Chapter 2
Interactive Intuitionistic Fuzzy Multi-Attribute Decision Making

During the last few years, more and more researchers have been applying IFSs to multi-attribute decision making under various different situations, and a lot of work has been done (Chen and Tan 1994; Hong and Choi 2000; Xu and Yager 2006; Boran et al. 2009; Xu 2010a, c; Zhao 2009; Zhao et al. 2010; Tan and Chen 2010; Wei 2010a; Xu and Xia 2012a; Xu and Cai 2010; Wu and Chen 2011, etc.). The considered decision making situations can be roughly classified as follows:

(1) Multi-attribute decision making situations where the attribute weights are completely known. Most of the existing research on intuitionistic fuzzy multi-attribute decision making has been focused on this issue. In early work, Chen and Tan (1994) developed an approach to intuitionistic fuzzy multi-attribute decision making by utilizing the minimum and maximum operations and the score function. Later, Hong and Choi (2000) improved Chen and Tan (1994)'s technique through adding an accuracy function. Considering that the minimum and maximum operations adopted in these two papers may produce the loss of considerable decision information, Xu and Yager (2006) proposed some geometric mean operators for aggregating intuitionistic fuzzy information, such as the intuitionistic fuzzy weighted geometric aggregation operator, the intuitionistic fuzzy ordered weighted geometric aggregation operator and the intuitionistic fuzzy hybrid aggregation operator, and applied them to develop a procedure for multi-attribute decision making with intuitionistic fuzzy information. Motivated by the well known TOPSIS, Boran et al. (2009) suggested a method to select the appropriate supplier in intuitionistic fuzzy group decision making environments, in which the intuitionistic fuzzy weighted averaging operator (Xu 2007e) was utilized to aggregate individual opinions of experts for rating the importance of attributes and alternatives. Zhao (2009) also used the the intuitionistic fuzzy weighted averaging operator to establish an evaluation model for intellectual capital with intuitionistic fuzzy information. On the basis of the idea of Yager (2004b)'s generalization aggregation, Zhao et al. (2010) introduced the generalized intuitionistic fuzzy weighted aggregation operator, the generalized intuitionistic fuzzy ordered weighted aggregation operator and the generalized

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intuitionistic fuzzy hybrid aggregation operator, and gave their application to intuitionistic fuzzy multi-attribute decision making. Inspired by the correlation properties of the traditional Choquet integral (Choquet 1953), Xu (2010c) proposed the intuitionistic fuzzy correlated averaging operators and the intuitionistic fuzzy correlated geometric operators, whose characteristic is that they can not only consider the importance of the elements or their ordered positions, but also reflect the correlations of the elements or their ordered positions. The developed operators were then applied to a practical decision making problem involving the prioritization of information technology improvement projects. Tan and Chen (2010) also studied the desirable characteristics of the intuitionistic fuzzy correlated averaging operators, based on which they gave an approach to multi-attribute decision making with intuitionistic fuzzy information, and applied it to solve a practical decision making problem where a manufacturing company wants to select the best global supplier according to the core competencies of potential suppliers. Based on the idea of Yager (2003)'s order induced aggregation, Wei (2010a) introduced an induced intuitionistic fuzzy ordered weighted geometric operator, and gave a corresponding technique for multi-attribute group decision making. Combining the order induced aggregation and generalized aggregation, Xu and Xia (2011) developed some new types of aggregation operators, including the induced generalized intuitionistic fuzzy Choquet integral operators and the induced generalized intuitionistic fuzzy Dempster-Shafer operators, and used them to financial decision making under intuitionistic fuzzy environments. Xu and Yager (2011) applied the weighted intuitionistic fuzzy Bonferroni mean to multi-attribute decision making, which can also reflect the interrelationship of the individual attributes, and thus can take the decision information into account as much as possible. Li (2010) developed a nonlinear programming methodology that is based on the TOPSIS to solve multi-attribute decision making problems with both the ratings of alternatives on attributes and the weights of attributes expressed with interval-valued intuitionistic fuzzy sets. Nayagama et al. (2011) focused on the technique for ranking IVIFSs in multi-attribute decision making. Wu and Chen (2011) developed an intuitionistic fuzzy ELECTRE (Elimination et choix traduisant la realité) method, for solving multi-attribute decision making problems. Devi (2011) extended VIKOR (the višekriterijumsko kompromisno rangiranje) method to solve multi-attribute decision making problems in which the weights of criteria and ratings of alternatives are taken as triangular intuitionistic fuzzy set, and applied it to robot selection problem for material handling task. Additionally, the Dempster–Shafer Theory of evidence (Baldwin 1987), has also been extended to multi-attribute decision making under intuitionsitic fuzzy environments (Dymova and Sevastjanov 2010).

(2) Multi-attribute decision making situations where the weight information on attributes is completely unknown. In these situations, it is necessary to determine the attribute weights in advance. Xu (2010a) first established an optimization model by which a straightforward formula for deriving attribute weights can be obtained, and then on the basis of information theory, the intuitionistic fuzzy

hybrid aggregation operator, the intuitionistic fuzzy weighted averaging operator, the score function and the accuracy function, he developed an approach to intuitionistic fuzzy group decision making. Xu and Cai (2010) established several nonlinear optimization models to determine the weights of the experts and the attributes, and utilized the simple additive weighting method to aggregate all the intuitionistic fuzzy information so as to rank and select the alternatives. Li et al. (2010) developed a linear programming methodology for solving multi-attribute group decision making problems using IFSs, in which the attribute weights are estimated using a new auxiliary linear programming model, which minimizes the group inconsistency index under some constraints, and the distances of the alternatives from the intuitionistic fuzzy positive ideal solution can be calculated to determine their ranking order.

(3) Multi-attribute decision making situations where the weight information on attributes is partially known. For a type of special situations where the weight information is expressed in interval ranges, Li (2005) constructed some linear programming models to generate the optimal weights for attributes, and proposed the corresponding multi-attribute decision making methods using IFSs. For more general situations where the information about attribute weights is partially known, which may be constructed by various forms, such as weak rankings, strict rankings, rankings with multiples, interval forms, and rankings of differences, Xu (2007h) established a linear programming model, a multi-objective optimization model and a single-objective optimization model to determine the optimal attribute weights. Based on which he used the intuitionistic fuzzy weighted geometric aggregation operator, the intuitionistic fuzzy hybrid geometric aggregation operator, the score function and the accuracy function to develop an approach to intuitionistic fuzzy multi-attribute group decision making, and gave its application to search the best global supplier for one of a manufacturing company's most critical parts used in assembling process. Xu (2007g) defined the concept of intuitionistic fuzzy ideal solution (IFIS), and then, based on the IFIS and the distance measure, he established some optimization models to derive the attribute weights. Furthermore, based on the established models, he developed some procedures for the rankings of alternatives under different situations, and extended the established models and procedures to interval-valued intuitionistic fuzzy environments. Park et al. (2009, 2011) also extended the TOPSIS method and the correlation coefficient method to solve multi-attribute group decision making problems in interval-valued intuitionistic fuzzy environment in which all the preference information provided by the experts is presented as interval-valued intuitionistic fuzzy decision matrices where each of the elements is characterized by IVIFVs, and the information about attribute weights is partially known. Wei (2010b) extended the grey relational analysis method to solve the intuitionistic fuzzy multi-attribute decision making problem with partial weight information.

(4) Dynamic multi-attribute decision making situations. In many situations, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation, information is

usually collected at different periods, in which the time has to be taken into account. For this issue, Xu and Yager (2008) presented the BUM function based method, the normal distribution based method, the exponential distribution based method and the average age method to determine the time series weights. They further developed the dynamic intuitionistic fuzzy weighted aggregation operator, based on which they proposed a procedure for dynamic intuitionistic fuzzy multi-attribute decision making, and used it to prioritize a set of agroecological regions in Hubei Province, China. Wang and Wei (2009) applied the intuitionistic fuzzy weighted geometric aggregation operator and the dynamic intuitionistic fuzzy weighted geometric operators to a supplier selection problem in supply chain management.

Nevertheless, in many real-life situations such as negotiation processes, a decision maker (or an expert) often needs to interact with group members (or analysts) in the process of decision making by providing and modifying his/her preference information gradually. All the existing methods introduced above are unsuitable for dealing with these situations, and thus, there is a necessity to develop some new techniques for interactive decision making under intuitionistic fuzzy envirements. To solve this issue, Xu and Xia (2012a) introduced the concept of dominated alternative, and gave a method to identify the dominated alternatives. Then they developed an interactive method for eliminating any dominated alternatives by updating the decision maker's preferences gradually so as to find out the optimal one eventually. A further extension of the interactive method to interval-valued intuitionistic fuzzy situations was given. Xu (2012b) defined the concepts of the overall attribute ideal solution and the overall attribute negative ideal solution of alternatives. Based on these two solutions, he defined the satisfaction degree of each alternative. After that, he established a multi-objective optimization model, and then transformed it into a single-objective optimization model. Furthermore, he established an interactive method for multi-attribute decision making with intuitionistic fuzzy information, and the extended results in interval-valued intuitionistic fuzzy situations were also pointed out.

2.1 Interactive Intuitionistic Fuzzy Multi-Attribute Decision Making by Identifying and Eliminating Dominated Alternatives

Consider a multi-attribute decision making problem, let Y, G and w be as defined previously, $w \in \Psi$, and Ψ is the set of the known weight information provided by the decision maker (or expert). If the weight information in the set Ψ is contradictory, and then Ψ is an empty set. In this case, the set Ψ should be returned to the decision maker for reconstruction until the revaluated weight information is not contradictory. The provided weight information may take one or some of the following forms (Park and Kim 1997; Park 2004; Xu and Chen 2007a; Kim and Ahn 1999; Kim et al. 1999; Xu 2006b, 2007a, c, 2010b), for $i \neq j$:

(1) A weak ranking: $\{w_i \geq w_j\}$;
(2) A strict ranking: $\{w_i - w_j \geq \alpha_i\}$;
(3) A ranking of differences: $\{w_i - w_j \geq w_k - w_l\}$, for $j \neq k \neq l$;
(4) A ranking with multiples: $\{w_i \geq \alpha_i w_j\}$;
(5) An interval form: $\{\alpha_i \leq w_i \leq \alpha_i + \varepsilon_i\}$,

where $\{\alpha_i\}$ and $\{\varepsilon_i\}$ are the non-negative constants. The forms (1), (2) and the forms (4), (5) are the well known types of imprecise information, and the form (3) is a ranking of differences of adjacent parameters obtained by weak rankings among the parameters, which can be subsequently constructed based on the form (1). Park (2004) gave a detailed interpretation as to when the incomplete weight information forms (1)–(5) could occur in practice: For some qualitative factors (e.g., level of management, skill level of labors, or ease of use), the decision maker may argue that the attribute G_i is the best, the attribute G_j is the second, and the attribute G_k is the worst. This information can be described with the corresponding weights of these three attributes in the form of weak orders $w_i \geq w_j \geq w_k$ or strict orders $w_i - w_j \geq \alpha_i$ and $w_j - w_k \geq \alpha_j$ with the positive constants α_i and α_j, i.e., the weight of the attribute G_i exceeds that of the attribute G_j by at least α_i, and the weight of the attribute G_j exceeds that of the attribute G_k by at least α_j (Cook and Kress 1991). The difference order $w_i - w_j \geq w_k - w_l$ is possible when the preference difference between w_i and w_j is greater than or equal to that between w_i and w_j, which is also referred to as strength of preference (Malakooti 2000). For the form (4), the decision maker may argue that the attribute G_j is the best (100 %), and the attribute G_i is in the level greater than or equal to $\alpha_i (0 \leq \alpha_i \leq 1)$ relative to the level of the attribute G_j. That is to say, the weight of the attribute G_i is greater than or equal to α_i times of that of the attribute G_j, which is expressed as $w_i \geq \alpha_i w_j$. The form (5) indicates that the crisp weight can not be specified but value range can be obtained, this type of weights is called interval weights. It is the most common form to describe the incomplete information about attribute weights, which can be provided by the decision maker directly (Yoon 1989; Bryson and Mobolurin 1997).

Let $B = (b_{ij})_{m \times n}$ be an intuitionistic fuzzy decision matrix, where $b_{ij} = (t_{ij}, f_{ij}, \pi_{ij})$ is an attribute value, which is expressed in an IFV, satisfying

$$t_{ij} \in [0, 1], \quad f_{ij} \in [0, 1], \quad t_{ij} + f_{ij} \leq 1, \quad \pi_{ij} = 1 - t_{ij} - f_{ij},$$
$$i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n \tag{2.1}$$

If all the attributes $G_i (i = 1, 2, \ldots, m)$ are of the same type, then the attribute values do not need normalization. Whereas, there are generally benefit attributes (i.e., the bigger the attribute values the better) and cost attributes (i.e., the smaller the attribute values the better) in multi-attribute decision making. In such cases, we may transform the attribute values of cost type into the attribute values of benefit type (Xu and Hu 2010), then $B = (b_{ij})_{m \times n}$ can be transformed into the intuitionistic fuzzy decision matrix $R = (r_{ij})_{m \times n}$, where

$$r_{ij} = (\mu_{ij}, v_{ij}, \pi_{ij}) = \begin{cases} b_{ij}, & \textit{for the benefit attribute } G_i \\ \left(b_{ij}\right)^{\mathrm{c}}, & \textit{for the } \cos t \textit{ attribute } G_i \end{cases}, \quad j = 1, 2, \ldots, n \tag{2.2}$$

where $\left(b_{ij}\right)^{\mathrm{c}}$ is the complement of b_{ij}, such that $\left(b_{ij}\right)^{\mathrm{c}} = (f_{ij}, t_{ij}, \pi_{ij})$, clearly, $\pi_{ij} = 1 - t_{ij} - f_{ij} = 1 - \mu_{ij} - v_{ij}, i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n.$

Since each atribute value r_{ij} must satisfy the condition $\mu_{ij} + v_{ij} \le 1$, i.e., $\mu_{ij} \le 1 - v_{ij}$, then we can transform the atribute value r_{ij} into the interval $r_{ij} = [\mu_{ij}, 1 - v_{ij}]$, whose expected value can be expressed as:

$$E(r_{ij}) = \frac{1}{2}\left(\mu_{ij} + 1 - v_{ij}\right) = \frac{1}{2}\left(\mu_{ij} + \mu_{ij} + \pi_{ij}\right) = \mu_{ij} + \frac{1}{2}\pi_{ij}$$
$$i = 1, 2, \ldots, m; \;\; j = 1, 2, \ldots, n \tag{2.3}$$

based on which we get the overall expected attribute value $E(r_j)$ corresponding to each alternative y_j by using the simple additive weighted method:

$$E(r_j) = \sum_{i=1}^{m} w_i E(r_{ij}) = \sum_{i=1}^{m} w_i \left(\mu_{ij} + \frac{1}{2}\pi_{ij}\right), \quad j = 1, 2, \ldots, n \tag{2.4}$$

The overall expected attribute values $E(r_j)(j = 1, 2, \ldots, n)$ are generally used to rank the corresponding alternatives $y_j(j = 1, 2, \ldots, n)$ (Saaty 1980; Chen and Hwang 1992; Xu and Chen 2007a). The larger the overall expected attribute value $E(r_j)$, the better the alternative y_j. Motivated by this idea and similar to Xu and Chen (2007a), we define the following:

Definition 2.1 (Xu and Xia 2012a). For an alternative $y_k \in Y$, if there exists $y_j \in Y$ such that $E(r_j) > E(r_k)$, then y_k is called the dominated alternative; Otherwise, the alternative y_k is called a non-dominated alternative.

To identify the dominated alternatives in Y, Xu and Xia (2012a) gave the following theorem, which can be proven similar to Xu and Chen (2007a):

Theorem 2.1 (Xu and Xia 2012a). The alternative y_k is the dominated alternative if and only if $J_k < 0$, where

MOD-2.1
$$J_k = \max_{w,\varsigma} \left(\sum_{i=1}^{m} w_i \left(\mu_{ik} + \frac{1}{2}\pi_{ik}\right) + \varsigma \right)$$
$$s.t. \quad \sum_{i=1}^{m} w_i \left(\mu_{ik} + \frac{1}{2}\pi_{ik}\right) + \varsigma \le 0, \quad j \ne k$$
$$w = (w_1, w_2, \ldots, w_m)^T \in \Psi$$
$$w_i \ge 0, \quad i = 1, 2, \ldots, m, \quad \sum_{i=1}^{m} w_i = 1$$

where ς is only an unconstrained auxiliary variable, which has no actual meaning.

It is noted that MOD-2.1 is a linear programming problem which can be solved easily by the Lingo or Matlab software.

Considering that the dominated alternatives are inferior to the non-dominated alternatives, they should be eliminated so as to diminish the given alternative set. In fact, the larger the value of J_k, the better the alternative y_k. If there are multiple no-dominated alternatives, then we can rank them according to the value of J_k. Although the weights in the vector $w = (w_1, w_2, \ldots, w_m)^T$ are allowed to be 0, i.e., $w_i \geq 0, \quad i = 1, 2, \ldots, m$, but they should satisfy the condition $\sum_{i=1}^{m} w_i = 1$, which guarantees that we can choose the optimal solution.

Based on Theorem 2.1, Xu and Xia (2012a) developed an interactive method for identifying and eliminating the dominated alternatives in intuitionistic fuzzy multi-attribute decision making with partial attribute weight information:

Method 2.1:

Step 1. Let Y, G, w, Ψ and B be defined as before. Utilize Eq. (2.2) to transform B into the normalized intuitionistic fuzzy decision matrix $R = (r_{ij})_{m \times n}$.

Step 2. Calculate the expected attribute value $E(r_{ij})$ of each attribute value r_{ij} in R by using Eq. (2.3), and then get the overall expected attribute value $E(r_j)$ corresponding to each alternative y_j by using the simple additive weighted method (2.4).

Step 3. Identify whether the alternative y_j is the dominated alternative or not through Theorem 2.1, and omit the dominated alternatives, and then get the set $\bar{Y}$, whose elements are the non-dominated ones. If the decision maker suggests that an alternative $y_k \in \bar{Y}$ be preferred to any other alternatives in $\bar{Y}$, or the alternative y_k is only one left in $\bar{Y}$, then the alternative y_k is the optimal one, go to Step 5; Otherwise, go to Step 4.

Step 4. Interact with the decision maker, and ask him/her to update his/her preferences, which are added as the new weight information to the set Ψ, if the updated information contradicts the weight information in Ψ, then return it to the decision maker for re-evaluation, and go to Step 3.

Step 5. End.

Higgins et al. (2008) applied a multi-objective greedy randomised adaptive search process as an evolutionary method to find solutions along the Pareto front in a non-linear integer-programming problem. The method is mainly used to deal with the usual fuzzy multi-objective decision making problems in which some objectives may be competitive, and without interaction with the decision maker. Compared to Higgins et al. (2008), the focus of Xu and Xia (2012a) is to propose an interactive method for solving intuitionistic fuzzy decision making problems. This interactive method can eliminate any dominated alternatives by updating the decision maker's preferences gradually so as to find out the optimal one eventually, and thus can make the decision result more reasonable. Moreover, MOD 2.1 is mainly to identify the non-dominated alternatives by maximizing the expected value of each alternative, which can be solved by a simple linear programming technique, and thus, it is much simpler than Higgins et al. (2008)'s method.

In what follows, we shall extend the developed interactive method to interval-valued intuitionistic fuzzy multi-attribute decision making environments.

In the cases where the decision maker can only provide interval numbers instead of the exact real numbers, and then the evaluation values given by the decision maker in the problem considered above are expressed as the IVIFVs (Xu and Hu 2010): $\tilde{b}_{ij} = (\tilde{t}_{ij}, \tilde{f}_{ij}, \tilde{\pi}_{ij})$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, which are called uncertain attribute values satisfying:

$$\begin{gathered}\tilde{t}_{ij} = [t_{ij}^-, t_{ij}^+] \subseteq [0,1], \quad \tilde{f}_{ij} = [f_{ij}^-, f_{ij}^+] \subseteq [0,1], \quad t_{ij}^+ + f_{ij}^+ \le 1, \\ \pi_{ij}^- = 1 - t_{ij}^+ - f_{ij}^+, \quad \pi_{ij}^+ = 1 - t_{ij}^- - f_{ij}^-, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n\end{gathered} \tag{2.5}$$

and they are contained in an interval-valued intuitionistic fuzzy decision matrix $\tilde{B} = (\tilde{b}_{ij})_{m\times n}$. Then we transform the attribute values of cost type into the attribute values of benefit type, i.e., transform $\tilde{B}$ into the normalized interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$, where

$$\tilde{r}_{ij} = (\tilde{\mu}_{ij}, \tilde{v}_{ij}, \tilde{\pi}_{ij}) = \begin{cases} \tilde{b}_{ij}, & \textit{for the benefit attribute } G_i \\ \left(\tilde{b}_{ij}\right)^{\mathrm{c}}, & \textit{for the } \cos t \textit{ attribute } G_i \end{cases}, \quad j = 1, 2, \ldots, n \tag{2.6}$$

where $\left(\tilde{b}_{ij}\right)^{\mathrm{c}}$ is the complement of $\tilde{b}_{ij}$, such that $\left(\tilde{b}_{ij}\right)^{\mathrm{c}} = \left(\tilde{f}_{ij}, \tilde{t}_{ij}, \tilde{\pi}_{ij}\right)$, and

$$\begin{gathered}\tilde{\mu}_{ij} = [\mu_{ij}^-, \mu_{ij}^+], \quad \tilde{v}_{ij} = [v_{ij}^-, v_{ij}^+], \quad \tilde{\pi}_{ij} = [\pi_{ij}^-, \pi_{ij}^+], \\ \pi_{ij}^- = 1 - t_{ij}^+ - f_{ij}^+ = 1 - \mu_{ij}^+ - v_{ij}^+, \quad \pi_{ij}^+ = 1 - t_{ij}^- - f_{ij}^- = 1 - \mu_{ij}^- - v_{ij}^-, \\ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n\end{gathered} \tag{2.7}$$

Similar to Eq. (2.3), we can calculate the expected value of the uncertain atribute value $\tilde{r}_{ij}$ as:

$$\begin{aligned}E(\tilde{r}_{ij}) &= \frac{1}{4}\left(\mu_{ij}^- + \mu_{ij}^+ + 2 - v_{ij}^- - v_{ij}^+\right) = \frac{1}{4}\left(\mu_{ij}^- + \mu_{ij}^+ + 1 - v_{ij}^- + 1 - v_{ij}^+\right) \\ &= \frac{1}{4}\left(2\mu_{ij}^- + 2\mu_{ij}^+ + \pi_{ij}^+ + \pi_{ij}^-\right), \quad i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n\end{aligned} \tag{2.8}$$

Then we use the simple additive weighted method:

$$
\begin{aligned}
E(\tilde{r}_j) &= \sum_{i=1}^{m} w_i E(\tilde{r}_{ij}) \\
&= \frac{1}{4}\sum_{i=1}^{m} w_i \left(2\mu_{ij}^{-} + 2\mu_{ij}^{+} + \pi_{ij}^{-} + \pi_{ij}^{+}\right), \quad j = 1, 2, \ldots, n
\end{aligned}
\tag{2.9}
$$

to get the overall expected uncertain attribute value $E(\tilde{r}_j)$ corresponding to each alternative y_j. The larger the value of $E(\tilde{r}_j)$, the better the alternative y_j.

If there exists $y_k \in Y$ such that $E(\tilde{r}_k) > E(\tilde{r}_j)$, then y_j is called a dominated alternative; Otherwise, the alternative y_j is called a non-dominated alternative.

Similar to Theorem 2.1, we have

Theorem 2.2 (Xu and Xia 2012a). The alternative y_j is the dominated alternative if and only if $\dot{J}_j < 0$, where

$$
\textbf{MOD-2.2} \quad \dot{J}_j = \max_{w,\varsigma} \left(\frac{1}{4}\sum_{i=1}^{m} w_i \left(2\mu_{ij}^{-} + 2\mu_{ij}^{+} + \pi_{ij}^{-} + \pi_{ij}^{+}\right) + \varsigma \right)
$$

$$
\begin{aligned}
s.\,t.\quad & \frac{1}{4}\sum_{i=1}^{m} w_i \left(2\mu_{ij}^{-} + 2\mu_{ij}^{+} + \pi_{ij}^{-} + \pi_{ij}^{+}\right) + \varsigma \le 0, \; k \ne j \\
& w = (w_1, w_2, \ldots, w_m)^T \in \Psi \\
& w_i \ge 0, \quad i = 1, 2, \ldots, m, \quad \sum_{i=1}^{m} w_i = 1
\end{aligned}
$$

where ς is only an unconstrained auxiliary variable, which has no actual meaning.

By Theorem 2.2, Xu and Xia (2012a) extended Method 2.1 to interval-valued intuitionistic fuzzy multi-attribute decision making situations:

Method 2.2:

Step 1. Let Y, G, w, Ψ and $\tilde{B}$ be as defined previously. Utilize Eq. (2.6) to transform $\tilde{B}$ into the normalized interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m \times n}$.

Step 2. Calculate the expected uncertain attribute value $E(\tilde{r}_{ij})$ of each uncertain attribute value $\tilde{r}_{ij}$ in $\tilde{R}$ by using Eq. (2.8), and then get the overall expected uncertain attribute value $E(\tilde{r}_j)$ corresponding to each alternative y_j by using the simple additive weighted method (2.9).

Step 3. See Method 2.1.

Step 4. See Method 2.1.

Step 5. End.

Now we show the application of the developed interactive method through a practical example:

Table 2.1 Intuitionistic fuzzy decision matrix $R = (r_{ij})_{5\times 5}$

	y_1	y_2	y_3	y_4	y_5
G_1	(0.3,0.1,0.6)	(0.8,0.1,0.1)	(0.5,0.3,0.2)	(0.5,0.2,0.3)	(0.3,0.6,0.1)
G_2	(0.4,0.3,0.3)	(0.3,0.4,0.3)	(0.6,0.2,0.2)	(0.3,0.5,0.2)	(0.4,0.2,0.2)
G_3	(0.5,0.1,0.4)	(0.6,0.3,0.1)	(0.3,0.5,0.2)	(0.6,0.1,0.3)	(0.7,0.1,0.2)
G_4	(0.7,0.2,0.1)	(0.4,0.5,0.1)	(0.5,0.4,0.1)	(0.4,0.5,0.1)	(0.5,0.3,0.2)
G_5	(0.4,0.4,0.2)	(0.6,0.3,0.1)	(0.7,0.2,0.1)	(0.6,0.2,0.2)	(0.5,0.4,0.1)

Example 2.1 (Chao and Chen 2009; Xu and Xia 2012a). Along with the advancement of information technology, the electronic learning (e-learning) has played an important role in teaching and learning, which has become more and more popular not only in different levels of schools but also in various commercial or industrial companies in Taiwan (Chao and Chen 2009). However, a successful e-learning depends on many factors (or criteria). In addition, based on the experience of teaching e-learning courses, e-learning material design, and real practice in Kao-Yuan University (KYU), Kaohsiung, Taiwan, Chao and Chen (2009) carefully examined and summarized five key factors (or criteria) to evaluate the effectiveness of an e-learning system. These five main factors are: (1) G_1: The e-learning material; (2) G_2: The quality of web learning platform; (3) G_3: The synchronous learning; (4) G_4: The learning record; and (5) G_5: The self-learning. A decision maker evaluates five e-learning systems $y_j (j = 1, 2, 3, 4, 5)$ with respect to these key factors $G_i (i = 1, 2, 3, 4, 5)$ by using the IFVs $r_{ij} = (\mu_{ij}, v_{ij}, \pi_{ij})$ $(i, j = 1, 2, 3, 4, 5)$, which are contained in the intuitionistic fuzzy decision matrix $R = (r_{ij})_{5\times 5}$ (see Table 2.1) and the weight information about attributes, given by the decision maker, is as follows:

$$\Psi = \Big\{ w_1 \le 0.1,\ 0.1 \le w_2 \le 0.5,\ 0.2 \le w_3 \le 0.3,\ w_4 \le 0.1,\ w_5 \le 0.4,\ w_4 \le w_5,$$
$$w_i \ge 0,\quad i = 1, 2, 3, 4, 5,\ \sum_{i=1}^{5} w_i = 1 \Big\} \tag{2.10}$$

Since all the attributes $G_i (i = 1, 2, 3, 4, 5)$ are the benefit attributes, then the attribute values in R do not need normalization.

In what follows, we can use Method 2.1 to derive the optimal e-learning system and show the solution process:

Step 1. Calculate the expected attribute value $E(r_{ij})$ of each attribute value r_{ij} in R by using Eq. (2.8):

$$
\begin{array}{llll}
E(r_{11}) = 0.60, & E(r_{12}) = 0.85, & E(r_{13}) = 0.60, & E(r_{14}) = 0.65 \\
E(r_{15}) = 0.35, & E(r_{21}) = 0.55, & E(r_{22}) = 0.45, & E(r_{23}) = 0.70 \\
E(r_{24}) = 0.40, & E(r_{25}) = 0.60, & E(r_{31}) = 0.70, & E(r_{32}) = 0.65 \\
E(r_{33}) = 0.40, & E(r_{34}) = 0.75, & E(r_{35}) = 0.80, & E(r_{41}) = 0.75 \\
E(r_{42}) = 0.45, & E(r_{43}) = 0.55, & E(r_{44}) = 0.45, & E(r_{45}) = 0.60 \\
E(r_{51}) = 0.50, & E(r_{52}) = 0.65, & E(r_{53}) = 0.75, & E(r_{54}) = 0.70
\end{array}
$$

$$E(r_{55}) = 0.55$$

and then get the overall expected attribute value $E(r_j)$ corresponding to each e-learning system y_j by using Eq. (2.9):

$$
\begin{aligned}
E(r_1) &= 0.60w_1 + 0.55w_2 + 0.70w_3 + 0.75w_4 + 0.50w_5 \\
E(r_2) &= 0.85w_1 + 0.45w_2 + 0.65w_3 + 0.45w_4 + 0.65w_5 \\
E(r_3) &= 0.60w_1 + 0.70w_2 + 0.40w_3 + 0.55w_4 + 0.75w_5 \\
E(r_4) &= 0.65w_1 + 0.40w_2 + 0.75w_3 + 0.45w_4 + 0.70w_5 \\
E(r_5) &= 0.35w_1 + 0.60w_2 + 0.80w_3 + 0.60w_4 + 0.55w_5
\end{aligned}
$$

By Theorem 2.1, we need to identify whether the e-learning system y_1 is a dominated alternative or not, and thus establish the following linear programming model:

$$
\begin{aligned}
J_1 = \max\ & (0.60w_1 + 0.55w_2 + 0.70w_3 + 0.75w_4 + 0.50w_5 + \varsigma_1 - \varsigma_2) \\
s.\,t.\ \ & 0.85w_1 + 0.45w_2 + 0.65w_3 + 0.45w_4 + 0.65w_5 + \varsigma_1 - \varsigma_2 \le 0 \\
& 0.60w_1 + 0.70w_2 + 0.40w_3 + 0.55w_4 + 0.75w_5 + \varsigma_1 - \varsigma_2 \le 0 \\
& 0.65w_1 + 0.40w_2 + 0.75w_3 + 0.45w_4 + 0.70w_5 + \varsigma_1 - \varsigma_2 \le 0 \\
& 0.35w_1 + 0.60w_2 + 0.80w_3 + 0.60w_4 + 0.55w_5 + \varsigma_1 - \varsigma_2 \le 0 \\
& w_1 \le 0.1,\ 0.1 \le w_2 \le 0.5,\ 0.2 \le w_3 \le 0.3, w_4 \le 0.1, w_5 \le 0.4, w_4 \le w_5 \\
& \varsigma_1 \ge 0,\ \varsigma_2 \ge 0,\ w_i \ge 0,\ \ i = 1, 2, 3, 4, 5,\ \ \sum_{i=1}^{5} w_i = 1
\end{aligned}
$$

where ς_1 and ς_2 are only the unconstrained auxiliary variables, which have no actual meaning. Solving this model, we get

$$\varsigma_1 = 0.0000, \quad \varsigma_2 = 0.6140, \quad J_1 = -0.0110$$

$$w_1 = 0.1000, \quad w_2 = 0.4800, \quad w_3 = 0.2200, \quad w_4 = 0.1000, \quad w_5 = 0.1000$$

Since J_1 is found to be a negative value, we determine that y_1 is the dominated e-learning system. Similarly, we have

(1) For the e-learning system y_2:

$$\varsigma_1 = 0.0000, \quad \varsigma_2 = 0.6180, \quad J_2 = -0.0090$$
$$w_1 = 0.1000, \quad w_2 = 0.3053, \quad w_3 = 0.2908, \quad w_4 = 0.0000, \quad w_5 = 0.3039$$

(2) For the e-learning system y_3:

$$\varsigma_1 = 0.0000, \quad \varsigma_2 = 0.5983, \quad J_3 = 0.0493$$
$$w_1 = 0.0976, \quad w_2 = 0.3561, \quad w_3 = 0.2000, \quad w_4 = 0.0000, \quad w_5 = 0.3463$$

(3) For the e-learning system y_4:

$$\varsigma_1 = 0.0000, \quad \varsigma_2 = 0.6208, \quad J_4 = 0.0285$$
$$w_1 = 0.0769, \quad w_2 = 0.1231, \quad w_3 = 0.3000, \quad w_4 = 0.1000, \quad w_5 = 0.4000$$

(4) For the e-learning system y_5:

$$\varsigma_1 = 0.0000, \quad \varsigma_2 = 0.6044, \quad J_5 = 0.0494$$
$$w_1 = 0.0000, \quad w_2 = 0.5000, \quad w_3 = 0.3000, \quad w_4 = 0.0778, \quad w_5 = 0.1222$$

Therefore, y_2 is also the dominated e-learning system, but y_3, y_4 and y_5 are the non-dominated e-learning systems. Then $\bar{Y} = \{y_3, y_4, y_5\}$.

Step 2. Interact with the decision maker, and suppose that he/she modifies the weight information $0.2 \leq w_3 \leq 0.3$ as $0.1 \leq w_3 \leq 0.2$, then it follows from Theorem 2.1 that

(1) For the e-learning system y_3:

$$\varsigma_1 = 0.0000, \quad \varsigma_2 = 0.5808, \quad J_5 = 0.1016$$
$$w_1 = 0.0769, \quad w_2 = 0.4231, \quad w_3 = 0.1000, \quad w_4 = 0.0000, \quad w_5 = 0.4000$$

(2) For the e-learning system y_4:

$$\varsigma_1 = 0.0000, \quad \varsigma_2 = 0.6350, \quad J_5 = -0.0150$$
$$w_1 = 0.1000, \quad w_2 = 0.2000, \quad w_3 = 0.2000, \quad w_4 = 0.1000, \quad w_5 = 0.4000$$

(3) For the e-learning system y_5:

$$\varsigma_1 = 0.0000, \quad \varsigma_2 = 0.6350, \quad J_5 = -0.0050$$
$$w_1 = 0.0000, \quad w_2 = 0.5000, \quad w_3 = 0.2000, \quad w_4 = 0.1000, \quad w_5 = 0.2000$$

from which we can see that y_3 is a non-dominated e-learning system, while y_4 and y_5 are the dominated e-learning systems, that is, there is only the e-learning system y_3 left in $\bar{Y}$, which indicates that the optimal e-learning system is y_3.

2.2 Interactive Intuitionistic Fuzzy Multi-Attribute Decision Making Based on Nonlinear Optimization Models

2.2.1 A Satisfaction-Degree-Based Method

Based on the intuitionistic fuzzy decision matrix $R = (r_{ij})_{m\times n}$, the relation between the overall attribute value $z_j(w)$ of the alternative y_j and the attribute weight w_i can be expressed as:

$$\begin{aligned} z_j(w) &= \sum_{i=1}^{m} w_i r_{ij} \\ &= \left(\sum_{i=1}^{m} w_i \mu_{ij},\ \sum_{i=1}^{m} w_i v_{ij},\ \sum_{i=1}^{m} w_i \pi_{ij} \right), \quad j = 1, 2, \ldots, n \end{aligned} \tag{2.11}$$

under the conditions $w_i \geq 0, \quad i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} w_i = 1$.

Since $\mu_{ij}, v_{ij} \in [0, 1], \mu_{ij} + v_{ij} \leq 1, \pi_{ij} = 1 - \mu_{ij} - v_{ij}$, for all $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$, then from Eq. (2.11), we have

$$\sum_{i=1}^{m} w_i \mu_{ij} \in [0, 1], \quad \sum_{i=1}^{m} w_i v_{ij} \in [0, 1], \text{ for all } \quad j = 1, 2, \ldots, n \tag{2.12}$$

and

$$\sum_{i=1}^{m} w_i \mu_{ij} + \sum_{i=1}^{m} w_i v_{ij} = \sum_{i=1}^{m} w_i \left(\mu_{ij} + v_{ij}\right) \leq \sum_{i=1}^{m} w_i = 1, \text{ for all } \quad j = 1, 2, \ldots, n \tag{2.13}$$

$$\sum_{i=1}^{m} w_i \pi_{ij} = \sum_{i=1}^{m} w_i (1 - \mu_{ij} - v_{ij}) = 1 - \sum_{i=1}^{m} \left(w_i \mu_{ij}\right) - \sum_{i=1}^{m} \left(w_i v_{ij}\right), \quad \text{for all } \quad j = 1, 2, \ldots, n \tag{2.14}$$

which indicates that the overall attribute value $z_j(w)$ is also an IFV. In general, the greater the value $z_j(w)$, the better the alternative y_j.

If we use Eq. (1.23) to rank IFVs, then we get the largest IFV $\beta^+ = (1, 0, 0)$ and the "smallest" IFV $\beta^- = (0, 0, 1)$ in the sense of the reliability of the information. In this case, we let $\beta_i^+ = (1, 0, 0)$ and $\beta_i^- = (0, 0, 1)$, for all $i = 1, 2, \ldots, m$, and then call $\beta^* = (\beta_1^+, \beta_2^+, \ldots, \beta_m^+)$ and $\beta_* = (\beta_1^-, \beta_2^-, \ldots, \beta_m^-)$ the intuitionistic fuzzy positive ideal solution and the intuitionistic fuzzy negative ideal solution, respectively.

Then by Eq. (2.11), we get the overall attribute values corresponding to the intuitionistic fuzzy positive ideal solution β^* and the intuitionistic fuzzy negative ideal solution β_* as follows, respectively (Xu 2012b):

$$z^*(w) = \sum_{i=1}^{m} w_i \beta_i^+ = \left(\sum_{i=1}^{m} w_i \times 1, \sum_{i=1}^{m} w_i \times 0, \sum_{i=1}^{m} w_i \times 0 \right) = (1, 0, 0) \quad (2.15)$$

$$z_*(w) = \sum_{i=1}^{m} w_i \beta_i^- = \left(\sum_{i=1}^{m} w_i \times 0, \sum_{i=1}^{m} w_i \times 0, \sum_{i=1}^{m} w_i \times 1 \right) = (0, 0, 1) \quad (2.16)$$

Based on Eqs. (2.11), (2.15) and (2.16), we get the distance between $z_j(w)$ and $z^*(w)$, and the distance between $z_j(w)$ and $z_*(w)$, respectively (Xu 2012b):

$$\begin{aligned} d\left(z_j(w), z^*(w)\right) &= \frac{1}{2}\left(\left| \sum_{i=1}^{m} w_i \mu_{ij} - 1 \right| + \left| \sum_{i=1}^{m} w_i v_{ij} - 0 \right| + \left| \sum_{i=1}^{m} w_i \pi_{ij} - 0 \right| \right) \\ &= \frac{1}{2}\left(\left(1 - \sum_{i=1}^{m} w_i \mu_{ij} \right) + \sum_{i=1}^{m} w_i v_{ij} + \sum_{i=1}^{m} w_i \pi_{ij} \right) \\ &= \frac{1}{2}\left(\left(1 - \sum_{i=1}^{m} w_i (1 - v_{ij} - \pi_{ij}) \right) + \sum_{i=1}^{m} w_i v_{ij} + \sum_{i=1}^{m} w_i \pi_{ij} \right) \\ &= \sum_{i=1}^{m} w_i (v_{ij} + \pi_{ij}) \end{aligned} \quad (2.17)$$

$$\begin{aligned} d\left(z_j(w), z_*(w)\right) &= \frac{1}{2}\left(\left| \sum_{i=1}^{m} w_i \mu_{ij} - 0 \right| + \left| \sum_{i=1}^{m} w_i v_{ij} - 0 \right| + \left| \sum_{i=1}^{m} w_i \pi_{ij} - 1 \right| \right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{m} w_i \mu_{ij} + \sum_{i=1}^{m} w_i v_{ij} + \left(1 - \sum_{i=1}^{m} w_i \pi_{ij} \right) \right) \\ &= \frac{1}{2}\left(1 + \sum_{i=1}^{m} w_i \mu_{ij} + \sum_{i=1}^{m} w_i v_{ij} - \sum_{i=1}^{m} w_i (1 - \mu_{ij} - v_{ij}) \right) \\ &= \sum_{i=1}^{m} w_i (\mu_{ij} + v_{ij}) \end{aligned} \quad (2.18)$$

The larger the distance $d\left(z_j(w), z_*(w)\right)$, the better the alternative y_j, while the smaller the distance $d\left(z_j(w), z^*(w)\right)$, the better the alternative y_j. Considering that a distance from the intuitionistic fuzzy positive ideal solution alone or a distance from the intuitionistic fuzzy negative ideal solution alone is not enough to conclude how good or bad is an alternative, by combining Eq. (2.17) with Eq. (2.18), Xu (2012b) defined the satisfaction degree (or closeness coefficient) of the alternative y_j as:

$$c\left(z_j(w)\right) = \frac{d\left(z_j(w), z_*(w)\right)}{d\left(z_j(w), z^*(w)\right) + d\left(z_j(w), z_*(w)\right)} = \frac{\sum_{i=1}^m w_i(\mu_{ij} + v_{ij})}{\sum_{i=1}^m w_i(v_{ij} + \pi_{ij}) + \sum_{i=1}^m w_i(\mu_{ij} + v_{ij})}$$
$$= \frac{\sum_{i=1}^m w_i(\mu_{ij} + v_{ij})}{\sum_{i=1}^m w_i(\mu_{ij} + v_{ij} + \pi_{ij}) + \sum_{i=1}^m w_i v_{ij}} = \frac{\sum_{i=1}^m w_i(\mu_{ij} + v_{ij})}{1 + \sum_{i=1}^m w_i v_{ij}} \tag{2.19}$$

Clearly, we have $c\left(z_j(w)\right) \in [0, 1]$. It follows from Eq. (2.19) that the larger the distance $d\left(z_j(w), z_*(w)\right)$ and the smaller the distance $d\left(z_j(w), z^*(w)\right)$, the higher the satisfaction degree $c\left(z_j(w)\right)$ of the alternative y_j, and thus, the better the alternative y_j. Consequently, Xu (2012a) established the following multi-objective optimization model:

MOD-2.3
$$\max \left(c\left(z_1(w)\right), c\left(z_2(w)\right), \ldots, c\left(z_n(w)\right)\right)$$
$$s.\,t.\quad w = (w_1, w_2, \ldots, w_m)^T \in \Psi$$
$$w_i \geq 0,\quad i = 1, 2, \ldots, m,\quad \sum_{i=1}^m w_i = 1$$

By the equal-weighted summation method (French et al. 1983), MOD-2.3 can be transformed into a single-objective optimization model:

MOD-2.4
$$\max \sum_{j=1}^n c\left(z_j(w)\right)$$
$$s.\,t.\quad w = (w_1, w_2, \ldots, w_m)^T \in \Psi$$
$$w_i \geq 0,\quad i = 1, 2, \ldots, m,\quad \sum_{i=1}^m w_i = 1$$

Combining Eq. (2.19) with MOD-2.4, we have

MOD-2.5
$$\max \sum_{j=1}^n \left(\frac{\sum_{i=1}^m w_i(\mu_{ij} + v_{ij})}{1 + \sum_{i=1}^m w_i v_{ij}}\right)$$
$$s.\,t.\quad w = (w_1, w_2, \ldots, w_m)^T \in \Psi$$
$$w_i \geq 0,\quad i = 1, 2, \ldots, m,\quad \sum_{i=1}^m w_i = 1$$

MOD-2.5 can be executed by using many efficient algorithms (Terlaky 1996) or using MATLAB 7.4.0 mathematics software package. The solution to the model (MOD 2.5) is the optimal weight vector $w^* = (w_1^*, w_2^*, \ldots, w_m^*)^T$ of the attributes $G_i (i = 1, 2, \ldots, m)$, and then by Eq. (2.11), we get the overall attribute value of each alternative y_j:

Table 2.2 Intuitionistic fuzzy decision matrix $R = (r_{ij})_{6\times 5}$

	y_1	y_2	y_3	y_4	y_5
G_1	(0.4,0.2,0.4)	(0.9,0.1,0.0)	(0.6,0.2,0.2)	(0.4,0.3,0.3)	(0.4,0.6,0.0)
G_2	(0.5,0.3,0.2)	(0.2,0.4,0.4)	(0.7,0.2,0.1)	(0.1,0.5,0.4)	(0.3,0.2,0.5)
G_3	(0.6,0.1,0.3)	(0.5,0.4,0.1)	(0.2,0.5,0.3)	(0.8,0.1,0.1)	(0.7,0.2,0.1)
G_4	(0.8,0.1,0.1)	(0.3,0.5,0.2)	(0.4,0.5,0.1)	(0.5,0.5,0.0)	(0.6,0.4,0.0)
G_5	(0.3,0.4,0.3)	(0.6,0.3,0.1)	(0.8,0.2,0.0)	(0.6,0.3,0.1)	(0.5,0.3,0.2)
G_6	(0.4,0.6,0.0)	(0.4,0.4,0.2)	(0.3,0.5,0.2)	(0.2,0.7,0.1)	(0.5,0.2,0.3)

$$z_j(w^*) = \sum_{i=1}^{m} w_i^* r_{ij} = \left(\sum_{i=1}^{m} w_i^* \mu_{ij},\ \sum_{i=1}^{m} w_i^* v_{ij},\ \sum_{i=1}^{m} w_i^* \pi_{ij} \right) \tag{2.20}$$

With the comparison method introduced previously, we can derive the ranking of the overall attribute values $z_j(w^*)$ $(j = 1, 2, \ldots, n)$, from which we can further rank and select the alternatives $y_j (j = 1, 2, \ldots, n)$.

Xu (2012b) used a practical example (adapted form Isıklar and Büyüközkan (2007)) to illustrate the solution process of the above method:

Example 2.2 (Isıklar and Büyüközkan 2007; Xu 2012b). The arrival of the mobile phone and its rapid and widespread growth may well be seen as one of the most significant developments in the fields of communication and information technology over the past two decades. At the end of the interview phase, when findings of whole research are turned into account, the essential attributes when selecting a mobile phone are decided to be gathered into six attributes:

(1) G_1 (Basic requirements): Reasonable cost/price, standard part used and standard process applied;
(2) G_2 (Physical characteristics): Design standards, weight, dimension, shape, water resistance, solidity, attractiveness and raw material properties;
(3) G_3 (Technical features): Talk time, standby time, international roaming and safety standards;
(4) G_4 (Functionality): ease of use;
(5) G_5 (Brand choice): Market vision and technical support;
(6) G_6 (Customer excitement): Games, ringing tones diversity, local language adaptability and business life facilitating services.

Suppose that there are five mobile phones $y_j (j = 1, 2, \ldots, 5)$ to be evaluated using the intuitionistic fuzzy information provided by the decision maker under the above six attributes. The evaluated attribute values are as listed in the intuitionistic fuzzy decision matrix $R = (r_{ij})_{6\times 5}$ (see Table 2.2) and the weight information about attributes is as follows:

$$\Psi = \left\{ w = (w_1, w_2, \ldots, w_6)^T | w_1 \leq 0.2,\ 0.1 \leq w_2 \leq 0.4,\ 0.1 \leq w_3 \leq 0.2, w_5 \leq 0.3, \right.$$
$$\left. w_5 - w_6 \leq 0.1,\ w_4 \leq w_5,\ w_2 - w_6 \geq w_4 - w_3,\ \ w_i \geq 0,\ \ i = 1, 2, \ldots, 6,\ \sum_{i=1}^{6} w_i = 1 \right\}$$

Since all the attributes $G_i (i = 1, 2, \ldots, 6)$ are the benefit attributes, then the attribute values in R do not need normalization.

Based on the evaluation information in R and Ψ, and using MOD-2.5, we establish the following model:

MOD-2.6
$$\max \left(\frac{0.6w_1 + 0.8w_2 + 0.7w_3 + 0.9w_4 + 0.7w_5 + w_6}{1 + 0.2w_1 + 0.3w_2 + 0.1w_3 + 0.1w_4 + 0.4w_5 + 0.6w_6} \right.$$
$$+ \frac{w_1 + 0.6w_2 + 0.9w_3 + 0.8w_4 + 0.9w_5 + 0.8w_6}{1 + 0.1w_1 + 0.4w_2 + 0.4w_3 + 0.5w_4 + 0.3w_5 + 0.4w_6}$$
$$+ \frac{0.8w_1 + 0.9w_2 + 0.7w_3 + 0.9w_4 + w_5 + 0.8w_6}{1 + 0.2w_1 + 0.2w_2 + 0.5w_3 + 0.5w_4 + 0.2w_5 + 0.5w_6}$$
$$+ \frac{0.7w_1 + 0.6w_2 + 0.9w_3 + w_4 + 0.9w_5 + 0.9w_6}{1 + 0.3w_1 + 0.5w_2 + 0.1w_3 + 0.5w_4 + 0.3w_5 + 0.7w_6}$$
$$\left. + \frac{w_1 + 0.5w_2 + 0.9w_3 + w_4 + 0.8w_5 + 0.7w_6}{1 + 0.6w_1 + 0.2w_2 + 0.2w_3 + 0.4w_4 + 0.3w_5 + 0.2w_6} \right)$$
$$s.t.\ \ w_1 \leq 0.2,\ 0.1 \leq w_2 \leq 0.4,\ \ 0.1 \leq w_3 \leq 0.2,\ w_5 \leq 0.3$$
$$w_5 - w_6 \leq 0.1,\ w_4 \leq w_5,\ \ w_2 - w_6 \geq w_4 - w_3$$
$$w_i \geq 0,\ i = 1, 2, \ldots, 6,\ \sum_{i=1}^{6} w_i = 1$$

By solving this model, we get the optimal weight vector $w^* = (0.200, 0.100, 0.200, 0.200, 0.200, 0.100)^T$, and then by Eq. (2.11), we get the overall attribute value of each alternative y_j:

$$z_1(w^*) = (0.510, 0.250, 0.240), \quad z_2(w^*) = (0.520, 0.340, 0.140)$$
$$z_3(w^*) = (0.500, 0.350, 0.150), \quad z_4(w^*) = (0.490, 0.360, 0.150)$$
$$z_5(w^*) = (0.520, 0.340, 0.140)$$

Without loss of generality, by using the ranking method (1.23), we get

$$L(z_1(w^*)) = 0.304, \quad L(z_2(w^*)) = 0.274, \quad L(z_3(w^*)) = 0.288$$
$$L(z_4(w^*)) = 0.293, \quad L(z_5(w^*)) = 0.274$$

and then rank $z_j(w^*)$ $(j = 1, 2, 3, 4, 5)$ in descending order of $L(z_j(w^*))(j = 1, 2, 3, 4, 5)$:

$$z_2(w^*) = z_5(w^*) > z_3(w^*) > z_4(w^*) > z_1(w^*)$$

from which we rank five mobile phones $y_j(j = 1, 2, 3, 4, 5)$ as: $y_2 \sim y_5 \succ y_3 \succ y_4 \succ y_1$. Thus, both y_2 and y_5 are the most desirable mobile phones.

2.2.2 An Interactive Method

However, the satisfaction degrees of some alternatives, drived by using the models (MOD 2.4 and MOD 2.5) maybe too high, which results in the very low satisfaction degrees of the other alternatives. In the process of decision making, the decision maker may hope to increase the satisfactory degrees of some alternatives, and decrease the satisfaction degrees of some other alternatives, by doing so, he/she can provide new preference information or can modify his/her previous preference information through interacting with the analyst gradually. To achieve this, we can utilize the max–min operator proposed by Zimmermann and Zysno (1980) to integrate the satisfaction degrees of all of the alternatives $y_j(j = 1, 2, \dots, n)$, i.e., based on MOD-2.3, Xu (2012b) got the following optimization model:

MOD-2.7 $\max \lambda$

$$\begin{aligned} s.\,t.\quad & c\left(z_j(w)\right) \geq \lambda, \quad j = 1, 2, \dots, n \\ & w = (w_1, w_2, \dots, w_m)^T \in \Psi \\ & w_i \geq 0, \quad i = 1, 2, \dots, m, \quad \sum_{i=1}^{m} w_i = 1 \end{aligned}$$

where $\lambda = \min_j c\left(z_j(w)\right)$.

By solving MOD-2.7, we obtain the original optimal weight vector $w^{(0)} = (w_1^{(0)}, w_2^{(0)}, \dots, w_m^{(0)})^T$, and then calculate the satisfaction degrees $c\left(z_j(w^{(0)})\right)$ $(j = 1, 2, \dots, n)$ of the alternatives $y_j(j = 1, 2, \dots, n)$. In the course of decision making, the decision maker provides the lower bounds $\lambda_j^{(0)}(j = 1, 2, \dots, n)$ of the satisfaction degrees of the alternatives $y_j(j = 1, 2, \dots, n)$ according to $c\left(z_j(w^{(0)})\right)$ $(j = 1, 2, \dots, n)$. Then, Xu (2012b) established the following optimization model:

MOD-2.8 $\max \sum_{j=1}^{n} \lambda_j$

$$\begin{aligned} s.\,t.\quad & c\left(z_j(w)\right) \geq \lambda_j \geq \lambda_j^{(0)}, \quad j = 1, 2, \dots, n \\ & w = (w_1, w_2, \dots, w_m)^T \in \Psi \\ & w_i \geq 0, \ i = 1, 2, \dots, m, \quad \sum_{i=1}^{m} w_i = 1 \end{aligned}$$

If there exists no optimal solution to MOD-2.8, which that some of the lower bounds, which are greater than the corresponding initial satisfaction degrees, should be decreased, then the decision maker needs to reconsider the lower bounds $\lambda_j^{(0)}(j = 1, 2, \ldots, n)$ of the satisfaction degrees of the alternatives $y_j(j = 1, 2, \ldots, n)$ till the optimal solution is obtained. Similar to Xu (2007c), we can prove the following result:

Theorem 2.3 (Xu 2012b). The optimal solution of MOD-2.8 is the Pareto solution of MOD-2.3.

Motivated by the idea of Xu (2007c), and based on MOD-2.7 and MOD-2.8, Xu (2012b) developed an interactive method for multi-attribute decision making with intuitionistic fuzzy information, which involves the following steps:

Step 1. Utilize MOD-2.7 to derive the original optimal weight vector $w^{(0)} = (w_1^{(0)}, w_2^{(0)}, \ldots, w_m^{(0)})^T$, and then calculate the satisfaction degrees $c\left(z_j(w^{(0)})\right)$ $(j = 1, 2, \ldots, n)$ of the alternatives $y_j(j = 1, 2, \ldots, n)$. The decision maker gives the lower bounds $\lambda_j^{(0)}(j = 1, 2, \ldots, n)$ of the satisfaction degrees of the alternatives $y_j(j = 1, 2, \ldots, n)$ according to the initial satisfaction degrees $c\left(z_j(w^{(0)})\right)$ $(j = 1, 2, \ldots, n)$. Let $t = 1$.

Step 2. Utilize MOD-2.8 to obtain the weight vector $w^{(t)} = (w_1^{(t)}, w_2^{(t)}, \ldots, w_m^{(t)})^T$ and calculate the satisfaction degrees $c\left(z_j(w^{(t)})\right)$ $(j = 1, 2, \ldots, n)$ of the alternatives $y_j(j = 1, 2, \ldots, n)$.

Step 3. If the decision maker is satisfied with the result obtained by Step 2, then calculate the overall attribute values $z_j(w^{(t)})$ $(j = 1, 2, \ldots, n)$ of the alternatives $y_j(j = 1, 2, \ldots, n)$ by using Eq. (2.9), and rank all alternatives according to $z_j(w^{(t)})$ $(j = 1, 2, \ldots, n)$, and then go to Step 4; if there exists no solution to MOD-2.8 (this indicates that some of the lower bounds, which are greater than the corresponding satisfaction degrees, should be decreased. In this case, the analyst shall inform the decision maker of the range that each of the lower bounds can change so as to ensure the existence of the solution of MOD-2.8 or the result does not satisfy the decision maker, then on the basis of the range of that each of the lower bounds can change, the decision maker decreases the satisfaction degrees of some alternatives, and increases the satisfaction degrees of some other alternatives. Let $t := t + 1$, and return to Step 2.

Step 4. End.

Xu (2012b) used Example 2.2 to illustrate the developed interactive method:

Step 1. Utilize MOD-2.7 to establish the following optimization model:

MOD-2.9 $\max \lambda$

$$
\begin{aligned}
s.t.\quad & \frac{0.6w_1 + 0.8w_2 + 0.7w_3 + 0.9w_4 + 0.7w_5 + w_6}{1 + 0.2w_1 + 0.3w_2 + 0.1w_3 + 0.1w_4 + 0.4w_5 + 0.6w_6} \geq \lambda \\
& \frac{w_1 + 0.6w_2 + 0.9w_3 + 0.8w_4 + 0.9w_5 + 0.8w_6}{1 + 0.1w_1 + 0.4w_2 + 0.4w_3 + 0.5w_4 + 0.3w_5 + 0.4w_6} \geq \lambda
\end{aligned}
$$

$$\frac{0.8w_1 + 0.9w_2 + 0.7w_3 + 0.9w_4 + w_5 + 0.8w_6}{1 + 0.2w_1 + 0.2w_2 + 0.5w_3 + 0.5w_4 + 0.2w_5 + 0.5w_6} \geq \lambda$$
$$\frac{0.7w_1 + 0.6w_2 + 0.9w_3 + w_4 + 0.9w_5 + 0.9w_6}{1 + 0.3w_1 + 0.5w_2 + 0.1w_3 + 0.5w_4 + 0.3w_5 + 0.7w_6} \geq \lambda$$
$$\frac{w_1 + 0.5w_2 + 0.9w_3 + w_4 + 0.8w_5 + 0.7w_6}{1 + 0.6w_1 + 0.2w_2 + 0.2w_3 + 0.4w_4 + 0.3w_5 + 0.2w_6} \geq \lambda$$
$$w_1 \leq 0.2,\ 0.1 \leq w_2 \leq 0.4, 0.1 \leq w_3 \leq 0.2,\ w_5 \leq 0.3$$
$$w_5 - w_6 \leq 0.1,\ w_4 \leq w_5, w_2 - w_6 \geq w_4 - w_3,$$
$$w_i \geq 0,\ i = 1, 2, \ldots, 6,\ \sum_{i=1}^{6} w_i = 1$$

By solving this model, we get the original optimal weight vector:

$$w^{(0)} = (0.126, 0.130, 0.200, 0.215, 0.215, 0.115)^T$$

and then calculate the satisfaction degrees $c\left(z_j(w^{(0)})\right)$ $(j = 1, 2, 3, 4, 5)$ of the alternatives $y_j (j = 1, 2, 3, 4, 5)$:

$$c\left(z_1(w^{(0)})\right) = 0.617, \quad c\left(z_2(w^{(0)})\right) = 0.617, \quad c\left(z_3(w^{(0)})\right) = 0.631$$
$$c\left(z_4(w^{(0)})\right) = 0.624, \quad c\left(z_5(w^{(0)})\right) = 0.637$$

from which the decision maker gives the lower bounds:

$$\lambda_1^{(0)} = 0.60, \quad \lambda_2^{(0)} = 0.63, \quad \lambda_3^{(0)} = 0.65, \quad \lambda_4^{(0)} = 0.65, \quad \lambda_5^{(0)} = 0.63$$

of the satisfaction degrees of the alternatives $y_j (j = 1, 2, 3, 4, 5)$.

Step 2. Based on MOD-2.8, we establish the following optimization model:

MOD-2.10 $$\max \sum_{j=1}^{5} \lambda_j$$

$s.\ t.$
$$\frac{0.6w_1 + 0.8w_2 + 0.7w_3 + 0.9w_4 + 0.7w_5 + w_6}{1 + 0.2w_1 + 0.3w_2 + 0.1w_3 + 0.1w_4 + 0.4w_5 + 0.6w_6} \geq \lambda_1 \geq 0.60$$
$$\frac{w_1 + 0.6w_2 + 0.9w_3 + 0.8w_4 + 0.9w_5 + 0.8w_6}{1 + 0.1w_1 + 0.4w_2 + 0.4w_3 + 0.5w_4 + 0.3w_5 + 0.4w_6} \geq \lambda_2 \geq 0.63$$
$$\frac{0.8w_1 + 0.9w_2 + 0.7w_3 + 0.9w_4 + w_5 + 0.8w_6}{1 + 0.2w_1 + 0.2w_2 + 0.5w_3 + 0.5w_4 + 0.2w_5 + 0.5w_6} \geq \lambda_3 \geq 0.65$$
$$\frac{0.7w_1 + 0.6w_2 + 0.9w_3 + w_4 + 0.9w_5 + 0.9w_6}{1 + 0.3w_1 + 0.5w_2 + 0.1w_3 + 0.5w_4 + 0.3w_5 + 0.7w_6} \geq \lambda_4 \geq 0.65$$
$$\frac{w_1 + 0.5w_2 + 0.9w_3 + w_4 + 0.8w_5 + 0.7w_6}{1 + 0.6w_1 + 0.2w_2 + 0.2w_3 + 0.4w_4 + 0.3w_5 + 0.2w_6} \geq \lambda_5 \geq 0.63$$
$$w_1 \leq 0.2,\ 0.1 \leq w_2 \leq 0.4,\ 0.1 \leq w_3 \leq 0.2,\ w_5 \leq 0.3$$

$$w_5 - w_6 \leq 0.1,\ w_4 \leq w_5, w_2 - w_6 \geq w_4 - w_3$$
$$w_i \geq 0,\ i = 1, 2, \ldots, 6,\ \sum_{i=1}^{6} w_i = 1$$

However, there is no solution to MOD-2.10. This indicates that some of the initial lower bounds $\lambda_j^{(0)} (j = 2, 3, 4)$, which are greater than the corresponding initial satisfaction degrees $c\left(z_j(w^{(0)})\right) (j = 2, 3, 4)$, should be decreased. By solving MOD-2.8 and MOD-2.10, we find that the initial lower bounds $\lambda_4^{(0)}$ can not be greater than their initial satisfaction degree $c\left(z_4(w^{(0)})\right)$, while the initial lower bounds $\lambda_2^{(0)}$ and $\lambda_3^{(0)}$ can be increased on the basis of the initial satisfaction degrees $c\left(z_2(w^{(0)})\right)$ and $c\left(z_3(w^{(0)})\right)$. Then we return this information to the decision maker as a reference, and the decision maker changes the lower bounds as: $\lambda_1^{(0)} = 0.60$, $\lambda_2^{(0)} = 0.64, \lambda_3^{(0)} = 0.63, \lambda_4^{(0)} = 0.62, \lambda_5^{(0)} = 0.63$, and correspondingly, MOD-2.10 is changed as:

MOD-2.11 $\max \sum_{j=1}^{5} \lambda_j$

$$\text{s.t.}\quad \frac{0.6w_1 + 0.8w_2 + 0.7w_3 + 0.9w_4 + 0.7w_5 + w_6}{1 + 0.2w_1 + 0.3w_2 + 0.1w_3 + 0.1w_4 + 0.4w_5 + 0.6w_6} \geq \lambda_1 \geq 0.60$$
$$\frac{w_1 + 0.6w_2 + 0.9w_3 + 0.8w_4 + 0.9w_5 + 0.8w_6}{1 + 0.1w_1 + 0.4w_2 + 0.4w_3 + 0.5w_4 + 0.3w_5 + 0.4w_6} \geq \lambda_2 \geq 0.64$$
$$\frac{0.8w_1 + 0.9w_2 + 0.7w_3 + 0.9w_4 + w_5 + 0.8w_6}{1 + 0.2w_1 + 0.2w_2 + 0.5w_3 + 0.5w_4 + 0.2w_5 + 0.5w_6} \geq \lambda_3 \geq 0.63$$
$$\frac{0.7w_1 + 0.6w_2 + 0.9w_3 + w_4 + 0.9w_5 + 0.9w_6}{1 + 0.3w_1 + 0.5w_2 + 0.1w_3 + 0.5w_4 + 0.3w_5 + 0.7w_6} \geq \lambda_4 \geq 0.62$$
$$\frac{1w_1 + 0.5w_2 + 0.9w_3 + 1w_4 + 0.8w_5 + 0.7w_6}{1 + 0.6w_1 + 0.2w_2 + 0.2w_3 + 0.4w_4 + 0.3w_5 + 0.2w_6} \geq \lambda_5 \geq 0.63$$
$$w_1 \leq 0.2,\ 0.1 \leq w_2 \leq 0.4, 0.1 \leq w_3 \leq 0.2,\ w_5 \leq 0.3$$
$$w_5 - w_6 \leq 0.1,\ w_4 \leq w_5,\ w_2 - w_6 \geq w_4 - w_3$$
$$w_i \geq 0,\ i = 1, 2, \ldots, 6,\ \sum_{i=1}^{6} w_i = 1$$

The solution to MOD-2.11 is the weight vector $w^{(1)} = (0.200, 0.100, 0.199, 0.197, 0.202, 0.102)^T$, and then we calculate the satisfactory degrees:

$$c\left(z_1(w^{(1)})\right) = 0.607,\quad c\left(z_2(w^{(1)})\right) = 0.642,\quad c\left(z_3(w^{(1)})\right) = 0.630$$
$$c\left(z_4(w^{(1)})\right) = 0.625,\quad c\left(z_5(w^{(1)})\right) = 0.641$$

Step 3. The decision maker is satisfied with the results $c\left(z_j(w^{(1)})\right)$ $(j = 1, 2, 3, 4, 5)$. In this case, we can use Eq. (2.11) to calculate the overall attribute values:

$$z_1(w^{(1)}) = (0.5084, 0.2516, 0.2400)$$
$$z_2(w^{(1)}) = (0.5206, 0.3395, 0.1399)$$
$$z_3(w^{(1)}) = (0.5008, 0.3494, 0.1498)$$
$$z_4(w^{(1)}) = (0.4893, 0.3604, 0.1503)$$
$$z_5(w^{(1)}) = (0.5195, 0.3396, 0.1409)$$

If we use the ranking method (1.23), then

$$L(z_1(w^{(1)})) = 0.3048, \quad L(z_2(w^{(1)})) = 0.2732, \quad L(z_3(w^{(1)})) = 0.2870$$
$$L(z_4(w^{(1)})) = 0.2937, \quad L(z_5(w^{(1)})) = 0.2741$$

and then rank $z_j(w^{(1)})(j = 1, 2, 3, 4, 5)$ in descending order of $L(z_j(w^{(1)}))(j = 1, 2, 3, 4, 5)$:

$$z_2(w^{(1)}) > z_5(w^{(1)}) > z_3(w^{(1)}) > z_4(w^{(1)}) > z_1(w^{(1)})$$

Therefore, we rank five mobile phones $y_j(j = 1, 2, 3, 4, 5)$ as: $y_2 \succ y_5 \succ y_3 \succ y_4 \succ y_1$, and then the most desirable mobile phone is y_2.

2.2.3 Extended Results in Interval-Valued Intuitionistic Fuzzy Situations

In this subsection, we extend the results introduced in Sects. 2.2.1 and 2.2.2 to interval-valued intuitionistic fuzzy situations. We first represent the interval-valued intuitionistic fuzzy multi-attribute decision making problem:

Let Y, G, w and Ψ be defined as in Sect. 2.2.1, and let $\tilde{B} = (\tilde{b}_{ij})_{m\times n}$ be an interval-valued intuitionistic fuzzy decision matrix, where $\tilde{b}_{ij} = (\tilde{t}_{ij}, \tilde{f}_{ij}, \tilde{\pi}_{ij})$ is an uncertain attribute value, which is expressed in an IVIFV, satisfying Eq. (2.5). By using Eq. (2.6), we can transform the uncertain attribute values of cost type into the uncertain attribute values of benefit type, i.e., $\tilde{B} = (\tilde{b}_{ij})_{m\times n}$ is transformed into the interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$, where $\tilde{r}_{ij} = (\tilde{\mu}_{ij}, \tilde{v}_{ij}, \tilde{\pi}_{ij})$, $\tilde{\mu}_{ij} = [\mu_{ij}^-, \mu_{ij}^+]$, $\tilde{v}_{ij} = [v_{ij}^-, v_{ij}^+]$, $\tilde{\pi}_{ij} = [\pi_{ij}^-, \pi_{ij}^+] = [1 - \mu_{ij}^+ - v_{ij}^+, 1 - \mu_{ij}^- - v_{ij}^-]$, $j = 1, 2, \ldots, n$.

On the basis of the interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$, the uncertain overall attribute value $\tilde{z}_j(w)$ of the alternative y_j can be expressed as:

$$\tilde{z}_j(w) = \sum_{i=1}^{m} w_i \tilde{r}_{ij} = \left(\left[\sum_{i=1}^{m} w_i \mu_{ij}^-, \sum_{i=1}^{m} w_i \mu_{ij}^+ \right], \left[\sum_{i=1}^{m} w_i v_{ij}^-, \sum_{i=1}^{m} w_i v_{ij}^+ \right], \left[\sum_{i=1}^{m} w_i \pi_{ij}^-, \sum_{i=1}^{m} w_i \pi_{ij}^+ \right] \right),$$
$$j = 1, 2, \ldots, n \tag{2.21}$$

Similar to Eqs. (2.12) and (2.13), we can prove that the uncertain overall attribute value $\tilde{z}(w)$ derived by Eq. (2.21) is also an IVIFVs. The greater the value $\tilde{z}(w)$, the better the alternative y_j. Some techniques have also been developed for comparing any two IVIFVs, for details, see Xu and Cai (2009), and Wang et al. (2009). Here we denote the largest IVIFV by $\tilde{\beta}^+ = ([1, 1], [0, 0], [0, 0])$ and denote the smallest IVIFVs by $\tilde{\beta}^- = ([0, 0], [0, 0], [1, 1])$ in the sense of the amount and reliability of information (Szmidit and Kacprzyk 2009a, b, c).

Let $\tilde{\beta}_i^+ = ([1, 1], [0, 0], [0, 0])$ and $\tilde{\beta}_i^- = ([0, 0], [0, 0], [1, 1])$, for all $i = 1, 2, \ldots, m$, then we call $\tilde{\beta}^* = (\tilde{\beta}_1^+, \tilde{\beta}_2^+, \ldots, \tilde{\beta}_m^+)$ and $\tilde{\beta}_* = (\tilde{\beta}_1^-, \tilde{\beta}_2^-, \ldots, \tilde{\beta}_m^-)$ the interval-valued intuitionistic fuzzy positive ideal solution and the interval-valued intuitionistic fuzzy negative ideal solution, respectively. Then by Eq. (2.21), we get the overall attribute values corresponding to the interval-valued intuitionistic fuzzy positive ideal solution $\tilde{\beta}^*$ and the interval-valued intuitionistic fuzzy negative ideal solution $\tilde{\beta}_*$ as follows:

$$\tilde{z}^*(w) = \sum_{i=1}^{m} w_i \tilde{\beta}_i^+ = ([1, 1], [0, 0], [0, 0]) \tag{2.22}$$

$$\tilde{z}_*(w) = \sum_{i=1}^{m} w_i \tilde{\beta}_i^- = ([0, 0], [0, 0], [1, 1]) \tag{2.23}$$

for any two interval-valued intuitionistic fuzzy values $\tilde{\beta}_i = (\tilde{\mu}_i, \tilde{v}_i, \tilde{\pi}_i) = ([\mu_i^-, \mu_i^+], [v_i^-, v_i^+], [\pi_i^-, \pi_i^+]) (i = 1, 2)$.

Based on Eqs. (2.21)–(2.23) and the distance measures (1.186), (1.187), we get the distance between $\tilde{z}_j(w)$ and $\tilde{z}^*(w)$, and the distance between $\tilde{z}_j(w)$ and $\tilde{z}_*(w)$, respectively:

$$\begin{aligned}
& d\left(\tilde{z}_j(w), \tilde{z}^*(w)\right) \\
&= \frac{1}{4}\left(2 - \sum_{i=1}^{m} w_i \mu_{ij}^- - \sum_{i=1}^{m} w_i \mu_{ij}^+ + \sum_{i=1}^{m} w_i v_{ij}^- + \sum_{i=1}^{m} w_i v_{ij}^+ + \sum_{i=1}^{m} w_i \pi_{ij}^- + \sum_{i=1}^{m} w_i \pi_{ij}^+\right) \\
&= \frac{1}{4}\left(2 - \sum_{i=1}^{m} w_i (1 - v_{ij}^+ - \pi_{ij}^+) - \sum_{i=1}^{m} w_i (1 - v_{ij}^- - \pi_{ij}^-)\right. \\
&\qquad \left. + \sum_{i=1}^{m} w_i v_{ij}^- + \sum_{i=1}^{m} w_i v_{ij}^+ + \sum_{i=1}^{m} w_i \pi_{ij}^- + \sum_{i=1}^{m} w_i \pi_{ij}^+\right) \\
&= \frac{1}{2} \sum_{i=1}^{m} \left(w_i (v_{ij}^- + v_{ij}^+ + \pi_{ij}^- + \pi_{ij}^+) \right)
\end{aligned} \tag{2.24}$$

$$
\begin{aligned}
& d\left(\tilde{z}_j(w), \tilde{z}_*(w)\right) \\
& = \frac{1}{4}\left(2+\sum_{i=1}^{m} w_i \mu_{ij}^{-}+\sum_{i=1}^{m} w_i \mu_{ij}^{+}+\sum_{i=1}^{m} w_i v_{ij}^{-}+\sum_{i=1}^{m} w_i v_{ij}^{+}-\sum_{i=1}^{m} w_i \pi_{ij}^{-}-\sum_{i=1}^{m} w_i \pi_{ij}^{+}\right) \\
& = \frac{1}{4}\left(2+\sum_{i=1}^{m} w_i \mu_{ij}^{-}+\sum_{i=1}^{m} w_i \mu_{ij}^{+}+\sum_{i=1}^{m} w_i v_{ij}^{-}+\sum_{i=1}^{m} w_i v_{ij}^{+}\right. \\
& \qquad \left.-\sum_{i=1}^{m} w_i\left(1-\mu_{ij}^{+}-v_{ij}^{+}\right)-\sum_{i=1}^{m} w_i\left(1-\mu_{ij}^{-}-v_{ij}^{-}\right)\right) \\
& = \frac{1}{2} \sum_{i=1}^{m} w_i\left(\mu_{ij}^{-}+\mu_{ij}^{+}+v_{ij}^{-}+v_{ij}^{+}\right) \qquad (2.25)
\end{aligned}
$$

Combining Eq. (2.24) with Eq. (2.25), Xu (2012b) defined the satisfaction degree of the alternative y_j as:

$$
\begin{aligned}
c\left(\tilde{z}_j(w)\right) & = \frac{d\left(\tilde{z}_j(w), \tilde{z}_*(w)\right)}{d\left(\tilde{z}_j(w), \tilde{z}^*(w)\right)+d\left(\tilde{z}_j(w), \tilde{z}_*(w)\right)} \\
& = \frac{\frac{1}{2} \sum_{i=1}^{m} w_i\left(\mu_{ij}^{-}+\mu_{ij}^{+}+v_{ij}^{-}+v_{ij}^{+}\right)}{\frac{1}{2} \sum_{i=1}^{m} w_i\left(v_{ij}^{-}+v_{ij}^{+}+\pi_{ij}^{-}+\pi_{ij}^{+}\right)+\frac{1}{2} \sum_{i=1}^{m} w_i\left(\mu_{ij}^{-}+\mu_{ij}^{+}+v_{ij}^{-}+v_{ij}^{+}\right)} \\
& = \frac{\sum_{i=1}^{m} w_i\left(\mu_{ij}^{-}+\mu_{ij}^{+}+v_{ij}^{-}+v_{ij}^{+}\right)}{2+\sum_{i=1}^{m} w_i\left(v_{ij}^{-}+v_{ij}^{+}\right)} \qquad (2.26)
\end{aligned}
$$

where $c\left(\tilde{z}_j(w)\right) \in [0, 1]$. It can be seen from Eq. (2.26) that the larger the distance $d\left(\tilde{z}_j(w), \tilde{z}_*(w)\right)$ and the smaller the distance $d\left(\tilde{z}_j(w), \tilde{z}^*(w)\right)$, the higher the satisfaction degree $c\left(\tilde{z}_j(w)\right)$ of the alternative y_j, and thus, the better the alternative y_j. Consequently, Xu (2012b) established the following multi-objective optimization model:

MOD-2.12

$$
\begin{aligned}
& \max \left(c\left(\tilde{z}_1(w)\right), c\left(\tilde{z}_2(w)\right), \ldots, c\left(\tilde{z}_n(w)\right)\right) \\
& s.\,t. \quad w=\left(w_1, w_2, \ldots, w_m\right)^T \in \Psi \\
& \qquad w_i \geq 0, \quad i=1,2, \ldots, m, \quad \sum_{i=1}^{m} w_i=1
\end{aligned}
$$

Similar to MOD-2.4, MOD-2.12 can be transformed into a single-objective optimization model:

MOD-2.13

$$\max \sum_{j=1}^{n} c\left(\tilde{z}_j(w)\right)$$
$$s.t.\quad w=(w_1,w_2,\ldots,w_m)^T \in \Psi$$
$$w_i \geq 0,\ i=1,2,\ldots,m,\quad \sum_{i=1}^{m} w_i = 1$$

Combining Eq. (2.26) with MOD-2.13, we have

MOD-2.14

$$\max \sum_{j=1}^{n} \left(\frac{\sum_{i=1}^{m} w_i \left(\mu_{ij}^- + \mu_{ij}^+ + v_{ij}^- + v_{ij}^+\right)}{2+\sum_{i=1}^{m} w_i \left(v_{ij}^- + v_{ij}^+\right)} \right)$$
$$s.t.\quad w=(w_1,w_2,\ldots,w_m)^T \in \Psi$$
$$w_i \geq 0,\quad i=1,2,\ldots,m,\quad \sum_{i=1}^{m} w_i = 1$$

The solution to MOD-2.14 is the optimal weight vector $w^*=(w_1^*,w_2^*,\ldots,w_m^*)^T$ of the attributes $G_i(i=1,2,\ldots,m)$, and then by Eq. (2.21), we get the uncertain overall attribute value of each alternative y_j:

$$\begin{aligned}\tilde{z}_j(w^*) &= \sum_{i=1}^{m} w_i^* \tilde{r}_{ij} \\ &= \left(\left[\sum_{i=1}^{m} w_i^*\mu_{ij}^-, \sum_{i=1}^{m} w_i^*\mu_{ij}^+\right], \left[\sum_{i=1}^{m} w_i^* v_{ij}^-, \sum_{i=1}^{m} w_i^* v_{ij}^+\right], \left[\sum_{i=1}^{m} w_i^*\pi_{ij}^-, \sum_{i=1}^{m} w_i^*\pi_{ij}^+\right] \right), \\ & \qquad j=1,2,\ldots,n \end{aligned} \tag{2.27}$$

By using Eq. (1.186), we can extend Eq. (1.23) to rank IVIFVs $\tilde{\beta}_i=(\tilde{\mu}_i,\tilde{v}_i,\tilde{\pi}_i)$ $(i=1,2,\ldots,n)$:

$$L(\tilde{\beta}_i)=0.5(1+E(\tilde{\pi}_i))d(\tilde{\beta}^+,\tilde{\beta}_i),\quad i=1,2,\ldots,n \tag{2.28}$$

where $E(\tilde{\pi}_{\beta_i})$ is the expected value of $\tilde{\pi}_i$. Then by Eq. (2.28), we can derive the ranking of the uncertain overall attribute values $\tilde{z}_j(w^*)(j=1,2,\ldots,n)$, from which we can further rank and select the alternatives $y_j(j=1,2,\ldots,n)$.

Nevertheless, in the process of decision making, the decision maker may hope to update his/her preference information by increasing the satisfaction degrees of some alternatives, and decreasing the satisfaction degrees of some other alternatives. Then similar to Sect. 2.2.2, Xu (2012b) developed an interactive method for decision making with interval-valued intuitionistic fuzzy information as follows:

Step 1. Utilize the max–min operator (Zimmermann and Zysno 1980) to integrate the satisfaction degrees of all of the alternatives $y_j (j = 1, 2, \ldots, n)$ by establishing the following optimization model:

MOD-2.15

$$\begin{aligned} & \max \dot{\lambda} \\ s.\,t. \quad & c\left(\tilde{z}_j(w)\right) \geq \dot{\lambda}, \quad j = 1, 2, \ldots, n \\ & w = (w_1, w_2, \ldots, w_m)^T \in \Psi \\ & w_i \geq 0, \quad i = 1, 2, \ldots, m, \quad \sum_{i=1}^{m} w_i = 1 \end{aligned}$$

where $\dot{\lambda} = \min_j c\left(\tilde{z}_j(w)\right)$, from which the original optimal weight vector $w^{(0)} = (w_1^{(0)}, w_2^{(0)}, \ldots, w_m^{(0)})^T$ is derived, and then calculate the satisfaction degrees $c\left(\tilde{z}_j(w^{(0)})\right) (j = 1, 2, \ldots, n)$ of the alternatives $y_j (j = 1, 2, \ldots, n)$. The decision maker gives the lower bounds $\dot{\lambda}_j^{(0)} (j = 1, 2, \ldots, n)$ of the satisfaction degrees of the alternatives $y_j (j = 1, 2, \ldots, n)$ according to the initial satisfaction degrees $c\left(\tilde{z}_j(w^{(0)})\right) (j = 1, 2, \ldots, n)$. Let $t = 1$.

Step 2. Establish the following optimization model:

MOD-2.16

$$\begin{aligned} & \max \sum_{j=1}^{n} \dot{\lambda}_j \\ s.\,t. \quad & c\left(\tilde{z}_j(w)\right) \geq \dot{\lambda}_j \geq \dot{\lambda}_j^{(0)}, \quad j = 1, 2, \ldots, n \\ & w = (w_1, w_2, \ldots, w_m)^T \in \Psi \\ & w_i \geq 0, \quad i = 1, 2, \ldots, m, \quad \sum_{i=1}^{m} w_i = 1 \end{aligned}$$

whose optimal solution is the Pareto solution of MOD-2.12. Then we utilize MOD-2.16 to obtain the weight vector $w^{(t)} = (w_1^{(t)}, w_2^{(t)}, \ldots, w_m^{(t)})^T$ and calculate the satisfaction degrees $c\left(\tilde{z}_j(w^{(t)})\right) (j = 1, 2, \ldots, n)$ of the alternatives $y_j (j = 1, 2, \ldots, n)$.

Step 3. If the decision maker is satisfied with the result obtained by Step 2, then we calculate the uncertain overall attribute values $\tilde{z}_j(w^{(t)})$ $(j = 1, 2, \ldots, n)$ of the alternatives $y_j (j = 1, 2, \ldots, n)$ by using Eq. (2.21), and rank all alternatives according to $\tilde{z}_j(w^{(t)})$ $(j = 1, 2, \ldots, n)$, and then go to Step 4; if there exists no solution to MOD-2.16 (this indicates that some of the lower bounds, which are greater than the corresponding initial satisfaction degrees, should be decreased. In this case, the analyst shall inform the decision maker of the range that each of the lower bounds can change so as to ensure the existence of the solution of MOD-2.16 or the result does not satisfy the decision maker, then on the basis of the range of that each of the lower bounds can change, the decision maker increases the satisfaction

Table 2.3 Interval-valued intuitionistic fuzzy decision matrix: $\tilde{R} = (\tilde{r}_{ij})_{6\times5}$

	y_1	y_2	y_3	y_4	y_5
G_1	([0.4,0.5], [0.1,0.2], [0.3,0.5])	([0.8,0.9], [0.0,0.1], [0.0,0.2])	([0.5,0.6], [0.1,0.2], [0.2,0.4])	([0.3,0.4], [0.4,0.5], [0.1,0.3])	([0.3,0.5], [0.3,0.4], [0.1,0.4])
G_2	([0.4,0.5], [0.2,0.4], [0.1,0.4])	([0.2,0.3], [0.4,0.6], [0.1,0.4])	([0.5,0.7], [0.1,0.2], [0.1,0.4])	([0.1,0.2], [0.5,0.6], [0.2,0.4])	([0.3,0.4], [0.2,0.3], [0.3,0.5])
G_3	([0.4,0.6], [0.1,0.2], [0.2,0.5])	([0.4,0.5], [0.3,0.4], [0.1,0.3])	([0.2,0.4], [0.5,0.6], [0.0,0.3])	([0.8,0.9], [0.0,0.1], [0.0,0.2])	([0.5,0.7], [0.1,0.2], [0.1,0.4])
G_4	([0.7,0.8], [0.1,0.2], [0.0,0.2])	([0.3,0.5], [0.4,0.5], [0.0,0.3])	([0.4,0.5], [0.3,0.4], [0.1,0.3])	([0.5,0.6], [0.4,0.4], [0.0,0.1])	([0.5,0.6], [0.3,0.4], [0.0,0.2])
G_5	([0.2,0.4], [0.3,0.4], [0.2,0.5])	([0.5,0.6], [0.3,0.3], [0.1,0.2])	([0.7,0.8], [0.1,0.2], [0.0,0.2])	([0.4,0.6], [0.2,0.3], [0.1,0.4])	([0.5,0.5], [0.2,0.3], [0.2,0.3])
G_6	([0.4,0.6], [0.3,0.4], [0.0,0.3])	([0.3,0.4], [0.3,0.4], [0.2,0.4])	([0.3,0.5], [0.4,0.5], [0.0,0.3])	([0.2,0.3], [0.6,0.7], [0.0,0.2])	([0.5,0.6], [0.1,0.2], [0.2,0.4])

degrees of some alternatives, and decreases the satisfaction degrees of some other alternatives. Let $t := t + 1$, and return to Step 2.

Step 4. End.

In Example 2.2, if the decision information provided by the decision maker is expressed in IVIFVs, as listed in the interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{6\times5}$ (see Table 2.3):

Then we can utilize the interactive method introduced in this subsection to derive the final decision results, which involves the following steps (Xu 2012b):

Step 1. Utilize MOD-2.15 to establish the following optimization model:

MOD-2.17 $\max \dot{\lambda}$

$$
\begin{aligned}
s.t.\ & \frac{1.2w_1 + 1.5w_2 + 1.3w_3 + 1.8w_4 + 1.3w_5 + 1.7w_6}{2 + 0.3w_1 + 0.6w_2 + 0.3w_3 + 0.3w_4 + 0.7w_5 + 0.7w_6} \geq \dot{\lambda} \\
& \frac{1.8w_1 + 1.5w_2 + 1.6w_3 + 1.7w_4 + 1.7w_5 + 1.4w_6}{2 + 0.1w_1 + w_2 + 0.7w_3 + 0.9w_4 + 0.6w_5 + 0.7w_6} \geq \dot{\lambda} \\
& \frac{1.4w_1 + 1.5w_2 + 1.7w_3 + 1.6w_4 + 1.8w_5 + 1.7w_6}{2 + 0.3w_1 + 0.3w_2 + 1.1w_3 + 0.7w_4 + 0.3w_5 + 0.9w_6} \geq \dot{\lambda} \\
& \frac{1.6w_1 + 1.4w_2 + 1.8w_3 + 1.9w_4 + 1.5w_5 + 1.8w_6}{2 + 0.9w_1 + 1.1w_2 + 0.1w_3 + 0.8w_4 + 0.5w_5 + 1.3w_6} \geq \dot{\lambda} \\
& \frac{1.5w_1 + 1.2w_2 + 1.5w_3 + 1.8w_4 + 1.5w_5 + 1.4w_6}{2 + 0.7w_1 + 0.5w_2 + 0.3w_3 + 0.7w_4 + 0.5w_5 + 0.3w_6} \geq \dot{\lambda}
\end{aligned}
$$

$$w_1 \leq 0.2,\ 0.1 \leq w_2 \leq 0.4, \quad 0.1 \leq w_3 \leq 0.2,\ w_5 \leq 0.3$$
$$w_5 - w_6 \leq 0.1,\ w_4 \leq w_5, \quad w_2 - w_6 \geq w_4 - w_3$$
$$w_i \geq 0,\ i = 1, 2, \ldots, 6,\ \sum_{i=1}^{6} w_i = 1$$

By solving this model, we get the original optimal weight vector:

$$w^{(0)} = (0.083, 0.147, 0.200, 0.223, 0.223, 0.123)^T$$

and then calculate the satisfaction degrees $c\left(\tilde{z}_j(w^{(0)})\right)(j = 1, 2, 3, 4, 5)$ of the alternatives $y_j (j = 1, 2, 3, 4, 5)$:

$$c\left(\tilde{z}_1(w^{(0)})\right) = 0.597, \quad c\left(\tilde{z}_2(w^{(0)})\right) = 0.597, \quad c\left(\tilde{z}_3(w^{(0)})\right) = 0.627$$
$$c\left(\tilde{z}_4(w^{(0)})\right) = 0.621, \quad c\left(\tilde{z}_5(w^{(0)})\right) = 0.605$$

based on which the decision maker provides the lower bounds:

$$\dot{\lambda}_1^{(0)} = 0.58, \quad \dot{\lambda}_2^{(0)} = 0.61, \quad \dot{\lambda}_3^{(0)} = 0.63, \quad \dot{\lambda}_4^{(0)} = 0.61, \quad \dot{\lambda}_5^{(0)} = 0.60$$

of the satisfaction degrees of the alternatives $y_j (j = 1, 2, 3, 4, 5)$.

Step 2. Using MOD-2.16, we establish the following optimization model:

MOD-2.18 $\max \sum_{j=1}^{5} \dot{\lambda}_j$

$$s.t. \quad \frac{1.2w_1 + 1.5w_2 + 1.3w_3 + 1.8w_4 + 1.3w_5 + 1.7w_6}{2 + 0.3w_1 + 0.6w_2 + 0.3w_3 + 0.3w_4 + 0.7w_5 + 0.7w_6} \geq \dot{\lambda}_1 \geq 0.58$$
$$\frac{1.8w_1 + 1.5w_2 + 1.6w_3 + 1.7w_4 + 1.7w_5 + 1.4w_6}{2 + 0.1w_1 + w_2 + 0.7w_3 + 0.9w_4 + 0.6w_5 + 0.7w_6} \geq \dot{\lambda}_2 \geq 0.61$$
$$\frac{1.4w_1 + 1.5w_2 + 1.7w_3 + 1.6\omega_4 + 1.8\omega_5 + 1.7\omega_6}{2 + 0.3w_1 + 0.3w_2 + 1.1w_3 + 0.7w_4 + 0.3w_5 + 0.9w_6} \geq \dot{\lambda}_3 \geq 0.63$$
$$\frac{1.6w_1 + 1.4w_2 + 1.8w_3 + 1.9w_4 + 1.5w_5 + 1.8w_6}{2 + 0.9w_1 + 1.1w_2 + 0.1w_3 + 0.8w_4 + 0.5w_5 + 1.3w_6} \geq \dot{\lambda}_4 \geq 0.61$$
$$\frac{1.5w_1 + 1.2w_2 + 1.5w_3 + 1.8w_4 + 1.5w_5 + 1.4w_6}{2 + 0.7w_1 + 0.5w_2 + 0.3w_3 + 0.7w_4 + 0.5w_5 + 0.3w_6} \geq \dot{\lambda}_5 \geq 0.60$$
$$w_1 \leq 0.2,\ 0.1 \leq w_2 \leq 0.4, 0.1 \leq w_3 \leq 0.2,\ w_5 \leq 0.3$$
$$w_5 - w_6 \leq 0.1,\ w_4 \leq w_5, w_2 - w_6 \geq w_4 - w_3$$
$$w_i \geq 0,\ i = 1, 2, \ldots, 6,\ \sum_{i=1}^{6} w_i = 1$$

The solution to MOD-2.18 is the weight vector $w^{(1)} = (0.144, 0.104, 0.200, 0.157, 0.248, 0.147)^T$, and then we calculate the satisfactory degrees:

$$c\left(\tilde{z}_1(\omega^{(1)})\right) = 0.580, \quad c\left(\tilde{z}_2(\omega^{(1)})\right) = 0.615, \quad c\left(\tilde{z}_3(\omega^{(1)})\right) = 0.630$$
$$c\left(\tilde{z}_4(\omega^{(1)})\right) = 0.618, \quad c\left(\tilde{z}_5(\omega^{(1)})\right) = 0.603$$

Step 3. Suppose that the decision maker is satisfied with the results $c\left(\tilde{z}_j(w^{(1)})\right)$ $(j = 1, 2, 3, 4, 5)$. In this case, we can use Eq. (2.21) to calculate the uncertain overall attribute values:

$$\tilde{z}_1(w^{(1)}) = ([0.3975, 0.5570], [0.1894, 0.2998], [0.1432, 0.4131])$$
$$\tilde{z}_2(w^{(1)}) = ([0.4312, 0.5469], [0.2829, 0.3685], [0.0846, 0.2859])$$
$$\tilde{z}_3(w^{(1)}) = ([0.4445, 0.5896], [0.2555, 0.3555], [0.0549, 0.3000])$$
$$\tilde{z}_4(w^{(1)}) = ([0.4207, 0.5455], [0.3102, 0.3945], [0.0600, 0.2691])$$
$$\tilde{z}_5(w^{(1)}) = ([0.4504, 0.5600], [0.1954, 0.2954], [0.1446, 0.3542])$$

By using the ranking method (2.28) (let $E(\tilde{\pi}_i) = 0.5(\pi_i^L + \pi_i^U)$), we have

$$L(\tilde{z}_1(w^{(1)})) = 0.3341, \quad L(\tilde{z}_2(w^{(1)})) = 0.3028, \quad L(\tilde{z}_3(w^{(1)})) = 0.2843$$
$$L(\tilde{z}_4(w^{(1)})) = 0.3010, \quad L(\tilde{z}_5(w^{(1)})) = 0.3091$$

by which we can rank $\tilde{z}_j(w^{(1)})(j = 1, 2, 3, 4, 5)$ in descending order:

$$\tilde{z}_3(w^{(1)}) > \tilde{z}_4(w^{(1)}) > \tilde{z}_2(w^{(1)}) > \tilde{z}_5(w^{(1)}) > \tilde{z}_1(w^{(1)})$$

Therefore, the ranking of the mobile phones $y_j (j = 1, 2, 3, 4, 5)$ is: $y_3 \succ y_4 \succ y_2 \succ y_5 \succ y_1$, and then the most desirable mobile phone is y_3. This ranking is different from the ones derived by the methods in Sects. 2.2.1 and 2.2.2, mainly due to the distinct change of the decision information.

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