Chapter 2
Conventional Integral Solutions of the GVF Equation

2.1 Introduction

The gradually-varied flow (GVF) equation for flow in open channels is normalized using the normal depth, h_n, before it can be analytically solved by the direct integration method. The right-hand side of the dimensionless GVF equation is a rational function expressed in terms of the dimensionless flow depth, h/h_n, to the two powers, M and N (called the hydraulic exponents), with the ratio of the critical depth, h_c, to h_n (or h_c/h_n) as a parameter. For simplicity, M and N are usually treated as constant, thus averting the impasse in integrating directly the GVF equation with the h/h_n-varying M and N. The direct-integration method is a conventional method used to analytically solve the GVF equation. The varied-flow function (VFF) needed in the direct-integration method has a drawback caused by the imprecise interpolation of the VFF-values. To overcome the drawback, we use the Gaussian hypergeometric functions (GHF) to analytically solve the GVF equation without recourse to the VFF. This chapter presents a review of conventional integral solutions of the GVF profiles by using the varied-flow function (VFF) and/or the elementary transcendental functions (ETF).

2.2 GVF Solution in Terms of Varied-Flow Function

The gradually varied flow (GVF) is a steady non-uniform flow in a prismatic channel with gradual changes in its water surface elevation. The computation of GVF profiles involves basically the solution of the dynamic equation of GVF. The main objective of the computation is to determine the shape of the flow profile. As shown by (1.41) in Chap. 1, the one-dimensional GVF dynamic equation for flow in open channels with arbitrary cross-sectional shapes can be expressed in terms only of the flow depth, h, if the section factor, Z, and the conveyance, K, of the channel section can be expressed in terms of the power of h with respective hydraulic exponents, M and N,

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as shown in (1.17) to (1.20). For the convenience of discussion, we re-write (1.41) for small slope angle in the form as

$$\frac{dh}{dx} = S_0 \frac{1 - (h_n/h)^N}{1 - (h_c/h)^M}. \tag{2.1}$$

Normalizing the flow depth h by the normal depth h_n, namely $u = h/h_n$, and substituting the dimensionless depth u into (2.1) yields

$$\frac{du}{dx} = \frac{S_0}{h_n} \frac{1 - u^{-N}}{1 - (h_c/h_n)^M u^{-M}}. \tag{2.2}$$

Even though it is commonly used to represent the velocity, the symbol u has been conventionally used to represent the dimensionless flow depth in the analysis of GVF profiles, as shown in some text books, such as Chow (1959), Subramanya (2009), among others. Hence, we will use the symbol u to represent the dimensionless flow depth in this book.Rearranging the equation into its reciprocal form, we can get an expression for dx in the form as

$$dx = \frac{h_n}{S_0}\left[1 - \frac{1}{1-u^N} + \left(\frac{h_c}{h_n}\right)^M \frac{u^{N-M}}{1-u^N}\right] du. \tag{2.3}$$

Given a cross-sectional shape other than wide rectangle, the M- and N-values generally vary with h/h_n; therefore, we cannot integrate the h_n-based dimensionless GVF equation over h/h_n. As long as the change in depth of a gradually varied flow is generally small, for simplicity, the hydraulic exponents are traditional assumed constant within the range of the limits of integration as was done by Chow (1959) and others. Integrating (2.3) yields

$$x = \frac{h_n}{S_0}\left[u - \int_0^u \frac{1}{1-u^N} du + \left(\frac{h_c}{h_n}\right)^M \int_0^u \frac{u^{N-M}}{1-u^N} du\right] + \text{Const.}. \tag{2.4}$$

The GVF equation so formulated as shown in (2.1) or (2.2) is a nonlinear differential equation of first order in h with the two exponents, M and N, and the ratio of the critical depth, h_c, to the normal depth, h_n, as three parameters. Even though Z and K can be expressed in terms of the power of h with the respective exponents, M and N, as shown in Chap. 1, one has not yet successfully integrated directly such a nonlinear differential equation with arbitrarily assumed real numbers of M and N due to the difficulty in integrating the two integrals of a proper fraction, which are expressed as reciprocals of the rational function representing the slope of the GVF profile, as shown in (2.4).

As shown in the text book of Chow (1959), the first integral on the right hand side of (2.4) is designated by $F(u, N)$ that is known as the varied-flow function (VFF), namely

$$F(u,N) = \int_0^u \frac{1}{1-u^N} du. \tag{2.5}$$

The second integral of the right hand side of (2.4) may also be expressed in the form of VFF after a variable transformation of u. Letting $v = u^{N/J}$ and $J = N/(N-M+1)$, we can express the second integral in the form as (Chow 1959)

$$\int_0^u \frac{u^{N-M}}{1-u^N} du = \frac{J}{N} \int_0^v \frac{1}{1-v^J} dv = \frac{J}{N} F(v,J), \tag{2.6}$$

where $F(v, J)$ is a varied flow function like $F(u, N)$, except the variable u and the exponent N are replaced by v and J, respectively. After (2.5) and (2.6) are substituted in (2.4), the solution of length along the channel for GVF profile can be expressed in terms of these two varied-flow functions, as shown below.

$$x = \frac{h_n}{S_0}\left[u - F(u,N) + \left(\frac{h_c}{h_n}\right)^M \frac{J}{N} F(v,J)\right] + \text{Const.}. \tag{2.7}$$

Equation (2.7) contains two VFF, and its solution can be simplified by the use of the VFF tables as given in the book of Chow (1959). It was said that the preparation of the VFF table was undertaken and performed for the first time during 1914–1915 by the Research Board of the then Russian Reclamation Service under the direction of Boris A. Bakhmeteff. The VFF table was published in 1932 when Bakhmeteff became Professor of Civil Engineering at Columbia University. More than 20 years later, the VFF table was extended and improved by Ven Te Chow during 1952–1954 and published in 1955. Chow's VFF table for positive and negative slopes is a big table having 15 pages as shown in the Appendix D in the book of Chow (1959). The literature review about the use of VFF in the solution of GVF profile will be shown in the following sections.

2.3 GVF Solution by the Bresse Method

For a wide rectangular channel, if the Chézy formula is used, the hydraulic exponents $M = N = 3$. Thus $J = N = 3$, $v = u$, and (2.7) becomes a simpler form.

$$x = \frac{h_n}{S_0}\left[u - \left(1 - \left(\frac{h_c}{h_n}\right)^3\right) F(u,3)\right] + \text{Const.}. \tag{2.8}$$

The function $F(u, 3)$ was first evaluated by Bresse (1860) in a closed form as

Table 2.1 The derivation of the Bresse solution of the varied-flow function $F(u, 3)$ as shown in Eq. (2.9)

Integral function: $F(u,3) := \int_0^u \frac{1}{1-u^3} du$
solution: $$\begin{aligned} \int_0^u \frac{1}{1-u^3} du & \\ &= \int_0^u \frac{1}{(1-u)\left(u^2+u+1\right)} du \\ &= \frac{1}{3}\int_0^u \frac{1}{1-u} du + \frac{1}{3}\int_0^u \frac{u+2}{\left(u^2+u+1\right)} du \\ &= \frac{1}{3}\int_0^u \frac{1}{1-u} du + \frac{1}{6}\int_0^u \frac{2u+1}{\left(u^2+u\right)+1} du + \frac{1}{2}\int_0^u \frac{1}{\left(u^2+u+1\right)} du \\ &= -\frac{1}{3}\ln(1-u) + \frac{1}{6}\ln\left(u^2+u+1\right) \\ &\quad + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{6}\ln\left(\frac{u^2+u+1}{(1-u)^2}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right) \end{aligned}$$

$$\begin{aligned} F(u,3) &= \int_0^u \frac{1}{1-u^3} du \\ &= \frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right), \end{aligned} \tag{2.9}$$

or in another form as

$$\begin{aligned} F(u,3) &= \int_0^u \frac{1}{1-u^3} du \\ &= \frac{1}{6}\ln\left(\frac{u^2+u+1}{(u-1)^2}\right) - \frac{1}{\sqrt{3}}\operatorname{arccot}\left(\frac{2u+1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\operatorname{arccot}\left(\frac{1}{\sqrt{3}}\right), \end{aligned} \tag{2.10}$$

Since there is a relation between two inverse trigonometric functions, i.e.,

$$\arctan\left(\frac{2u+1}{\sqrt{3}}\right) + \operatorname{arccot}\left(\frac{2u+1}{\sqrt{3}}\right) = \frac{\pi}{2}. \tag{2.11}$$

The details of the derivation of (2.9) are shown in Table 2.1. A determination of GVF profile by using (2.8) and (2.9) [or (2.10)] is widely known as the Bresse method (Chow 1959). Obviously, the GVF solution by the Bresse method is only limited for flow in a wide rectangular channel and its flow resistance can be described by the Chézy formula. It will be discussed later that the Bresse solution is one of the solution using elementary transcendental functions.

2.4 GVF Solution by the Bakhmeteff-Chow Procedure

In general cases, the hydraulic exponents are some positive real numbers, depending channel geometry and flow depth, and this causes the difficuly in the integration of two integrals in (2.4). To overcome such difficulty in the integration, Bakhmeteff (1932) prepared an integration table of the varied-flow function (VFF), by which such two integrals for the fixed M- and N-values could be approximately evaluated. Decades later, Chow (1955) refined and extended the VFF table for computing GVF profiles numerically in both sustaining and adverse channels with all kinds of regular cross-sectional shapes. The solution of the length x of the flow profile can be simplified by the use of the table of VFF (Chow 1959).

Hydraulic engineers resorted to a procedure proposed by Bakhmeteff (1932) and Chow (1959), who evaluated the two integrals using the VFF table, for more than a half century for there has been no method better than the Bakhmeteff-Chow procedure based on the VFF table to evaluate such two integrals, from which GVF profiles are computed numerically for channels with h-dependent M- and N-values. Thus, if we want to evaluate these two integrals more precisely by some means other than the VFF table, we should first examine how Bakhmeteff and Chow computed the VFF-values, which enabled them to construct the VFF table. We undertake this scrutiny next.

In the approach taken by Bakhmeteff (1932), because his GVF equation is equivalent to the one for GVF in wide channels with the flow resistance expressed by the Chézy formula, namely $M = N = 3$, the second integral can merge into the first integral (similiar to 2.8) to form a sole integral, which he called the VFF. Four methods were proposed by him in the computation of the VFF-values, as explained in Appendix II of his book. In all the four methods except for the third one in which he used the approximate integration formula, he expanded the integrand of the VFF into an infinite series, thereby numerically integrating and calculating a sum of the first few terms of the expanded infinite series for each fixed value of N. The variable and the parameter of the VFF are the h_n-based dimensionless depth, h/h_n, and N, respectively. He determined a number of terms needed in the computation of the infinite series for each method by a specific convergence criterion established for each infinite series. His fourth method was particularly designed to safeguard the convergence of the computed VFF-value for h/h_n near unity in case that his third method failed to yield the satisfactory result. As claimed by Bakhmeteff, the precision of the computed VFF-values in his table was to keep their numerical errors less than 0.0005. It was also reported that the VFF-values so computed for the specific N-values in the range between 2.8 and 5.4 were tabulated by a team of Russian scientists led by Bakhmeteff before the Russian revolution in 1917, while the range of the N-values in the VFF table, especially its lower end, was slightly extended after the Russian revolution.

For integrating the GVF equation, Chow (1955) essentially used Bakhmeteff (1932) four methods in the computation of the VFF-values for the first integral. The integrand in the second integral was first transposed to a form of the VFF of a composite variable with a modified parameter expressed in terms of M and N before it was evaluated using the same four methods to compute the VFF-values in the evaluation of the first integral, as shown in (2.6). After recalculating the VFF-values for the two integrals, Chow (1955) corrected several errors found in Bakhmeteff's (1932) VFF table for the N-values ranging from 2.8 to 5.4 and then extended them to cover the N-values ranging from 2.2 to 9.8 for sustaining slopes. Besides, Chow (1957) prepared the VFF table for adverse slopes for the N-values in the range between 2.0 and 5.5. For dissemination of the Bakhmeteff-Chow procedure based on such two VFF tables, one for sustaining slopes and the other for adverse slopes, Chow (1959) further published both refined VFF tables in his widely-circulated book on open-channel hydraulics.

2.5 Drawbacks on the VFF Table for GVF Solution

Unfortunately, the Bakhmeteff-Chow procedure based on the VFF table suffers from two major drawbacks, which have impeded the progress of the 1-D approach in the GVF profile computation. The first drawback is caused by the imprecise interpolation of the VFF-values for a range of the dimensionless flow depth near unity or between the contiguous N-values as the VFF parameter. In other words, the VFF-values provided in the table are not sufficiently precise to evaluate such two integrals, especially for the dimensionless flow depth near unity, where the VFF-value changes rapidly with a small change in the dimensionless flow depth or between the contiguous N-values as the VFF parameter. On the other hand, the second drawback has resulted from the incompleteness of a method proposed by Bakhmeteff (1932); Chow (1955, 1959) to render the exact M- and N-values. Rigorously speaking, to use their method, the computed M- and N-values, which are supposed to vary considerably with the depth of flow in a channel with cross-sectional shape other than wide rectangle, are inaccurate. In fact, to overcome the first drawback, we have undertaken a novel approach to integrate the two integrals using the Gaussian hypergeometric function (GHF) without recourse to the VFF table as shown in Jan and Chen (2012) as well as in Jan and Chen (2013). The elaboration of this technique using the h_n-based GHF is reported in Chap. 3 and that using the h_c-based GHF in Chap. 4. However, before presenting such a technique, we review briefly all worthwhile efforts made by previous investigators to overcome both drawbacks in the following.

2.6 Attempts Made on M and N by Previous Investigators

In the past, attention has been focused on how to overcome the second drawback, but all attempts made by previous researchers to cope with it have not produced very fruitful results. Though overcoming the second drawback is not the main objective of the study of this book, it deserves a cursory review because knowledge of our failure to fix the second drawback may help counter the first drawback. Mononobe (1938) introduced two assumptions for the M- and N-values, thereby taking into account the possible effects of changes in velocity, friction head, and channel shape without dividing the channel length into short reaches. To compute the M- and N-values, various methods similar to that proposed by Bakhmeteff (1932); Chow (1955, 1959) or slight modifications thereof were adopted by many investigators, such as Woodward and Posey (1941), Kirpich (1948), Lee (1947), Von Seggern (1950), Subramanya and Ramamurthy (1974), Patil et al. (2001, 2003), Zaghloul and Anwar (1991), and Srivastava (2003), among others.

All the methods used by the previous investigators assume constant M and N. To remove the constant assumption of M and N, Chow (1981) formulated the two linear differential equations of first order, one for M and the other for N, thereby solving them for the exact expressions of the M- and N-values, which vary with the flow depth, h. Though Chow (1981) succeeded in removing the unwanted assumption of constant M and N from his earlier method, one can readily prove that the formulation of the two linear differential equations of first order for M and N are redundant, for the solutions of such differential equations are indeed identical to the M- and N-expressions that can be derived from the respective definitions of M and N. Chow (1981), aside from a lack of the theoretical basis to justify the derivation of his linear differential equations, could not even analytically solve the equations for h-dependent M and N for channels with any regular cross-sectional shape except for triangle and wide rectangles. In practical applications of Chow (1981) method, for example, to partially full flows in closed conduits, such as circular pipes, where the M- and N-values vary rapidly as the flow depth approaches the crown, Zaghloul (1998) numerically solved linear differential equations numerically for M and N proposed by Chow (1981) using the Runge–Kutta method. Although Zaghloul (1998) applied Chow's (1981) method to express the h-dependent M- and N-values for gravity flow in pipes, Shivareddy and Viswanadh (2008) a decade later still did not take into account the respective variations of M and N with h for flow in channels with D-shaped tunnel section.

2.7 Previous Studies on Integrating the GVF Equation

As for the first drawback, one cannot completely fix it until a new technique is developed to evaluate the two integrals of the proper fraction without resort to the VFF table. Apparently, the difficulty in overcoming the first drawback has resulted from a lack of our mathematical knowledge to find the exact functions to integrate such two integrals, which are transposed from the GVF equation, i.e. the nonlinear differential equation of first order in h with M, N, and h_c/h_n as the three parameters. Because the rational function representing the reciprocal of the slope of the GVF profile is improper (i.e., the degree of the denominator is less than or equal to the degree of the numerator), it can be rewritten by the process of long division (i.e., division algorithm) as a polynomial plus a proper fraction. A literature survey reveals that many studies have been undertaken by previous investigators to integrate such a proper fraction. For expedience, this proper fraction can be separated into two component fractions and the integration of the two component fractions has hitherto been referred to as the two integrals. The integration of both integrals can be performed in two ways. The first way is to have each integrand in the two integrals expanded into a finite set of partial fractions so that every term of them can be integrated separately, using the elementary transcendental functions (ETF), such as trigonometric, inverse trigonometric, hyperbolic, inverse hyperbolic, exponential, and logarithmic functions. In the second way, each of the two integrals was integrated using a number of various infinite series, one in a specified range of the dimensionless depth, thereby computing the values of both integrals. Obviously, the four methods used by Bakhmeteff (1932); Chow (1955, 1957, 1959) in the computation of the VFF-values belong to the second way, and other typical studies attempted by previous investigators are reviewed chronologically in the following.

The Bresse solution of GVF profile as shown in (2.8) and (2.9) [or (2.10)], was probably the earliest ETF-based solution for $M = N = 3$ that one can find in the literature. In fact, the reciprocal of the dimensionless GVF equation for $M = N = 3$, which can be exactly integrated to yield the Bresse solution, was used by Bakhmeteff (1932) to formulate the two integrals. Such two integrals for $M = N = 3$ can be transposed and merged into a sole integral, which Bakhmeteff (1932) called the VFF. Later, in an analogous approach, Gibson (1934) obtained the same sole integral, which he called the backwater function instead of the VFF and then tabulated it for computing GVF profiles in channels with cross-sectional shape other than wide rectangle. Gunder (1943) was among the earliest persons who applied the method of partial-fraction expansion in the evaluation of the two integrals for $M = 3$ and $N = 10/3$. The two integrals upon expansion in partial fractions were integrated by him using the ETF. In fact, the two integrals so evaluated appears to be a sum of those terms expressed later in Equations (D.2) and (D.6) as shown in Appendix D for flow in wide channels with flow resistance expressed using the Manning formula.

The two integrals formulated by Von Seggern (1950) look slightly different from those derived by Bakhmeteff (1932); Chow (1955, 1959), but in fact the integrands of both integrals prove to be identical if the former is converted to the latter, or vice versa, because he adopted the reciprocal of the dimensionless depth used by Bakhmeteff and Chow. Von Seggern also used an infinite series to compute the values of the two integrals. The infinite series used by Von Seggern is similar to that adopted by Bakhmeteff except with a slight modification made to account for the effect of the parameter, M, contained in one of the integrals. To meet the conversion criterion imposed on the integration of the infinite series, Von Seggern transposed the integrands of the two integrals into several plausible forms so that the values of both integrals at various ranges of his dimensionless depth can be approximately computed without inducing many terms needed in the numerical computation to warrant the convergence of the infinite series.

Instead of evaluating the two integrals, Pickard (1963) transposed the GVF equation into a form suitable for integration, i.e., the twelve flow profiles (three in mild, two in critical, three in steep, two in horizontal, and two in adverse channels) being arranged into four different integrals of two terms each. Any of the integrals was simply expressed in terms of one of three higher transcendental functions, which Pickard called the backwater integrals of the first, second, and third kinds. Among the three kinds, the backwater integral of the second kind was integrated in closed form, but the backwater of the first and third kinds were more conveniently expanded into an infinite series. The latter on integration yielded a solution with sort of the GHF before being reduced to a finite series of polylogarithms and polynomials of moderate order.

To integrate directly the GVF equation, most previous investigators preferred to express the rational function representing the reciprocal of the slope of the GVF profile in terms of the dimensionless flow depth rather than to substitute the geometric elements of a given channel section into the section factor, Z, and the conveyance, K, of the channel section, thereby evaluating the two integrals, but some investigators did it in reverse by substituting their geometric elements of a channel section under study into Z and K in the GVF equation before evaluating the two integrals. For example, Keifer and Chu (1955) computed backwater curves in closed conduits by substituting the geometric elements of a circular conduit section into Z and K in the GVF equation before integrating it by use of Simpsons rule. In another example, Allen and Enever (1968) substituted the geometric elements of a rectangular, triangular, or trapezoidal section into Z and K in the GVF equation before evaluating the two integrals. Allen and Enever used two power-law flow resistance formulas, the Chézy formula and the Blasius formula in their formulation of the GVF equation. They subsequently evaluated the two indefinite integrals via the partial-fraction expansion and then attempted to get the ETF-based solution for each of the cross-sectional geometries under study. Because a generalized form of the two integrals formulated using the 1/7th-power flow resistance formula for such regular channel shapes was too complex to be integrated directly, they only obtained a few simple ETF-based solutions, such as the Bresse solution for wide rectangular channels and a solution based on logarithmic function for narrow rectangular channels, among others.

In another approach analogous to the procedure used by Chow (1955, 1957, 1959) in the computation of the VFF, Gill (1976). evaluated the two indefinite integrals in two steps. Gill directly integrated the first integral using the partial-fraction expansion for 6 different N-values, i.e., $N = 2.5, 3, 10/3, 3.5, 4$ and 5 for each case of sustaining and adverse slopes, thereby obtaining the ETF-based solution of the first integral for each N-value studied. Upon acquiring all the ETF-based solutions of the first integrals for such N-values studied, Gill proceeded to evaluate the second integral by transposition to a form of the VFF of a composite variable with a modified parameter expressed in terms of M and N, by which he was able to express the second integral in the form of the first integral.

In an approach similar to that taken by Allen and Enever (1968), Kuma (1978) substituted their geometric elements of a rectangular or triangular channel section into Z and K in the GVF equation before evaluating the two integrals. However, unlike the two flow resistance formulas adopted by Allen and Enever for comparison, the Chézy formula was the only one used by Kumar in the GVF equation. Subsequently, Kumar also evaluated the two integrals via the partial-fraction expansion and then obtained their ETF-based solutions, thereby constructing two tables of his exclusively defined VFF, one for rectangular channels and the other for triangular channels, which are similar in concept to, but not refined as, the VFF tables developed by Bakhmeteff (1932); Chow (1955, 1957, 1959).

An approach taken by Zaghloul (1990, 1992) to integrate the GVF equation was the same as that used by Keifer and Chu (1955) to compute GVF profiles in circular channels by substituting the geometric elements of a circular conduit section into Z and K in the GVF equation before integrating it by use of Simpson's rule or the direct step and integration methods. Zaghloul formulated a computer model in such a way to handle all GVF computations in sustaining, horizontal, and adverse pipe slopes.

Instead of expressing the rational function representing the reciprocal of the slope of the GVF profile in the form of a polynomial plus a proper fraction, which was further separated into the two integrals, Ramamurthy et al. (2000) transformed (2.1) to (2.12) and then integrated directly.

$$S_0 \frac{dx}{dh} = \frac{1 - (h_c/h)^M}{1 - (h_n/h)^N}. \tag{2.12}$$

The right-hand side of (2.12) was expressed as a series in h_n/h, using the binomial series expansion to faciliate its direct integration. That is to say, the rational function in (2.12) was expanded in the form of a binomial series, i.e., an infinite series similar to that expanded by Bakhmeteff (1932) in his first method to compute the VFF. This binomial series merits attention because the infinite number of terms in the binomial series can be rearranged into two groups, each of which may be expressed in the form of an infinite series resembling the one that is defined using the GHF, as shown in Appendix A. Because the binomial series so expanded is valid only if the dimensionless flow depth or its reciprocal is less than unity, the binomial series expansion was applied by Ramamurthy et al. only in two zones of the GVF profiles in mild (M) channels corresponding to the M1 and M2 profiles. The CVF solutions obtained by Ramamurthy et al. are summarized as follows.

For $h_n/h < 1$, the expansion of (2.12) in the form as

$$\begin{aligned} S_0 \frac{dx}{dh} &= \frac{1 - (h_c/h)^M}{1 - (h_n/h)^N} \\ &= \left[1 - \left(\frac{h_c}{h}\right)^M\right]\left[1 + \left(\frac{h_n}{h}\right)^N + \left(\frac{h_n}{h}\right)^{2N} + \ldots\right]. \end{aligned} \tag{2.13}$$

The direct integration of (2.13) yields

$$x = \frac{h}{S_0}\left[\sum_{i=0}^{\infty} \frac{(h_n/h)^{iN}}{1 - iN} - \left(\frac{h_c}{h}\right)^M \sum_{i=0}^{\infty} \frac{(h_n/h)^{iN}}{1 - M - iN}\right] + \text{Const.}. \tag{2.14}$$

For $h_n/h > 1$, the expansion of (2.12) in the form as

$$S_0 \frac{dx}{dh} = \frac{1-(h_c/h)^M}{1-(h_n/h)^N} = \left(\frac{h}{h_n}\right)^N \left[\left(\frac{h_c}{h}\right)^M - 1\right]\left[1 + \left(\frac{h}{h_n}\right)^N + \left(\frac{h}{h_n}\right)^{2N} + \ldots\right]. \tag{2.15}$$

The direct integration of (2.15) yields

$$x = \frac{h}{S_0}\left[-\sum_{i=1}^{\infty} \frac{(h/h_n)^{iN}}{1+iN} + \left(\frac{h_c}{h}\right)^M \sum_{i=1}^{\infty} \frac{(h/h_n)^{iN}}{1-M+iN}\right] + \text{Const.}. \tag{2.16}$$

As reviewed briefly above, there have been two methods used by previous investigators to evaluate the two integrals. To assess the practical usefulness of the two methods to evaluate the two integrals, we compare the advantages and disadvantage of two solutions obtained from both methods in hopes of keeping abreast of the status quo, progress, and perspective of such evaluations as well as of finding some novel techniques to bring such evaluations to a justifiable end. As far as the accuracy of the computation is concerned, the ETF-based solution is superior to the infinite-series-based solution because the former is exact in contrast to the latter, which is only approximate pending the convergence rule to decide how many terms are required in the infinite series so as to be computed within the prescribed tolerance. On the other hand, if we assess the usefulness of both solutions from the practical point of view, the infinite-series-based solution should be more useful than the ETF-based solution because one can formulate a table, such as the VFF table, using data obtained from the infinite-series-based solution for a specified range of various N-values. Such a VFF-like table is useful in a numerical procedure for computing GVF profiles in channels with cross-sectional shape other than wide rectangle, whose M- and N-values vary with h. Unfortunately, it is too tedious to construct such a VFF-like table from the ETF-based solutions, though Gill (1976) obtained a number of ETF-based solutions for such purposes.

Nowadays, the extensive and rapidly growing use of digital computers can facilitate the evaluation of the two integrals, one way or the other, by use of the mathematics software, such as the Mathematica software (Wolfram 1996). As for the first way, for example, Venutelli (2004) found from the Mathematica software the ETF-based solutions of the two indefinite integrals for $M = 3$ and $N = 10/3$ (provided the Manning formula is used as flow resistance), which are expanded into two finite sets of partial fractions, each term of which can be integrated separately by use of the ETF. Equation (2.4) on substitution of $M = 3$ and $N = 10/3$ yields

$$x = \frac{h_n}{S_0}\left[u - \int_0^u \frac{1}{1-u^{10/3}}du + \left(\frac{h_c}{h_n}\right)^3 \int_0^u \frac{u^{1/3}}{1-u^{10/3}}du\right] + \text{Const.}. \tag{2.17}$$

Venutelli did not directly solve (2.17), but made a substitution of $\eta = u^{1/3}$ on the two integral terms of (2.17) first so as to transform the equation in the following form.

$$x = \frac{h_n}{S_0}\left[u - \int_0^{\eta} \frac{3\eta^2}{1-\eta^{10}} d\eta + \left(\frac{h_c}{h_n}\right)^3 \int_0^{\eta} \frac{3\eta^3}{1-\eta^{10}} d\eta\right] + \text{Const.}. \tag{2.18}$$

Venutelli then solved the two integral terms of (2.18) by using the Mathematica software so as to obtain an ETF-based solution of GVF profiles for subcritical and supercritical flows in mild and steep wide channels. In fact, the transformation from (2.17) to (2.18) is not necessary. Nowadays, to get an ETF-based solution of the dimensionless GVF profile for flow in a wide channel with hydraulic exponents $M = 3$ and $N = 10/3$, we can directly from (2.17) by using the Mathematica software, as shown in Table 2.2. More ETF-based solutions of the two indefinite integrals having $M = 3$ and five different N-values obtained from the Mathematica software are displayed in Appendix D. It is obvious that these ETF-based analytical solutions, as shown in Appendix D, are lengthy and disorgannized, except Bresse's solution for the case of $M = N = 3$.

As for the second way to express the solutions of the two integrals using an infinite series, one can also readily obtain the GHF-based solutions of the two integrals from the Mathematica software, as will be elaborated later in Chap. 3. However, it is unlikely that one can relate the two infinite series expressed in terms of the hypergeometric series (see Appendix A for detail) with those adopted by Bakhmeteff (1932),Chow (1955, 1957, 1959) in their computation of the VFF or derived by Ramamurthy et al. (2000) using solely the binomial expansion of the rational function. The software **Mathematica** is a computational software program used in many scientific, engineering, mathematical and computing fields, see Wolfram (1996). Except of the **Mathematica** software, the GHF-based GVF solutions discussed in this book can be solved by other similar mathematical programs, such as the **Maple** software (Bernardin et al. 2011) and the **Matlab** software (Houcque 2005).

In addition, there are some researchers have tried to directly solve the dynamic GVF Eq. i.e., (1.6), instead of (1.41), without using the hydraulic exponents M and N, under some approximation techniques. Dubin (1999) ever proposed an approximate semi-analytical solution for GVF in rectangular channels using the Manning formula. Vatankhah (2010) proposed an analytical solution for GVF in triangular channels using the Manning formula. Vatankhah and Easa (2011) derived a semi-analytical solution for GVF in general rectangular channels using the Manning formula. Vatankhah (2011) presented a direction integration method to derive a semi-analytical solution of the Manning-based GVF profiles in parabolic. Vatankhah (2012) also derived a direction integration method to derive a semi-analytical solution of the Manning-based GVF profiles in trapezoidal channels. Note that these kinds of semi-analytical solutions, similar to the above-mentioned ETF-based solutions, are lengthy and disorganized.

Table 2.2 The ETF-based solution of the dimensionless GVF profile for flow in a wide sustaining channel with hydraulic exponents $M = 3$ and $N = 10/3$

Equation: $x_* = u - \int \frac{1}{1-u^{10/3}} du + \lambda^3 \int \frac{u^{1/3}}{1-u^{10/3}} + \text{Const.}$.

Solution (for $0 \le u < 1$ or $u > 1$):

$$\begin{aligned}
x_* = u - \frac{3}{40} & \left\{ -2\sqrt{2\left(5+\sqrt{5}\right)} \arctan\left[\frac{1+\sqrt{5}-4u^{1/3}}{\sqrt{10-2\sqrt{5}}}\right] - 2\sqrt{10-2\sqrt{5}} \arctan\left[\frac{1-\sqrt{5}+4u^{1/3}}{\sqrt{10+2\sqrt{5}}}\right] \right. \\
& - 2\sqrt{10-2\sqrt{5}} \arctan\left[\frac{-1+\sqrt{5}+4u^{1/3}}{\sqrt{10+2\sqrt{5}}}\right] + 2\sqrt{10-2\sqrt{5}} \arctan\left[\frac{1+\sqrt{5}+4u^{1/3}}{\sqrt{10-2\sqrt{5}}}\right] - 4\ln\left[-1+u^{1/3}\right] + 4\ln\left[1+u^{1/3}\right] \\
& + (1+\sqrt{5})\ln\left[1-\frac{1}{2}(-1+\sqrt{5})u^{1/3}+u^{2/3}\right] - (1+\sqrt{5})\ln\left[1+\frac{1}{2}(-1+\sqrt{5})u^{1/3}+u^{2/3}\right] \\
& \left. + (-1+\sqrt{5})\ln\left[1-\frac{1}{2}(1+\sqrt{5})u^{1/3}+u^{2/3}\right] - (-1+\sqrt{5})\ln\left[1+\frac{1}{2}(1+\sqrt{5})u^{1/3}+u^{2/3}\right] \right\} \\
+ \frac{3\lambda^3}{40} & \left\{ -2\sqrt{2\left(5+\sqrt{5}\right)} \arctan\left[\frac{1+\sqrt{5}-4u^{1/3}}{\sqrt{10-2\sqrt{5}}}\right] - 2\sqrt{10-2\sqrt{5}} \arctan\left[\frac{1-\sqrt{5}+4u^{1/3}}{\sqrt{10+2\sqrt{5}}}\right] \right. \\
& - 2\sqrt{10-2\sqrt{5}} \arctan\left[\frac{-1+\sqrt{5}+4u^{1/3}}{\sqrt{10+2\sqrt{5}}}\right] - 2\sqrt{10-2\sqrt{5}} \arctan\left[\frac{1+\sqrt{5}+4u^{1/3}}{\sqrt{10-2\sqrt{5}}}\right] - 4\ln\left[-1+u^{1/3}\right] \\
& - 4\ln\left[1+u^{1/3}\right] - (-1+\sqrt{5})\ln\left[1-\frac{1}{2}\left(-1+\sqrt{5}\right)u^{1/3}+u^{2/3}\right] - (-1+\sqrt{5})\ln\left[1+\frac{1}{2}\left(-1+\sqrt{5}\right)u^{1/3}+u^{2/3}\right] \\
& \left. + (1+\sqrt{5})\ln\left[1-\frac{1}{2}(1+\sqrt{5})u^{1/3}+u^{2/3}\right] + (1+\sqrt{5})\ln\left[1+\frac{1}{2}\left(1+\sqrt{5}\right)u^{1/3}+u^{2/3}\right] \right\} + \text{Const.}.
\end{aligned}$$

2.8 Summary

The direct-integration method is a conventional method used to analytically solve the GVF equation. This chapter has introduced the conventional methods used to find the direct integration solutions of the GVF equation by using the varied-flow functions (VFF) and the elementary transcendental functions (ETF). The GVF equation is usually normalized using the normal depth, h_n before it can be analytically solved by the direct integration method. The hydraulic exponents M and N, and the ratio of the critical depth to normal depth (h_c/h_n) are the three key parameters in the normalized GVF equation. For simplicity, M and N are usually treated as constant, thus averting the impasse in integrating directly the GVF equation. The GVF solution by the Bakhmeteff-Chow procedure with VFF-tables is reviewed. These VFF-tables needed in the direct-integration method has a drawback caused by the imprecise interpolation of the VFF-values. The ETF-based solutions are lengthy and disorganized, except for the special case of $M = N = 3$ (Bresse's solution). To overcome these drawbacks, we will use the Gaussian hypergeometric functions (GHF) to analytically solve the GVF equation without recourse to the VFF tables and the lengthy elementary transcendental functions, as shown in the following chapters.

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