electric field and $\alpha = -\sqrt{e^3/(4\pi\varepsilon_0)}$ is the proportionality constant.

(c) Determine the magnitude of the external electric field $\mathcal{E}$ required to reduce the work function of tungsten by 1 %. The work function $e\phi$ of tungsten is 4.52 eV.

(d) Determine the relative change in the thermionic emission current density j for a tungsten filament at temperature $T = 2300$ K, if the temperature T increases by 1 % and the work function $e\phi$ decreases by 1 %.

SOLUTION:

(a) Figure 1.21 shows a sketch of potential energy E_P of an electron against its distance x from the surface of a metal. The dotted curve is for metal with no external electric field $\mathcal{E}$ applied, the dashed curve shows the potential $-e\mathcal{E}x$ of the applied external electric field $\mathcal{E}$, and the solid curve shows the combined potential energy forming the Schottky potential barrier as a result of the applied external electric field $\mathcal{E}$. Also indicated are x_S, the location of the maximum potential energy E_P of the Schottky barrier and the reduction of the work function $e\phi$ as a result of the applied external electric field $\mathcal{E}$.

Using the electrostatic method of images, the Coulomb force exerted on an electron with charge $-e$ at a distance x from a metallic emitter (cathode) is the same as the Coulomb force of attraction between the electron and its image with positive charge $+e$ separated by a distance $2x$. Coulomb force on the electron is thus expressed as

$$F_{\text{Coul}} = -\frac{e^2}{4\pi\varepsilon_0(2x)^2} = -\frac{e^2}{16\pi\varepsilon_0 x^2}, \tag{1.270}$$

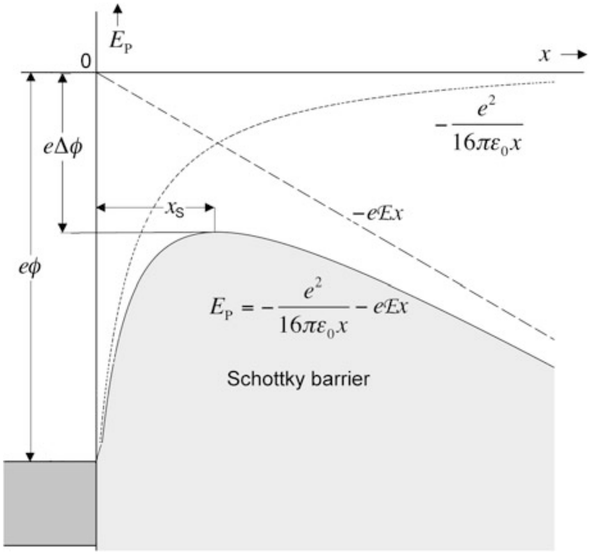


Fig. 1.21 Schematic representation of the Schottky barrier in thermionic emission enhanced by external electric field $\mathcal{E}$ showing the potential energy E_P of an electron as a function of its distance x from the surface of a metal. The *dotted curve* is for metal with no external electric field $\mathcal{E}$ applied, the *dashed curve* shows the potential $-e\mathcal{E}x$ of the applied external electric field $\mathcal{E}$, and the *solid curve* shows the combined potential energy forming the Schottky potential barrier as a result of the applied external electric field $\mathcal{E}$

resulting in potential energy $E_P(x, \mathcal{E}=0)$

$$E_P(x, \mathcal{E}=0) = \int_x^{\infty} F_{\text{Coul}}\,dx = -\frac{e^2}{16\pi\varepsilon_0}\int_x^{\infty}\frac{dx}{x^2} = \frac{e^2}{16\pi\varepsilon_0}\left[\frac{1}{x}\right]_x^{\infty} = -\frac{e^2}{16\pi\varepsilon_0 x} \tag{1.271}$$

associated with the Coulomb force F_{Coul} given in (1.270).

(b) If an external electric field $\mathcal{E}$ is applied, there is an additional contribution $-e\mathcal{E}x$ to the potential energy $E_P(x, \mathcal{E}=0)$ of the electron resulting in total potential energy $E_P(x, \mathcal{E})$ expressed as

$$E_P(x, \mathcal{E}) = -\frac{e^2}{16\pi\varepsilon_0 x} - e\mathcal{E}x. \tag{1.272}$$

The extra potential energy term generated by the external electric field $\mathcal{E}$ causes a lowering of the work function $e\phi$ by a small amount $e\Delta\phi$ and an effective work function $(e\phi)_{\text{eff}} = e\phi - e\Delta\phi$ (Schottky effect). Because of the presence of electric field $\mathcal{E}$, potential energy E_P exhibits not only a maximum but also has the shape of a potential barrier that is referred to as the Schottky barrier. As shown in Fig. 1.21, for a given electric field $\mathcal{E}$ the work function reduction $e\Delta\phi$ occurs at position x_S where E_P exhibits a maximum, in contrast to the behavior of E_P for $\mathcal{E}=0$ where maximum of E_P occurs at $x=\infty$.

We find x_S by setting $\mathrm{d}E_P/\mathrm{d}x|_{x=x_S} = 0$ and get

$$\frac{\mathrm{d}E_P(x = x_S, \mathcal{E})}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{e^2}{16\pi\varepsilon_0 x} - e\mathcal{E}x\right]\bigg|_{x=x_S} = \frac{e^2}{16\pi\varepsilon_0 x_S^2} - e\mathcal{E} = 0. \quad (1.273)$$

Solving (1.273) for x_S, we get the following expression for the position of the maximum x_S of the Schottky barrier

$$x_S = \sqrt{\frac{e}{16\pi\varepsilon_0\mathcal{E}}} = \frac{1}{2}\sqrt{\frac{e}{4\pi\varepsilon_0\mathcal{E}}}. \quad (1.274)$$

Since by definition $E_P(x = x_S, \mathcal{E}) = e\Delta\phi$, we insert x_S of (1.274) into (1.273) and get the following expression for the reduction in the Schottky work function $e\Delta\phi$

$$\begin{aligned} e\Delta\phi &= -\frac{e^2}{16\pi\varepsilon_0 x_S} - e\mathcal{E}x_S = -\frac{e^2}{16\pi\varepsilon_0}\sqrt{\frac{16\pi\varepsilon_0\mathcal{E}}{e}} - e\mathcal{E}\sqrt{\frac{e}{16\pi\varepsilon_0\mathcal{E}}} \\ &= -e\sqrt{\frac{e}{4\pi\varepsilon_0}}\sqrt{\mathcal{E}} = -e\sqrt{\frac{1.602\times10^{-19}\ \mathrm{A\cdot s}}{4\pi\times8.85\times10^{-12}\ \mathrm{A\cdot s/(V\cdot m)}}}\sqrt{\mathcal{E}} \\ &= -\left(e\sqrt{0.144\times10^{-8}\ \mathrm{V\cdot m}}\right)\sqrt{\mathcal{E}} = \alpha\sqrt{\mathcal{E}}. \end{aligned} \quad (1.275)$$

According to (1.275) the magnitude of the Schottky work function reduction $e\Delta\phi$ is proportional to $\sqrt{\mathcal{E}}$, the square root of the external electric field $\mathcal{E}$, and the proportionality constant α is equal to $-e\sqrt{0.144\times10^{-8}\ \mathrm{V\cdot m}}$.

(c) To calculate to magnitude of the external field required to reduce the work function of tungsten by 1 % we use (1.275) with $e\Delta\phi = -0.01\times(4.52\ \mathrm{eV}) = -0.0452\ \mathrm{eV}$. Rearranging (1.275) we get

$$\mathcal{E} = \frac{(e\Delta\phi)^2}{e^2(0.144\times10^{-8}\ \mathrm{V\cdot m})} = \frac{(-0.0452\ \mathrm{eV})^2}{e^2(0.144\times10^{-8}\ \mathrm{V\cdot m})} = 3.14\times10^7\ \mathrm{V/m}. \quad (1.276)$$

Thus, it takes a very strong external electric field $\mathcal{E}$ to reduce the work function of tungsten by a relatively small amount of 1 %.

(d) The current density j, temperature T, and work function $e\phi$ in thermionic emission are related to one another by the Richardson-Dushman equation stated in (1.268). The change in the current density Δj can be expressed in terms of the change in temperature ΔT and the change in work function $\Delta(e\phi)$ as

$$\Delta j = \frac{\partial j}{\partial T}\Delta T + \frac{\partial j}{\partial(e\phi)}\Delta(e\phi) \quad \text{or} \quad \frac{\Delta j}{j} = \frac{1}{j}\left[\frac{\partial j}{\partial T}\Delta T + \frac{\partial j}{\partial(e\phi)}\Delta(e\phi)\right], \quad (1.277)$$

where $\Delta j/j$ represents the relative change in the current density, i.e., the relative change in the number of electrons emitted from the surface of the metal.

Using the Richardson-Dushman equation (1.268) we evaluate the partial derivatives $\partial j/\partial T$ and $\partial j/\partial(e\phi)$, respectively, as

$$\frac{\partial j}{\partial T} = 2A_{\mathrm{R}}Te^{-\frac{e\phi}{kT}} + A_{\mathrm{R}}T^2\left(\frac{e\phi}{kT^2}\right)e^{-\frac{e\phi}{kT}} = \frac{2kT+e\phi}{kT^2}j \tag{1.278}$$

and

$$\frac{\partial j}{\partial(e\phi)} = -A_{\mathrm{R}}T^2\left(\frac{1}{kT}\right)e^{-\frac{e\phi}{kT}} = -\frac{1}{kT}j. \tag{1.279}$$

Inserting (1.278) and (1.279) into (1.277) we get the following expression for the relative change $\Delta j/j$

$$\frac{\Delta j}{j} = \frac{2kT+e\phi}{kT}\left(\frac{DT}{T}\right) - \frac{e\phi}{kT}\left(\frac{eD\phi}{e\phi}\right). \tag{1.280}$$

For tungsten filament ($e\phi = 4.52$ eV) at temperature $T = 2300$ K the product kT amounts to $kT = (0.8617\times10^{-4}\ \mathrm{eV}\cdot\mathrm{K}^{-1})\times(2300\ \mathrm{K}) = 0.1982\ \mathrm{eV} \approx 0.2\ \mathrm{eV}$ and the relative change in thermionic current density $\Delta j/j$ for a relative change in temperature of $\Delta T/T = 0.01$ and a relative change in the work function of $\Delta(e\phi)/e\phi = -0.01$ is from (1.280) calculated as

$$\begin{aligned}\frac{\Delta j}{j} &= \frac{2\times(0.2\ \mathrm{eV}) + (4.52\ \mathrm{eV})}{0.2\ \mathrm{eV}}\times0.01 + \frac{(4.52\ \mathrm{eV})}{0.2\ \mathrm{eV}}\times0.01 \\ &= 0.246 + 0.226 = 0.472.\end{aligned} \tag{1.281}$$

The result of (1.281) shows that at $T = 2300$ K a 1 % increase in temperature T has roughly the same effect as a 1 % decrease in tungsten work function $e\phi$ (as a result of Schottky effect), both effects working together to increase the number of electrons emitted from the surface of the tungsten metal.

1.28 Tunneling

1.28.Q1 **(54)**

A general model for alpha decay tunneling and alpha barrier penetration is shown schematically in Fig. 1.22 for nucleus with atomic number Z and atomic mass number A.

For the α decay of radium-226 into radon-222 with a half-life $t_{1/2}$ of 1602 years and α-particle kinetic energy E_{K} of 4.78 MeV, define, calculate or estimate the following pertinent parameters of α decay:

(a) R_{sep}; **(b)** $U_{\mathrm{C}}(r)$; **(c)** $U(R_{\mathrm{sep}})$; **(d)** $D_{\alpha\text{–N}}$; **(e)** w; **(f)** υ_α; **(g)** t_0, and **(h)** ν

(i) Summarize your calculated answers in a figure similar to Fig. 1.22.

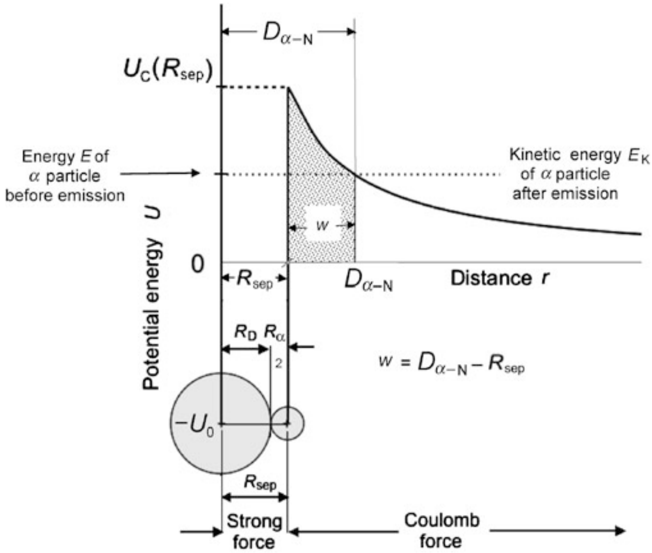


Fig. 1.22 Schematic representation of parameters in alpha decay tunneling

SOLUTION:

The parameters of α decay are defined as follows:

R_{sep}	is the separation distance between the centers of the daughter nucleus D with radius R_{A-4} and the α particle with radius R_α when they are just touching one another.
$U(r)$	is the Coulomb repulsion potential between the daughter nucleus D and the α particle as a function of distance r for $r \geq R_{sep}$.
$U(R_{sep})$	is the Coulomb repulsion potential between daughter nucleus D and α particle at distance $r = R_{sep}$.
$D_{\alpha\text{–N}}$	is the distance r at which the Coulomb potential $U(r)$ equals the kinetic energy of the emitted α-particle, i.e., $U(r_0) = E_K(\alpha)$.
w	is the width of the barrier at the level $U(r_0)$.
υ_α	is the velocity of the α-particle with kinetic energy E_K.
$\bar{t}_\alpha$	is the mean time for traversal of a nucleus by α particle.
ν_α	is the frequency of the α particle hitting the barrier wall inside the nucleus.

(a) Separation distance R_{sep} between ${}^{222}_{86}\text{Rn}$ nucleus and α particle, is determined with the standard relationship between the nuclear radius R and atomic mass number A given as $R(A) = R_0\sqrt[3]{A}$, where R_0 is the nuclear radius constant (1.25 fm)

$$R_{sep} = R_D + R_\alpha = R_0\left(\sqrt[3]{A-4} + \sqrt[3]{A_\alpha}\right) = (1.25\ \text{fm})\times\left(\sqrt[3]{222} + \sqrt[3]{4}\right)$$
$$= 7.57\ \text{fm} + 1.98\ \text{fm} = 9.55\ \text{fm}. \qquad (1.282)$$

(b) Coulomb potential energy $U_C(r)$ between the daughter nucleus D and the α particle is

$$U_C(r) = \frac{2(Z-2)e^2}{4\pi\varepsilon_0 r}. \tag{1.283}$$

(c) Height $U_C(R_{sep})$ of the Coulomb barrier $U_C(r)$ at $r = R_{sep}$ is given as

$$\begin{aligned} U_C(R_{sep}) &= \frac{2(Z-2)e^2}{4\pi\varepsilon_0 R_{sep}} \\ &= \frac{2\times 86e\times(1.602\times10^{-19}\ \mathrm{A\cdot s})}{4\pi\times(8.85\times10^{-12}\ \mathrm{A\cdot s\cdot V^{-1}\cdot m^{-1}})\times(9.6\times10^{-15}\ \mathrm{m})} \\ &= 25.81\ \mathrm{MeV}. \end{aligned} \tag{1.284}$$

(d) Distance $D_{\alpha\text{–}N}$ is the distance at which the Coulomb potential $U_C(r)$ equals the kinetic energy E_K attained by the α particle at very large distance from the nucleus. In α-particle scattering this distance is known as the distance of closest approach in a head-on collision between the α particle and the nucleus. For a nucleus with atomic number $Z-2$ distance $D_{\alpha\text{–}N}$ is expressed as follows

$$\begin{aligned} D_{\alpha\text{–}N} &= \frac{2(Z-2)e^2}{4\pi\varepsilon_0 E_K} \\ &= \frac{2\times 86e\times(1.602\times10^{-19}\ \mathrm{A\cdot s})}{4\pi\times(8.85\times10^{-12}\ \mathrm{A\cdot s\cdot V^{-1}\cdot m^{-1}})\times(4.78\times10^{6}\ \mathrm{eV})} \\ &= 5.2\times10^{-14}\ \mathrm{m} = 52\ \mathrm{fm}. \end{aligned} \tag{1.285}$$

(e) Width w of the potential barrier at $U_C(r_0) = E_K$ is

$$w = D_{\alpha\text{–}N} - R_{sep} = 52\ \mathrm{fm} - 9.6\ \mathrm{fm} = 42.4\ \mathrm{fm}. \tag{1.286}$$

(f) Velocity υ_α of the α particle is calculated using the relativistic expression (T1.58) and (T2.7) to get

$$\begin{aligned} \upsilon_\alpha &= c\sqrt{1-\frac{1}{\left(1+\frac{E_K}{m_\alpha c^2}\right)^2}} = c\sqrt{1-\frac{1}{\left(1+\frac{4.78\ \mathrm{MeV}}{3727\ \mathrm{MeV}}\right)^2}} = 0.05c \\ &= 1.52\times10^{7}\ \mathrm{m/s}, \end{aligned} \tag{1.287}$$

where $m_\alpha c^2$ is the rest energy of the α particle (3727.3 MeV).

(g) Average time $\bar{t}_\alpha$ for the α particle to traverse the nucleus

$$\bar{t}_\alpha = \frac{2R_{sep}}{\upsilon_\alpha} = \frac{2\times(9.55\times10^{-15}\ \mathrm{m})}{1.52\times10^{7}\ \mathrm{m}} = 1.26\times10^{-21}\ \mathrm{s}. \tag{1.288}$$

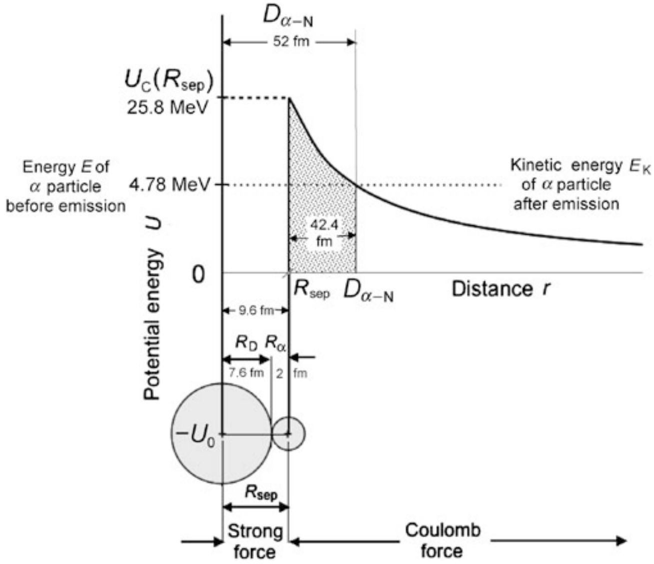


Fig. 1.23 Summary of results for tunneling of alpha particles in alpha decay of radium-226 into radon-222 **(a)** Separation distance: $R_{sep} = 9.6$ fm; **(b)** Coulomb potential energy: $U_C(r) = 2(Z-2)e^2/(4\pi\varepsilon_0 r)$; **(c)** Height of Coulomb barrier at separation distance: $U_C(R_{sep}) = 25.8$ MeV; **(d)** Distance $D_{\alpha-N}$ at which Coulomb potential equals kinetic energy of α particle: $D_{\alpha-N} = 52$ fm; **(e)** Width w of the potential barrier at the level of tunneling: $w = 42.4$ fm; **(f)** Velocity υ_α of the α particle: $\upsilon_\alpha = 0.05c$; **(g)** Mean time $\bar{t}_\alpha$ for the α particle to traverse the nucleus: $\bar{t}_\alpha = 1.3\times10^{-21}$ s; and **(h)** Frequency ν_α of α particle hitting the potential barrier: $\nu_\alpha = 8\times10^{20}\ \text{s}^{-1}$

(h) Frequency ν_α of hitting the potential barrier wall inside the nucleus

$$\nu_\alpha = \frac{1}{t_0} = \frac{\upsilon}{2R_{sep}} = \frac{1.52\times10^{7}\ \text{m}\cdot\text{s}^{-1}}{2\times9.55\times10^{-15}\ \text{m}} = 7.96\times10^{20}\ \text{s}^{-1}. \qquad (1.289)$$

(i) Summary of results **(a)** through **(h)** is given in Fig. 1.23.

1.28.Q2 **(55)**

In classical physics, a particle striking a potential barrier will only be repelled by the barrier; in quantum physics, however, the particle striking a potential barrier may be repelled by the barrier or transmitted through the barrier in a peculiar phenomenon referred to as quantum tunneling.

The tunneling transmission factor T is used to describe the probability of a particle tunneling through a potential barrier. It is given as the ratio of the transmitted probability current density j_{trans} to the incident probability current density j_{inc} and approximated as

$$T = \frac{j_{\text{trans}}}{j_{\text{inc}}} \approx \exp\left\{-\frac{2\sqrt{2m}}{\hbar}\int_a^b \sqrt{U(z)-E}\,\mathrm{d}z\right\}, \tag{1.290}$$

where E, m, and $U(z)$ are the particle energy, particle mass, and barrier potential, respectively, and the integration limits a and b represent the classical limits of the potential barrier. Inside the barrier the following condition applies: $E < U(z)$.

The barrier transmission coefficient T_α for α decay is given by the Gamow expression

$$\ln T_\alpha = -4\pi(Z-2)\sqrt{\frac{E_{\text{G}}}{E_{\text{K}}}} + 8\sqrt{\frac{(Z-2)R_{\text{sep}}}{a_{\text{G}}}}, \tag{1.291}$$

where

$$a_{\text{G}} = \frac{4\pi\varepsilon_0}{e^2}\frac{\hbar^2c^2}{m_\alpha c^2} = 7.256\ \text{fm}$$

and

$$E_{\text{G}} = \frac{1}{2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\frac{m_\alpha c^2}{\hbar^2c^2} = 0.09927\ \text{MeV}.$$

(a) List and sketch at least three examples of a quantum potential barrier, provide expressions for $U(z)$ for each barrier, and state the classical limit a and b for each barrier.

(b) Use the general approximation for potential barrier transmission coefficient T given as

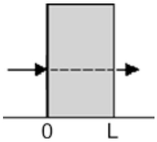
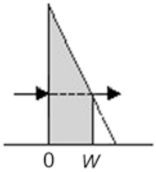
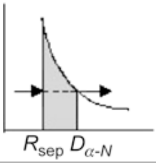
$$T = \exp\left\{-\frac{2\sqrt{2m_\alpha}}{\hbar}\int \sqrt{U(z)-E}\,\mathrm{d}z\right\} \tag{1.292}$$

to derive the general expression for the Gamow potential barrier transmission coefficient T_α in α decay. General parameters of α decay of importance to calculation of the barrier transmission factor are given in Fig. 1.22.

(c) Use (1.291) to calculate potential barrier transmission coefficient T_α for radium-226 (Ra-226) α decay into radon-222 (Rn-222). Relevant parameters for Ra-226 α decay were determined in Prob. 54.

(d) Use T_α determined in **(b)** to calculate the decay constant λ and half-life $t_{1/2}$ for α decay of Ra-226.

Table 1.20 Various tunneling phenomena in physics with their associated charged particle, functional shape of potential energy, and classical limits

Tunneling potential	Tunneling particle	Potential diagram	Potential $U(z)$		Classical limits a and b
Square potential	Arbitrary charge q		$U(z) = qV$ constant	(1.293)	0 and L
Field emission	Electron charge e		$U(z) = e\mathcal{E}z$ linear	(1.294)	0 and w
Alpha decay	Alpha particle charge $2e$		$U(z) = \frac{2(Z-2)e^2}{4\pi\varepsilon_0 z}$	(1.295)	R_{sep} and r_0 [see Prob. 54(**b**)]

SOLUTION:

(a) Examples of quantum potential barrier are provided in Table 1.20 that provides for each example the tunneling particle, sketch of the potential barrier, mathematical expression for the potential barrier, and classical limits for each barrier function.

(b) The transmission coefficient T_α for α-particle tunneling through the nuclear potential barrier is approximated with the so-called Gamow formula as follows (see Prob. 54)

$$\ln T_\alpha \approx -\frac{2\sqrt{2m_\alpha c^2}}{\hbar c}\int_{R_{sep}}^{D_{\alpha\text{–N}}}\sqrt{\left(\frac{2(Z-2)e^2}{4\pi\varepsilon_0 r} - E_K\right)}\,dr$$
$$= -\frac{2\sqrt{2m_\alpha c^2 E_K}}{\hbar c}\int_{R_{sep}}^{D_{\alpha\text{–N}}}\sqrt{\left(\frac{D_{\alpha\text{–N}}}{r} - 1\right)}\,dr, \qquad (1.296)$$

where Z is the atomic number of the parent nucleus and $m_\alpha c^2$ is the rest energy of the α particle (3727.3 MeV).

As indicated in Fig. 1.23, the two classical distances which set the upper and lower limit in the barrier transmission integral are: $D_{\alpha\text{–N}}$, the distance of closest approach in head-on collision α-particle scattering on daughter nucleus with atomic number $(Z-2)$ and R_{sep}, the separation between the daughter nucleus and the α-particle when the two nuclei are just touching each other. $D_{\alpha\text{–N}}$ is given as: $D_{\alpha\text{–N}} = 2(Z-2)e^2/(4\pi\varepsilon_0 E_K)$ with E_K the kinetic energy of the α particle.

To simplify the integral in (1.296) we now introduce a new variable: $u = r/D_{\alpha\text{–N}}$, recognize that $\mathrm{d}r = D_{\alpha\text{–N}}\,\mathrm{d}u$ and get

$$\ln T_\alpha = -\frac{2\sqrt{2m_\alpha c^2 E_\mathrm{K}}}{\hbar c} D_{\alpha\text{–N}} \int_{R_\mathrm{sep}/D_{\alpha\text{–N}}}^{1} \sqrt{\left(\frac{1}{u}-1\right)}\,\mathrm{d}u. \tag{1.297}$$

To solve the integral in (1.297) we introduce the following new variable $u = \sin^2\theta$, recognize that $\mathrm{d}u = 2\sin\theta\cos\theta\,\mathrm{d}\theta$, and obtain the following solution

$$\int_{R_\mathrm{sep}/D_{\alpha\text{–N}}}^{1} \sqrt{\left(\frac{1}{u}-1\right)}\,\mathrm{d}u = 2\int_{\sqrt{R_\mathrm{sep}/D_{\alpha\text{–N}}}}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta = [\sin\theta\cos\theta+\theta]_{\sqrt{R_\mathrm{sep}/D_{\alpha\text{–N}}}}^{\pi/2}, \tag{1.298}$$

with the upper integration limit equal to $\frac{1}{2}\pi$ and the lower integration limit approximated to read $\sqrt{R_\mathrm{sep}/D_{\alpha\text{–N}}}$ based on an assumption that $R_\mathrm{sep}/D_{\alpha\text{–N}}$ is very small, allowing the use of the approximation $\sin\theta \approx \theta$.

Inserting the upper and lower integration limits into (1.298) we get the following solution

$$\int_{R_\mathrm{sep}/D_{\alpha\text{–N}}}^{1} \sqrt{\left(\frac{1}{u}-1\right)}\,\mathrm{d}u = \{\sin\theta\cos\theta+\theta\}_{\sqrt{R_\mathrm{sep}/D_{\alpha\text{–N}}}}^{\pi/2} = \frac{\pi}{2} - 2\sqrt{\frac{R_\mathrm{sep}}{D_{\alpha\text{–N}}}}. \tag{1.299}$$

Inserting (1.299) into (1.297) results in the following general expression for transmission of the nuclear potential barrier in α decay

$$\begin{aligned}\ln T_\alpha &= -\frac{2\sqrt{2m_\alpha c^2 E_\mathrm{K}}}{\hbar c} D_{\alpha\text{–N}}\left(\frac{\pi}{2} - 2\sqrt{\frac{R_\mathrm{sep}}{D_{\alpha\text{–N}}}}\right)\\ &= -4\pi(Z-2)\sqrt{\frac{1}{2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\frac{m_\alpha c^2}{E_\mathrm{K}}} + 8\sqrt{\frac{m_\alpha c^2}{\hbar^2 c^2}\frac{e^2}{4\pi\varepsilon_0}(Z-2)R_\mathrm{sep}}\\ &= -4\pi(Z-2)\sqrt{\frac{E_\mathrm{G}}{E_\mathrm{K}}} + 8\sqrt{\frac{(Z-2)R_\mathrm{sep}}{a_\mathrm{G}}},\end{aligned} \tag{1.300}$$

where we introduced two constants: a_G and E_G relevant to α decay. The two constants are modeled after the well-known atomic constants: the Bohr radius constant $a_0 = 0.5292$ Å and the Rydberg energy $E_\mathrm{R} = 13.61$ eV, except that the electron mass m_e in the Bohr radius constant and in the Rydberg energy is replaced by the mass of the α particle m_α. The two nuclear constants are thus expressed as follows

$$a_\mathrm{G} = \frac{4\pi\varepsilon_0}{e^2}\frac{\hbar^2 c^2}{m_\alpha c^2} = a_0\frac{m_\mathrm{e}c^2}{m_\alpha c^2} = (0.5292\ \text{Å})\times\frac{0.511}{3727} = 7.256\ \mathrm{fm} \tag{1.301}$$

and

$$E_G = \frac{1}{2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{m_\alpha c^2}{\hbar^2 c^2} = E_R \frac{m_\alpha c^2}{m_e c^2} = (13.61\text{ eV}) \times \frac{3727.3}{0.511} = 0.09927\text{ MeV}. \tag{1.302}$$

(c) From Prob. 54 we get the following relevant parameters for the radium-226 decay: $Z = 88$, $E_K = 4.78$ MeV and $R_{sep} = 9.55$ fm. Inserting these parameters into the general expression for T_α results in the following barrier transmission factor T_α for Ra-226 α decay

$$\begin{aligned} T_\alpha &= \exp\left\{-4\pi \times 86\sqrt{\frac{0.09927\text{ MeV}}{4.78\text{ MeV}}} + 8\sqrt{86\frac{9.55\text{ fm}}{7.256\text{ fm}}}\right\} \\ &= \exp\{-155.74 + 85.1\} = 2.1 \times 10^{-31}. \end{aligned} \tag{1.303}$$

(d) The barrier transmission coefficient T_α is the probability for the α particle to tunnel through the potential barrier upon striking the barrier. In Prob. 54 we calculated the frequency (repetition rate) ν_α of the α particle striking the potential barrier and obtained $\nu_\alpha = 7.96 \times 10^{20}\text{ s}^{-1}$ for the α decay of Ra-226. The α decay constant λ_α of Ra-226 can be expressed as the product of the barrier transmission coefficient T_α as well as the repetition rate ν_α and the half-life $t_{1/2}$ is then given as

$$t_{1/2} = \frac{\ln 2}{\lambda_\alpha} = \frac{\ln 2}{T_\alpha \nu_\alpha} = \frac{\ln 2}{(2.1 \times 10^{-31}) \times (7.96 \times 10^{20}\text{ s}^{-1})} = 4.15 \times 10^9\text{ s} = 131.5\text{ a}. \tag{1.304}$$

This result is more than an order of magnitude smaller than the measured half-life of 1602 years for radium-226. However, the estimate is satisfactory, considering the series of approximations that were involved in the derivation of the tunneling transmission coefficient T_α. Moreover, T_α depends heavily on kinetic energy E_K of the α particle as well as on the initial separation R_{sep} between the α particle and the daughter nucleus and a minute change in one or both of these parameters results in a large change in T_α, since both parameters appear in the exponential.

1.28.Q3 **(56)**

Thermionic emission (TE) and field emission (FE) are physical phenomena of importance not only in theoretical physics but also in practical production of x rays.

(a) In point form compare thermionic emission (TE) from a metal with field emission (FE) from a metal.

(b) Provide schematic diagrams for potential energy of electron against its distance from metal surface for thermionic emission and field emission.

SOLUTION:

(a) In both the thermionic emission (TE) and field emission (FE) electrons are emitted from the surface of a metal. On a microscopic scale, however, the two effects are different:

(1) In TE electrons surmount the potential barrier, while in FE electrons tunnel through the potential barrier.

(2) In TE the temperature T of the metal increases the energy of electrons to allow them to surmount the potential barrier; in FE a strong external electric field $\mathcal{E}$ narrows the potential barrier to make the tunneling of electrons through the barrier possible.

(3) In TE the work function remains constant with temperature but the kinetic energy of electrons rises with metal temperature, allowing electrons to escape from the metal. In FE the barrier thickness diminishes with an increasing applied electric field, making it easier for electrons to tunnel through the barrier.

(4) The higher the temperature T of the metal, the stronger is the emission of electrons in TE; the stronger is the external electric field $\mathcal{E}$, the stronger is the emission of electrons in FE.

(5) TE is of importance in the fields of electronics and communications in general. In medical physics TE plays an important role in hot cathode x-ray tubes (Coolidge tubes) and electron guns in linear accelerators. FE shows promise in the development of cold cathode x-ray tubes.

(6) The form of the functional dependence of electron current density j on T in TE is the same as that on $\mathcal{E}$ in FE.

(7) In TE the electron current density $j(T)$ is expressed by the Richardson-Dushman equation as follows

$$j_{\mathrm{RD}}(T) = A_{\mathrm{R}} T^2 e^{-\frac{e\phi}{kT}}, \tag{1.305}$$

where A_{R} is the Richardson constant of the electron-emitting metal; k is the Boltzmann constant; and $e\phi$ is the work function of the metal (of the order of 2 eV to 5 eV). The Richardson-Dushman plot of Arrhenius diagram requires plotting $\ln(j/T^2)$ against $1/T$ and results in a straight line with slope $e\phi/k$ allowing determination of the work function $e\phi$ and ordinate intercept $\ln A$ allowing determination of the Richardson constant A.

(8) In FE the electron current density $j(\mathcal{E})$ is expressed by the Fowler-Nordheim equation as follows

$$j_{\mathrm{FN}}(\mathcal{E}) = \alpha \mathcal{E}^2 e^{-\frac{\beta}{\mathcal{E}}}, \tag{1.306}$$

where α and β are constants specific to the metallic electron emitter. The Fowler-Nordheim plot of Arrhenius diagram requires plotting $\ln(j/\mathcal{E}^2)$ against $1/\mathcal{E}$ and results in a straight line with slope β and ordinate intercept $\ln \alpha$ allowing determination of α.

(9) FE is in principle similar to the Schottky effect; however, it is generally accepted that Schottky effect influences electron emission from metallic surfaces

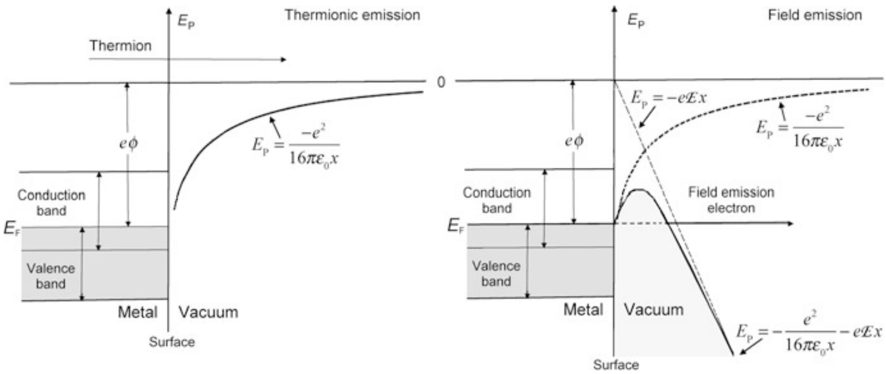


Fig. 1.24 Schematic diagram for thermionic emission (TE) *on the left* and field emission (TE) tunneling of electrons through the surface potential barrier *on the right*

at relatively low electric fields and field emission deals with extremely high electric fields.

(b) Schematic diagrams for potential energy E_P of electron against its distance from surface of the metal emitter for thermionic emission (left side) and field emission (right side) are depicted in Fig. 1.24.

1.29 Maxwell Equations

1.29.Q1 **(57)**

All electromagnetic phenomena are governed by Maxwell equations which form a set of four partial differential equations that are invariant with respect to Lorentz transformation (see Prob. 26) and relate the electric field $\mathcal{E}$ as well as the magnetic field $\mathcal{B}$ to their source: the charge density ρ and current density $\mathbf{j}$, respectively.

(a) State the four Maxwell equations for vacuum in the differential form and in the integral form. Define the physical quantities pertaining to each of the four Maxwell equations and state their units in the SI system. For each of the four equations, in addition to Maxwell, give the name of the physicist who is associated with the given equation.

(b) State the two theorems that link the Maxwell equations in the differential form with Maxwell equations in the integral form and indicate how the two theorems are used to modify the Maxwell equations from the differential into integral form.

(c) Modify the general Maxwell equations in (**a**) for use in evacuated waveguides.

SOLUTION:

(a) The four general Maxwell equations are as follows:

Table 1.21 The four general Maxwell equations in differential and integral form

	Equation	Differential form	Integral form	
1	Maxwell *Gauss law for electricity*	$\nabla \cdot \boldsymbol{\mathcal{E}} = \frac{\rho}{\varepsilon_0}$	$\oiint_S \boldsymbol{\mathcal{E}} \cdot d\mathbf{S} = \frac{q}{\varepsilon_0}$	(1.307)
2	Maxwell *Gauss law for magnetism*	$\nabla \cdot \boldsymbol{\mathcal{B}} = 0$	$\oiint_S \boldsymbol{\mathcal{B}} \cdot d\mathbf{S} = 0$	(1.308)
3	Maxwell *Faraday law of induction*	$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}$	$\oint \boldsymbol{\mathcal{E}} \cdot d\boldsymbol{\ell} = -\frac{\partial}{\partial t} \iint \boldsymbol{\mathcal{B}} \cdot d\mathbf{S}$	(1.309)
4	Maxwell *Ampère law* extended by Maxwell	$\nabla \times \boldsymbol{\mathcal{B}} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t}$	$\oint \boldsymbol{\mathcal{B}} \cdot d\boldsymbol{\ell} = \mu_0 I + \frac{1}{c^2} \frac{\partial}{\partial t} \iint \boldsymbol{\mathcal{E}} \cdot d\mathbf{S}$	(1.310)

where

$\mathcal{E}$ is electric field in V/m.
$\mathcal{B}$ is magnetic field in tesla ($1\ \text{T} = 1\ \text{V} \cdot \text{s/m}^2$).
ρ is total charge density in C/m^3.
q is total charge in volume $\mathcal{V}$ given in C.
$\mathbf{j}$ is current density in A/m^2.
I is current in A.
$d\mathbf{S}$ is differential vector element of surface area S with direction normal to the surface.
$d\boldsymbol{\ell}$ is differential vector element of path length tangential to the path.
ε_0 is the electric constant ($8.85 \times 10^{-12}\ \text{A} \cdot \text{s} \cdot \text{V}^{-1} \cdot \text{m}^{-1}$).
μ_0 is the magnetic constant ($4\pi \times 10^{-7}\ \text{V} \cdot \text{s} \cdot \text{A}^{-1} \cdot \text{m}^{-1}$).

(b) The two theorems are the Stokes-Kelvin curl theorem and the Gauss-Ostrogradski divergence theorem:

Stokes-Kelvin curl theorem:

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \iint_S \text{curl}\, \mathbf{A} \cdot d\mathbf{S} = \oint_\ell \mathbf{A} \cdot d\boldsymbol{\ell} \tag{1.311}$$

relates the surface integral of the curl ($\nabla \times$) of a vector field $\mathbf{A}$ over surface S to the closed loop line integral of the vector field $\mathbf{A}$ over the boundary of the surface S given by closed loop ℓ.

Gauss-Ostrogradski divergence theorem:

$$\iiint_{\mathcal{V}} \nabla \cdot \mathbf{A}\, d\mathcal{V} = \iiint_{\mathcal{V}} \text{div}\, \mathbf{A}\, d\mathcal{V} = \oiint_S \mathbf{A} \cdot d\mathbf{S} \tag{1.312}$$

relates the volume integral of the divergence $(\nabla\cdot)$ of a vector field $\mathbf{A}$ over volume $\mathcal{V}$ to the closed surface integral of the vector field $\mathbf{A}$ over the boundary of the volume $\mathcal{V}$ given by closed surface S.

Maxwell equation #1 $(\nabla\cdot\boldsymbol{\mathcal{E}} = \rho/\varepsilon_0)$ is also known as the *Gauss law of electrostatics* and can be expressed in integral form through application of the Gauss divergence theorem as follows

$$\iiint_{\mathcal{V}} \nabla\cdot\boldsymbol{\mathcal{E}}\,\mathrm{d}\mathcal{V} = \oiint_S \boldsymbol{\mathcal{E}}\cdot\mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\iiint_{\mathcal{V}} \rho\,\mathrm{d}\mathcal{V} = \frac{q}{\varepsilon_0}. \tag{1.313}$$

The Gauss law states that the electric flux $\oiint_S \boldsymbol{\mathcal{E}}\cdot\mathrm{d}\mathbf{S}$ through any closed surface S is proportional to the total charge q enclosed in volume $\mathcal{V}$ by the closed surface S.

Maxwell equation #2 $(\nabla\cdot\boldsymbol{\mathcal{B}} = 0)$ is also known as the *Gauss law of magnetism* and can be expressed in integral form through application of the Gauss divergence theorem

$$\iiint_{\mathcal{V}} \nabla\cdot\boldsymbol{\mathcal{B}}\,\mathrm{d}\mathcal{V} = \oiint_S \boldsymbol{\mathcal{B}}\cdot\mathrm{d}\mathbf{S} = 0. \tag{1.314}$$

The Gauss law states that the magnetic flux $\oiint_S \boldsymbol{\mathcal{B}}\cdot\mathrm{d}\mathbf{S}$ through any closed surface S is equal to zero implying that there are no magnetic monopoles.

Maxwell equation #3 $(\nabla\times\boldsymbol{\mathcal{E}} = -\partial\boldsymbol{\mathcal{B}}/\partial t)$ is also known as the *Faraday law of induction*. It can be expressed in integral form through application of the Stokes curl theorem

$$\iint_S (\nabla\times\boldsymbol{\mathcal{E}})\cdot\mathrm{d}\mathbf{S} = \oint_\ell \boldsymbol{\mathcal{E}}\cdot\mathrm{d}\boldsymbol{\ell} = -\frac{\partial}{\partial t}\iint_S \boldsymbol{\mathcal{B}}\cdot\mathrm{d}\mathbf{S} = -\frac{\partial\phi_\mathrm{B}}{\partial t} = U_\mathrm{ind}, \tag{1.315}$$

where $\phi_\mathrm{B} = \iint_S \boldsymbol{\mathcal{B}}\,\mathrm{d}\mathbf{S}$ is the magnetic flux and U_ind is the induced voltage.

The Faraday law of induction states that the line integral of the electric field $\mathcal{E}$ over a closed loop ℓ is equal to the negative of the rate of change of the magnetic flux ϕ_B through the area enclosed by the closed loop ℓ.

Maxwell equation #4 $(\nabla\times\boldsymbol{\mathcal{B}} = \mu_0\mathbf{j} + c^{-2}\partial\boldsymbol{\mathcal{E}}/\partial t)$ is also known as the *Ampère law* extended by Maxwell. It can be expressed in integral form through application of the Stokes curl theorem

$$\iint_S (\nabla\times\boldsymbol{\mathcal{B}})\cdot\mathrm{d}\mathbf{S} = \oint_\ell \boldsymbol{\mathcal{B}}\cdot\mathrm{d}\boldsymbol{\ell} = \mu_0\iint_S \mathbf{j}\cdot\mathrm{d}\mathbf{S} + \frac{1}{c^2}\frac{\partial}{\partial t}\iint_S \boldsymbol{\mathcal{E}}\cdot\mathrm{d}\mathbf{S} = \mu_0 I + \frac{1}{c^2}\frac{\partial\phi_\mathrm{E}}{\partial t}, \tag{1.316}$$

where $\phi_\mathrm{E} = \iint_S \boldsymbol{\mathcal{E}}\cdot\mathrm{d}\mathbf{S}$ is the electric flux and I is the current.

The Ampère law states that the line integral of the magnetic field $\mathcal{B}$ over a closed loop ℓ is equal to sum of the net free current $I = \iint_S \mathbf{j}\cdot\mathrm{d}\mathbf{S}$ passing through the surface enclosed by the closed loop ℓ and the rate of change of the electric flux ϕ_E through the same surface.

(c) Based on Maxwell equations in **(a)** above and assuming that there are no electric currents or charges present in vacuum, Maxwell equations are simplified to read, as shown in Table 1.17.

Table 1.22 The four Maxwell equations for free space in differential and integral form

	Equation	Differential form	Integral form	
1	Maxwell *Gauss law for electricity*	$\nabla \cdot \mathcal{E} = 0$	$\oint\!\!\oint_S \mathcal{E} \cdot d\mathbf{S} = 0$	(1.317)
2	Maxwell *Gauss law for magnetism*	$\nabla \cdot \mathcal{B} = 0$	$\oint\!\!\oint_S \mathcal{B} \cdot d\mathbf{S} = 0$	(1.318)
3	Maxwell *Faraday law of induction*	$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$	$\oint_\ell \mathcal{E} \cdot d\boldsymbol{\ell} = -\frac{\partial}{\partial t} \iint_S \mathcal{B} \cdot d\mathbf{S} = -\frac{\partial \phi_M}{\partial t}$	(1.319)
4	Maxwell *Ampère law* extended by Maxwell	$\nabla \times \mathcal{B} = \frac{1}{c^2} \frac{\partial \mathcal{E}}{\partial t}$	$\oint_\ell \mathcal{B} \cdot d\boldsymbol{\ell} = \frac{1}{c^2} \frac{\partial}{\partial t} \iint_S \mathcal{E} \cdot d\mathbf{S} = \frac{1}{c^2} \frac{\partial \phi_E}{\partial t}$	(1.320)

1.30 Poynting Theorem and Poynting Vector

1.30.Q1 **(58)**

In 1884 English physicist John Henry Poynting used the Lorentz equation for a moving charge in an electromagnetic (EM) field and Maxwell equations for electromagnetism to derive a theorem that expresses the conservation of energy for EM fields. The theorem relates the rate of change of the energy u stored in the EM field and energy flow expressed by the Poynting vector $\mathbf{S}$. The Poynting vector $\mathbf{S}$ points in the direction of motion of the EM wave, coincides with the direction of energy flow, and is generally expressed as

$$\mathbf{S} = \frac{\mathcal{E} \times \mathcal{B}}{\mu_0} = \varepsilon_0 c^2 \mathcal{E} \times \mathcal{B} \tag{1.321}$$

where c, the speed of light in vacuum, is given by the standard expression $c = 1/\sqrt{\varepsilon_0 \mu_0}$

(a) Express the Poynting vector $\mathbf{S}$ of (1.321) for electromagnetic radiation.
(b) Determine the intensity I of electromagnetic radiation.
(c) Pressure p_{rad} that EM radiation exerts on an absorbing target.

Note: Both I and p_{rad} are related to the mean Poynting vector $\overline{|\mathbf{S}|} = \bar{S}$ of the EM radiation.

SOLUTION:

(a) The magnitude of the Poynting vector $|\mathbf{S}| = S$ equals to the power per unit area crossing a surface normal to the direction of $\mathbf{S}$. Electric field $\mathcal{E}$ and magnetic field $\mathcal{B}$ are perpendicular to one another as well as to the direction of wave propagation and expressed generically as

$$\mathcal{E} = \mathcal{E}_0 f(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad \text{and} \quad \mathcal{B} = \mathcal{B}_0 f(\mathbf{k} \cdot \mathbf{r} - \omega t), \tag{1.322}$$

where

$\mathbf{k}$ is the wave vector in direction of wave propagation.
ω is the angular frequency of the plane wave.
$\mathcal{E}_0$ is the constant amplitude vector for the electric field $\mathcal{E}$.
$\mathcal{B}_0$ is the constant amplitude vector for the magnetic field $\mathcal{B}$.

In (1.322) function f satisfies the wave equation $\nabla^2 f = c^{-2}\partial^2 f/\partial t^2$ that is derived from the four Maxwell equations for free space (see Prob. 58). For electric field $\boldsymbol{\mathcal{E}}$ and magnetic field $\boldsymbol{\mathcal{B}}$ function f is usually given as

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad \text{and} \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{1.323}$$

or

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t) \quad \text{and} \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t). \tag{1.324}$$

The Poynting vector (1.321) can thus be expressed as follows

$$\mathbf{S} = \frac{\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{B}}}{\mu_0} = \varepsilon_0 c^2 \boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{B}} = \varepsilon_0 c^2 \boldsymbol{\mathcal{E}}_0\times\boldsymbol{\mathcal{B}}_0 \cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t). \tag{1.325}$$

(b) In practice S varies at exceedingly large frequencies, making it more practical to use the mean value of S averaged over one period of oscillation or over a multiple integer number of periods. The mean value of S over one period is thus the mean power per unit area in the wave and is expressed as

$$\bar{S} = \overline{|\mathbf{S}|} = \varepsilon_0 c^2 |\boldsymbol{\mathcal{E}}_0\times\boldsymbol{\mathcal{B}}_0| \overline{\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{1.326}$$

The mean value of $\cos^2(\mathbf{kr}-\omega t)$ is evaluated as follows

$$\begin{aligned}\overline{\cos^2 x} &= \frac{1}{2\pi}\int_0^{2\pi} \cos^2 x \,\mathrm{d}x = \frac{1}{4\pi}\int_0^{2\pi} \cos 2x \,\mathrm{d}x + \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}x \\ &= \frac{1}{8\pi}[\sin 2x + 2x]_0^{2\pi} = \frac{1}{2},\end{aligned} \tag{1.327}$$

providing the following expression for the mean Poynting vector $\bar{S}$ that is also referred to as the EM intensity or irradiance I

$$\bar{S} = I = \frac{1}{2}\varepsilon_0 c^2 \mathcal{E}_0 \mathcal{B}_0 = \frac{1}{2}\varepsilon_0 c \mathcal{E}_0^2 = \frac{1}{2}\frac{c}{\mu_0}\mathcal{B}_0^2, \tag{1.328}$$

where we used the following equality: $\mathcal{E}_0 = c\mathcal{B}_0$. A plot of $\cos^2 x$ against x is given in Fig. 1.25 which also shows that $\overline{\cos^2 x}$ equals to 0.5 for one period or integer number of periods.

(c) Electromagnetic radiation pressure p_{rad} results from the momentum carried by radiation and is defined as the force per unit area exerted by an EM wave upon

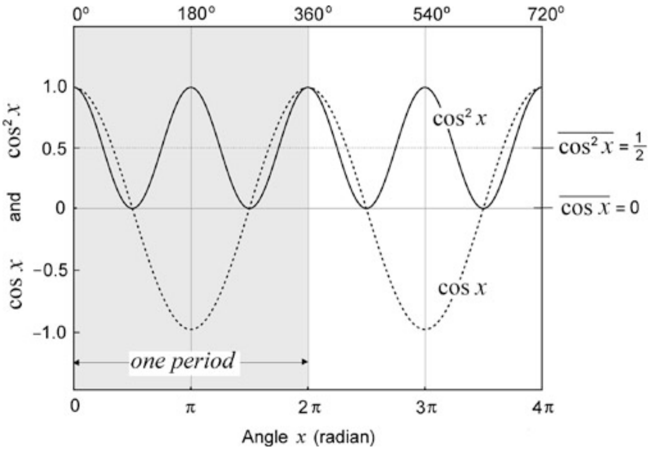


Fig. 1.25 Trigonometric functions $\cos x$ (*dotted curve*) and $\cos^2 x$ (*solid curve*) against angle x. For one period or for an integer number of periods $\overline{\cos x} = 0$ and $\overline{\cos^2 x} = 0.5$

the surface of a target exposed to EM radiation or energy per unit volume (energy density) carried by radiation. It acts in the direction of the Poynting vector **S**.

Since the mean Poynting vector $\bar{S}$ is the mean power per unit area A in the wave, we can say that the power P in the wave is $P = \bar{S}A$ and the force F of the wave when it hits a target is power P over velocity c of the wave, i.e., $F = \bar{S}A/c$. Therefore, the radiation pressure $p_{\rm rad}$ is given as

$$p_{\rm rad} = \frac{F}{A} = \frac{\bar{S}}{c} \tag{1.329}$$

when the target fully absorbs the incident radiation and $p_{\rm rad} = 2\bar{S}/c$ when the target fully reflects the incident radiation. Radiation pressure is thus proportional to the EM intensity I (mean Poynting vector $\bar{S}$) and inversely proportional to the speed of light c in vacuum

$$p_{\rm rad} = \frac{I}{c} = \frac{1}{2}\varepsilon_0 \mathcal{E}_0^2 = \frac{1}{2\mu_0}\mathcal{B}_0^2. \tag{1.330}$$

1.31 Normal Probability Distribution

1.31.Q1 **(59)**

Random variation in natural processes most commonly follows the probability distribution generally known in mathematics as the normal probability distribution but also referred to as Gaussian distribution in physics and "bell curve" in social science. The function describing the normal distribution has a long tradition in mathematics and physics. De Moivre used it in 18th century as an approximation to the binomial distribution, Laplace used it to study measurement errors, and Gauss used it in his analysis of astronomical data.

For the probability density function

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \tag{1.331}$$

with the mean value $\bar{x} = 0$ (also referred to as the expectation value $\bar{x} = 0$) and for the following values of standard deviation $\sigma = 0.5; 1; 2; 3; 5;$ and 10:

(a) Plot $P(x)$ for the following values of σ: 0.5; 1; 2; 3; 5; and 10.
(b) Show that $\int_{-\infty}^{\infty} P(x)\,\mathrm{d}x = 1$ for all σ.
(c) Determine $P(0)$ as a function of σ.
(d) Determine $P(\sigma)$ as a function of σ.
(e) Determine the general expression for the full-width-at-half-maximum (FWHM) as a function of σ.
(f) Summarize your results of **(a)** through **(e)** in Table 1.23A.

Table 1.23A Summary of results **(a)** through **(e)**

1	σ	0.5	1.0	2.0	3.0	5.0	10.0
2	$P(0) =$						
3	$P(\sigma) =$						
4	FWHM =						

SOLUTION:

(a) A plot of the probability density function $P(x)$ against x for various values of the standard deviation σ is given in Fig. 1.26; a plot of $\sigma P(x)$ against x/σ is shown in Fig. 1.27.

(b)

$$\int_{-\infty}^{\infty} P(x)\,\mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x = \frac{2\sigma\sqrt{2}}{\sigma\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{x^2}{2\sigma^2}} \frac{\mathrm{d}x}{\sigma\sqrt{2}}$$
$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2}\,\mathrm{d}u = \mathrm{erf}(\infty) = 1. \tag{1.332}$$

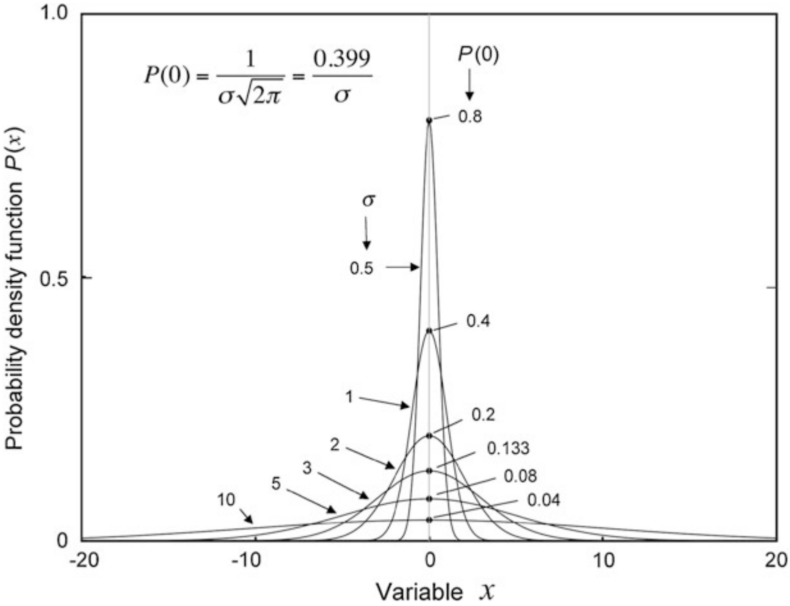


Fig. 1.26 Probability density function $P(x)$ against coordinate x for various values of the standard deviation σ in the range from $\sigma = 0.5$ to $\sigma = 10$

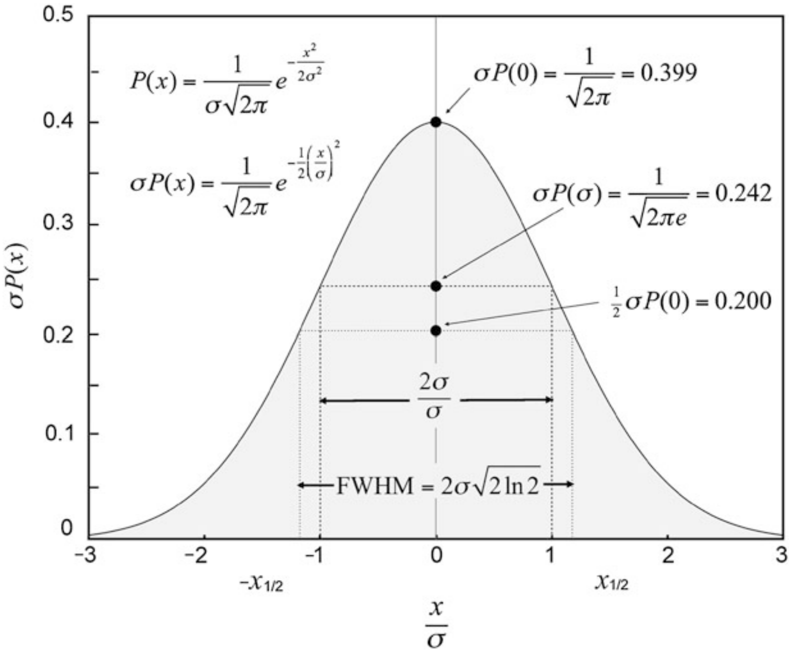


Fig. 1.27 Plot of $\sigma P(x)$, the probability density function $P(x)$ multiplied by standard deviation $P(x)$ against x/σ, the coordinate x divided by σ

Values for the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,\mathrm{d}u$ are available in standard mathematical tables and for $x = \infty$ we get $\mathrm{erf}(\infty) = 1$. Thus, integral $\int_{-\infty}^{\infty} P(x)\,\mathrm{d}x = 1$ for all σ.

(c)

$$P(0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\bigg|_{x=0} = \frac{1}{\sigma\sqrt{2\pi}} = \frac{0.399}{\sigma}. \tag{1.333}$$

(d)

$$P(\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\bigg|_{x=\sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sigma\sqrt{2\pi e}} = \frac{0.242}{\sigma}$$
$$= \frac{P(0)}{\sqrt{e}} = 0.6065 P(0). \tag{1.334}$$

(e) The maximum in $P(x)$ occurs at $x = 0$ and the value of $P(0)$ is given in (**c**) above. The full-width-at-half-maximum (FWHM) occurs at $x = \pm x_{1/2}$ where

$$P(x_{1/2}) = \frac{1}{2} P(0) = \frac{1}{2\sigma\sqrt{2\pi}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x_{1/2}^2}{2\sigma^2}} \tag{1.335}$$

or

$$\ln\frac{1}{2} = -\frac{x_{1/2}^2}{2\sigma^2} \quad \text{and} \quad x_{1/2} = \sigma\sqrt{2\ln 2} = 1.177\sigma. \tag{1.336}$$

Since by definition the full-width-at-half-maximum (FWHM) equals to $2x_{1/2}$, we get

$$\mathrm{FWHM} = 2x_{1/2} = 2\sigma\sqrt{2\ln 2} = 2.355\sigma. \tag{1.337}$$

(f) Results of **(a)** through **(e)** are summarized in Table 1.23B.

Table 1.23B Summary table for Prob. 59

1	σ	0.5	1.0	2.0	3.0	5.0	10.0
2	$P(0) = \frac{0.399}{\sigma}$	0.800	0.400	0.200	0.133	0.080	0.040
3	$P(\sigma) = \frac{0.242}{\sigma}$	0.484	0.242	0.121	0.081	0.048	0.024
4	$\mathrm{FWHM} = 2.355\sigma$	1.178	2.355	4.710	7.064	11.77	23.55

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