

Chapter 3

The SFT Compactness Results

We will describe now the compactification of the moduli space of holomorphic curves. The elements which have to be added are called ‘holomorphic buildings’. In order to keep the notation more manageable we will first present the case of holomorphic curves without boundary (‘the closed case’) as in the article [12]. We will later present a modification of the construction in order to accommodate holomorphic curves with boundary.

3.1 Holomorphic Buildings for Curves Without Boundary

3.1.1 Holomorphic Buildings of Height 1

Let

$$\mathbf{S} = (S, j, M \cup Z, D)$$

be a noded Riemann surface. The set of marked points consists of a set Z , called *punctures* and a set M which we will call *marked points*. This is a slight change in notation from Chap. 1 where we denoted the set of marked points simply by M . In this chapter we want to distinguish between different kinds of marked points: The domains of the holomorphic curves will be the surface S with some marked points removed. We will call these points ‘punctures’ and use the notation Z for them. The set of the remaining marked points will be denoted by M . The set

$$D = \{\{\bar{d}_1, \underline{d}_1\}, \dots, \{\bar{d}_k, \underline{d}_k\}\} \subset \mathring{S}$$

is called the sets of *nodal pairs* or *special marked points*. Each pair $\{\bar{d}_j, \underline{d}_j\}$ may belong to the same or to different components of S . The set D is usually considered unordered. We consider the set of marked points and punctures to be ordered. We assume that $\mathbb{R} \times V$ is the symplectization of a closed manifold V endowed with some contact form λ , and where J is a $d\lambda$ compatible complex structure on $\xi = \ker \lambda$.

Definition 3.1 A holomorphic curve

$$\tilde{u} = (a, u) : S \setminus Z \longrightarrow \mathbb{R} \times V$$

is called a *trivial cylinder* if $S = S^2$, $M = D = \emptyset$, Z consists of exactly two points, and u maps S onto a periodic orbit of the Reeb vector field.

Definition 3.2 (Holomorphic building of height 1) A holomorphic curve

$$\tilde{u} = (a, u) : S \setminus Z \longrightarrow \mathbb{R} \times V$$

is called a *holomorphic building of height 1* if it has finite energy, and if for all nodal pairs

$$\tilde{u}(\overline{d}_j) = \tilde{u}(\underline{d}_j).$$

Such a curve is called *stable* if the following conditions are satisfied:

- at least one component of the curve is not a trivial cylinder.
- If S' is a connected component of S on which \tilde{u} is constant, then the Riemann surface S' with all its marked points, punctures and nodal pairs is stable in the sense of Definition 1.17.

Assume that p is a puncture. Under some nondegeneracy assumptions on the set of periodic orbits of the Reeb vector field, the asymptotic behavior of \tilde{u} is given by Proposition 2.47. We may then associate a particular periodic orbit of the Reeb vector field to each such puncture.

As in [12] we will call two such curves $(\tilde{u} = (a, u), j, \mathbf{S})$ and $(\tilde{u}' = (a', u'), j', \mathbf{S}')$ *equivalent* if there is a diffeomorphism $\phi : S \rightarrow S'$ such that

- $(\phi_*)^{-1} \circ j' \circ \phi_* = j$
- ϕ sends the sets M, Z, D onto the sets M', Z', D' preserving the ordering of the sets
- $u' \circ \phi = u$
- $a' \circ \phi - a$ is locally constant.

The last condition means that we identify curves which differ by translation in the \mathbb{R} -direction. The \mathbb{R} -component at each puncture either tends to $+\infty$ or to $-\infty$ (see Theorem 2.74). We will use the notation Z^\pm to distinguish between positive and negative punctures. The question of ordering sets of marked points and the notion of equivalence of noded surfaces or equivalence of holomorphic curves only becomes relevant if algebraic invariants are constructed from the compactified moduli spaces of holomorphic curves. This is not the subject of this text, we only deal with the compactness theory.

3.1.2 Holomorphic Buildings of Height N

Assume, we have N stable possibly disconnected holomorphic buildings of height 1 $(\tilde{u}_k, j_k, \mathbf{S}_k)_{1 \leq k \leq N}$ where $\tilde{u}_k = (a_k, u_k) : S_k \setminus Z_k \rightarrow \mathbb{R} \times V$ and $\mathbf{S}_k = (S_k, M_k \cup Z_k, D_k)$ are noded Riemann surfaces. We denote by Γ_k^\pm the sets of boundary circles added to $S_k \setminus Z_k$ by compactifying the punctures in Z_k^\pm . We assume that

$$\#Z_{k+1}^- = \#Z_k^+,$$

i.e. the building on level $k+1$ has as many negative punctures as the building on level k positive punctures. We assume furthermore that there are orientation reversing diffeomorphisms

$$\Phi_k : \Gamma_k^+ \rightarrow \Gamma_{k+1}^-.$$

Using these we may construct a piecewise smooth surface with boundary

$$S^{Z, \Phi} = S_1^{Z_1} \cup_{\Phi_1} \dots \cup_{\Phi_{k-1}} S_k^{Z_k}.$$

Definition 3.3 (Holomorphic building of height N) The finite sequence $(\tilde{u}_k, j_k, \mathbf{S}_k)_{1 \leq k \leq N}$ of holomorphic buildings of height 1 together with the decoration maps $\{\Phi_1, \dots, \Phi_{N-1}\}$ is called a *holomorphic building of height N (or with N levels)* if the following condition is satisfied: The maps $u_k : S_k^{Z_k} \rightarrow V$, $1 \leq k \leq N$, fit together to a continuous map $u : S^{Z, \Phi} \rightarrow V$. This means that the curve u_k is asymptotic at its negative punctures to the same periodic orbits as u_{k-1} at its positive punctures (see Fig. 3.1).

Two holomorphic buildings of height N $(\tilde{u}_k, j_k, \mathbf{S}_k, \Phi_k)_{1 \leq k \leq N}$ and $(\tilde{u}'_k, j'_k, \mathbf{S}'_k, \Phi'_k)_{1 \leq k \leq N}$ are called equivalent if there are diffeomorphisms $\phi = \{\phi_1, \dots, \phi_N\}$, $\phi_k : S_k \rightarrow S'_k$ which renders the corresponding buildings of height 1 equivalent. Moreover, the diffeomorphisms are compatible with the attaching maps, i.e.

$$\Phi'_k \circ \phi_{k-1} = \phi_k \circ \Phi_{k-1}.$$

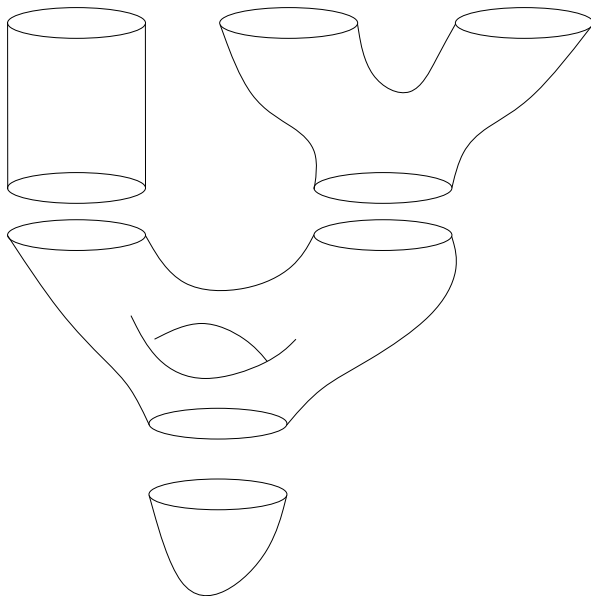
Definition 3.4 The signature of a level N holomorphic building is the four-tuplet $\alpha = (g, \mu, p^+, p^-)$ where g is the arithmetic genus of $S^{\phi, Z}$, $\mu = \#M$, $p^+ = \#Z_N^+$ and $p^- = \#Z_1^-$. We denote the set of all holomorphic buildings of height N and signature α by $\mathcal{M}^{\alpha, N}(V)$. We set

$$\overline{\mathcal{M}}^\alpha(V) := \bigcup_{N=1}^{\infty} \mathcal{M}^{\alpha, N}(V).$$

Definition 3.5 (Convergence of holomorphic buildings) Assume that

$$(\tilde{u}_m, j_m, \mathbf{S}_m)_{m \in \mathbb{N}} \subset \overline{\mathcal{M}}^\alpha(V)$$

Fig. 3.1 A holomorphic building of height three



is a sequence of holomorphic buildings of height less or equal to $N \in \mathbb{N}$.¹ We say that this sequence converges to a building $(\tilde{u}, j, \mathbf{S}) \in \mathcal{M}^{N, \alpha}(V)$ of height N if the following conditions are satisfied:

- (1) There are sequences of extra marked points for the surfaces \mathbf{S}_m and \mathbf{S} such that all the surfaces have the same number of boundary and interior marked points and the underlying Riemann surfaces become stable.
- (2) The surfaces \mathbf{S}_m converge to the surface \mathbf{S} in the Deligne–Mumford sense (after having introduced the extra marked points as in (1) above) as described in Sect. 1.3 of the chapter on Riemann surfaces (see Definition 1.87). Denote the diffeomorphisms between the underlying surfaces $S^{Z, \Phi} \rightarrow S^{Z_m, \Phi_m}$ by φ_m .
- (3) The sequence of the projections into V , $u_m \circ \varphi_m : S^{Z, \Phi} \rightarrow V$, converge to $u : S^{Z, \Phi} \rightarrow V$ uniformly (up to the ends).
- (4) Denote by C_l the union of all the components of $S^{Z, \Phi} \setminus \bigcup_k \Gamma_k$ ² which correspond to a particular level l of the building $(\tilde{u} = (a, u), j, \mathbf{S})$. Then there is a sequence of real numbers $(c_m^l)_{m \in \mathbb{N}}$ for each l such that

$$(a_m \circ \varphi_m - a - c_m^l)|_{C_l} \longrightarrow 0 \quad \text{in } C_{\text{loc}}^0.$$

¹Recall that by definition each $(\tilde{u}_m, j_m, \mathbf{S}_m)$ is itself a sequence $(\tilde{u}_m^n, j_m^n, \mathbf{S}_m^n)_{1 \leq n \leq N_m}$, $N_m \leq N$ of holomorphic buildings of height 1.

²The circles and arcs used to compactify the surfaces are removed here.

3.2 Adding Additional Marked Points

In this section we are considering a convergent sequence of noded marked Riemann surfaces $\mathbf{S}_n = (S_n, j_n, M_n, D_n)$ of signature (g, μ) . In this section we are using the notation of Chap. 2 where we do not use any particular notation for the punctures. See Definitions 1.10 and 1.87 for the notions of signature and convergence of surfaces. If these surfaces have nonempty boundary, we write $M_n = M_n^{\text{int}} \cup M_n^{\partial}$, $D_n = D_n^{\text{int}} \cup D_n^{\partial}$ where M_n^{int} is the set of interior marked points while M_n^{∂} denotes the set of the marked points on the boundary of S_n (similarly for the special marked points or nodal pairs D_n). In the case of nonempty boundary we have $\mu = (\mu^{\text{int}}, \mu^{\partial})$ where $\mu^{\text{int}} = \#M^{\text{int}}$ and $\mu^{\partial} = \#M^{\partial}$. Denote the limit by $\mathbf{S} = (S, j, M, D)$ (its signature is also (g, μ) , and we write again $M = M^{\text{int}} \cup M^{\partial}$, $D = D^{\text{int}} \cup D^{\partial}$ if $\partial S \neq \emptyset$). We will now change the sequence by adding two additional marked points to each surface S_n : Let $Y_n = \{y_n^{(1)}, y_n^{(2)}\}$ be a pair of points on $S_n \setminus (M_n \cup D_n)$ such that

$$d_n(y_n^{(1)}, y_n^{(2)}) \xrightarrow{n \rightarrow \infty} 0$$

where d_n is the distance function with respect to the Poincaré metric $h^{j_n, M_n \cup D_n}$ on $\dot{S}_n := S_n \setminus (M_n \cup D_n)$. We also assume that there is a sequence of positive real numbers $R_n \rightarrow \infty$ and injective holomorphic maps

$$\psi_n : D_{R_n} \rightarrow \dot{S}_n, \quad D_{R_n} := \{z \in \mathbb{C} \mid |z| \leq R_n\}$$

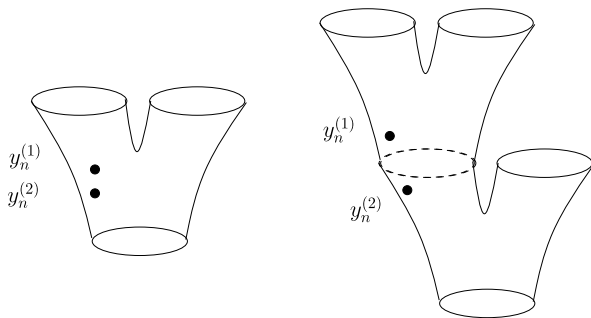
such that

$$\psi_n(0) = y_n^{(1)} \quad \text{and} \quad \psi_n(1) = y_n^{(2)}$$

if the points $y_n^{(1)}, y_n^{(2)}$ are in the interior of S_n . The other case we will consider is when both lie on the boundary ∂S_n . Then the injective holomorphic maps ψ_n will be defined on the upper half-disk $D_{R_n}^+$. After passing to a subsequence we may assume that the sequence $\mathbf{S}'_n := (S_n, j_n, M_n \cup Y_n, D_n)$ also converges to some limit $\mathbf{S}' = (S', j', M', D')$. The purpose of this section is to find how \mathbf{S}' and \mathbf{S} are related. In a nutshell, \mathbf{S}' is obtained from \mathbf{S} by attaching no, one or two spheres (or disks) to \mathbf{S} . This will be very important for the SFT compactness result later since this will help us analyze bubbling phenomena. The result we are going to discuss here is Proposition 4.3 in [12] and its generalization for the case of surfaces with nonempty boundary. It would be very cumbersome to precisely formulate the result, we rather describe and illustrate the phenomena (hoping this may be more illuminating). We will first consider the case of surfaces without boundary as in [12]. There are additional complications if the surfaces S_n have boundary, and we will address these later.

Recall that the Deligne–Mumford compactness result, Theorem 1.91, was proved along the following lines: Using Bers' theorem we may decompose the surfaces $S_n \setminus (M_n \cup D_n)$ into pairs of pants so that the corresponding Fenchel–Nielsen parameters converge after passing to a subsequence. Because of the assumption $d_n(y_n^{(1)}, y_n^{(2)}) \rightarrow 0$ we may at this point assume that the surfaces $S_n \setminus (M_n \cup D_n)$ consist of either one or two pairs of pants (see Fig. 3.2). Introducing the additional

Fig. 3.2 The surface $S_n \setminus (M_n \cup D_n)$



marked points Y_n amounts to changing the hyperbolic metric and therefore the pair of pants decomposition (see Fig. 3.3 for the case where both additional points are contained in the same pairs of pants). We invite the reader to draw the corresponding pictures for the case where $S_n \setminus (M_n \cup D_n)$ consists of two pairs of pants (and to construct case (iii) from Fig. 5 in the paper [12]). Because the surfaces \hat{S}_n contain annuli $\psi_n(D_{R_n} \setminus D_2)$ with larger and larger moduli and $y_n^{(1)}, y_n^{(2)} \in \psi_n(D_2)$ the last pair of pants decomposition depicted in Fig. 3.3 (and similar scenarios) cannot occur by Bers' theorem. In fact, we must have $\ell(\gamma_n) \rightarrow 0$, and the limit surface \mathbf{S}' contains an additional spherical component with two marked points on it corresponding to the sequences $\{y_n^{(1)}\}$ and $\{y_n^{(2)}\}$. In the case where $\inf_n \ell(\delta_n) > 0$ the limit \mathbf{S}' differs from \mathbf{S} by exactly one additional spherical component.³ If in addition $\ell(\delta_n) \rightarrow 0$ then a second spherical component may appear as well (see Fig. 3.4).⁴ If we replace the marked point m in Fig. 3.4 by a nondegenerate boundary component then we may obtain limits \mathbf{S}' which differ from \mathbf{S} by adding one spherical component and pinching a closed curve to a point. See Fig. 3.5 for an illustration where the sequence \mathbf{S}_n is assumed to be constant for simplicity and $\ell(\gamma_n), \ell(\delta_n) \rightarrow 0$. This case is missing in the paper [12].

Let us consider the boundary case where the sequences $\{y_n^{(1)}\}, \{y_n^{(2)}\}$ lie in $\partial S_n \setminus (M_n^\partial \cup D_n^\partial)$. Recall from our discussion of the Deligne–Mumford result that we double the surfaces S_n and then decompose the doubled surfaces into pairs of pants. As before, we will assume that $y_n^{(1)}$ and $y_n^{(2)}$ lie in the same pair of pants Y . The intersection $\partial S_n \cap Y$ consists of a union of geodesic arcs connecting two boundary components and maybe some boundary components of the pair of pants Y (see Figs. 1.52 and 1.55). Figure 3.6 illustrates some of the a priori possible scenarios. We then need to repeat the previous discussion and keep track of the boundary curve(s). Pictures (IV) and (V) actually lead to unstable components in the limiting nodal surface which we wish to avoid. We therefore require that the points $y_n^{(1)}$ and $y_n^{(2)}$ lie in the same component of $\partial S_n \cap Y$. Adding two marked points to the boundary has the effect of adding one or two additional components to the limit surface which are disks. Figure 3.7 depicts the situation for picture (II).

³In the paper [12] this is case (i) in Fig. 5 and the second case in Fig. 3.4.

⁴This is case (ii) in the paper [12].

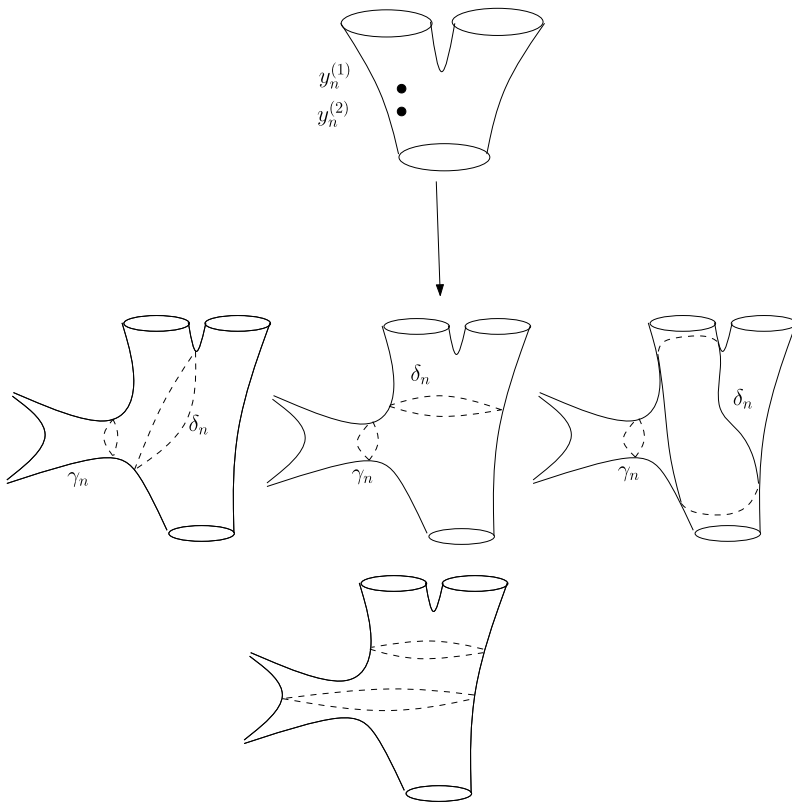


Fig. 3.3 The surface $S_n \setminus (M_n \cup D_n \cup Y_n)$. The last picture displays an impossible configuration by Bers' theorem

3.3 The Compactness Result for the Case Without Boundary

3.3.1 Statement of the Result

We will formulate and prove the SFT compactness result for holomorphic curves without boundary where the target is a cylindrical symplectic manifold (W, ω) , i.e. $W = \mathbb{R} \times M$ where M is a closed contact manifold and \tilde{J} equals the usual \mathbb{R} -invariant almost complex structure (with $d\lambda$ -compatible $J : \ker \lambda \rightarrow \ker \lambda$). We also assume throughout that the contact form on M is non-degenerate.

Theorem 3.6 (SFT compactness theorem for curves without boundary in the cylindrical case) *Consider a sequence of pseudoholomorphic curves*

$$\mathcal{C}_n = (\tilde{u}_n, S_n, j_n, M_n, \underline{Z}_n \cup \overline{Z}_n),$$

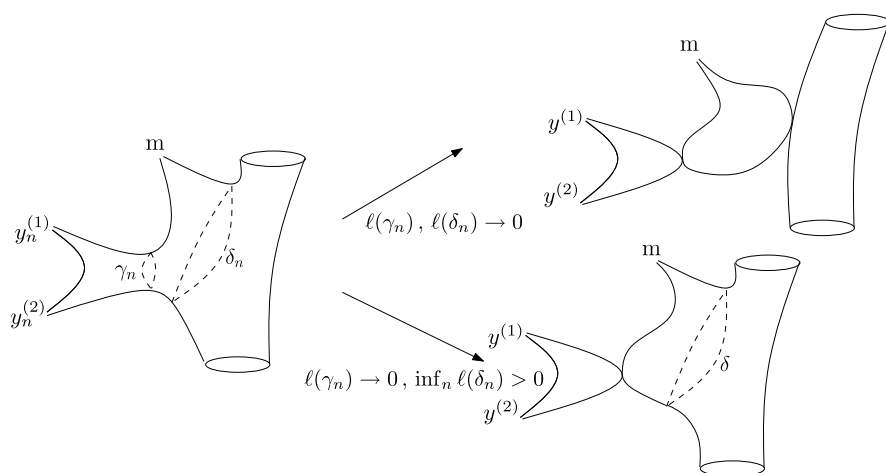


Fig. 3.4 Adding two points to a cylinder with a *marked point* m , viewed as a degenerate boundary component: Two possible limit surfaces

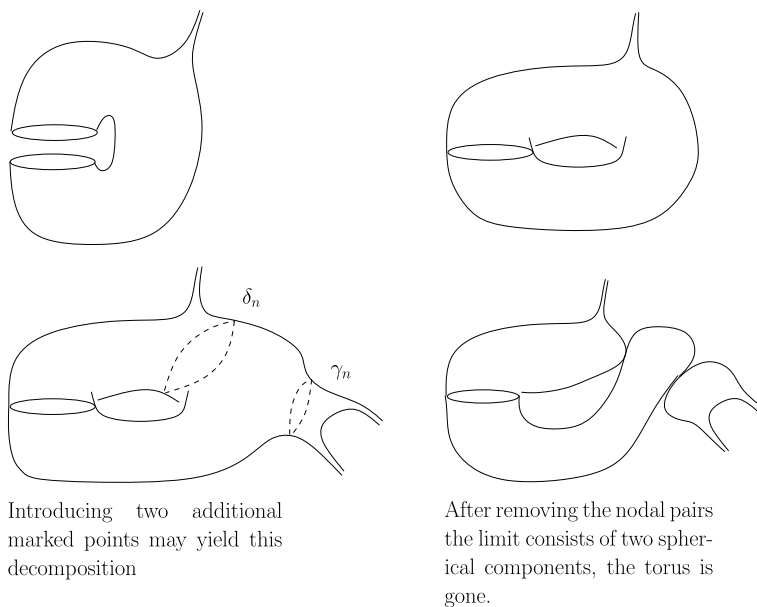


Fig. 3.5 Starting with a 1-punctured torus

where (S_n, j_n) is a compact stable Riemann surface without boundary, with marked points M_n , punctures $\underline{Z}_n \cup \overline{Z}_n$ (note that j_n extends over all of them) and all surfaces have the same signature. The maps $\tilde{u}_n : S_n \setminus (\underline{Z}_n \cup \overline{Z}_n) \rightarrow W$ are pseudoholomorphic curves with energy bounded by some constant $E_0 > 0$, and the punctures

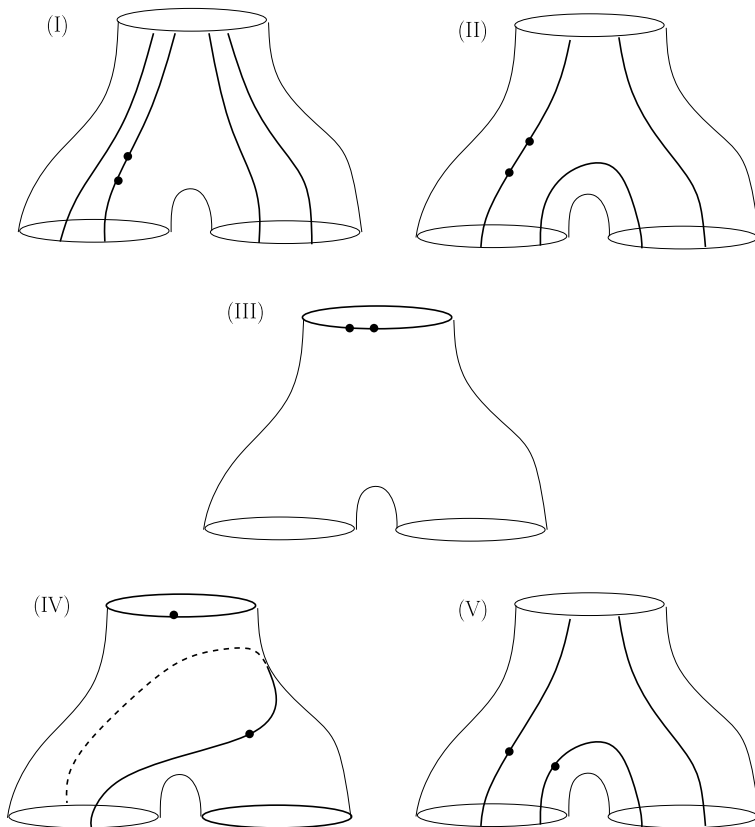


Fig. 3.6 Adding two marked points on the boundary. The intersection of the boundary curve ∂S_n and the pair of pants is depicted by the fat curve(s)

$\underline{Z}_n = \{z_1^{(n)}, \dots, z_{p_-}^{(n)}\}$ are all negative while the punctures $\overline{Z}_n = \{\bar{z}_1^{(n)}, \dots, \bar{z}_{p_+}^{(n)}\}$ are all positive. We also assume that all the curves \tilde{u}_n are asymptotic at the corresponding punctures to the same periodic orbits.⁵ Then there exists a subsequence of $\{C_n\}$ which converges to a stable holomorphic building of height $k \geq 1$.

It is not a restriction to treat only convergence of smooth curves (smooth meaning that S_n is not noded) because we may just include nodal points in the sets M_n of marked points. We divide the proof into several steps.

⁵This is not a real assumption since it can be achieved by merely passing to a suitable subsequence due to the uniform bound on the energy and the nondegeneracy of the contact form. In the paper [12] the authors also consider the Morse–Bott case for curves without boundary. Then one has to assume that all the curves \tilde{u}_n are asymptotic at the corresponding punctures to periodic orbits lying in the same connected component in the space of periodic orbits.

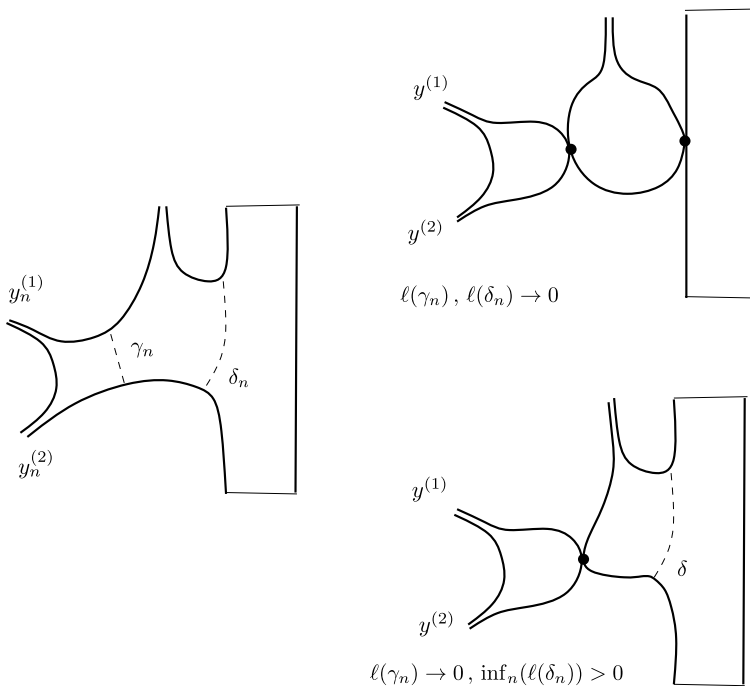


Fig. 3.7 Possible limit surfaces after adding two marked points to the boundary in (II) of Fig. 3.6

3.3.2 Gradient Bounds

Let $\{C_n\}$ be a sequence of pseudoholomorphic curves as in Theorem 3.6. We will establish a uniform gradient bound on the ε -thick parts of S_n after removing finitely many points. This statement is more subtle than it seems at first sight. The magnitude of the gradient $\|\nabla \tilde{u}_n(z)\| = \sup_{|\zeta|=1} \|Du_n(z)\zeta\|$ depends on the metric on the surface S_n . On the other hand, removing points from the surface S_n alters the metric on S_n . The statement made here is that the procedure of successively removing pairs of points from the surface whenever the gradient blows up necessarily terminates after finitely many steps resulting in a uniform gradient bound. Here is the precise statement:

Proposition 3.7 (Gradient bounds after removing finitely many points) *There is an integer $K \geq 0$ and a constant $C > 0$ which only depend on E_0 and points*

$$Y_n = \{y_n^{(1)}, u_n^{(1)}, \dots, y_n^{(K)}, u_n^{(K)}\} \subset \dot{S}_n := S_n \setminus (M_n \cup \underline{Z}_n \cup \overline{Z}_n)$$

such that

$$\|\nabla \tilde{u}_n(z)\| \leq \frac{C}{\rho_n(z)} \quad \forall z \in \dot{S}_n \setminus Y_n, \quad (3.1)$$

where ρ_n denotes the injectivity radius with respect to the Poincaré metric $h^{\dot{S}_n \setminus Y_n}$ on $(\dot{S}_n \setminus Y_n, j_n)$. The gradient is computed with respect to $h^{\dot{S}_n \setminus Y_n}$ and the obvious ω -compatible metric on W , i.e.

$$\|\nabla \tilde{u}_n(z)\| := \sup\{\|D\tilde{u}_n(z)\zeta\| : \zeta \in T_z S_n, |\zeta|_{h^{\dot{S}_n \setminus Y_n}} = 1\}.$$

Proof By the Deligne–Mumford compactness result we may assume after passing to a subsequence that the domains converge to some noded surface,

$$(S_n, j_n, M_n, \underline{Z}_n \cup \overline{Z}_n) \rightarrow (S, j, M, D, \underline{Z} \cup \overline{Z}).$$

The proof uses some kind of iteration argument. Assume that $z_n \in \dot{S}_n$ is a sequence such that

$$\lim_{n \rightarrow \infty} \rho_n(z_n) \|\nabla \tilde{u}_n(z_n)\| = +\infty \quad (3.2)$$

where ρ_n denotes the injectivity radius with respect to the hyperbolic metric on $\dot{S}_n = S_n \setminus (M_n \cup \underline{Z}_n \cup \overline{Z}_n)$, i.e. $Y_n = \emptyset$. Let us first consider the case where $W = \mathbb{R} \times M$. Writing $\tilde{u}_n = (a_n, u_n)$ and replacing a_n with $a_n - a_n(z_n)$ we may assume that $a_n(z_n) = 0$ for all n .

We now claim that we can find complex coordinate charts near the points z_n with some amount of control on the gradient which is the contents of the following lemma.

Lemma 3.8 *There are holomorphic charts $\psi_n : D \rightarrow U_n \subset (\dot{S}_n, j_n)$ with $\psi_n(0) = z_n$ if $z_n \notin \partial \dot{S}_n$ and positive constants C_1, C_2 such that for all $z \in D$ and all large n*

$$C_1 \rho_n(z_n) \leq \|\nabla \psi_n(z)\| \leq C_2 \rho_n(z_n). \quad (3.3)$$

Here, $D \subset \mathbb{C}$ is the open unit disk and U_n is some open neighborhood of z_n . The gradient is computed with respect to the Euclidean metric on D (or D^+) and the Poincaré metric h_n on \dot{S}_n so that

$$\|\nabla \psi_n(z)\| = \sup\{|D\psi_n(z)\zeta|_{h_n} : \zeta \in T_z S_n, |\zeta|_{\text{Eucl}} = 1\}.$$

Remark 3.9 We will later consider the case where $\partial S_n \neq \emptyset$. If $z_n \in \partial \dot{S}_n$ then the same statement is true for a suitable holomorphic chart $\psi_n : D^+ \rightarrow U_n$ with $\psi_n(0) = z_n$, $\psi_n(D^+ \cap \mathbb{R}) \subset \partial \dot{S}_n$ and $D^+ = \{z \in D \mid \text{Im}(z) \geq 0\}$. We note that the proof of Lemma 3.8 in the boundary case is not any different from the interior case because we double Riemann surfaces along their boundary.

Proof We consider the following two cases for subsequences $\{z_{n_k}\} \subset \{z_n\}$ and $0 < \varepsilon \leq \sinh(1/2) < \sinh^{-1}(1)$ (we use the notation $\{z_n\}$ for a subsequence for simplicity):

- (1) $\rho_n(z_n) \geq \varepsilon$
- (2) $\rho_n(z_n) < \varepsilon$,

and we will establish the estimate (3.3) for a suitable number ε with constants $C_1, C_2 > 0$ not depending on n .

Let $0 < \varepsilon \leq \sinh(1/2)$. In case (1) we have $z_n \in \text{Thick}_\varepsilon(\dot{S}_n)$, and the claim holds for this subsequence (with constants depending on ε only and not on the particular choice of the subsequence): Using suitable diffeomorphisms we may consider S instead of S_n with the hyperbolic metrics induced by the metrics h_n (we denote these induced metrics again by h_n), and we have $h_n \rightarrow h$ and also $\rho_n \rightarrow \rho$ on $\text{Thick}_{\varepsilon'}(\dot{S})$ for $0 < \varepsilon' < \varepsilon$. Hence we may view $\text{Thick}_\varepsilon(\dot{S}_n)$ for large n as a proper subset of $\text{Thick}_{\varepsilon'}(\dot{S})$. We then cover the compact set $\text{Thick}_{\varepsilon'}(\dot{S})$ with finitely many holomorphic coordinate charts⁶ $\phi_k : D_2 \rightarrow S$ or $\phi_k : D_2^+ \rightarrow S$ such that $\{\phi_k(D), \phi_k(D^+)\}$ cover $\text{Thick}_{\varepsilon'}(\dot{S})$. We have $z_n = \phi_k(z_0)$ for some k and $|z_0| \leq 1$. Then take $\psi_n := \phi_k|_{D(z_0)}$ and the claim holds with constants C_1, C_2 depending on ε but not on n . While discussing case (2) we will impose further restrictions on the possible choices for ε .

Consider now the case (2) where $z_n \in \text{Thin}_\varepsilon(\dot{S}_n)$. By the Thick–thin decomposition, Theorem 1.85, every component of $\text{Thin}_\varepsilon(\dot{S}_n)$ is isometric either to a cusp or to a piece of a hyperbolic cylinder because $\varepsilon \leq \sinh^{-1}(1)$. Let us assume first that the sequence $\{z_n\}$ lies in a component of the ε -thin part of the surface which is isometric to a cusp. By Lemma 1.83 the injectivity radius $\rho(p)$ at a point p of the surface is given by half the length of the shortest geodesic loop through that point. If $z = s + it \in \mathcal{C} = \{z \in \mathbb{H} : \text{Im}(z) \geq 1/2\} / \{z \sim z + 1\}$ is the standard cusp, we obtain

$$\rho(z) = \frac{1}{2} \text{dist}(z, z + 1) = \sinh^{-1} \left(\frac{1}{2t} \right)$$

using the distance formula

$$\sinh \left(\frac{1}{2} \text{dist}(z, w) \right) = \frac{|z - w|}{2\sqrt{\text{Im}(z)\text{Im}(w)}}, \quad z, w \in \mathbb{H}.$$

Any component of the ε -thin part isometric to a cusp is then isometric to the following subset of the standard cusp:

$$\mathcal{C}_\varepsilon := \left\{ z \in \mathbb{H} : \text{Im}(z) \geq \frac{1}{2 \sinh^{-1} \varepsilon} \right\} / \{z \sim z + 1\}.$$

We define now

$$\begin{aligned} \psi_n : D &\longrightarrow \mathcal{C} \\ \psi_n(z) &:= \frac{1}{2}z + z_n. \end{aligned}$$

⁶Here $D_r = \{z \in \mathbb{C} : |z| < r\}$ and $D_r^+ = \{z \in D_r \mid \text{Im}(z) \geq 0\}$.

Note that this is well-defined since with $z_n = s_n + it_n$

$$\begin{aligned} \operatorname{Im}\left(\frac{z}{2} + z_n\right) &\geq t_n - \frac{1}{2} \\ &\geq \frac{1}{2 \sinh^{-1} \varepsilon} - \frac{1}{2} \\ &\geq \frac{1}{2} \end{aligned}$$

in view of $\sinh^{-1} \varepsilon \leq 1/2$. If $h(s + it) = \frac{1}{t^2} g_{\text{eucl}}$ is the hyperbolic metric on \mathbb{H} and $z = s + it$ then

$$\|\nabla \psi_n(z)\| = \sup_{\zeta_{\text{eucl}}=1} |D\psi_n(z)\zeta|_h = \frac{1}{t + 2t_n}.$$

In order to prove the claim we have to show that there are $0 < C_1 < C_2$ such that for all $t \in [-1, 1]$ and for all n

$$C_1 \leq (t + 2t_n) \sinh^{-1}\left(\frac{1}{2t_n}\right) \leq C_2. \quad (3.4)$$

If $f(t, \tau) := (t + \tau) \sinh^{-1}(\tau^{-1})$ then

$$\lim_{\tau \rightarrow +\infty} f(t, \tau) = 1$$

so that (3.4) holds for

$$C_1 := \min\{f(t, \tau) \mid -1 \leq t \leq +1, \tau \geq 2\}$$

and

$$C_2 := \max\{f(t, \tau) \mid -1 \leq t \leq +1, \tau \geq 2\}.$$

Note that these constants do not even depend on the choice of ε . Recall that

$$t_n \geq \frac{1}{2 \sinh^{-1} \varepsilon} \geq 1$$

in view of $\varepsilon \leq \sinh \frac{1}{2}$ such that $\tau = 2t_n \geq 2$.

We assume now that $\{z_n\} \subset \text{Thin}_\varepsilon(\dot{S}_n)$ where $\text{Thin}_\varepsilon(\dot{S}_n)$ is isometric to a subset of a collar neighborhood $\mathcal{C}(\beta_n)$ of a short closed geodesic β_n , i.e.

$$\mathcal{C}(\beta_n) = \{z \in \mathbb{H} \mid \text{dist}(z, \beta_n) \leq w_n\} / \{z \sim e^{\ell_n} z\}$$

where

$$w_n := \sinh^{-1}\left(\frac{1}{\sinh \frac{\ell_n}{2}}\right), \quad \beta_n(t) = i e^{t\ell_n}, \quad t \in [0, 1], \quad (3.5)$$

and $\ell_n \leq 2 \sinh^{-1}(1)$ is the length of β_n with respect to the hyperbolic metric on \mathbb{H} . The injectivity radius at some point $z \in \mathcal{C}(\beta_n)$ is given by

$$\sinh(\rho_n(z)) = \sinh\left(\frac{1}{2}d(z, e^{\ell_n}z)\right) = \frac{|z|}{\operatorname{Im}(z)} \sinh \frac{\ell_n}{2} = \frac{|z|}{\operatorname{Im}(z) \sinh w_n}. \quad (3.6)$$

Hence we may assume that the sequence $\{z_n\} = \{s_n + it_n\}$ satisfies

$$1 \leq |z_n| \leq e^{\ell_n}, \quad \frac{|z_n|}{t_n} \sinh \frac{\ell_n}{2} \leq \sinh \varepsilon. \quad (3.7)$$

We define now the maps

$$\begin{aligned} \psi_n : D &\longrightarrow \mathcal{C}(\beta_n) \\ z &\longmapsto z_n + rz, \quad r = \tanh \frac{\ell_n}{2}. \end{aligned}$$

We first verify that indeed $\psi_n(D) \subset \mathcal{C}(\beta_n)$ provided ε is chosen sufficiently small, i.e. we have to show that $\operatorname{dist}(z_n + rz, i\mathbb{R}) \leq w_n$ if $z \in D$. Recalling the definition of w_n in (3.5) and the fact that the lengths $\{\ell_n\}$ are bounded from above by Bers' theorem we define $\delta := \inf_n w_n > 0$ and we pick

$$0 < \varepsilon \leq \inf_n \sinh^{-1}\left(\frac{2 \sinh^2 \frac{w_n}{2}}{\sqrt{1 + \sinh^2 w_n}}\right) = \inf_n \sinh^{-1}\left(1 - \frac{1}{\cosh w_n}\right).$$

Note that with $z = s + it$

$$\sinh \frac{1}{2}d(z, i\mathbb{R}) = \frac{|z - i|z||}{2\sqrt{t|z|}} = \sqrt{\frac{|z|}{2t}} \quad (3.8)$$

and that we have the inequalities⁷

$$\frac{t_n - \frac{rs_n}{|z_n|}}{\sqrt{|z_n|^2 + r^2}} \leq \frac{t}{|z|} \leq \frac{t_n + \frac{rs_n}{|z_n|}}{\sqrt{|z_n|^2 + r^2}} \quad \forall z = (s, t) \in B_r(z_n).$$

We estimate using (3.7)

$$\begin{aligned} \frac{|z|}{t} &\leq \frac{\sqrt{|z_n|^2 + \tanh^2 \frac{\ell_n}{2}}}{t_n - \frac{s_n}{|z_n|} \tanh \frac{\ell_n}{2}} \\ &\leq \sqrt{\frac{|z_n|^2}{t_n^2} + \frac{1}{t_n^2} \tanh^2 \frac{\ell_n}{2}} \end{aligned}$$

⁷The points $z_{\pm} = (s_{\pm}, t_{\pm}) \in \partial B_r(z_n)$ with $t_{\pm} = t_n \pm \frac{rs_n}{|z_n|}$ and $|z_{\pm}| = \sqrt{|z_n|^2 + r^2}$ are characterized by the condition that the lines $\{\tau z_{\pm} \mid \tau \in \mathbb{R}\}$ are tangent to the boundary of $B_r(z_n)$.

$$\begin{aligned}
&\leq \sqrt{\frac{\sinh^2 \varepsilon}{\sinh^2 \frac{\ell_n}{2}} + \frac{\sinh^2 \varepsilon}{\cosh^2 \frac{\ell_n}{2}}} \\
&\leq \sinh \varepsilon \sqrt{\sinh^2 w_n + 1} \\
&\leq 2 \sinh^2 \frac{w_n}{2}
\end{aligned}$$

by our choice of ε . Combining this with (3.8) we see that the maps ψ_n satisfy

$$d(\psi_n(z), \beta_n) \leq w_n \quad \forall z \in D.$$

In order to show that the maps ψ_n are injective we have to make sure that

$$B_r(z_n) \cap e^{\ell_n} B_r(z_n) = \emptyset,$$

which is the same as $|z_n| + r \leq e^{\ell_n}(|z_n| - r)$. This inequality follows from

$$r = \tanh \frac{\ell_n}{2} \leq |z_n| \tanh \frac{\ell_n}{2} = |z_n| \frac{e^{\ell_n} - 1}{e^{\ell_n} + 1}.$$

We have

$$\|\nabla \psi_n(z)\| = \frac{r}{t_n + rt}, \quad z \in D$$

and therefore with (3.6)

$$\begin{aligned}
\frac{\rho_n(z_n)}{\|\nabla \psi_n(z)\|} &= \left(\frac{t_n}{r} + t \right) \sinh^{-1} \left(\frac{|z_n|}{t_n} \sinh \frac{\ell_n}{2} \right) \\
&= \left(\frac{t_n}{\sinh \frac{\ell_n}{2}} \cosh \frac{\ell_n}{2} + t \right) \sinh^{-1} \left(\frac{|z_n|}{t_n} \sinh \frac{\ell_n}{2} \right).
\end{aligned}$$

We recall that

$$\frac{\sinh \frac{\ell_n}{2}}{t_n} \leq \sinh \varepsilon$$

so that

$$\begin{aligned}
\frac{\rho_n(z_n)}{\|\nabla \psi_n(z)\|} &\geq \left(\frac{t_n}{\sinh \frac{\ell_n}{2}} - 1 \right) \sinh^{-1} \left(\frac{\sinh \frac{\ell_n}{2}}{t_n} \right) \\
&\geq \frac{\varepsilon}{\sinh \varepsilon} - \varepsilon
\end{aligned}$$

and similarly

$$\begin{aligned} \frac{\rho_n(z_n)}{\|\nabla\psi_n(z)\|} &\leq \left(\frac{t_n}{\sinh \frac{\ell_n}{2}} + 1 \right) \sinh^{-1} \left(\frac{\sinh \frac{\ell_n}{2}}{t_n} \right) \\ &\leq 1 + \varepsilon. \end{aligned}$$

This finally completes the proof of the lemma and (3.3). \square

The following statement is left as an exercise to the reader.

Exercise 3.10 Assume (S, j) is a Riemann surface, where j is induced by some metric and some orientation, and let $|\cdot|$ be the corresponding norm on TS . Let $u : S \rightarrow (W, \omega, J)$ be a J -holomorphic curve where J is ω -compatible, and where $\|v\|^2 := \omega_q(v, Jv)$ for $v \in T_q W$. With the notation

$$\|\nabla u(z)\| := \sup\{\|Du(z)\zeta\| : \zeta \in T_z S, |\zeta| = 1\}$$

show that for any $\zeta \in T_z S$ with $|\zeta| = 1$ we have

$$\frac{1}{2}\|\nabla u(z)\| \leq \|Du(z)\zeta\|.$$

We resume the proof of Proposition 3.7. Consider now the pseudoholomorphic disks $\tilde{u}_n \circ \psi_n$. Because of Lemma 3.8, Exercise 3.10 above and because \tilde{u}_n is \tilde{J} -holomorphic, we have for all unit vectors $\zeta \in \mathbb{C}$:

$$\begin{aligned} \|D(\tilde{u}_n \circ \psi_n)(0)\zeta\| &= \left\| D\tilde{u}_n(z_n) \frac{D\psi_n(0)\zeta}{\|D\psi_n(0)\zeta\|} \right\| \|D\psi_n(0)\zeta\| \\ &\geq C_1 \frac{1}{4} \|\nabla \tilde{u}_n(z_n)\| \rho_n(z_n) \\ &\rightarrow \infty \end{aligned}$$

Using Lemma 2.39 and the usual bubbling-off analysis, we can find sequences $\{w_n\} \subset D$, $w_n \rightarrow 0$, $R_n = \|\nabla(\tilde{u}_n \circ \psi_n)(w_n)\| \rightarrow +\infty$, $\varepsilon_n \searrow 0$ such that $\varepsilon_n R_n \rightarrow +\infty$ and

$$\|\nabla(\tilde{u}_n \circ \psi_n)(w)\| \leq 2\|\nabla(\tilde{u}_n \circ \psi_n)(w_n)\| \quad \forall |w - w_n| \leq \varepsilon_n.$$

As before we define rescaled maps

$$\tilde{v}_n(w) := (\tilde{u}_n \circ \psi_n) \left(w_n + \frac{w}{R_n} \right) \quad \text{for } w \in B_{\varepsilon_n R_n}(0)$$

which converge uniformly on compact sets with all derivatives to a nontrivial \tilde{J} -holomorphic finite energy plane which we denote by \tilde{v}_∞ . We claim that the energy of any such \tilde{v}_∞ is bounded from below by a constant $\hbar > 0$. We identify the domain of \tilde{v}_∞ with $S^2 \setminus \{\text{point}\}$. If the puncture is not removable then \tilde{v}_∞ is asymptotic to a periodic orbit of the Reeb vector field. Then the smallest period among

all periodic orbits of the Reeb vector fields at the cylindrical ends serves as a lower bound. Otherwise, in the case of a removable puncture, we apply the mean value inequality (Proposition 2.59). It implies that a \tilde{J} -holomorphic plane with image contained in a compact subset C of the target symplectic manifold W and with energy less than a certain constant $\varepsilon_0 = \varepsilon_0(C)$ must be constant. Since W is the symplectization of a closed contact manifold the constant ε_0 actually does not depend on C because the almost complex structure is \mathbb{R} -invariant (look at Remark 2.60).

Remark 3.11 We insert a few brief remarks on how to modify the above argument in the case of surfaces with boundary for later reference. Some of the notions mentioned here will be defined in the next section. In this case the maps $\tilde{u}_n \circ \psi_n$ are defined on the half-disk, and we have boundary values in some Lagrangian submanifold $L \subset W$ which is of the form $\mathbb{R} \times \mathcal{L}$ in the cylindrical part of W with some Legendrian $\mathcal{L} \subset M$. The limit \tilde{v}_∞ may also be a finite energy half-plane with boundary values in L . We identify the domain of \tilde{v}_∞ with $D \setminus \{1\}$. If the puncture is not removable then \tilde{v}_∞ is asymptotic to a characteristic chord for the Legendrian \mathcal{L} . Then the smallest length among all characteristic chords serves as a lower bound for the energy of \tilde{v}_∞ . The mean value inequality can also be applied to a half-plane. Since we only consider noncompact symplectic manifolds W with cylindrical ends (or manifolds which arise from the splitting construction) the constant ε_0 actually does not depend on C because the almost complex structure on the cylindrical part is \mathbb{R} -invariant (see Remark 2.60).

We define now sequences

$$\bar{w}_n := \psi_n(w_n), \quad \bar{u}_n := \psi_n\left(w_n + \frac{1}{R_n}\right)$$

which both lie in $\psi_n(D) = U_n \subset \dot{S}_n$ and which satisfy $d_n(\bar{w}_n, \bar{u}_n) \rightarrow 0$ where d_n denotes the distance with respect to the Poincaré metric on \dot{S}_n . Indeed,

$$\begin{aligned} d_n(\bar{w}_n, \bar{u}_n) &\leq \int_0^1 \left\| \frac{d}{dt} \psi_n\left(w_n + \frac{t}{R_n}\right) \right\| dt \\ &\leq \frac{1}{R_n} \int_0^1 \left\| \nabla \psi_n\left(w_n + \frac{t}{R_n}\right) \right\| dt \\ &\leq \frac{C_2 \rho_n(z_n)}{R_n} \end{aligned}$$

using (3.3). Following Sect. 3.2 we now add $\{\bar{w}_n, \bar{u}_n\}$ as additional marked points to the Riemann surfaces $(S_n, j_n, M_n, \underline{Z}_n \cup \overline{Z}_n)$. After passing to a subsequence we may assume that the sequence $(S_n, j_n, M_n \cup \{\bar{w}_n, \bar{u}_n\}, \underline{Z}_n \cup \overline{Z}_n)$ converges to a noded Riemann surface $(S', j', M', D', \underline{Z}' \cup \overline{Z}')$ which differs from $(S, j, M, D, \underline{Z} \cup \overline{Z})$ by one or two additional spherical components. Recalling the construction in Sect. 3.2 and the one of \tilde{v}_∞ we see that one of the spherical components is the domain of \tilde{v}_∞ , and therefore has energy bounded from below by some positive constant \hbar .

If inequality (3.1) holds for the sequence $(\tilde{u}_n, S_n, j_n, M_n \cup \{\bar{w}_n, \bar{u}_n\}, \underline{Z}_n \cup \bar{Z}_n)$ then the proof is complete. Otherwise, there is a sequence $\{z'_n\} \subset \dot{S}_n \setminus \{\bar{w}_n, \bar{u}_n\}$ such that

$$\lim_{n \rightarrow +\infty} \rho_n(z'_n) \|\nabla \tilde{u}_n(z'_n)\| = +\infty$$

where ρ_n this time denotes the injectivity radii with respect to the Poincaré metrics on the surface $(S_n, j_n, M_n \cup \{\bar{w}_n, \bar{u}_n\})$. As before, there are holomorphic charts $\psi'_n : D \rightarrow U'_n \subset \dot{S}_n \setminus \{\bar{w}_n, \bar{u}_n\}$ (or charts defined on D^+) with $\psi'_n(0) = z'_n$ and positive constants C'_1, C'_2 such that for all $z \in D$ and all large n

$$C'_1 \rho_n(z'_n) \leq \|\nabla \psi'_n(z)\| \leq C'_2 \rho_n(z'_n). \quad (3.9)$$

We now repeat the above bubbling-off analysis generating again a \tilde{J} -holomorphic sphere or plane with energy bounded below by the same positive constant $\hbar > 0$ as before. Recalling that the energy of the original sequence was uniformly bounded we conclude that the above iteration must terminate after finitely many steps. \square

3.3.3 Convergence in the Thick Part

Assume that $\mathcal{C}_n = (\tilde{u}_n, S_n, j_n, M_n, \underline{Z}_n \cup \bar{Z}_n)$ is a sequence as in Theorem 3.6. Invoking Proposition 3.7, after adding more marked points, we may assume that the gradient bound (3.1) holds:

$$\|\nabla \tilde{u}_n(z)\| \leq \frac{C}{\rho(z, h_{j_n})} \quad \forall z \in \dot{S}_n,$$

where $\rho(*, h_{j_n})$ denotes the injectivity radius with respect to the hyperbolic metric h_{j_n} on $\dot{S}_n = S_n \setminus (M_n \cup \underline{Z}_n \cup \bar{Z}_n)$. Again, by the Deligne–Mumford compactness result we may assume after passing to a subsequence that the domains converge,

$$(S_n, j_n, M_n, \underline{Z}_n \cup \bar{Z}_n) \rightarrow (S, j, M, D, \underline{Z} \cup \bar{Z}).$$

Let $\varepsilon > 0$, and let

$$\text{Thick}_\varepsilon(\dot{S}) = \{z \in \dot{S} \mid \rho(z) \geq \varepsilon\}$$

where ρ is the injectivity radius with respect to the Poincaré metric h on $\dot{S} \setminus \bigcup_j \Gamma_j$, and where $\{\Gamma_j\}$ is a finite collection of disjoint embedded closed curves and embedded arcs starting and ending at $\partial \dot{S}$.

By Definition 1.87 there is a sequence of diffeomorphisms $\varphi_n : S \rightarrow S_n$ with $h_n := (\varphi_n)^* h_{j_n} \rightarrow h$ in $C_{\text{loc}}^\infty(\dot{S} \setminus \bigcup_j \Gamma_j)$. We have

$$\|\nabla(\tilde{u}_n \circ \varphi_n)(z)\| \leq \frac{C}{\rho_n(z)} \quad \forall z \in \dot{S} \setminus \bigcup_j \Gamma_j. \quad (3.10)$$

where this time $\rho_n = \text{inrad}_{h_n}$. For each $\varepsilon > 0$ we then have a uniform gradient bound on the maps $\tilde{u}_n \circ \varphi_n$. Here are some details: There is an integer $N = N(\varepsilon) > 0$ such that

$$\sup\{|\rho_n(z) - \rho(z)| : z \in \text{Thick}_\varepsilon(\dot{S})\} \leq \frac{\varepsilon}{4} \quad \forall n \geq N.$$

Then $\rho_n(z) \geq \frac{3}{4}\varepsilon$ for $z \in \text{Thick}_\varepsilon(\dot{S})$ which results in

$$\begin{aligned} \|\nabla(\tilde{u}_n \circ \varphi_n)(z)\| &\leq \left| \frac{C}{\rho_n(z)} - \frac{C}{\rho(z)} \right| + \frac{C}{\rho(z)} \\ &= \frac{C|\rho(z) - \rho_n(z)|}{\rho(z)\rho_n(z)} + \frac{C}{\rho(z)} \\ &\leq \frac{4}{3} \cdot \frac{C}{\rho(z)} \end{aligned}$$

so that we may replace the injectivity radii ρ_n in the gradient estimate with ρ for sufficiently large n at the expense of choosing a larger constant C .

The results in Sect. 2.2.3 then yield a uniform bound on all derivatives on $\text{Thick}_\varepsilon(\dot{S})$ for any $\varepsilon > 0$. The Ascoli–Arzela theorem guarantees the existence of a subsequence which converges in

$$C_{\text{loc}}^\infty\left(\dot{S} \setminus \bigcup_j \Gamma_j\right) = C_{\text{loc}}^\infty\left(\bigcup_\varepsilon \text{Thick}_\varepsilon(\dot{S})\right).$$

In order to obtain such a convergent subsequence we actually have to shift the \mathbb{R} -components of the curves such that the gradient bound implies a C^0 -bound. If $\dot{S} \setminus \bigcup_j \Gamma_j$ has several connected components we may have to shift the curves on the individual components by different constants.

Remark 3.12 In the case where $\partial S_n \neq \emptyset$ the set $\{\Gamma_j\}$ is a finite collection of disjoint embedded closed curves and embedded arcs starting and ending at $\partial \dot{S}$. The convergence statement in the thick part is not affected by this modification.

3.3.4 Convergence in the Thin Part and Level Structure

In this section we will cover curves with and without boundary at the same time since the case of curves with boundary only requires a few minor additional remarks which does not complicate the exposition. In the last section we have extracted a subsequence from $\{\mathcal{C}_n\}$ which converges on compact subsets of $\bigcup_\varepsilon \text{Thick}_\varepsilon(\dot{S})$, and we denote its various connected components by C_1, \dots, C_m . We denote the limit by

$$\mathcal{C} = (\tilde{u} = (a, u), S, j, M, \underline{Z} \cup \overline{Z}).$$

Fig. 3.8 The geodesics Γ_j shrink to a point. The *left picture* depicts a geodesic arc between two boundary points of the surface

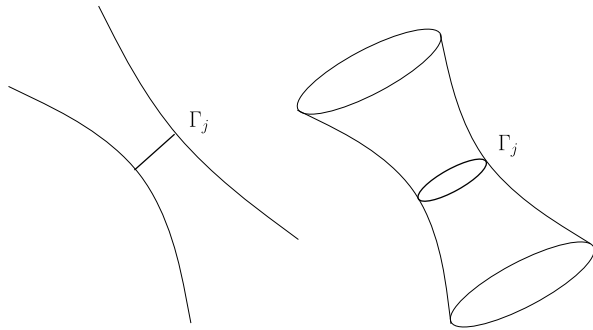
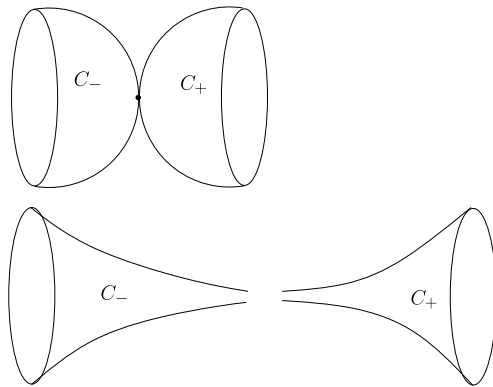


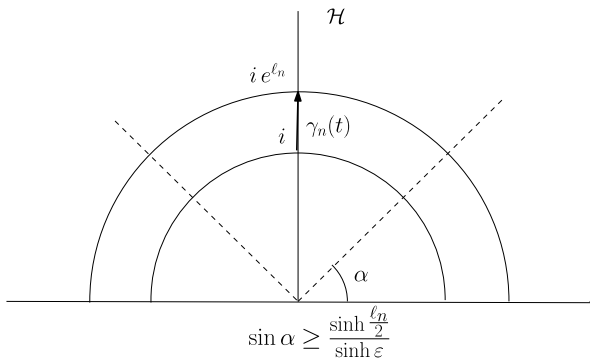
Fig. 3.9 After passing to the limit: Two components C_- and C_+ adjacent to a node



For sufficiently small $\varepsilon > 0$ the ε -thin part of \dot{S} consists of neighborhoods of punctures isometric to standard cusps.⁸ We have to distinguish between two kinds of punctures of $\dot{S} = \bigcup_{j=1}^m C_j$. Some of the punctures are actually nodes, they were generated in the limit process $n \rightarrow \infty$ by shrinking certain closed geodesics (or geodesic arcs between boundary points in the case of nonempty boundary) to points, and they come in pairs (see Figs. 3.8, 3.9). The other punctures were already there before we passed to the limit. The aim of this section is to understand the asymptotic behavior of the curve \tilde{u} near the punctures. If \tilde{u} is bounded near a puncture then we may extend it smoothly across the puncture by the removal of singularities theorem, Theorem 2.68. Otherwise, \tilde{u} approaches a cylinder over a periodic Reeb orbit by Proposition 2.47 or a strip over a characteristic chord by Theorem 2.57. We will first consider the situation near a node adjacent to two components C_- and C_+ as in Fig. 3.9. We will denote the asymptotic

⁸Recall from the definition of convergence of surfaces that there are diffeomorphisms $\varphi_n : S \rightarrow S_n$ and disjoint embedded loops $\{\Gamma_j\} \subset \dot{S}$ such that $\varphi_n(\Gamma_j)$ are geodesics in S_n . The lengths of these geodesics tend to zero as $n \rightarrow \infty$. It follows from the proof of the collar lemma that the circles Γ_j are actually degenerate boundary components for the limit metric $h = \lim_{n \rightarrow \infty} \varphi_n^* h_n$, and these are isometric to standard cusps. This also explains why removal of singularities theorem and Proposition 2.47 can be applied near a node.

Fig. 3.10 A model of a component of the ε -thin part of $(\dot{S}, \varphi_n^* h_n)$ due to the collar lemma



limit of $u|_{C_-}$ by γ_- , and the limit of $u|_{C_+}$ by γ_+ . These may be points, characteristic chords or periodic orbits of the Reeb vector field. We wish to show that if one is a point then so is the other, and we would like to explore how they ‘fit together’.

We consider now components of $\text{Thin}_\varepsilon(\dot{S}, \varphi_n^* h_n)$. These are either isometric to cusps or to half-cusps. In the latter case we double the surface so that we may consider cusps only. More precisely, by the collar lemma we may assume that each such component of $\text{Thin}_\varepsilon(\dot{S}, \varphi_n^* h_n)$, which will eventually mutate into a node, is isometric to the following set:

$$\mathcal{C}(\gamma_n) := \{z \in \mathbb{H} \mid \text{dist}(z, \gamma_n) \leq w(\gamma_n), \rho_n(z) \leq \varepsilon\} / \{z \sim e^{\ell_n} z\},$$

where ‘dist’ refers to the usual distance in the hyperbolic plane,

$$\gamma_n(t) := i e^t \quad \text{for } 0 \leq t \leq \ell_n, \ell_n \searrow 0$$

is a closed geodesic corresponding to $\varphi_n(\Gamma_j)$ and where

$$w(\gamma_n) := \sinh^{-1} \left(\frac{1}{\sinh \frac{\ell_n}{2}} \right).$$

We have the following formula for the injectivity radius at a point $z = r e^{i\alpha}$:

$$\sinh(\rho_n(r e^{i\alpha})) = \frac{|z|}{\text{Im}(z)} \sinh \frac{\ell_n}{2} = \frac{1}{\sin \alpha} \sinh \frac{\ell_n}{2}. \quad (3.11)$$

If ε is kept fixed then the width of the cylinder increases as $n \rightarrow \infty$, and the angle between the dotted rays in Fig. 3.10 opens up.⁹ We will use another model which

⁹Note that a simple calculation yields

$$\varepsilon \geq \min \{ \rho_n(z) \mid z \in \mathcal{C}(\gamma_n) \} = \frac{\ell_n}{2}.$$

describes the degeneration process $n \rightarrow \infty$ better: Consider the conformal maps

$$\begin{aligned}\phi_n : [-\sigma_n^\varepsilon, \sigma_n^\varepsilon] \times S^1 &\longrightarrow \mathbb{H}, \quad S^1 = \mathbb{R}/\mathbb{Z} \\ \phi_n(s, t) &:= i e^{-i\ell_n(s+it)}.\end{aligned}$$

They map $\{0\} \times S^1$ onto the geodesics γ_n . Since $i e^{-i\ell_n s} = i \cos(\ell_n s) + \sin(\ell_n s)$ the image will be in the ε -thin part if and only if we choose

$$\sigma_n^\varepsilon = \frac{1}{\ell_n} \cos^{-1} \left(\frac{\sinh \frac{\ell_n}{2}}{\sinh \varepsilon} \right) \quad (3.12)$$

(which tends to $+\infty$ as $n \rightarrow \infty$). Moreover,

$$|D\phi_n(s, t)(\sigma, \tau)|_{\text{hyper}}^2 = \frac{\ell_n^2}{\cos^2(\ell_n s)} (\sigma^2 + \tau^2).$$

We estimate using (3.10), (3.11) and the metric $ds^2 + dt^2$ on the cylinders $Z_n^\varepsilon := [-\sigma_n^\varepsilon, \sigma_n^\varepsilon] \times S^1$

$$\begin{aligned}\sinh(\rho_n(\phi_n(s, t))) &= \frac{\sinh \frac{\ell_n}{2}}{\cos(\ell_n s)} \\ &= \frac{\sinh \frac{\ell_n}{2}}{\ell_n} \|\nabla \phi_n(s, t)\|\end{aligned}$$

where

$$\|\nabla \phi_n(s, t)\| := \sup_{\sigma^2 + \tau^2 = 1} |D\phi_n(s, t)(\sigma, \tau)|_{\text{hyper}} = \frac{\ell_n}{\cos(\ell_n s)}$$

and

$$\begin{aligned}\|\nabla(\tilde{u}_n \circ \varphi_n \circ \phi_n)(s, t)\| &\leq \frac{C}{\rho_n(\phi_n(s, t))} \|\nabla \phi_n(s, t)\| \\ &= C \frac{\sinh(\rho_n(\phi_n(s, t)))}{\rho_n(\phi_n(s, t))} \frac{\ell_n}{\sinh \frac{\ell_n}{2}}\end{aligned}$$

which is uniformly bounded in n , $0 < \varepsilon < \sinh^{-1}(1)$ and in (s, t) . We summarize

$$\sup_{n, \varepsilon} \|\nabla(\tilde{u}_n \circ \varphi_n \circ \phi_n)\|_{C^0(Z_n^\varepsilon)} < \infty. \quad (3.13)$$

Note that the pull-back of the hyperbolic metric under ϕ_n is given by

$$\frac{\ell_n^2}{\cos^2(\ell_n s)} (ds^2 + dt^2).$$

The gradient bound (3.13) would not hold if we chose this metric on the cylinders Z_n^ε since $\|\nabla \phi_n(s, t)\|$ would then equal 1.

Pick a sequence $\varepsilon_k \searrow 0$. We have by convergence in the thick part

$$\lim_{n \rightarrow \infty} u_n(\phi_n(\phi_n(\pm \sigma_n^{\varepsilon_k}, t))) = u(\gamma_\pm^{\varepsilon_k}(t))$$

where $\gamma_\pm^{\varepsilon_k}$ are closed curves in $C_\pm \cap \text{Thick}_{\varepsilon_k}(\dot{S}, \varphi_n^* h_n)$. We then obtain

$$\lim_{k \rightarrow \infty} u(\gamma_\pm^{\varepsilon_k}(t)) = \gamma_\pm(T_\pm t),$$

where T_\pm are the periods of the orbits γ_\pm , but not necessarily the minimal periods. In the boundary case γ_\pm are characteristic chords of lengths T_\pm , and the maps ϕ_n are defined on $[-\sigma_n^\varepsilon, \sigma_n^\varepsilon] \times [0, 1]$ instead. In both cases, after choosing a diagonal sequence which we denote by $\varepsilon_n \searrow 0$, $\sigma_n = \sigma_n^{\varepsilon_n} \rightarrow \infty$ we may assume that

$$\lim_{n \rightarrow \infty} (u_n \circ \varphi_n \circ \phi_n)(\pm \sigma_n, t) = \gamma_\pm(T_\pm t).$$

and also

$$\lim_{n \rightarrow \infty} (u_n \circ \varphi_n \circ \phi_n)(\pm \sigma_n \mp h, t) = \gamma_\pm(T_\pm t) \quad (3.14)$$

for any positive constant h . This follows from the following calculation for the injectivity radius:

$$\begin{aligned} & \sinh(\rho_n(\phi_n(\sigma_n^{\varepsilon_k} - h, t))) \\ &= \frac{|\phi_n(\sigma_n^{\varepsilon_k} - h, t)|}{\text{Im } \phi_n(\sigma_n^{\varepsilon_k} - h, t)} \sinh \frac{\ell_n}{2} \\ &= \frac{1}{\cos(\ell_n \sigma_n^{\varepsilon_k} - \ell_n h)} \sinh \frac{\ell_n}{2} \\ &= \sinh \varepsilon_k \frac{\sinh(\ell_n/2)}{\sinh(\ell_n/2) \cos(\ell_n h) + \sin(\ell_n h) \sqrt{\sinh^2 \varepsilon_k - \sinh^2(\ell_n/2)}}. \end{aligned}$$

We note that

$$\lim_{n \rightarrow \infty} \sinh(\rho_n(\phi_n(\sigma_n^{\varepsilon_k} - h, t))) = \frac{\sinh \varepsilon_k}{1 + 2h \sinh \varepsilon_k}$$

and the same diagonal sequence $\{\varepsilon_n\}$ yields $\lim_{n \rightarrow \infty} (u_n \circ \varphi_n \circ \phi_n)(\sigma_n - h, t) = \gamma_+(T_+ t)$.

Because of the gradient bound (3.13) we may, after maybe adding suitable constants to the \mathbb{R} -components, assume that a subsequence of $(\tilde{u}_n \circ \varphi_n \circ \phi_n)_{n \in \mathbb{N}}$ converges in $C_{\text{loc}}^\infty(\mathbb{R} \times S^1)$ (or $C_{\text{loc}}^\infty(\mathbb{R} \times [0, 1])$) to a finite energy pseudoholomorphic cylinder or strip $\tilde{w} = (c, w)$ which has asymptotic limits $\delta_\pm(t) := \lim_{s \rightarrow \pm \infty} w(s, t)$.

They may or may not coincide with $\gamma_{\pm}(T_{\pm}t)$. In any case, they must satisfy

$$\int_{\gamma_-} \lambda \leq \int_{\delta_-} \lambda \leq \int_{\delta_+} \lambda \leq \int_{\gamma_+} \lambda. \quad (3.15)$$

We introduce the more convenient notation $\tilde{v}_n := (b_n, v_n) := u_n \circ \varphi_n \circ \phi_n$ and if $c \in \mathbb{R}$ we write $\tilde{v}_n - c$ for $(b_n - c, v_n)$. What may happen geometrically is that the images $v_n([- \sigma_n, \sigma_n] \times S^1)$ (or $v_n([- \sigma_n, \sigma_n] \times [0, 1])$) break up at several periodic orbits (or characteristic chords), different from γ_{\pm} . The aim is to capture this in the resulting holomorphic building. If break-up does not happen then we may declare \tilde{w} to be the limit of the sequence (\tilde{v}_n) .

Since for large n the maps $v_n(\pm \sigma_n, *) = (u_n \circ \varphi_n \circ \phi_n)(\pm \sigma_n, *)$ are arbitrarily close to the loops $\gamma_{\pm}(T_{\pm}t)$, we can associate to this setup a unique homotopy class of maps $F : [-1, 1] \times S^1 \rightarrow M$ with $F(\pm 1, t) = \gamma_{\pm}(T_{\pm}t)$. We set

$$S(\gamma_-, \gamma_+, [F]) := \int_{[0, 1] \times S^1} F^* d\lambda = \int_{\tilde{\gamma}_+} \lambda - \int_{\tilde{\gamma}_-} \lambda = T_+ - T_-,$$

where we use the notation $\tilde{\gamma}_{\pm}(t) := \gamma_{\pm}(T_{\pm}t)$. Since we assumed nondegeneracy, there is a positive constant \hbar such that $S(\gamma_-, \gamma_+, [F]) \geq \hbar$ whenever $S(\gamma_-, \gamma_+, [F])$ is not zero, and \hbar is the smallest gap in the action spectrum.¹⁰ In the case where γ_{\pm} are characteristic chords we use a relative homotopy class $F : [-1, 1] \times [0, 1] \rightarrow M$, $F([-1, 1] \times \{0, 1\}) \subset \mathcal{L}$. The following two propositions describe the breaking up of the thin part. The cases of cylinder and strip are largely identical, so we will focus on one of them. The first deals with the case where the ‘neck’ has no energy.

Proposition 3.13 (No energy in the neck-“bubbles connect”) *If $S(\gamma_-, \gamma_+, [F]) = 0$ then $\gamma_- \equiv \gamma_+$, and there is a sequence $\{b_n\} \subset \mathbb{R}$ such that the sequence $\tilde{v}_n - b_n$ converges in C_{loc}^{∞} to either a constant or a trivial cylinder over the Reeb orbit $\tilde{\gamma}_{\pm}(t) = \gamma_{\pm}(T_{\pm}t)$.*

Proof We have

$$\int_{[-\sigma_n, \sigma_n] \times S^1} v_n^* d\lambda \rightarrow 0, \quad \sup_n E(v_n) < +\infty.$$

Because the asymptotic limits γ_{\pm} have the same action they are either both points or both periodic orbits of the Reeb vector field. We wish to show that $\gamma_+(t) = \gamma_-(t)$ and that for any C^{∞} -neighborhood U of $\tilde{\gamma}_{\pm}$, in the case where γ_{\pm} is a periodic orbit, or for any neighborhood U of γ_{\pm} in the case where γ_{\pm} is a point, there exists an integer $N \geq 1$ such that

$$v_n(s, t) \in U \quad \forall -\sigma_n \leq s \leq \sigma_n. \quad (3.16)$$

¹⁰Also note that $S(\gamma_-, \gamma_+, [F]) \geq 0$ if the homotopy class $[F]$ has a pseudoholomorphic representative.

Theorems 2.81 and 2.85 describe the behavior of long holomorphic cylinders and strips with small $d\lambda$ -energy. Then for every $\delta > 0$ there exist constants $n_0, h > 0$ such that for all $n \geq n_0$ one of the following two possibilities holds:

- (a) $\tilde{v}_n([-\sigma_n + h, \sigma_n - h] \times S^1) \subset B_\delta(\tilde{v}_n(0, 0))$
- (b) $v_n(s, *) \in U_\delta(x_T) \ \forall s \in [-\sigma_n + h, \sigma_n - h]$ for a suitable T -periodic orbit $x = x(\sigma)$ of the Reeb vector field, where $U_\delta(x_T)$ is the δ -ball around $x_T : t \mapsto x(Tt)$ in $C^\infty(S^1, M)$.

If γ_\pm are both periodic orbits then (b) above and (3.14) imply the assertion (3.16) and that $\gamma_-(t) = \gamma_+(t)$. Moreover, every subsequence of $(v_n)_{n \in \mathbb{N}}$ has a subsequence which converges in $C_{\text{loc}}^\infty(\mathbb{R} \times S^1)$ to $\gamma_- \equiv \gamma_+$, and therefore the whole sequence converges. The case of characteristic chords is treated similarly.

In the case where γ_\pm are points we proceed as follows: The sequence $\tilde{v}_n(0, 0) - b_n(0, 0)$ has a subsequence which converges to some point $(0, p) \in \{0\} \times M$. Then, by statement (a) above and by (3.14) the two points γ_\pm must be equal to p and the assertion follows. \square

We now consider the case where the actions of γ_- and γ_+ are different.

Proposition 3.14 *If $S(\gamma_-, \gamma_+, [F]) \geq \hbar$ then, after passing to a suitable subsequence, we can find finitely many points, characteristic chords or periodic orbits $\gamma_- = \gamma_0, \dots, \gamma_N = \gamma_+$, $N \geq 1$ with*

$$\int_{\gamma_{k-1}} \lambda \leq \int_{\gamma_k} \lambda, \quad k = 1, \dots, N$$

and sequences of real numbers $(s_{n,k})_{n \in \mathbb{N}}, k = 1, \dots, N$, with $\lim_{n \rightarrow \infty} s_{n,k} = +\infty$ such that the translated maps

$$(s, t) \mapsto (b_n(s + s_{n,k}, t) - b_n(s_{n,k}, 0), v_n(s + s_{n,k}, t))$$

converge in C_{loc}^∞ to pseudoholomorphic cylinders or strips $\tilde{w}_k = (c_k, w_k)$ with

$$\lim_{s \rightarrow -\infty} w_k(s, t) = \bar{\gamma}_{k-1}(t) = \gamma_{k-1}(T_{k-1}t)$$

and

$$\lim_{s \rightarrow +\infty} w_k(s, t) = \bar{\gamma}_k(t) = \gamma_k(T_k t).$$

Proof Because of $S(\gamma_-, \gamma_+, [F]) \geq \hbar$ we have

$$\tilde{v}_n = (b_n, v_n) : [-\sigma_n, \sigma_n] \times S^1 \longrightarrow \mathbb{R} \times M$$

$$\lim_{n \rightarrow \infty} v_n|_{\{\pm\sigma_n\} \times S^1} = \gamma_\pm, \quad \gamma_- \neq \gamma_+ \quad \text{and} \quad \int_{[-\sigma_n, \sigma_n] \times S^1} v_n^* d\lambda \geq \hbar$$

for sufficiently large n . Recall that the functions

$$\begin{aligned}\psi_n &: [-\sigma_n, \sigma_n] \times S^1 \rightarrow [0, \infty) \\ \psi_n(s) &:= \int_{\{s\} \times S^1} v_n^* \lambda\end{aligned}$$

are monotone increasing with

$$\lim_{n \rightarrow \infty} \psi_n(\pm \sigma_n) = \int_{\gamma_{\pm}} \lambda.$$

By non-degeneracy there are at most finitely many periodic orbits δ with

$$\int_{\gamma_-} \lambda \leq \int_{\delta} \lambda \leq \int_{\gamma_+} \lambda$$

which are the only periodic orbits at which the holomorphic cylinders may break up. We may arrange them by the magnitude of their actions $\gamma_- = \delta_0, \dots, \delta_N = \gamma_+$. We will conduct the proof by an iteration argument which will terminate after finitely many steps. We first pick a sequence $s_n \in [-\sigma_n, \sigma_n]$ so that

$$\int_{\{s_n\} \times S^1} v_n^* \lambda = \frac{1}{2} \left(\int_{\delta_{N'}} \lambda + \int_{\delta_N} \lambda \right) =: \frac{1}{2} (T_{N'} + T_N)$$

where N' is the largest integer $0 \leq N' < N$ for which $T_{N'} < T_N$. In view of (3.14) the sequences $|\sigma_n - s_n|$ and $|\sigma_n + s_n|$ are both unbounded. We then consider

$$\tilde{v}'_n(s, t) := (b'_n(s, t), v'_n(s, t)) = (b_n(s + s_n, t), v_n(s + s_n, t)).$$

As we pointed out before, because of the gradient bound (3.13), we may find a subsequence $(\tilde{v}'_{n_k})_{k \in \mathbb{N}}$ so that the sequence $\tilde{v}'_{n_k} - b_{n_k}(s_{n_k}, 0)$ converges in $C_{\text{loc}}^\infty(\mathbb{R} \times S^1)$ to a finite energy pseudoholomorphic cylinder $\tilde{w} = (c, w)$ with central action

$$\int_{\{0\} \times S^1} w^* \lambda = \frac{1}{2} (T_{N'} + T_N)$$

which has asymptotic limits

$$\lim_{s \rightarrow +\infty} w(s, t) = \delta(T_N t) = \gamma_+(T_+ t)$$

and

$$\lim_{s \rightarrow -\infty} w(s, t) = \delta_m(T_m t)$$

where m may be any integer between 0 and N' . If $m > 0$ we repeat the procedure by picking a sequence $s'_{n_k} \in [-\sigma_n, \sigma_n]$ with

$$\int_{\{s'_{n_k}\} \times S^1} v_{n_k}^* \lambda = \frac{1}{2} (T_{N''} + T_{N'})$$

where N'' is the largest integer $0 \leq N'' < N'$ for which $T_{N''} < T_{N'}$. The iteration procedure has to stop after finitely many steps, and we get the assertion of the proposition. \square

Adding the marked point $(\varphi_{n_k} \circ \phi_{n_k})^{-1}(s_n, 0)$ at each iteration step we create an additional spherical component for the limiting Riemann surface which removes a positive amount of energy no smaller than \hbar from $S(\gamma_-, \gamma_+, [F])$.

If the ε -thin part consists of a cusp near a puncture, the analysis is similar as before. We have to distinguish the cases where the quantity $S(\gamma_-, \gamma_+, [F])$ is zero or greater than \hbar .

Having analyzed the behavior of the sequence in the thin part we introduce an ordering of the components C_i of $S \setminus \bigcup \Gamma_j$ in the following way: We say $C_i \leq C_j$ if

$$\limsup_{n \rightarrow \infty} (a_n(\varphi_n(x_i)) - a_n(\varphi_n(x_j))) < +\infty.$$

This definition does not depend on the choice of the points x_i and x_j . In the case where the symplectic manifold under consideration is $\mathbb{R} \times M$ we proceed as follows: We define the components C_i which are minimal with respect to the above ordering as level-one components. After removing those, we declare the minimal remaining ones to be of level two and so on. If W is a manifold with cylindrical ends we declare the components whose remains in the compact part to be of level zero. We then use the above ordering to define levels $\pm 1, \pm 2, \pm 3, \dots$. If the levels of two components adjacent to a node differ by more than one then we introduce trivial cylinders to fill up the levels in between. The situation is the same for holomorphic buildings with boundary.

3.4 More General Holomorphic Buildings and Compactness Results

Let $\mathbf{S} = (S, j, M, D, M_\partial, D_\partial)$ be a noded Riemann surface with nonempty boundary (as in Fig. 1.51 on page 96). Here, $M_\partial \subset \partial S$ is a finite set (marked points on the boundary), and we assume that $D_\partial \subset \partial S$ is another set of nodal pairs. Given an ordering of the boundary components of ∂S the orientation of the boundary then induces a natural ordering of the set M_∂ once a ‘first’ point is chosen. Recall that we used to compactify a surface with interior marked points by adding circles. As for boundary marked points, an arc is added (see Fig. 1.47), and we denote the compactified surface by S^D .

3.4.1 Holomorphic Buildings of Height 1

Let

$$\mathbf{S} = (S, j, M \cup Z, D, M_\partial \cup Z_\partial, D_\partial)$$

be a noded Riemann surface with possibly nonempty boundary. The set of marked points consists of the sets Z, Z_∂ , called *interior punctures* and *boundary punctures*, and the sets M, M_∂ which we will call *marked points*. When we discussed convergence of Riemann surfaces with boundary we saw that a boundary component may degenerate by shrinking to a point. It is then indistinguishable from an interior puncture, and we may either remember the ‘past history’ of this puncture or not. In the latter case we would just include it in the set of interior punctures Z , otherwise we would have to distinguish between ‘degenerate’ boundary components Z' and interior punctures Z'' . One way of bookkeeping may be better than the other from the algebraic point of view if it comes to constructing Symplectic Field Theory for curves with boundary, but this is not the subject of these lectures. The sets

$$D = \{\{\bar{d}_1, \underline{d}_1\}, \dots, \{\bar{d}_k, \underline{d}_k\}\} \subset \overset{\circ}{S}$$

and

$$D_\partial = \{\{\bar{\delta}_1, \underline{\delta}_1\}, \dots, \{\bar{\delta}_l, \underline{\delta}_l\}\} \subset \partial S$$

are called the sets of interior and boundary *nodal pairs*. Each pair $\{\bar{d}_j, \underline{d}_j\}$ may belong to the same or to different components of S , and the same is true for pairs $\{\bar{\delta}_j, \underline{\delta}_j\}$. The set D is usually considered unordered, but the set D_∂ carries a natural ordering induced by the orientation of ∂S , once the components of ∂S are ordered. We consider the set of marked points and punctures to be ordered. We denote the components of $\partial S \setminus Z_\partial$ by $\partial_1 S, \dots, \partial_m S$, and we assume that $\mathcal{L}_1, \dots, \mathcal{L}_m \subset \mathbb{R} \times V$ are totally real submanifolds where $\mathbb{R} \times V$ is the symplectization of a closed manifold V endowed with some contact form λ , and where J is a $d\lambda$ compatible complex structure on $\xi = \ker \lambda$. We will cast the definition of a holomorphic building in a rather general framework, but we will only prove the compactness result in the case where the boundary condition is $\mathbb{R} \times \mathcal{L}$ where $\mathcal{L} \subset M$ is a Legendrian submanifold.

Definition 3.15 A holomorphic curve

$$\tilde{u} = (a, u) : S \setminus (Z \cup Z_\partial) \longrightarrow \mathbb{R} \times V$$

- is called a *trivial cylinder* if $S = S^2$, $M = D = \emptyset$, Z consists of exactly two points, and u maps S onto a periodic orbit of the Reeb vector field
- and it is called a *trivial strip* if S is the closed unit disk in \mathbb{C} , $M = D = M_\partial = D_\partial = Z = \emptyset$, Z_∂ consists of exactly two points, and u maps S onto a trajectory $\gamma(t)$, $0 \leq t \leq 1$ of the Reeb vector field with $\gamma(0) \in L_0$, $\gamma(1) \in L_1$ and $L_0, L_1 \in \{\mathcal{L}_1, \dots, \mathcal{L}_m\}$

Definition 3.16 (Holomorphic building of height 1) A holomorphic curve

$$\tilde{u} = (a, u) : S \setminus (Z \cup Z_\partial) \longrightarrow \mathbb{R} \times V$$

is called a *holomorphic building of height 1* if it has finite energy, if it satisfies the boundary conditions $\tilde{u}(\partial_j S) \subset \mathcal{L}_j$, and if for all pairs of special marked points

$$\tilde{u}(\bar{d}_j) = \tilde{u}(\underline{d}_j) \quad \text{and} \quad \tilde{u}(\bar{\delta}_j) = \tilde{u}(\underline{\delta}_j).$$

Such a curve is called *stable* if the following conditions are satisfied:

- At least one component of the curve is not a trivial cylinder or a trivial strip.
- If S' is a connected component of S with $\partial S' = \emptyset$ on which \tilde{u} is constant, then the Riemann surface S' with all its marked points, punctures and special marked points is stable in the sense of Definition 1.17. If S' is such a component with nonempty boundary then we demand that its double $S' \# S'$ is stable as above.

Assume that p is a puncture. If $p \in Z_\partial$ let $\partial_j S, \partial_{j+1} S$ be the components of $\partial S \setminus Z_\partial$ adjacent to p . We consider the asymptotic behavior of \tilde{u} near p in the following cases:

- (1) $p \in Z$ is an interior puncture: Under some nondegeneracy assumptions on the set of periodic orbits of the Reeb vector field, the asymptotic behavior of \tilde{u} is given by Proposition 2.47. We may then associate a particular periodic orbit of the Reeb vector field to each interior puncture.
- (2) If $p \in Z_\partial$ and if

$$L_0 = \mathbb{R} \times \mathcal{L}_0, \quad L_1 = \mathbb{R} \times \mathcal{L}_1,$$

where $\mathcal{L}_0, \mathcal{L}_1 \subset V$ are Legendrian submanifolds (i.e. $\lambda|_{T\mathcal{L}_i} \equiv 0$) then \tilde{u} approaches a trivial strip, see Theorem 2.57. Again, under suitable nondegeneracy assumptions, we may assume to each boundary puncture a characteristic chord, i.e. a trajectory connecting two given Legendrian submanifolds of V .

- (3) If p is a boundary puncture, V is three-dimensional, and if one boundary condition adjacent to p is $\mathbb{R} \times \mathcal{L}$ where $\mathcal{L} \subset V$ is a Legendrian link while the other is of the form $\{0\} \times F$ where F is a Seifert surface for \mathcal{L} then the asymptotics of \tilde{u} is described by Theorem 2.58. In this case we may associate to each such puncture a point on the knot \mathcal{L} .

As in [12] we will call two such curves $(\tilde{u} = (a, u), j, \mathbf{S})$ and $(\tilde{u}' = (a', u'), j', \mathbf{S}')$ *equivalent* if there is a diffeomorphism $\phi : S \rightarrow S'$ such that

- $(\phi_*)^{-1} \circ j' \circ \phi_* = j$
- ϕ sends the sets $M, Z, D, M_\partial, Z_\partial, D_\partial$ onto the sets $M', Z', D', M'_\partial, Z'_\partial, D'_\partial$ preserving the ordering (if applicable) and mapping nodal pairs to nodal pairs.
- $u' \circ \phi = u$
- $a' \circ \phi - a$ is locally constant

The last condition means that we identify curves which differ by translation in the \mathbb{R} -direction. Then the boundary condition of course has to be \mathbb{R} -invariant as well (otherwise we ask for $\tilde{u}' \circ \phi = \tilde{u}$). In cases (1) and (2) above the \mathbb{R} -component at each puncture either tends to $+\infty$ or to $-\infty$ (see Theorem 2.74) while in case (3)

the \mathbb{R} -component tends to zero at the puncture. In cases (1) and (2) we will use the notation Z^\pm and Z_∂^\pm to distinguish between positive and negative punctures.

3.4.2 Holomorphic Buildings of Height N

Assume, we have N stable possibly disconnected holomorphic buildings of height 1 $(\tilde{u}_k, j_k, \mathbf{S}_k)_{1 \leq k \leq N}$ where $\tilde{u}_k = (a_k, u_k) : S_k \setminus (Z_k \cup Z_{\partial,k}) \rightarrow \mathbb{R} \times V$ and $\mathbf{S}_k = (S_k, M_k \cup Z_k, D_k, M_{\partial,k} \cup Z_{\partial,k}, D_{\partial,k})$ are noded Riemann surfaces with possibly nonempty boundary. As in [12] we assume that the union of all the interior marked points $\bigcup_{k=1}^N M_k$ and the union of all the boundary marked points $\bigcup_{k=1}^N M_{\partial,k}$ are equipped with some ordering respecting the orderings of the individual sets M_k and $M_{\partial,k}$. We denote by Γ_k^\pm and $\Gamma_{\partial,k}^\pm$ the sets of boundary circles and boundary arcs added to $S_k \setminus (Z_k \cup Z_{\partial,k})$ by compactifying the punctures in Z_k^\pm and in $Z_{\partial,k}^\pm$. We assume that

$$\#Z_{k+1}^- = \#Z_k^+ \quad \text{and} \quad \#Z_{\partial,k+1}^- = \#Z_{\partial,k}^+,$$

i.e. the building on level $k+1$ has as many interior or boundary negative punctures as the building on level k positive punctures. We assume furthermore that there are orientation reversing diffeomorphisms

$$\Phi_k : \Gamma_k^+ \rightarrow \Gamma_{k+1}^- \quad \text{and} \quad \Phi_{\partial,k} : \Gamma_{\partial,k}^+ \rightarrow \Gamma_{\partial,k+1}^-.$$

Using these we may construct a piecewise smooth surface with boundary

$$S^{Z,\Phi} = S_1^{Z_1, Z_{\partial,1}} \cup_{\Phi_1, \Phi_{\partial,1}} \cdots \cup_{\Phi_{k-1}, \Phi_{\partial,k-1}} S_k^{Z_k, Z_{\partial,k}}.$$

Definition 3.17 (Holomorphic building of height N) The sequence $(\tilde{u}_k, j_k, \mathbf{S}_k)_{1 \leq k \leq N}$ of holomorphic buildings of height 1 together with the decoration maps $\{\Phi_1, \Phi_{\partial,1}, \dots, \Phi_{N-1}, \Phi_{\partial,N-1}\}$ and the orderings of the sets $\bigcup_{k=1}^N M_k$ and $\bigcup_{k=1}^N M_{\partial,k}$ is called a *holomorphic building of height N (or with N levels)* if the following condition is satisfied: The maps $u_k : S_k^{Z_k, Z_{\partial,k}} \rightarrow V$, $1 \leq k \leq N$, fit together to a continuous map $u : S^{Z,\Phi} \rightarrow V$. This means that the curve u_k is asymptotic at its negative punctures to the same periodic orbits and characteristic chords as u_{k-1} at its positive punctures (see Fig. 3.11).

Two holomorphic buildings of height N $(\tilde{u}_k, j_k, \mathbf{S}_k, \Phi_k, \Phi_{\partial,k})_{1 \leq k \leq N}$ and $(\tilde{u}'_k, j'_k, \mathbf{S}'_k, \Phi'_k, \Phi'_{\partial,k})_{1 \leq k \leq N}$ are called equivalent if there are diffeomorphisms $\phi = \{\phi_1, \dots, \phi_N\}$, $\phi_k : S_k \rightarrow S'_k$ which renders the corresponding buildings of height 1 equivalent. Moreover, the diffeomorphisms are compatible with the attaching maps, i.e.

$$\Phi'_k \circ \phi_{k-1} = \phi_k \circ \Phi_{k-1} \quad \text{and} \quad \Phi'_{\partial,k} \circ \phi_{k-1} = \phi_k \circ \Phi_{\partial,k-1}$$

and they respect the orderings of the sets of marked points. We also identify holomorphic buildings which differ by a synchronous re-ordering of the punctures $Z_{\partial,k}^+$

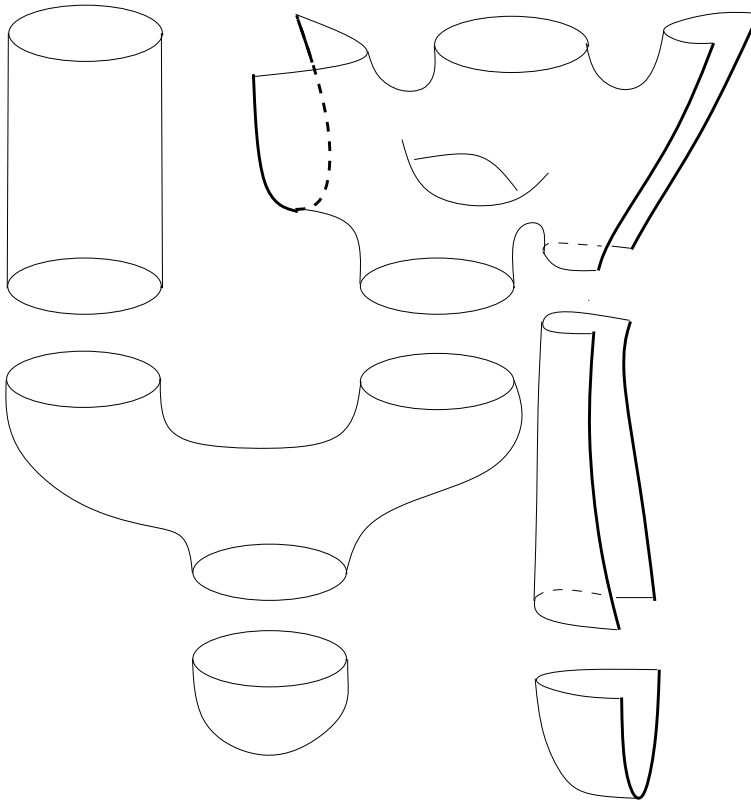


Fig. 3.11 A holomorphic building of height three

and $Z_{\partial, k+1}^-$ (and Z_k^+ and Z_{k+1}^-), i.e. the ordering of the punctures does not matter except for the positive punctures on the highest level and the negative punctures on the lowest level.

Definition 3.18 The signature of a level N holomorphic building is the seven-tuplet $\alpha = (g, \mu, \mu_{\partial}, p^+, p_{\partial}^+, p^-, p_{\partial}^-)$ where g is the arithmetic genus of $S^{\phi, Z}$, $\mu = \#M$, $\mu_{\partial} = \#M_{\partial}$, $p^+ = \#Z_N^+$, $p_{\partial}^+ = \#Z_{\partial, N}^+$, $p^- = \#Z_1^-$ and $p_{\partial}^- = \#Z_{\partial, 1}^-$. We denote the set of all holomorphic buildings of height N and signature α by $\mathcal{M}^{\alpha, N}(V)$. We set

$$\overline{\mathcal{M}}^{\alpha}(V) := \bigcup_{N=1}^{\infty} \mathcal{M}^{\alpha, N}(V).$$

Definition 3.19 (Convergence of holomorphic buildings) Assume that

$$(\tilde{u}_m, j_m, \mathbf{S}_m)_{m \in \mathbb{N}} \subset \overline{\mathcal{M}}^{\alpha}(V)$$

is a sequence of holomorphic buildings of height less or equal to $N \in \mathbb{N}$.¹¹ We say that this sequence converges to a building $(\tilde{u}, j, \mathbf{S}) \in \mathcal{M}^{N, \alpha}(V)$ of height N if the following conditions are satisfied:

- There are sequences of extra marked points for the surfaces \mathbf{S}_m and \mathbf{S} such that all the surfaces have the same number of boundary and interior marked points and the underlying Riemann surfaces become stable.
- The surfaces \mathbf{S}_m converge to the surface \mathbf{S} in the Deligne–Mumford sense (after having introduced the extra marked points) as described in Sect. 1.3 of the chapter on Riemann surfaces. Denote the diffeomorphisms between the underlying surfaces $S^{Z, \Phi} \rightarrow S^{Z_m, \Phi_m}$ by φ_m .
- The sequence of the projections into V , $u_m \circ \varphi_m : S^{Z, \Phi} \rightarrow V$, converge to $u : S^{Z, \Phi} \rightarrow V$ uniformly (up to the ends).
- Denote by C_l the union of all the components of $S^{Z, \Phi} \setminus \bigcup_k \Gamma_k$ ¹² which correspond to a particular level l of the building $(\tilde{u} = (a, u), j, \mathbf{S})$. Then there is a sequence of real numbers $(c_m^l)_{m \in \mathbb{N}}$ for each l such that

$$(a_m \circ \varphi_m - a - c_m^l)|_{C_l} \longrightarrow 0 \quad \text{in } C_{\text{loc}}^0.$$

3.4.3 Holomorphic Buildings in Manifolds with Cylindrical Ends

In this section we will introduce the notion of holomorphic buildings in manifolds with cylindrical ends. More precisely, we will consider symplectic manifolds of the following types:

- (A) W is a symplectic manifold with contact type boundary $V = V^+ \cup V^-$, and cylindrical ends $E_{\pm} = \mathbb{R}^{\pm} \times V^{\pm}$ are attached to the contact type boundary as explained on page 111. We may consider a relative version of this as follows: Let $\mathcal{L}^{\pm} \subset V^{\pm}$ be Legendrian submanifolds, and let $L \subset W$ be a Lagrangian submanifold such that

$$L \cap V = (\mathbb{R} \times \mathcal{L}^+) \cup (\mathbb{R} \times \mathcal{L}^-).$$

- (B) W is obtained from another symplectic manifold W' by cutting W' open along a contact type hypersurface V and then attaching a positive and a negative end to the newly created boundary ('splitting construction', see page 111). As for the relative version we consider a Lagrangian submanifold $L \subset W$ such that $V \cap L = \mathcal{L}$ is a Legendrian submanifold of V .

In the first case we define a *holomorphic building of height $k_-|1|k_+$* where k_{\pm} are non-negative integers. It consists of the following ingredients:

¹¹Recall that by definition each $(\tilde{u}_m, j_m, \mathbf{S}_m)$ is itself a sequence $(\tilde{u}_m^n, j_m^n, \mathbf{S}_m^n)_{1 \leq n \leq N_m}$, $N_m \leq N$ of holomorphic buildings of height 1.

¹²The circles and arcs used to compactify the surfaces are removed here.

- (1) A *holomorphic building of height 1* which is a proper noded holomorphic curve

$$v : S \setminus (Z_0 \cup Z_{\partial,0}) \rightarrow W$$

of finite energy satisfying the boundary condition $v(\partial S \setminus Z_{\partial,0}) \subset L$ and also

$$v(\bar{d}_j) = v(\underline{d}_j) \quad \text{and} \quad v(\bar{\delta}_j) = v(\underline{\delta}_j).$$

We denote the ‘data’ of this holomorphic curve by $(S_0, M_0 \cup Z_0, D_0, M_{\partial,0} \cup Z_{\partial,0}, D_{\partial,0})$.

- (2) A holomorphic building of height k_+ in the cylindrical manifold $\mathbb{R} \times V_+$ and another of height k_- in the cylindrical manifold $\mathbb{R} \times V_-$ which we denote by $(\tilde{u}_k, j_k, \mathbf{S}_k)$ with $1 \leq k \leq k_+$ or $-k_- \leq k \leq -1$ and with decoration maps $\{\Phi_k, \Phi_{\partial,k}\}$, $1 \leq k \leq k_+ - 1$ or $-k_- \leq k \leq -2$ gluing the positive punctures in $Z_k \cup Z_{\partial,k}$ to the negative punctures in $Z_{k+1} \cup Z_{\partial,k+1}$, and where $\mathbf{S}_k = (S_k, M_k \cup Z_k, D_k, M_{\partial,k} \cup Z_{\partial,k}, D_{\partial,k})$ keeps track of all the marked points, punctures and nodes in the interior and on the boundary.
- (3) An ordering of the total set of marked points $\bigcup_{-k_- \leq j \leq k_+} M_j$ compatible with the ordering on each M_j (the set of marked points on each individual level 1 curve), but not necessarily preserving the numbering of the sets M_{-k_-}, \dots, M_{k_+} . The same for the set of marked points $\bigcup_{-k_- \leq j \leq k_+} M_{\partial,j}$ on the boundary.

We also need to assume the following:

- The numbers $p_0^+, p_{\partial,0}^+$ of positive punctures of v equals the numbers of negative punctures $p_1^-, p_{\partial,1}^-$ of \tilde{u}_1 .
- The numbers $p_0^-, p_{\partial,0}^-$ of negative punctures of v equals the numbers of positive punctures $p_{-1}^+, p_{\partial,-1}^+$ of \tilde{u}_{-1} .
- For $k = -1, 0$ there are also (orientation reversing) diffeomorphisms $\Phi_k, \Phi_{\partial,k} : \Gamma_k^+ \rightarrow \Gamma_{k+1}^-$, where Γ_k^\pm are the circles (or arcs, respectively) added for each interior puncture (or boundary puncture, respectively).

Denote by S^0 the oriented blow-up of S_0 at the punctures $Z_0 \cup Z_{\partial,0}$, and let

$$S^+ = S_1 \cup_{\Phi_1, \Phi_{\partial,1}} \cdots S_{k_+-1} \cup_{\Phi_{k_+-1}, \Phi_{\partial,k_+-1}} S_{k_+}$$

$$S^- = S_{k_-} \cup_{\Phi_{k_-}, \Phi_{\partial,k_-}} \cdots S_{-2} \cup_{\Phi_{-2}, \Phi_{\partial,-2}} S_{-1}$$

be the surfaces obtained from gluing together the various levels of the holomorphic buildings at their punctures. Gluing S^- with S^0 using $\Phi_{-1}, \Phi_{\partial,-1}$, and gluing S^+ with S^0 using $\Phi_0, \Phi_{\partial,0}$ we obtain a piecewise smooth surface $\bar{S} = S^- \cup S^0 \cup S^+$. We finally demand that for sufficiently small $\varepsilon > 0$ the maps

$$u^- : S^- \rightarrow V^-, \quad u^+ : S^+ \rightarrow V^+$$

induced by u_1, \dots, u_{k_+} and u_{k_-}, \dots, u_{-1} , respectively and the map $G^\varepsilon \circ v : S^0 \rightarrow \overset{\circ}{W}$ (see (2.3) on page 111 for the definition of G^ε) fit together to a continuous map $\bar{S} \rightarrow \bar{W}$. Figure 3.12 illustrates a holomorphic building in a mani-

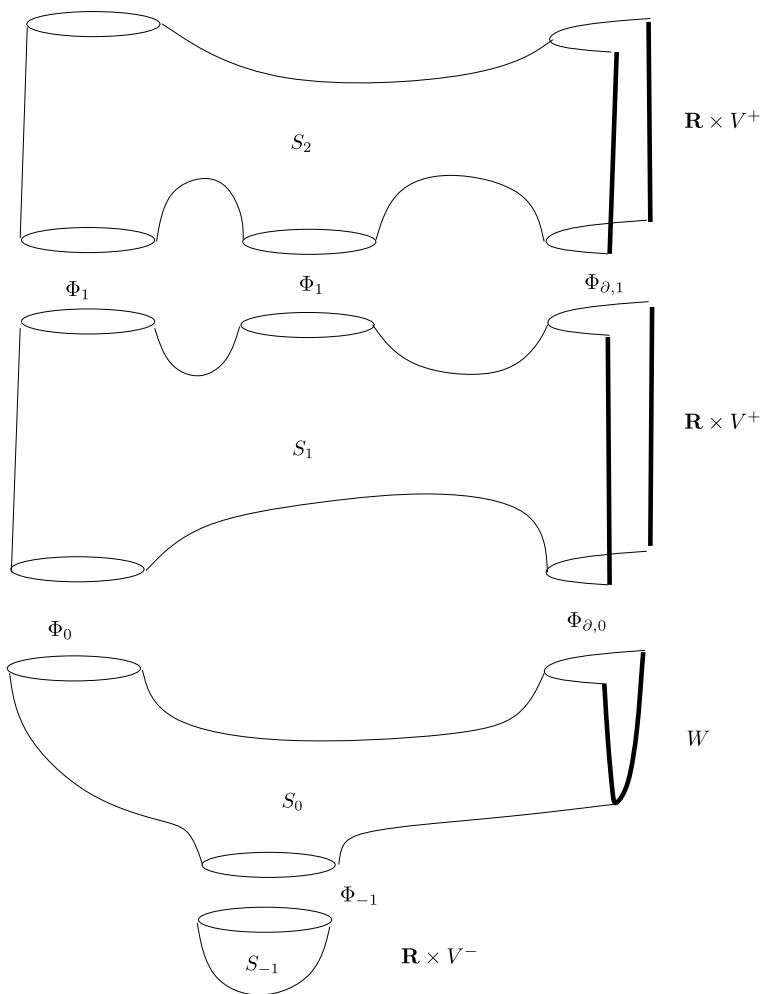


Fig. 3.12 A holomorphic building of height $2|1|1$

fold with cylindrical ends. The signature of such a building consists of the data $(g, \mu, \mu_\partial, p^-, p_\partial^-, p^+, p_\partial^+)$ where $\mu = \#M$, $\mu_\partial = \#M_\partial$ are the numbers of all marked points, $p_\partial^+ = p_{\partial, k_+}^+$, $p^+ = p_{k_+}^+$ and $p_\partial^- = p_{\partial, k_-}^-$, $p^- = p_{k_-}^-$ are the total number of punctures at the highest and the lowest level (interior and boundary punctures recorded separately).

Given a holomorphic building $(\tilde{u}^-, v, \tilde{u}^+)$ of height $k_-|1|k_+$, we will sometimes say that it consists of three layers with the one in the middle being the *main layer*. The equivalence of two such buildings is defined similarly to the equivalence of buildings of height N except that there is no \mathbb{R} -translation in the main layer to factor out. The definition of convergence is also evident. We say a holomorphic building of height $k_-|1|k_+$ is *stable* if all three layers are. In the case of the main layer

this means the following: For every component C of the Riemann surface we want $F_0|_C$ either to be not constant or otherwise we want C with all its punctures and marked points $D_0, D_{\partial,0}, M_0, M_{\partial,0}, Z_0, Z_{\partial,0}$ to be stable, i.e. $\#\text{Aut}(C) < \infty$. We denote the moduli space of holomorphic buildings of height $k_-|k_+$ and signature $(g, \mu, \mu_{\partial}, p^-, p_{\partial}^-, p^+, p_{\partial}^+)$ by

$${}^{k_-,k_+}\mathcal{M}_{(g,\mu,\mu_{\partial},p^-,p_{\partial}^-,p^+,p_{\partial}^+)}(W, L, J).$$

We use the notation ${}^{k_-,k_+}\mathcal{M}_{(g,\mu,\mu_{\partial})}(W, L, J)$ for the union of the above spaces over all $p^-, p_{\partial}^-, p^+, p_{\partial}^+ \geq 0$. We write $\overline{\mathcal{M}}_{(g,\mu,\mu_{\partial})}(W, L, J)$ for the union over all $k_{\pm} \geq 0$.

The case of the splitting construction is not much different from the case of a manifold with cylindrical ends. The appropriate notion is the one of a holomorphic building of height $\bigwedge_1^{k_0}$ which consists of a holomorphic building of height 1 in the main layer and a holomorphic building of height k_0 in the cut-open region. We refrain from bothering the reader with another heap of notation, and refer to Fig. 3.13 instead. The relevant notation is contained in the paper [12] for the case of surfaces without boundary.

3.4.4 A More General Compactness Result

We will now consider a sequence of pseudoholomorphic curves in a symplectic manifold (W, ω) with almost complex structure J , so that W is one of the following:

- W is cylindrical, i.e. $W = \mathbb{R} \times M$ where M is a closed contact manifold and $J = \tilde{J}$ is the usual \mathbb{R} -invariant almost complex structure.
- $W = E_- \cup \overline{W} \cup E_+$ is a symplectic manifold with cylindrical ends as explained in Sect. 2.1 of Chap. 2.¹³
- W is a symplectic manifold created by splitting along a contact type hypersurface.

We always assume that the almost complex structure on W is compatible with the symplectic structure, and that it equals the \mathbb{R} -invariant almost complex structure \tilde{J} (with $d\lambda$ -compatible $J : \ker \lambda \rightarrow \ker \lambda$) in the cylindrical ends. We also assume throughout that the contact forms in the ends are non-degenerate. In the paper [12] the authors introduce the notion of an ‘adjusted almost complex structure’. Our requirements are a special case of this. The boundary condition considered here is a Lagrangian submanifold $L \subset W$ with $L = \mathbb{R} \times \mathcal{L}$ in the cylindrical ends where $\mathcal{L} \subset V$ is a Legendrian submanifold.

The more general compactness result below does not need a new proof. We inserted remarks into the proofs of the previous results in order to indicate the necessary additions and modifications.

¹³Note that the ends may be disconnected.

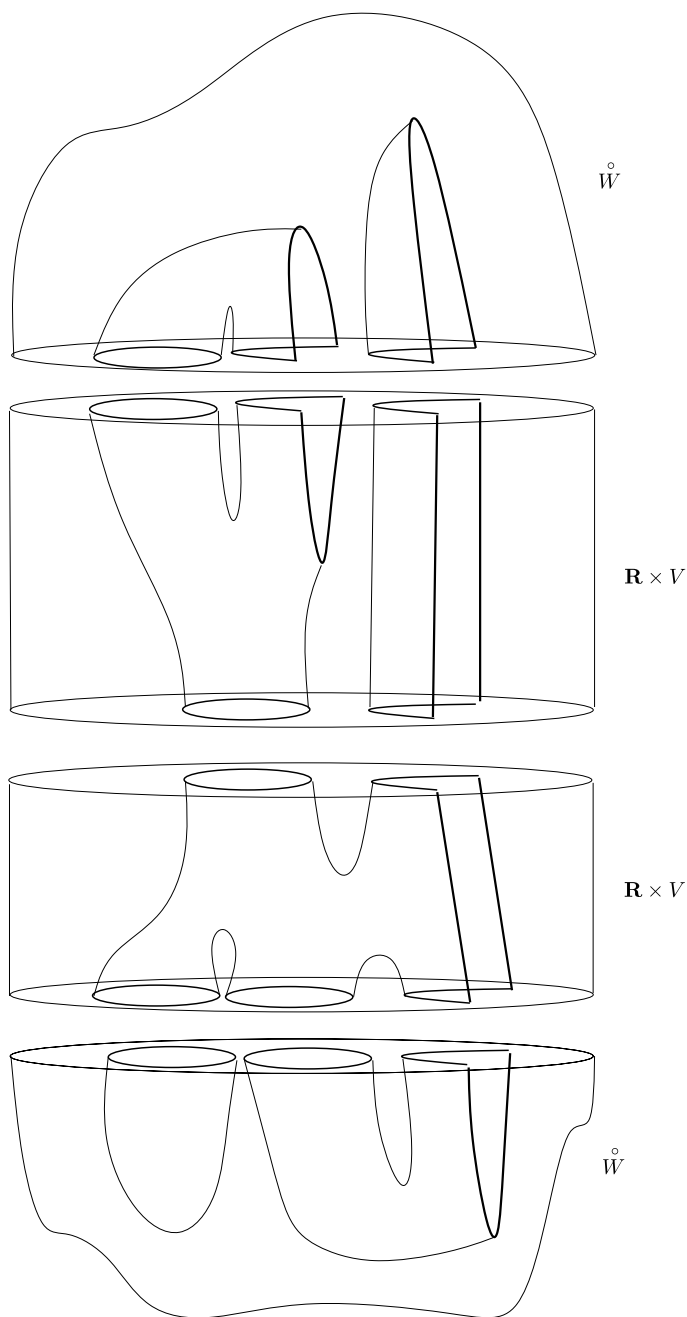


Fig. 3.13 A holomorphic building of height \bigwedge_1^2

Theorem 3.20 (SFT compactness theorem) *Consider a sequence of pseudoholomorphic curves*

$$\mathcal{C}_n = (\tilde{u}_n, S_n, j_n, M_n, \underline{Z}_n \cup \overline{Z}_n),$$

where (S_n, j_n) is a compact stable Riemann surface, possibly with boundary, with marked points $M_n = M_n^{\text{int}} \cup M_n^{\partial}$ in the interior and on the boundary, punctures $\underline{Z}_n \cup \overline{Z}_n = \underline{Z}_n^{\text{int}} \cup \underline{Z}_n^{\partial} \cup \overline{Z}_n^{\text{int}} \cup \overline{Z}_n^{\partial}$ (note that j_n extends over all of them) and all surfaces have the same signature. The maps

$$\tilde{u}_n : S_n \setminus (\underline{Z}_n \cup \overline{Z}_n) \longrightarrow W$$

are pseudoholomorphic curves with energy bounded by some constant $E_0 > 0$ and boundary condition in $L \subset W$ as described above, and the punctures

$$\underline{Z}_n^{\text{int}} = \{\underline{z}_1^{(n)}, \dots, \underline{z}_{p_-}^{(n)}\}, \quad \underline{Z}_n^{\partial} = \{\underline{\zeta}_1^{(n)}, \dots, \underline{\zeta}_{q_-}^{(n)}\}$$

are all negative while the punctures

$$\overline{Z}_n^{\text{int}} = \{\overline{z}_1^{(n)}, \dots, \overline{z}_{p_+}^{(n)}\}, \quad \overline{Z}_n^{\partial} = \{\overline{\zeta}_1^{(n)}, \dots, \overline{\zeta}_{q_+}^{(n)}\}$$

are all positive. We also assume that all the curves \tilde{u}_n are asymptotic at the corresponding punctures to the same periodic orbits/characteristic chords (i.e. to each interior puncture we associate a certain periodic orbit and to each boundary puncture a certain characteristic chord).¹⁴ Then there exists a subsequence of $\{\mathcal{C}_n\}$ which converges to

- a stable holomorphic building of height $k \geq 1$ in the cylindrical case,
- a stable holomorphic building of height $k_-|1|k_+$ in the case with cylindrical ends,
- a stable holomorphic building of height $\bigwedge_1^{k_0}$ in the splitting case.

¹⁴This is not a real assumption since it can be achieved by merely passing to a suitable subsequence due to the uniform bound on the energy and the nondegeneracy of the contact form. In the paper [12] the authors also consider the Morse–Bott case for curves without boundary. Then one has to assume that all the curves \tilde{u}_n are asymptotic at the corresponding punctures to periodic orbits lying in the same connected component in the space of periodic orbits.

An Introduction to Compactness Results in Symplectic
Field Theory

Abbas, C.

2014, VIII, 252 p. 73 illus., Hardcover

ISBN: 978-3-642-31542-8