2 From Riemann Manifolds to Euclidean Manifolds

Mapping from a left two-dimensional Riemann manifold to a right two-dimensional Euclidean manifold, Cauchy–Green and Euler–Lagrange deformation tensors, equivalence theorem for equiareal mappings, conformeomorphism and areomorphism, Korn–Lichtenstein equations and Cauchy–Riemann equations, Mollweide projection, canonical criteria for (conformal, equiareal, isometric, equidistant) mappings, polar decomposition and simultaneous diagonalization for more than two matrices.

Let there be given the left two-dimensional Riemann manifold $\{\mathbb{M}_l^2, G_{MN}\}$ as well as the right two-dimensional Euclidean manifold $\{\mathbb{M}_r^2, g_{\mu\nu}\} = \{\mathbb{R}^2, \delta_{\mu\nu}\} = \mathbb{E}^2$. In many applications, the choice of $\{\mathbb{R}^2, \delta_{\mu\nu}\}$ is the "plane manifold", for instance, (i) the equatorial plane of the sphere or the ellipsoid, (ii) the meta-equatorial, also called *oblique equatorial plane* of the sphere or the ellipsoid, (iii) the plane generated by developing the cylinder, the cone, a ruled surface (namely surfaces which are "Gauss flat"), (iv) the tangent space $T_{\boldsymbol{U}_0}\mathbb{M}_l^2$ of the left two-dimensional Riemann manifold fixed to the point $\boldsymbol{U}_0 := \{U_0^1, U_0^2\}$ being covered by Cartesian coordinates. (Refer to all previous examples.) We shall not repeat the various *deformation measures* of type *multiplicative* and *additive* for the special case of the right two-dimensional Euclidean manifold $\{\mathbb{R}^2, \delta_{\mu\nu}\}$. Instead, we present to you (i) the left and right eigenspace analysis and synthesis of the Cauchy–Green deformation tensor, special case $\{\mathbb{M}_r^2, g_{\mu\nu}\} = \{\mathbb{R}^2, \delta_{\mu\nu}\}$, (ii) the left and right eigenspace analysis and synthesis of the Euler–Lagrange deformation tensor, special case $\{\mathbb{M}_r^2, g_{\mu\nu}\} = \{\mathbb{R}^2, \delta_{\mu\nu}\}$. (iii) Conformeomorphism, conformal mapping, special case $\{\mathbb{M}_r^2, g_{\mu\nu}\} = \{\mathbb{R}^2, \delta_{\mu\nu}\}$; Korn–Lichtenstein equations, special case Cauchy–Riemann equations (d'Alembert–Euler equations).

2-1 Eigenspace Analysis, Cauchy–Green Deformation Tensor

Left and right eigenspace analysis and synthesis of the Cauchy–Green deformation tensor, special case $\{\mathbb{M}_r^2, g_{\mu\nu}\} = \{\mathbb{R}^2, \delta_{\mu\nu}\}$.

First, let us confront you with Lemma 2.1, where we present detailed results of the left and right eigenspace analysis and synthesis of the Cauchy–Green deformation tensor for the special case of

E.W. Grafarend et al., *Map Projections*, DOI 10.1007/978-3-642-36494-5_2,

a right Euclidean manifold. Second, we focus on an interpretation of the results and additionally discuss a short example.

Lemma 2.1 (Left and right eigenspace analysis and synthesis of the Cauchy–Green deformation tensor, special case $\{\mathbb{M}^2_{\mathrm{r}},\, g_{\mu\nu}\} = \{\mathbb{R}^2,\, \delta_{\mu\nu}\}$).

(i) Synthesis.

For the matrix pair of positive-definite and symmetric matrices $\{\mathrm{C}_l,\, \mathrm{G}_l\}$ or $\{\mathrm{C}_\mathrm{r},\, \mathrm{G}_\mathrm{r}\}$, a simultaneous diagonalization is (the right Frobenius matrix F_r is an orthonormal matrix)

$$\begin{aligned} \mathrm{C}_l = \mathrm{J}_l^\mathrm{T}\mathrm{J}_l,\ \mathrm{F}_l^\mathrm{T}\mathrm{C}_l\mathrm{F}_l = \mathrm{diag}[\Lambda_1^2,\ \Lambda_2^2],\ \mathrm{F}_l^\mathrm{T}\mathrm{G}_l\mathrm{F}_l \\ = \mathrm{I} \text{ versus } \mathrm{F}_\mathrm{r}^\mathrm{T}\mathrm{C}_\mathrm{r}\mathrm{F}_\mathrm{r} = \mathrm{diag}\,[\lambda_1^2,\ \lambda_2^2],\ \mathrm{F}_\mathrm{r}^\mathrm{T}\mathrm{F}_\mathrm{r} = \mathrm{I}. \end{aligned} \tag{2.1}$$

(ii) Analysis.

Left eigenvalues or left principal stretches:

$$\begin{gathered} |\mathrm{C}_l - \Lambda_i^2\mathrm{G}_l| = 0, \\ \Lambda_{1,2}^2 = \Lambda_\pm^2 = \frac{1}{2}\left(\mathrm{tr}[\mathrm{C}_l\mathrm{G}_l^{-1}] \pm \sqrt{(\mathrm{tr}[\mathrm{C}_l\mathrm{G}_l^{-1}])^2 - 4\mathrm{det}[\mathrm{C}_l\mathrm{G}_l^{-1}]}\right). \end{gathered} \tag{2.2}$$

Left eigencolumns:

$$\begin{aligned} \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} &= \frac{1}{\sqrt{G_{11}(c_{22} - \Lambda_1^2 G_{22})^2 - 2G_{12}(c_{12} - \Lambda_1^2 G_{12})(c_{22} - \Lambda_1^2 G_{22}) + G_{22}(c_{12} - \Lambda_1^2 G_{12})^2}} \times \\ &\quad \times \begin{bmatrix} +(c_{22} - \Lambda_1^2 G_{22}) \\ -(c_{12} - \Lambda_1^2 G_{12}) \end{bmatrix}, \\ \begin{bmatrix} F_{12} \\ F_{22} \end{bmatrix} &= \frac{1}{\sqrt{G_{22}(c_{11} - \Lambda_2^2 G_{11})^2 - 2G_{12}(c_{11} - \Lambda_2^2 G_{11})(c_{12} - \Lambda_2^2 G_{12}) + G_{11}(c_{12} - \Lambda_2^2 G_{12})^2}} \times \\ &\quad \times \begin{bmatrix} -(c_{12} - \Lambda_2^2 G_{12}) \\ +(c_{11} - \Lambda_2^2 G_{11}) \end{bmatrix}. \end{aligned} \tag{2.3}$$

Right eigenvalues or right principal stretches

(the right general eigenvalue problem reduces to the right special eigenvalue problem):

$$\begin{gathered} |\mathrm{C}_\mathrm{r} - \lambda_i^2\mathrm{G}_\mathrm{r}| = |\mathrm{C}_\mathrm{r} - \lambda_i\mathrm{I}_2| = 0\,\forall\, i \in \{1, 2\}, \\ \lambda_{1,2}^2 = \lambda_\pm^2 = \frac{1}{2}\left(\mathrm{tr}[\mathrm{C}_\mathrm{r}\mathrm{G}_\mathrm{r}^{-1}] \pm \sqrt{\mathrm{tr}[\mathrm{C}_\mathrm{r}\mathrm{G}_\mathrm{r}^{-1}])^2 - 4\mathrm{det}\,[\mathrm{C}_\mathrm{r}\mathrm{G}_\mathrm{r}^{-1}]}\right) = \\ = \frac{1}{2}\left(C_{11} + C_{22} \pm \sqrt{(C_{11} + C_{22})^2(2C_{12})^2}\right). \end{gathered} \tag{2.4}$$

Right eigencolumns:

$$
\mathrm{F_r} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{cases} \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix} = \frac{1}{\sqrt{(C_{22}-\lambda_1^2)^2 + C_{12}^2}} \begin{bmatrix} C_{22} - \lambda_1^2 \\ -C_{12} \end{bmatrix}, \\ \\ \begin{bmatrix} f_{12} \\ f_{22} \end{bmatrix} = \frac{1}{\sqrt{(C_{11}-\lambda_2^2)^2 + C_{12}^2}} \begin{bmatrix} -C_{12} \\ C_{11} - \lambda_2^2 \end{bmatrix}. \end{cases} \tag{2.5}
$$

Since the right Frobenius matrix $\mathrm{F_r}$ is an orthonormal matrix, it can be represented by

$$
\begin{gathered}
\mathrm{F_r} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \ \forall \varphi \in [0, 2\pi], \\
\tan\varphi = \frac{C_{12}}{C_{11} - \lambda_-^2}, \quad \tan 2\varphi = \frac{2C_{12}}{C_{11} - C_{22}}.
\end{gathered} \tag{2.6}
$$

End of Lemma.

The proof of Lemma 2.1 is straightforward from Lemma 1.6 as soon as we specialize $\mathrm{G_r} = \mathrm{I}_2$. Of special interest is the right eigenspace analysis. Here, the right Frobenius matrix $\mathrm{F_r}$ is orthonormal. As an orthonormal matrix (also called "proper rotation matrix"), it can be parameterized by a rotation angle φ. Such an angle of rotation orientates the right eigenvectors $\{\boldsymbol{f}_1, \boldsymbol{f}_2 | \mathcal{O}\}$ with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2 | \mathcal{O}\}$, $\mathbb{R}^2 = \operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. Indeed, the "$\tan 2\varphi$ identity" leads to an easy computation of the orientation of the right eigenvectors. We proceed to a short example.

Example 2.1 (Orthogonal projection of points of the sphere $\mathbb{S}^2_{R+}$ onto the equatorial plane $\mathbb{P}^2_{\mathcal{O}}$ through the origin $\mathcal{O}$).

In Example 1.6, we presented already to you the special map projection of the hemisphere $\mathbb{S}^2_{R+}$ onto the central equatorial plane $\mathbb{P}^2_{\mathcal{O}}$ by computing its characteristic right Cauchy–Green deformation tensor as well as its right eigenspace. Here, we aim at testing the right Frobenius matrix $\mathrm{F_r}$ on orthonormality. Let us transfer the right eigencolumns to build up

$$
\mathrm{F_r} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = -\frac{1}{\sqrt{x^2 + y^2}} \begin{bmatrix} x & y \\ y & -x \end{bmatrix}. \tag{2.7}
$$

Is this Frobenius matrix of integrating factors an orthonormal matrix? Please test $\mathrm{F_r^* F_r} = \mathrm{I}_2$ to convince yourself. Here, we generate

$$
\mathrm{F_r} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = -\frac{1}{\sqrt{x^2 + y^2}} \begin{bmatrix} x & y \\ y & -x \end{bmatrix}, \tag{2.8}
$$

$$
\tan\varphi = -\frac{y}{x}, \quad \tan 2\varphi = \frac{2\tan\alpha}{1 - \tan^2\alpha} = -\frac{2xy}{x^2 - y^2}, \tag{2.9}
$$

$$
C_{12} = \frac{xy}{R^2 - (x^2 + y^2)}, \quad C_{11} - C_{22} = \frac{x^2 - y^2}{R^2 - (x^2 + y^2)}, \tag{2.10}
$$

$$\tan 2\varphi = \frac{2C_{12}}{C_{11} - C_{22}} = \frac{2xy}{x^2 - y^2}. \tag{2.11}$$

$$\text{If } x = y, \text{ then } \tan\varphi = -1,\ \tan 2\varphi \to \pm\infty,\ \varphi = \mp 45°.$$

End of Example.

2-2 Eigenspace Analysis, Euler–Lagrange Deformation Tensor

Left and right eigenspace analysis and synthesis of the Euler–Lagrange deformation tensor, special case $\{\mathbb{M}_r^2,\, g_{\mu\nu}\} = \{\mathbb{R}^2,\, \delta_{\mu\nu}\}$.

First, let us confront you with Lemma 2.2, where we present detailed results of the left and right eigenspace analysis and synthesis of the Euler–Lagrange deformation tensor for the special case of a right Euclidean manifold. Second, we focus on an interpretation of the results.

Lemma 2.2 (Left and right eigenspace analysis and synthesis of the Euler–Lagrange deformation tensor, special case $\{\mathbb{M}_r^2,\ g_{\mu\nu}\} = \{\mathbb{R}^2,\ \delta_{\mu\nu}\}$).

(i) Synthesis.

For the pair of symmetric matrices $\{\mathrm{E}_l,\, \mathrm{G}_l\}$ or $\{\mathrm{E}_\mathrm{r},\, \mathrm{G}_\mathrm{r}\}$, where the matrices $\{\mathrm{G}_l,\ \mathrm{G}_\mathrm{r}\}$ are positive definite, a simultaneous diagonalization is (the right Frobenius matrix F_r is an orthonormal matrix)

$$\mathrm{F}_l^\mathrm{T}\mathrm{E}_l\mathrm{F}_l = \mathrm{diag}[K_1,\, K_2], \quad \mathrm{F}_l^\mathrm{T}\mathrm{G}_i\mathrm{F}_l = \mathrm{I} \quad \text{versus} \quad \begin{array}{l}\mathrm{F}_\mathrm{r}^\mathrm{T}\mathrm{E}_\mathrm{r}\mathrm{F}_\mathrm{r} = \mathrm{diag}\ [\kappa_1,\, \kappa_2],\\ \mathrm{F}_\mathrm{r}^\mathrm{T}\mathrm{F}_1 = \mathrm{I}.\end{array} \tag{2.12}$$

(ii) Analysis.

Left eigenvalues:

$$\begin{aligned}|\mathrm{E}_l - K_i\mathrm{G}_l| = 0, \quad K_{1,2} &= K_\pm \\ &= \frac{1}{2}\left(\mathrm{tr}[\mathrm{E}_l\mathrm{G}_l^{-1}] \pm \sqrt{(\mathrm{tr}[\mathrm{E}_l\mathrm{G}_l^{-1}])^2 - 4\mathrm{det}\ [\mathrm{E}_l\mathrm{G}_l^{-1}]}\right).\end{aligned} \tag{2.13}$$

Left eigencolumns:

$$\begin{aligned}\begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} = &\frac{1}{\sqrt{G_{11}(e_{22} - K_1G_{22})^2 - 2G_{12}(e_{12} - K_1G_{12})(e_{22} - K_1G_{22}) + G_{22}(e_{12} - K_1G_{12})^2}} \times \\ &\times \begin{bmatrix} e_{22} - K_1G_{22} \\ -(e_{12} - K_1G_{12}) \end{bmatrix},\end{aligned} \tag{2.14}$$

$$\begin{bmatrix} F_{12} \\ F_{22} \end{bmatrix} = \frac{1}{\sqrt{G_{22}(e_{11}-K_2G_{11})^2 - 2G_{12}(e_{11}-K_2G_{11})(e_{12}-K_2G_{12}) + G_{11}(e_{12}-K_2G_{12})^2}} \times$$

$$\times \begin{bmatrix} -(e_{12}-K_2G_{12}) \\ e_{11}-K_2G_{11} \end{bmatrix}.$$

Right eigenvalues

(the right general eigenvalue problem reduces to the right special eigenvalue problem):

$$|\mathrm{E_r} - \kappa_i \mathrm{I_r}| = 0,$$

$$\begin{aligned} \kappa_{1,2} = \kappa_\pm &= \frac{1}{2}\left(\mathrm{tr}[\mathrm{E_r}] \pm \sqrt{(\mathrm{tr}[\mathrm{E_r}])^2 - 4\det[\mathrm{E_r}]}\right) = \\ &= \frac{1}{2}\left(E_{11} + E_{22} \pm \sqrt{(E_{11}+E_{22})^2 + (2E_{12})^2}\right). \end{aligned} \tag{2.15}$$

Right eigencolumns:

$$\mathrm{F_r} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{cases} \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix} = \frac{1}{\sqrt{(E_{22}-k_1)^2 + E_{12}^2}} \begin{bmatrix} E_{22}-k_1 \\ -E_{12} \end{bmatrix}, \\ \begin{bmatrix} f_{12} \\ f_{22} \end{bmatrix} = \frac{1}{\sqrt{(E_{11}-k_2)^2 + E_{12}^2}} \begin{bmatrix} -E_{12} \\ E_{11}-k_2 \end{bmatrix} \end{cases} \tag{2.16}$$

Since the right Frobenius matrix $\mathrm{F_r}$ is an orthonormal matrix, it can be represented by

$$\begin{aligned} \mathrm{F_r} &= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \ \forall\phi \in [0, 2\pi], \\ \tan\phi &= \frac{E_{12}}{E_{11}-\kappa_-},\ \tan 2\phi = \frac{2E_{12}}{E_{11}-E_{22}}. \end{aligned} \tag{2.17}$$

End of Lemma.

Lemma 1.7 is the basis of the proof if we specialize $\mathrm{G_r} = \mathrm{I}_2$. Again, we emphasize that within the right eigenspace analysis the right Frobenius matrix is orthonormal. As an orthonormal matrix, i.e. $\mathrm{F_r} \in \mathrm{SO}(2) := \{\mathrm{F_r} \in \mathbb{R}^{2\times 2} \,|\, \mathrm{F_r^T F_r} = \mathrm{I}_2 \text{ and } \det[\mathrm{F_r}] = +1\}$, it can be properly parameterized by a rotation angle ϕ. Such an angle of rotation orientates the right eigenvectors $\{\boldsymbol{f}_1, \boldsymbol{f}_2|\mathcal{O}\}$ with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2|\mathcal{O}\}$, $\mathbb{R}^2 = \mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. Indeed, the "$\tan 2\phi$ identity" leads to an easy computation of the orientation of the right eigenvectors.

2-3 The Equivalence Theorem for Conformal Mappings

The equivalence theorem for conformal mappings from the left two-dimensional Riemann manifold to the right two-dimensional Euclidean manifold (conformeomorphism), Korn–Lichtenstein equations and Cauchy–Riemann equations (d'Alembert–Euler equations).

The previous equivalence theorem for a conformeomorphism is specialized for the case of the two-dimensional right Euclidean manifold $\{\mathbb{M}^2_{\mathrm{r}}, g_{\mu\nu}\} = \{\mathbb{R}^2,\ \delta_{\mu\nu}\} =: \mathbb{E}^2$. In many applications, the choice of $\{\mathbb{R}^2,\ \delta_{\mu\nu}\}$ is the planar manifold, for instance, the tangent space $T_{\boldsymbol{U}_o}\mathbb{M}^2_l$ of the left two-dimensional Riemann manifold fixed to the point $\boldsymbol{U}_0 = \{U^1_0,\ U^2_0\}$, being covered by Cartesian or polar coordinates. For an illustration of such a setup of a "planar manifold", go back to our previous examples.

2-31 Conformeomorphism

First, let us confront you with Lemma 2.3. The proof based upon Theorem 1.11 is straightforward. Examples are given in the following chapters.

Lemma 2.3 (Conformeomorphism, conformal mapping, special case $\{\mathbb{M}^2_{\mathrm{r}}, g_{\mu\nu}\} = \{\mathbb{R}^2,\ \delta_{\mu\nu}\}$).

Let $\overline{f} : \mathbb{M}^2_l \rightarrow \{\mathbb{R}^2, \delta_{\mu\nu}\}$ be an orientation preserving conformal mapping. Then the following conditions are equivalent.

$$\text{(i) } \Psi_l(\dot{\boldsymbol{U}}_1, \dot{\boldsymbol{U}}_2) = \Psi_{\mathrm{r}}(\dot{\boldsymbol{u}}_1, \dot{\boldsymbol{u}}_2) \tag{2.18}$$

for all tangent vectors $\dot{\boldsymbol{U}}_1, \dot{\boldsymbol{U}}_2$ and their images $\dot{\boldsymbol{u}}_1, \dot{\boldsymbol{u}}_2$, respectively.

$$\begin{gathered}
\text{(ii) } \mathrm{C}_l = \lambda^2(\boldsymbol{U}_0)\mathrm{G}_l \ \text{ versus } \ \mathrm{C}_{\mathrm{r}} = \lambda^2\mathrm{I}_2, \mathrm{C}_{\mathrm{r}}^{-1} = \mathrm{I}_2/\lambda^2,\\
C_{11} = C_{22} = \lambda^2, \quad C_{12} = C_{21} = 0, \quad C^{11} = C^{22} = \lambda^{-2}, \quad C^{12} = C^{21} = 0;\\[1em]
\mathrm{E}_l = K(\boldsymbol{U}_0)\mathrm{G}_l \quad \text{versus} \quad \mathrm{E}_{\mathrm{r}} = \kappa\mathrm{I}_2, \quad \mathrm{E}_{\mathrm{r}}^{-1} = \mathrm{I}_2/\kappa,\\
E_{11} = E_{22} = \kappa, \quad E_{12} = E_{21} = 0, \quad E^{11} = E^{22} = \kappa^{-1}, \quad E^{12} = E^{21} = 0.
\end{gathered} \tag{2.19}$$

$$\begin{aligned}
\text{(iii) } \begin{bmatrix} K = (\Lambda^2 - 1)/2 \\ \Lambda^2 = 2K + 1 \end{bmatrix} \quad &\text{versus} \quad \begin{bmatrix} (\lambda^2 - 1)/2 = \kappa \\ 2\kappa + 1 = \lambda^2 \end{bmatrix},\\
\Lambda_1 = \Lambda_2 = \Lambda(\boldsymbol{U}_0) \ &\text{versus} \ \lambda_1 = \lambda_2 = \lambda(\boldsymbol{u}_0),\\
K_1 = K_2 = K(\boldsymbol{U}_0) \quad &\text{versus} \quad \kappa_1 = \kappa_2 = \kappa(\boldsymbol{u}_0),\\
\Lambda^2(U_0) = \mathrm{tr}[\mathrm{C}_l\mathrm{G}_l^{-1}]/2 \quad &\text{versus} \quad \lambda^2(\boldsymbol{u}_0) = \mathrm{tr}[\mathrm{C}_{\mathrm{r}}]/2;\\
\text{(left dilatation)} K = \mathrm{tr}[\mathrm{E}_l\mathrm{G}_l^{-1}]/2 \quad &\text{versus} \ \text{(right dilatation)} \kappa = \mathrm{tr}[\mathrm{E}_{\mathrm{r}}]/2,
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
\mathrm{tr}[\mathrm{C}_l\mathrm{G}_l^{-1}] = 2\sqrt{\det[\mathrm{C}_l\mathrm{G}_l^{-1}]} \quad &\text{versus} \quad \mathrm{tr}[\mathrm{C}_l\mathrm{G}_l^{-1}] = 2\sqrt{\det[\mathrm{C}_{\mathrm{r}}]},\\
\mathrm{tr}[\mathrm{E}_l\mathrm{G}_l^{-1}] = 2\sqrt{\det[\mathrm{E}_l\mathrm{G}_l^{-1}]} \quad &\text{versus} \quad \mathrm{tr}[\mathrm{E}_{\mathrm{r}}] = 2\sqrt{\det[\mathrm{E}_{\mathrm{r}}]}.
\end{aligned} \tag{2.21}$$

(iv) (Generalized Korn–Lichtenstein equations, Cauchy–Riemann equations, subject to the integrability conditions $u_{UV} = u_{VU}$ and $v_{UV} = v_{VU}$)

$$\begin{bmatrix} u_U \\ u_V \end{bmatrix} = \frac{1}{\sqrt{G_{11}G_{22} - G_{12}^2}} \begin{bmatrix} -G_{12} & G_{11} \\ -G_{22} & G_{12} \end{bmatrix} \begin{bmatrix} u_U \\ u_V \end{bmatrix}. \tag{2.22}$$

End of Lemma.

2-32 Higher-Dimensional Conformal Mapping

In order to develop the theory of a higher-dimensional conformal diffeomorphism (in Gauss's words: "in kleinsten Teilen ähnlich"), we first derive the Korn–Lichtenstein equations of a two-dimensional conformal mapping $\mathbb{M}_l^2 \to \mathbb{M}_\mathrm{r}^2 := \{\mathbb{R}^2, \delta_{\mu\nu}\} = \mathbb{E}^2$ by means of *exterior calculus*, namely by means of the *Hodge star operator*. With such an experience built up, second, we derive the *Zund equations* of a three-dimensional conformal mapping $\mathbb{M}_l^3 \to \mathbb{M}_\mathrm{r}^3 := \{\mathbb{R}^3,\ \delta_{\mu\nu}\} = \mathbb{E}^3$ by means of exterior calculus taking advantage of the Hodge star operator in $\mathbb{R}^3$. Note that the Hodge star operator generalizes the *vector product*, also called *cross product* or *outer product*, to any dimension. Indeed, the classical vector product serves us only in $\mathbb{R}^3$. Box 2.1 summarizes the various steps to produce a conformal diffeomorphism $\mathbb{M}_l^2 \to \mathbb{M}_\mathrm{r}^2 = \{\mathbb{R}^2, \delta_{\mu\nu}\} = \mathbb{E}^2$ in terms of exterior calculus. First, we introduce the left Jacobi map $\{\mathrm{d}x,\, \mathrm{d}y\} \to \{\mathrm{d}U,\, \mathrm{d}V\}$ and the right Jacobi map $\{\mathrm{d}U,\, \mathrm{d}V\} \to \{\mathrm{d}x,\, \mathrm{d}y\}$. Second, we compute the right Cauchy–Green matrix $\mathrm{C_r}$ subject to its conformal structure $\mathrm{C_r} = \lambda^2 \mathrm{I}_2$ and $\mathrm{C_r^{-1}} = \lambda^{-2}\mathrm{I}_2$. We are led to a representation of the conformal right Cauchy–Green matrix $\mathrm{C_r} = \mathrm{J_r^T G}_l \mathrm{J_r} = \lambda^2 \mathrm{I}_2$ or $\mathrm{C_r^{-1}} = \mathrm{J}_l^\mathrm{T} \mathrm{G}_l^{-1} \mathrm{J}_l = \lambda^{-2}\mathrm{I}_2$ in terms of the Jacobi matrices J_l and $\mathrm{J_r}$. The rows of the left Jacobi matrix can be interpreted as "G_l^{-1} orthogonal", while the right Jacobi matrix can be interpreted as "G_l orthogonal". Third, this result of conformal geometry is used by the Hodge star operator. One-by-one, we define $\mathrm{d}x$, x_1, x_2, and $\mathrm{d}y^*$. Here, we make use of the two-dimensional permutation symbol $e_{LM} \in \mathbb{R}^{2\times 2}(L,\ M \in \{1, 2\})$. Fourth, we explicitly represent the exterior form $\mathrm{d}x = \mathrm{d}y^*$ of the Korn–Lichtenstein equations: compare with Lemma 2.4.

Lemma 2.4 (Grafarend and Syffus (1998d, p. 292), conformeomorphism $\mathbb{M}_l^2 \to \mathbb{M}_\mathrm{r}^2 := \{\mathbb{R}^2, \delta_{\mu\nu}\}$, Korn–Lichtenstein equations).

The following formulations of the Korn–Lichtenstein equations producing a conformal diffeomorphism $\mathbb{M}_l^2 \to \mathbb{M}_\mathrm{r}^2 := \{\mathbb{R}^2, \delta_{\mu\nu}\}$ are equivalent.

Formulation (i):

$$\mathrm{d}x = *\mathrm{d}y. \tag{2.23}$$

Formulation (ii):

$$\frac{\partial x}{\partial U^L} = e_{LM} \sqrt{\det[\mathrm{G_l}]}\, G^{MN} \frac{\partial y}{\partial U^N}. \tag{2.24}$$

Formulation (iii):

$$x_U = \frac{1}{\sqrt{|\mathrm{G}_l|}}(-G_{12 yU} + G_{11 yV}), \quad x_V = \frac{1}{\sqrt{|\mathrm{G}_l|}}(-G_{22 yU} + G_{12 yV}), \tag{2.25}$$

$$\mathrm{G}_l = [G_{MN}] = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} \Leftrightarrow \frac{1}{|\mathrm{G}_l|} \begin{bmatrix} G_{22} & -G_{12} \\ -G_{12} & G_{11} \end{bmatrix} = [G^{LM}] = \mathrm{G}_l^{-1}, \tag{2.26}$$

subject to the integrability conditions

$$\frac{\partial^2 x}{\partial U \partial V} = \frac{\partial^2 x}{\partial V \partial U}, \quad \frac{\partial^2 y}{\partial U \partial V} = \frac{\partial^2 y}{\partial V \partial U}. \tag{2.27}$$

End of Lemma.

Box 2.1 (Conformal diffeomorphism $\mathbb{M}_l^2 \to \mathbb{M}_\mathrm{r}^2 = \{\mathbb{R}^2, \delta_{\mu\nu}\} = \mathbb{E}^2$, exterior calculus).

Diffeomorphism :

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \mathrm{J}_l \begin{bmatrix} \mathrm{d}U \\ \mathrm{d}V \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathrm{d}U \\ \mathrm{d}V \end{bmatrix} = \mathrm{J}_\mathrm{r} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}$$

$$\Leftrightarrow$$

$$\mathrm{J}_l = \mathrm{J}_r^{-1} \tag{2.28}$$

$$\Leftrightarrow$$

$$\mathrm{J}_r = \mathrm{J}_l^{-1}.$$

Right Cauchy–Green matrix for a conformal diffeomorphism:

$$\mathrm{C}_\mathrm{r} = \mathrm{J}_\mathrm{r}^\mathrm{T} \mathrm{G}_l \mathrm{J}_\mathrm{r} = \lambda^2 \mathrm{I}_2 \tag{2.29}$$

$$\Leftrightarrow$$

$$\mathrm{C}_\mathrm{r}^{-1} = \mathrm{J}_l \mathrm{G}_l^{-1} \mathrm{J}_l^\mathrm{T} = \lambda^{-2} \mathrm{I}_2.$$

The rows of the left Jacobi matrix are G_l^{-1} orthogonal :

$$\mathrm{d}x = x_U \mathrm{d}U + x_{\mathrm{d}V} \mathrm{d}V = \sum_{M=1}^{2} x_M \mathrm{d}U^M, \quad x_1 := D_U x = x_U, \quad x_2 := D_V x = x_V. \tag{2.30}$$

Hodge star operator:

$$*\mathrm{d}y \ := \sum_{L,M,N=1}^{2} e_{LM} \sqrt{\det[\mathrm{G}_l]} G^{MN} y_N \mathrm{d}U^L, \tag{2.31}$$

subject to

$$y_1 := D_{Uy} = y_U, \quad y_2 := D_{Vy} = y_V.$$

Permutation symbol:

$$e_{LM} = \begin{cases} +1 \text{ for an even permutation of the indices } L, M \in \{1,2\} \\ -1 \text{ for an odd permutation of the indices } L, M \in \{1,2\}. \\ 0 \quad \text{otherwise} \end{cases} \tag{2.32}$$

Korn–Lichtenstein equations in exterior calculus:

$$\mathrm{d}x = \sum_{M=1}^{2} x_M \mathrm{d}U^\mathrm{M} = \sum_{L,M,N=1}^{2} e_{LM} \sqrt{\det[\mathrm{G}_l]} G^{MN} y_N \mathrm{d}U^L = \mathrm{d}y^*$$

$$\Leftrightarrow \tag{2.33}$$

$$\frac{\partial x}{\partial U^L} = e_{LM} \sqrt{\det[\mathrm{G}_l]} G^{MN} \frac{\partial y}{\partial U^N}, \quad \mathrm{d}x = \mathrm{d}y^*.$$

Box 2.2 summarizes the operational procedure for generating a conformal diffeomorphism, also called *conformeomorphism*, $\mathbb{M}^2_l \to \mathbb{M}^2_{\mathrm{r}} = \{\mathbb{R}^2, \delta_{\mu\nu}\} = \mathbb{E}^3$, again in terms of exterior calculus. First, we introduce the differential one-forms, the differential two-forms, and the differential three-forms. Second, we apply the Hodge star operator (i) to $*\mathrm{d}x$ etc., (ii) to $*(\mathrm{d}y \wedge \mathrm{d}z)$ etc., and (iii) to $*(\mathrm{d}y \wedge \mathrm{d}y \wedge \mathrm{d}z)$. The columns $[x_1,\ x_2,\ x_3]^{\mathrm{T}}$, $[y_1,\ y_2,\ y_3]^{\mathrm{T}}$, and $[z_1,\ z_2,\ z_3]^{\mathrm{T}}$ may be considered orthogonal. Third, we represent the expression $*(\mathrm{d}y\ \wedge \mathrm{d}z)$ as an example explicitly. Again, the three-dimensional permutation symbol $e_{LM_1M_2} \in \mathbb{R}^{3\times3\times3}(L,\ M_1, M_2 \in \{1,\ 2\})$ as a three-dimensional array is defined. Fourth, we explicitly compute the expression $\mathrm{d}x = *(\mathrm{d}y \wedge \mathrm{d})z$, the *Zund equations* of a three-dimensional conformal mapping $\mathbb{M}^3_l \to \mathbb{M}^3_{\mathrm{r}} = \mathbb{E}^3$: compare with Lemma 2.5.

Box 2.2 (Conformal diffeomorphism $\mathbb{M}^3_l \to \mathbb{M}^3_{\mathrm{r}} = \{\mathbb{R}^3, \delta_{\mu\nu}\} = \mathbb{E}^3$, exterior calculus).

Differential frame:

$$\left.\begin{aligned} \mathrm{d}x &= x_1\mathrm{d}U + x_2\mathrm{d}V + x_3\mathrm{d}W \\ \text{(i) } \mathrm{d}y &= y_1\mathrm{d}U + y_2\mathrm{d}V + y_3\mathrm{d}W \\ \mathrm{d}z &= z_1dU + z_2dV + z_3\mathrm{d}W \end{aligned}\right] \text{ (one-forms)}, \tag{2.34}$$

$$\text{(ii) } \mathrm{d}y \wedge \mathrm{d}z, \quad \mathrm{d}z \wedge \mathrm{d}x, \quad \mathrm{d}x \wedge \mathrm{d}y, \text{ (two-forms)},$$
$$\text{(iii) } \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \qquad \text{(three-form)}.$$

Hodge star operator:

$$\begin{gathered} \text{(i) } * \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z, \quad *\mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}x, \quad *\mathrm{d} = \mathrm{d}x\ \wedge \mathrm{d}y; \\ \text{(ii) } * (\mathrm{d}y \wedge \mathrm{d}z) = \mathrm{d}x, \quad *(\mathrm{d}z \wedge \mathrm{d}x) = \mathrm{d}y, \quad *(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}z; \\ \text{(iii) } * (\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = 1. \end{gathered} \tag{2.35}$$

Example :

$$\forall\ L,\ M_1,\ M_2,\ N_1,\ N_2 \in \{1,2,3\}\ :$$

$$*(\mathrm{d}y \wedge \mathrm{d}z) = \sum_{L,M_1,M_2,N_1,N_2=1}^{3} e_{LM_1M_2}\sqrt{|\mathrm{G}_l|}G^{M_1N_1}G^{M_2N_2}\frac{\partial y}{\partial U^{N_1}}\frac{\partial z}{\partial U^{N_2}}\mathrm{d}U^L. \tag{2.36}$$

Permutation symbol:

$$e_{LM_1M_2} = \begin{cases} +1 \text{ for an even permutation of the indices } L, M_1, M_2 \in \{1,2,3\} \\ -1 \text{ for an odd permutation of the indices } \ L, M_1, M_2 \in \{1,2,3\}. \\ 0 \quad \text{otherwise} \end{cases} \tag{2.37}$$

Zund equations of a two-dimensional conformal diffeomorphism in exterior calculus:

$$\mathrm{d}x = \sum_{M=1}^{3} x_M\mathrm{d}U^M =$$

$$= \sum_{L,M_1,M_2,N_1,N_2=1}^{3} e_{LM_1M_2}\sqrt{|\mathrm{G}_l|}G^{\mathrm{M}_1N_1}G^{M_2N_2}\frac{\partial y}{\partial U^{N_1}}\frac{\partial z}{\partial U^{N_2}}\mathrm{d}U^L = *(\mathrm{d}y \wedge \mathrm{d}z) \tag{2.38}$$

$$\Leftrightarrow$$

$$\frac{\partial x}{\partial U^L} = e_{LM_1M_2}\sqrt{\det\left[\mathrm{G}_l\right]}G^{M_1N_1}G^{M_2N_2}\frac{\partial y}{\partial U^{N_1}}\frac{\partial z}{\partial U^{N_2}},$$

$$\mathrm{d}x = *(\mathrm{d}y \wedge \mathrm{d}z).$$

Lemma 2.5 (Zund (1987), Grafarend and Syffus (1998d, p. 292), the Zund equations of a three-dimensional conformeomorphism $\mathbb{M}_l^3 \to \mathbb{M}_\mathrm{r}^3 = \{\mathbb{R}^3, \delta_{\mu\nu}\} = \mathbb{E}^3$).

Equivalent formulations of the equations producing a conformal mapping $\mathbb{M}_l^3 \to \mathbb{M}_\mathrm{r}^3 = \mathbb{E}^3$ are provided by the following formulations.

Formulation (i):

$$\mathrm{d}x = *(\mathrm{d}y \wedge \mathrm{d}z). \tag{2.39}$$

Formulation (ii):

$$\forall I,\ J_1,\ J_2,\ K_1,\ K_2 \in \{1,2,3\} : \frac{\partial x}{\partial U^I} = \frac{1}{2}e_{IJ_1J_2}\sqrt{|\mathrm{G}_l|}G^{J_1K_1}G^{J_2K_2}\frac{\partial y}{\partial U^{k_1}}\frac{\partial z}{\partial U^{K_2}}. \tag{2.40}$$

Formulation (iii):

$$\frac{\partial x}{\partial U} = \frac{1}{2}\sqrt{|\mathrm{G}_l|}\begin{array}{l}\big[(G^{21}G^{32} - G^{31}G^{12})\frac{\partial y}{\partial U}\frac{\partial z}{\partial V} + (G^{21}G^{33} - G^{31}G^{23})\frac{\partial y}{\partial U}\frac{\partial z}{\partial W} + \\ +(G^{22}G^{31} - G^{32}G^{21})\frac{\partial y}{\partial V}\frac{\partial z}{\partial U} + (G^{22}G^{33} - G^{32}G^{23})\frac{\partial y}{\partial V}\frac{\partial z}{\partial W} + \\ +(G^{23}G^{31} - G^{33}G^{21})\frac{\partial y}{\partial W}\frac{\partial z}{\partial U} + (G^{23}G^{32} - G^{33}G^{22})\frac{\partial y}{\partial W}\frac{\partial z}{\partial V}\big]\,,\end{array} \tag{2.41}$$

$$\frac{\partial x}{\partial V} = \frac{1}{2}\sqrt{|\mathrm{G}_l|}\begin{array}{l}\big[(G^{31}G^{12} - G^{11}G^{32})\frac{\partial y}{\partial U}\frac{\partial z}{\partial V} + (G^{31}G^{13} - G^{11}G^{33})\frac{\partial y}{\partial U}\frac{\partial z}{\partial W} + \\ +(G^{32}G^{11} - G^{12}G^{31})\frac{\partial y}{\partial V}\frac{\partial z}{\partial U} + (G^{32}G^{13} - G^{12}G^{33})\frac{\partial y}{\partial V}\frac{\partial z}{\partial W} + \\ +(G^{33}G^{11} - G^{13}G^{31})\frac{\partial y}{\partial W}\frac{\partial z}{\partial U} + (G^{33}G^{12} - G^{13}G^{32})\frac{\partial y}{\partial W}\frac{\partial z}{\partial V}\big]\,,\end{array} \tag{2.42}$$

$$\frac{\partial x}{\partial W} = \frac{1}{2}\sqrt{|\mathrm{G}_l|}\begin{array}{l}\big[(G^{11}G^{22} - G^{21}G^{12})\frac{\partial y}{\partial U}\frac{\partial z}{\partial V} + (G^{11}G^{23} - G^{21}G^{13})\frac{\partial y}{\partial U}\frac{\partial z}{\partial W} + \\ +(G^{12}G^{21} - G^{22}G^{11})\frac{\partial y}{\partial V}\frac{\partial z}{\partial U} + (G^{12}G^{23} - G^{22}G^{13})\frac{\partial y}{\partial V}\frac{\partial z}{\partial W} + \\ +(G^{13}G^{21} - G^{23}G^{11})\frac{\partial y}{\partial W}\frac{\partial z}{\partial U} + (G^{13}G^{22} - G^{23}G^{12})\frac{\partial y}{\partial W}\frac{\partial z}{\partial V}\big]\,,\end{array} \tag{2.43}$$

subject to

$$\begin{aligned}
G^{11} &= \frac{1}{|\mathrm{G}_l|}(G_{22}G_{33} - G_{23}G_{32}), \quad G^{12} = \frac{1}{|\mathrm{G}_l|}(G_{13}G_{32} - G_{12}G_{33}),\\
G^{13} &= \frac{1}{|\mathrm{G}_l|}(G_{12}G_{23} - G_{13}G_{22}), \quad G^{22} = \frac{1}{|\mathrm{G}_l|}(G_{11}G_{33} - G_{13}G_{31}),\\
G^{23} &= \frac{1}{|\mathrm{G}_l|}(G_{12}G_{31} - G_{11}G_{32}), \quad G^{33} = \frac{1}{|\mathrm{G}_l|}(G_{11}G_{22} - G_{12}G_{21}).
\end{aligned} \tag{2.44}$$

Formulation (iv):

$$
\begin{aligned}
\frac{\partial x}{\partial U} &= \frac{1}{\sqrt{|\mathrm{G}_l|}}\left[G_{11}\left(\frac{\partial y}{\partial V}\frac{\partial y}{\partial W}-\frac{\partial y}{\partial W}\frac{\partial y}{\partial V}\right)+G_{12}\left(\frac{\partial y}{\partial W}\frac{\partial z}{\partial U}-\frac{\partial y}{\partial U}\frac{\partial z}{\partial W}\right)\right.\\
&\qquad\left.+G_{13}\left(\frac{\partial y}{\partial U}\frac{\partial z}{\partial V}-\frac{\partial y}{\partial V}\frac{\partial z}{\partial U}\right)\right],\\
\frac{\partial x}{\partial V} &= \frac{1}{\sqrt{|\mathrm{G}_l|}}\left[G_{12}\left(\frac{\partial y}{\partial V}\frac{\partial z}{\partial W}-\frac{\partial y}{\partial W}\frac{\partial z}{\partial V}\right)+G_{22}\left(\frac{\partial y}{\partial W}\frac{\partial z}{\partial U}-\frac{\partial y}{\partial U}\frac{\partial z}{\partial W}\right)\right.\\
&\qquad\left.+G_{23}\left(\frac{\partial y}{\partial U}\frac{\partial z}{\partial V}-\frac{\partial y}{\partial V}\frac{\partial z}{\partial U}\right)\right],\\
\frac{\partial x}{\partial W} &= \frac{1}{\sqrt{|\mathrm{G}_l|}}\left[G_{13}\left(\frac{\partial y}{\partial V}\frac{\partial z}{\partial W}-\frac{\partial y}{\partial W}\frac{\partial z}{\partial V}\right)+G_{23}\left(\frac{\partial y}{\partial W}\frac{\partial z}{\partial U}-\frac{\partial y}{\partial U}\frac{\partial z}{\partial W}\right)\right.\\
&\qquad\left.+G_{33}\left(\frac{\partial y}{\partial U}\frac{\partial z}{\partial V}-\frac{\partial y}{\partial V}\frac{\partial z}{\partial U}\right)\right],
\end{aligned}
\tag{2.45}
$$

subject to the integrability conditions $\frac{\partial^2 x}{\partial U \partial V}=\frac{\partial^2 x}{\partial V \partial U}$, $\frac{\partial^2 x}{\partial U \partial W}=\frac{\partial x^2}{\partial W \partial U}$, $\frac{\partial^2 x}{\partial V \partial W}=\frac{\partial^2 x}{\partial W \partial V}$.

End of Lemma.

Question.

Question: "Why did we bother you with the three-dimensional conformal mapping of a three-dimensional Riemann manifold to a three-dimensional Euclidean manifold?" Answer: "One of the main reasons is the inability of the theory of complex manifolds to work conformally with odd-dimensional real manifolds. Only even-dimensional real manifolds $\mathbb{M}^{2n}(\mathbb{R})$ can be transformed to complex manifolds $\mathbb{M}^n(\mathbb{C})$".

Finally, Lemma 2.6 presents the partial differential equations of a conformeomorphism if it exists from a left n-dimensional (pseudo-)Riemann manifold $\mathbb{M}_l^n$ of signature l to a right n-dimensional (pseudo-)Riemann manifold $\mathbb{M}_\mathrm{r}^n = \mathrm{E}^n$ of signature r.

Lemma 2.6 (Grafarend and Syffus (1998d, p. 293), conformeomorphism).

Equivalent formulations of the equations producing a conformal mapping $\mathbb{M}_l^n \to \mathbb{M}_\mathrm{r}^n = \mathbb{E}^n$ are provided by the following formulations.

Formulation (i):

$$\mathrm{d}x^1 = *(\mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n). \tag{2.46}$$

Formulation (ii):

$$
\begin{gathered}
\forall L,\ M_1,\ \ldots,\ M_p,\ N_1,\ \ldots,\ N_p \in \{1,\ \ldots,\ n\}\\
(p = n\ -1):\\
\frac{\partial x}{\partial U^L} = \frac{1}{p!} e_{LM_1\ldots M_p}\sqrt{\det[\mathrm{G_l}]}\,G^{M_1N_1}\ldots G^{M_pN_p}\frac{\partial x^2}{\partial U^{N_1}}\ldots\frac{\partial x^n}{\partial U^{N_p}},
\end{gathered}
\tag{2.47}
$$

subject to the integrability conditions

$$\frac{\partial^2 x^1}{\partial U^L \partial U^N} = \frac{\partial^2 x^1}{\partial U^N \partial U^L}. \tag{2.48}$$

End of Lemma.

2-4 The Equivalence Theorem for Equiareal Mappings

The equivalence theorem for equiareal mappings from the left two-dimensional Riemann manifold to the right two-dimensional Euclidean manifold (areomorphism), Mollweide projection of the ellipsoid-of-revolution, principal stretches.

The previous equivalence theorem for an areomorphism is specialized for the case of the two-dimensional right Euclidean manifold $\{\mathbb{M}_\mathrm{r}^2,\ g_{\mu\nu}\} = \{\mathbb{R}^2,\ \delta_{\mu\nu}\} =: \mathbb{E}^2$. In many applications, the choice of $\{\mathbb{R}^2,\ \delta_{\mu\nu}\}$ is the planar manifold, for instance, the tangent space $T_{\boldsymbol{U}_0}\mathbb{M}_l^2$ of the left two-dimensional Riemann manifold fixed to the point $\boldsymbol{U}_0 = \{U_0^1,\ U_0^2\}$, being covered by Cartesian or polar coordinates. For an illustration of such a setup of a "planar manifold", go back to our previous examples. Here, we focus on the equivalence theorem, namely the differential equations which govern an equiareal mapping $\mathbb{M}_l^2 \to \{\mathbb{R}^2,\ \delta_{\mu\nu}\}$.

Theorem 2.7 (Areomorphism, $\mathbb{M}_l^2 \to \{\mathbb{R}^2, \delta_{\mu\nu}\}$, equiareal mapping).

Let $\overline{f} : \mathbb{M}_\mathrm{r}^2 := \{\mathbb{R}^2,\ \delta_{\mu\nu}\} =: \mathbb{E}^2$ be an orientation preserving equiareal mapping. Then the following conditions are equivalent.

Condition (i):

$$\sqrt{\det[G_l]}\mathrm{d}U \wedge \mathrm{d}V = \mathrm{d}u \wedge \mathrm{d}v. \tag{2.49}$$

Condition (ii):

$$\begin{gathered}\det[\mathrm{C}_l] = 1 \text{ and } \det[\mathrm{C}_l\mathrm{G}_l^{-1}] = 1,\\ \det[\mathrm{I}_2 - 2\mathrm{E}_\mathrm{r}] = 1 \text{ and } \det[2\mathrm{E}_l + \mathrm{G}_l] = \det[\mathrm{G}_l].\end{gathered} \tag{2.50}$$

Condition (iii):

$$\Lambda_1\Lambda_2 = 1 \text{ and } \lambda_1\lambda_2 = 1. \tag{2.51}$$

Condition (iv):

$$\begin{aligned}U_uV_u - U_vV_u &= 1/\sqrt{\det[\mathrm{G}_l]} =\\ &= 1/\sqrt{G_{11}G_{22} - G_{12}^2},\\ u_Uv_V - u_Vv_U &= \sqrt{\det[\mathrm{G}_l]} =\\ &= \sqrt{G_{11}G_{22} - G_{12}^2}.\end{aligned} \tag{2.52}$$

End of Theorem.

Here, we only have specialized Theorem 1.14 to $\mathbb{M}^2_{\mathrm{r}} := \{\mathbb{R}^2,\ \delta_{\mu\nu}\} =: \mathbb{E}^2$. One of the most popular equiareal mappings $\mathbb{E}^2_{A_1,A_1,A_2} \to \{\mathbb{R}^2,\ \delta_{\mu\nu}\}$ is the *Mollweide projection of the ellipsoid-of-revolution* to the plane, which is presented in Example 2.2 and is illustrated in Fig. 2.1.

Example 2.2 (Mollweide projection of the ellipsoid-of-revolution, with reference to Grafarend et al. (1995a)).

Let us assume that we have found a solution of the right characteristic equation, which generates an equiareal mapping of the ellipsoid-of-revolution $\mathbb{E}^2_{A_1,A_1,A_2}$ parameterized by the two coordinates $\{\Lambda,\ \Phi\}$ (called {*Gauss surface normal longitude, Gauss surface normal latitude*}) as outlined in Box 2.3, also called *generalized Mollweide projection*. Such a generalized Mollweide projection is classified as "pseudo-cylindric" and equiareal, mapping the circular equator equidistantly. Its mapping equations $x(\Lambda,\ \Phi)$ and $y(\Phi)$, where $\{x, y\}$ are Cartesian coordinates that cover $\{\mathbb{R}^2,\ \delta_{\mu\nu}\} = \mathbb{E}^2$, depend on $\cos t(\Phi)$ and $\sin t(\Phi)$. The auxiliary function $t(\Phi)$ is a solution of the *generalized Kepler equation* since for relative eccentricity $E^2 = (A_1^2 - A_2^2)/A_1^2 \to 0$ the generalized Kepler equations reduces to the Kepler equation. Such a Kepler equation is known from the *classical Mollweide projection* of the sphere or from solving the Kepler two-body problem in mechanics.

End of Example.

We pose two problems. (i) Prove that the generalized Mollweide projection of the ellipsoid-of-revolution is equiareal. For this purpose, observe the postulate $\det[C_l G_l^{-1}] = 1$. (ii) Determine the left principal stretches Λ_1 and Λ_2 by setting up the characteristic equations of the left eigenvalue problem that is presented in Box 2.4.

Solution (the first problem).

Here, we set up the test of an equiareal mapping to be based upon the postulate $\det[\mathrm{C}_l \mathrm{G}_l^{-1}] = 1$. First, by means of Box 2.5, we compute the left Jacobi matrix substituted by $D_\Lambda x, D_\Phi x,\ D_\Lambda y$, and $D_\Phi y$. Second, we set up the left Cauchy–Green matrix $\mathrm{C}_l = \mathrm{J}^* \mathrm{G}_r \mathrm{J}_l$ subject to $\mathrm{G}_\mathrm{r} = \mathrm{I}_2$. We have to emphasize that C_l is not a diagonal matrix. Third, we adopt the left matrix of the metric G_l. Fourth, given the left Cauchy–Green matrix, C_l, and the left matrix of the metric, G_l, we derive the determinantal identity $\det[\mathrm{C}_l \mathrm{G}_l^{-1}] = 1$. By means of *implicit differentiation of the generalized Kepler equation*, we compute (t'), $(t')^2$, $(t')^2 \cos^4 t, a^2 b^2$ and $1/G_{11}G_{22}$ in step five. Sixth, taking all individual terms into one, we have proven $\det[\mathrm{C}_l \mathrm{G}_l^{-1}] = 1$.

End of Solution (the first problem).

Solution (the second problem).

First, we set up the characteristic equations of the left general eigenvalue problem of Box 2.4 in order to compute the left principal stretches Λ_1 and Λ_2, respectively. Second, the solution of the left characteristic equation subject to the condition of an equiareal mapping, namely $\det[\mathrm{C}_l \mathrm{G}_l^{-1}] = 1$, accounts for computing the first left invariant $\mathrm{tr}\,[\mathrm{C}_l \mathrm{G}_l^{-1}]$. Indeed, a simple form of such an invariant is not available. Accordingly, we left $\mathrm{tr}[\mathrm{C}_l \mathrm{G}_l^{-1}]$ with a formula for $(t')^2$ and $1/G_{11}G_{22}$, respectively.

End of Solution (the second problem).

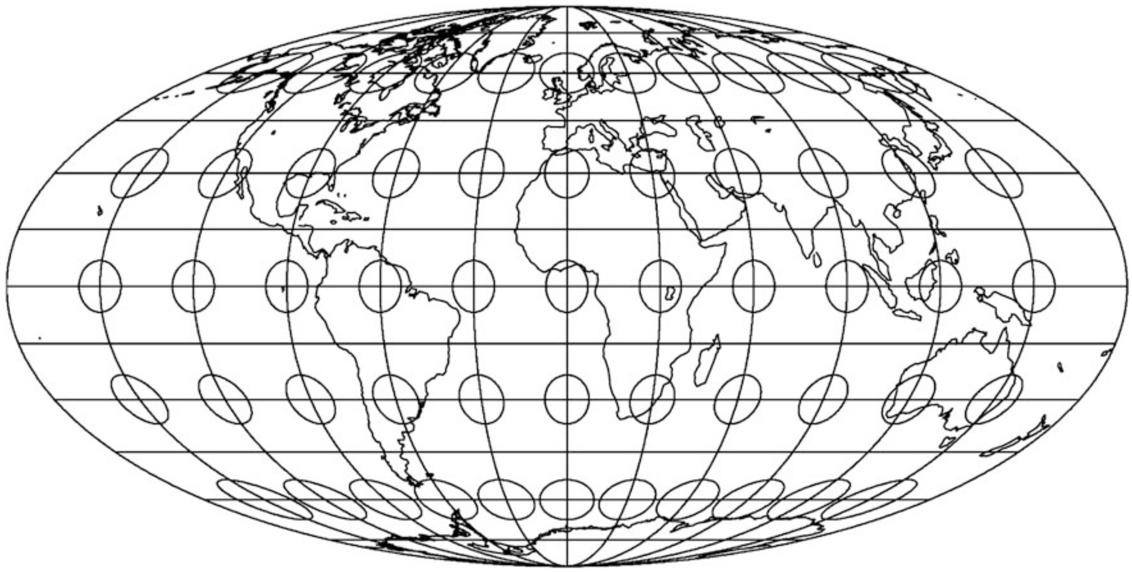

Fig. 2.1. Mollweide projection of an ellipsoid-of-revolution, Grafarend et al. (1995a)

Box 2.3 (The Mollweide projection of $\mathbb{E}_{A_1,A_1,A_2}$; the pseudo-cylindric, equiareal, equidistant mapping of the circular equator).

Mapping equations:

$$x(\Lambda,\ \Phi) = a\Lambda\ \cos t(\Phi),$$
$$y(\Lambda,\ \Phi) = b\sin t(\Phi). \tag{2.53}$$

Generalized Kepler equations:

$$2t + \sin 2t = \pi \frac{\ln \frac{1+E\sin\Phi}{1-E\sin\phi} + \frac{2E\sin\Phi}{1-E^2\sin^2\Phi}}{\ln\frac{1+E}{1-E} + \ln\frac{2E}{1-E^2}}. \tag{2.54}$$

Scales:

$$a\ = A_1,$$
$$b = \frac{A_1(1-E^2)}{\pi E}\left(\ln\frac{1+E}{1-E} + \frac{2E}{1-E^2}\right). \tag{2.55}$$

Box 2.4 ([The left principal stretches, the left eigenvalues, and the generalized Mollweide projection of the ellipsoid-of-revolution).

Characteristic equation of the left general eigenvalue problem:

$$\Lambda^4 - \mathrm{tr}\,[\mathrm{C}_l\mathrm{G}_l^{-1}]\Lambda^2 + \det[\mathrm{C}_l\mathrm{G}_l^{-1}] = 0 \quad \text{subject to} \quad \det[\mathrm{C}_l\mathrm{G}_l^{-1}] = 1 \tag{2.56}$$

$$\Lambda_{1,2}^2 = \frac{1}{2}\left[\mathrm{tr}\,[\mathrm{C}_l\mathrm{G}_l^{-1}] \pm \sqrt{\left(\mathrm{tr}\,[\,\mathrm{C}_l\mathrm{G}_l^{-1}]\right)^2 - 4}\right]. \tag{2.57}$$

Computation of the first invariant $\mathrm{tr}\,[\mathrm{C}_l\mathrm{G}_l^{-1}]$:

$$\mathrm{tr}\,[\mathrm{C}_l\mathrm{G}_l^{-1}] = \frac{a^2\cos^2 t}{G_{11}} + (t')^2\frac{a^2\Lambda^2\sin^2 t + b^2\cos^2 t}{G_{22}},$$

$$\cos^4 t(t')^2 = \frac{E^2\cos^2\Phi}{(1-E^2\sin^2\Phi)^4}\frac{\pi^2}{\left(\ln\frac{1+E}{1-E} + \frac{2E}{1-E^2}\right)^2},$$

$$\mathrm{tr}\,[\mathrm{C}_l\mathrm{G}_l^{-1}] = \frac{1}{G_{11}G_{22}}[a^2G_{22}\cos^2 t + (t')^2G_{11}(a^2\Lambda^2\sin^2 t + b^2\cos^2 t)], \tag{2.58}$$

$$\frac{1}{G_{11}G_{22}} = \frac{(1-E^2\sin^2\Phi)^4}{A_1^4(1-E^2)^2\cos^2\Phi},$$

$$G_{11} = \frac{A_1^2\cos^2\Phi}{1-E^2\sin^2\Phi},$$

$$G_{22} = \frac{A_1^2(1-E^2)^2}{(1-E^2\sin^2\Phi)^3}.$$

Box 2.5 (Left Cauchy–Green matrix, generalized Mollweide projection of the ellipsoid-of-revolution).

Left Jacobi matrix:

$$\mathrm{J}_l := \begin{bmatrix} D_\Lambda x \; D_\Phi x \\ D_\Lambda y \; D_\Phi y \end{bmatrix}, \tag{2.59}$$

$$D_\Lambda x = a\ \cos t, \quad D_\Phi x = D_t x D_\Phi t = -a\Lambda\sin t\, t',$$

$$D_{\Lambda y} = 0, \quad D_{\Phi y} = D_t y D_\Phi t = +b\cos t\, t'.$$

Left Cauchy–Green matrix:

$$\mathrm{C}_l := \mathrm{J}_l^*\mathrm{G}_\mathrm{r}\mathrm{J}_l, \quad \mathrm{G}_\mathrm{r} = \mathrm{I}_2 \Rightarrow \mathrm{C}_l = \mathrm{J}_l^*\mathrm{J}_l,$$

$$\mathrm{C}_l = \begin{bmatrix} a^2\cos^2 t & -a\Lambda^2\cos t\sin t\, t' \\ -a\Lambda^2\cos t\sin t t' & (a^2\Lambda^2\sin^2 t + b^2\cos^2 t)(t')^2 \end{bmatrix}. \tag{2.60}$$

Left matrix of the metric:

$$\mathrm{G}_l = \begin{bmatrix} N^2(\Phi) & 0 \\ 0 & M^2(\Phi) \end{bmatrix} \quad (N(\Phi)\,\text{and}\,M(\Phi):\ \text{see Example 1.3}). \tag{2.61}$$

$$\det\,[\mathrm{C}_l\mathrm{G}_l^{-1}] = 1:$$

$$\det\ [\mathrm{C}_l\mathrm{G}_l^{-}1] = \frac{a^2\cos^2 t}{G_{11}}\frac{a^2\Lambda^2\sin^2 t + b^2\cos^2 t}{G_{22}}(t')^2 - \frac{a^4\Lambda^2\cos^2 t\sin^2 t}{G_{11}G_{22}}(t')^2 = \tag{2.62}$$

$$= \frac{\cos^4 t}{G_{11}G_{22}}a^2b^2(t')^2.$$

$$(t'):$$

$$2(1 + \cos 2t)\mathrm{d}t = \frac{\pi}{\ln \frac{1+E}{1-E} + \frac{2E}{1-E^2}} \times$$

$$\times \left[\frac{1 - E\sin\Phi}{1 + E\sin\Phi} \frac{E\cos\Phi}{1 - E\sin\Phi} + \frac{E\cos\Phi(1 + E\sin\Phi)}{(1 - E\sin\Phi)^2} \right)$$

$$+ \frac{2E\cos\Phi}{1 - E^2\sin^2\Phi} + \frac{4E^3\sin^2\Phi\cos\Phi}{(1 - E^2\sin^2\Phi)^2} \right] \mathrm{d}\Phi,$$

$$1 + \cos 2t = 2\cos^2 t, \tag{2.63}$$

$$\cos^2 t(t') = \frac{E\cos\Phi}{(1 - E^2\sin^2\Phi)^2} \frac{\pi}{\ln \frac{1+E}{1-E} + \frac{2E}{1-E^2}},$$

$$\cos^4 t(t')^2 = \frac{E^2\cos^2\Phi}{(1 - E^2\sin^2\Phi)^4} \frac{\pi^2}{\left(\ln \frac{1+E}{1-E} + \frac{2E}{1-E^2}\right)^2},$$

$$\frac{1}{G_{11}G_{22}} = \frac{(1 - E^2\sin^2\Phi)^4}{A_1^2\cos^2\Phi A_1^2(1-E)^2}, \quad a^2b^2 = \frac{A_1^4(1-E^2)^2}{\pi^2 E^2}\left(\ln\frac{1+E}{1-E} + \frac{2E}{1-E^2}\right)^2.$$

(6th) Determinantal identity:

$$\det[\mathrm{C}_l \mathrm{G}_l^{-1}] = 1. \tag{2.64}$$

2-5 Canonical Criteria for Conformal, Equiareal, and Other Mappings

Canonical criteria for conformal, equiareal, and isometric mappings as well as equidistant mappings $\mathbb{M}_l^2 \to \{\mathbb{R}^2, \delta_{\mu\nu}\}$, Hilbert invariants.

Question.

Question: "How can we generalize those canonical criteria for a conformal, an equiareal, or an isometric mapping $\mathbb{M}_l^2 \to \mathbb{M}_\mathrm{r}^2 := \{\mathbb{R}^2, \delta_{\mu\nu}\} = \mathbb{E}^2$ if we restrict the right two-dimensional Riemann manifold to be two-dimensional Euclidean?" Answer: "Let us refer to Boxes 1.46 and 1.47 in order to formulate the answer. As it is outlined in Box 2.6, the fundamental four Hilbert invariants I_1 and I_2 or i_1 and i_2 become dependent, typically called 'syzygetic', as soon as we are dealing with a conformal mapping $\mathbb{M}_l^2 \to \{\mathbb{R}^2, \delta_{\mu\nu}\}$."

Box 2.6 (Canonical representation of Hilbert invariants, $\mathbb{M}_l^2 \to \{\mathbb{R}^2,\ \delta_{\mu\nu}\}$).

$$\begin{aligned} I_1(\mathrm{C}_l) &:= \Lambda_1^2 + \Lambda_2^2 = \mathrm{tr}\,[\mathrm{C}_l\mathrm{G}_l^{-1}] \quad \text{versus} \quad i_1(\mathrm{C}_\mathrm{r}) := \lambda_1^2 + \lambda_2^2 = \mathrm{tr}\,[\mathrm{C}_\mathrm{r}],\\ I_2(\mathrm{C}_l) &:= \Lambda_1^2\Lambda_2^2 = \det\,[\mathrm{C}_l\mathrm{G}_l^{-1}] \quad \text{versus} \quad i_2(\mathrm{C}_\mathrm{r}) := \lambda_1^2\lambda_2^2 = \det\,[\mathrm{C}_\mathrm{r}], \end{aligned} \tag{2.65}$$

or

$$\begin{aligned} I_1(\mathrm{E}_l) &:= K_1 + K_2 = \mathrm{tr}\,[\mathrm{E}_l\mathrm{G}_l^{-1}] \quad \text{versus} \quad i_1(\mathrm{E}_\mathrm{r}) := \kappa_1 + \kappa_2 = \mathrm{tr}\,[\mathrm{E}_\mathrm{r}],\\ I_2(\mathrm{E}_l) &:= K_1K_2 = \det\,[\mathrm{E}_l\mathrm{G}_l^{-1}] \quad \text{versus} \quad i_2(\mathrm{E}_\mathrm{r}) := \kappa_1\kappa_2 = \det\,[\mathrm{E}_\mathrm{r}]. \end{aligned} \tag{2.66}$$

Special case: conformal mapping (syzygy).

$$I_1 = 2\sqrt{I_2} \quad \text{versus} \quad i_1 = 2\sqrt{i_2}. \tag{2.67}$$

Note that for a general diffeomorphism, namely $\overline{f} : \{\mathbb{M}^2,\ G_{MN}\} \to \{\mathbb{R}^2,\ \delta_{\mu\nu}\}$, the first two Hilbert invariants $I_1(\mathrm{E}_l)$ and $i_1(\mathrm{E}_\mathrm{r})$ are also called *left* and *right dilatation*. They measure the *isotropic part* of a deformation, while the following *shear components* its *anisotropic part*:

$$\begin{aligned} \Gamma_1(\mathrm{C}_l) &:= C_{22} - C_{11} \quad \text{versus} \quad \gamma_1(\mathrm{C}_\mathrm{r}) := c_{22} - c_{11},\\ \Gamma_1(\mathrm{E}_l) &:= E_{22} - E_{11} \quad \text{versus} \quad \gamma_1(\mathrm{E}_\mathrm{r}) := e_{22} - e_{11},\\ \Gamma_2(\mathrm{C}_l) &:= 2C_{12} \quad \text{versus} \quad \gamma_2(\mathrm{C}_\mathrm{r}) := 2c_{12},\\ \Gamma_2(\mathrm{E}_l) &:= 2E_{12} \quad \text{versus} \quad \gamma_2(\mathrm{E}_\mathrm{r}) := 2e_{12}. \end{aligned} \tag{2.68}$$

2-6 Polar Decomposition and Simultaneous Diagonalization of Three Matrices

Polar decomposition and simultaneous diagonalization of three matrices: $\{\mathrm{E}_l,\ \mathrm{C}_l,\ \mathrm{G}_l\}$ versus $\{\mathrm{E}_\mathrm{r},\ \mathrm{C}_\mathrm{r},\ \mathrm{G}_\mathrm{r}\}$, stretch matrices.

A first remark has to be made towards the group theoretical representation of the left F_l and the right F_r matrix of eigenvectors. In case of $\{\mathbb{M}_\mathrm{r}^2, g_{\mu\nu}\} = \{\mathbb{R}^2,\ \delta_{\mu\nu}\}$, we took advantage of the fact that the right matrix F_r of eigenvectors is an orthonormal matrix R. In the general case $\{\mathbb{M}_l^2,\ G_{MN}\} = \{\mathbb{M}_\mathrm{r}^2,\ g_{\mu\nu}\}$, the left F_l and right the F_r matrix of eigenvectors enjoy the polar decomposition

$$\begin{array}{ccc} \mathrm{F}_l = \mathrm{R}_1\mathrm{S}_1 & \text{versus} & \mathrm{F}_\mathrm{r} = \mathrm{R}_3\mathrm{S}_3 \\ \text{versus} & & \text{versus} \\ \mathrm{F}_l = \mathrm{S}_2\mathrm{R}_2 & \text{versus} & \mathrm{F}_\mathrm{r} = \mathrm{S}_4\mathrm{R}_4 \end{array}, \tag{2.69}$$

where the matrices R_i are orthonormal, $\mathrm{R}_i^{-1} = \mathrm{R}_i^\mathrm{T}$, while the matrices S_i are by definition symmetric, $\mathrm{S}_i = \mathrm{S}_\mathrm{i}^\mathrm{T}$. These symmetric matrices S_i are sometimes called *stretch matrices*. or more details including numerical examples, we refer to Marsden and Hughes (1983, pp. 51–55), Ogden (1984, pp. 92–94), Simo and Taylor (1991), and Ting (1985). Here, we conclude with a second remark relating again to the simultaneous diagonalization of two matrices, e.g. the pairs of Cauchy–Green deformation tensors $\{C_l, G_l\}$ or $\{C_r, G_r\}$ and the pairs of Euler–Lagrange deformation tensors $\{E_l, G_l\}$ or $\{E_r, G_r\}$. respectively. Of course, we could also aim at a simultaneous diagonalization of three matrices, e.g. the triplets

$$\{E_l, C_l, G_l\} \quad \text{versus} \quad \{E_r, C_r, G_r\}, \tag{2.70}$$

in particular

$$U_l^T G_l X_l = S_l^1 \Leftrightarrow G_l = U_l S_l^1 X_i^{-1} \quad \text{versus} \quad G_r = U_r S_r^1 X_r^{-1} \Leftrightarrow U_r^T G_r X_r = S_r^1, \tag{2.71}$$

$$X_l^T C_l Y_l = S_l^2 \Leftrightarrow C_l = (X_l^{-1})^T S_l^2 Y_l^{-1} \text{ versus } C_r = (X_r^{-1})^T S_r^2 Y_r^{-1} \Leftrightarrow X_r^T C_r Y_r = S_r^2, \tag{2.72}$$

$$Y_l^T E_l V_l = S_l^3 \Leftrightarrow E_l = (Y_l^{-1})^T S_l^3 V_l^T \text{ versus } E_r = (Y_r^{-1})^T S_r^3 V_r^T \Leftrightarrow Y_r^T E_r V_r = S_r^3, \tag{2.73}$$

where S^1, S^2, and S^3 are certain quasi-diagonal matrices, where V and U are unitary matrices. and non-singular matrices are X_l, Y_l and X_r, Y_r, respectively. But we are not able to diagonalize G_l and G_r, respectively, to unity. The diagonalization of G_l and G_r, respectively, to unit matrices is by all means recommendable since accordingly all other tensors, e.g. C_l and C_r, respectively, or E_l and E_r, alternatively, refer to unit vectors which span the local tangent space of $\mathbb{M}_l^2$ or $\mathbb{M}_r^2$, respectively. Before we proceed to the next chapter, let us here additionally note that a tree of generalization of the ordinary singular value decompositions has been developed by Chu (1991a,b), De Moor and Zha (1991), Zha (1991), and others to which we refer.

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Map Projections
Cartographic Information Systems
Grafarend, E.W.; You, R.-J.; Syffus, R.
2014, XXVI, 935 p. 286 illus., 3 illus. in color. In 2 volumes, not available separately., Hardcover
ISBN: 978-3-642-36493-8