

Chapter 2

Parameter Estimation for an i.i.d. Model

Оценивание параметров в модели с независимыми одинаково
распределёнными наблюдениями

Кадры, овладевшие техникой, решают всё!

*Personnels that became proficient in technique decide
everything!*

Joseph Stalin

Exercise 2.1 (Glivenko-Cantelli theorem). *Let F be the distribution function of a random variable X and let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from F . Define the edf as*

$$F_n(x) \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n \mathbf{1}(X_i \leq x).$$

Prove that

$$\sup_x |F_n(x) - F(x)| \xrightarrow{a.s.} 0, \quad n \rightarrow \infty$$

1. *If F is a continuous distribution function;*
2. *If F is a discrete distribution function.*

1. Consider first the case when the function F is continuous in y . Fix any integer N and define with $\varepsilon = 1/N$ the points $t_1 < t_2 < \dots < t_N = +\infty$ such that

$$F(t_j) - F(t_{j-1}) = \varepsilon \text{ for } j = 2, \dots, N. \quad (2.1)$$

For every j , by the law of large numbers: $F_n(t_j) \xrightarrow{a.s.} F(t_j)$. This implies that for some $n(\varepsilon)$, it holds for all $n \geq n(\varepsilon)$

$$|F_n(t_j) - F(t_j)| \leq \varepsilon, \quad j = 1, \dots, N. \quad (2.2)$$

$F(t)$ and $F_n(t)$ are nondecreasing functions. This implies that for every $t \in [t_{j-1}, t_j]$ it holds

$$F(t_{j-1}) \leq F(t) \leq F(t_j), \quad F_n(t_{j-1}) \leq F_n(t) \leq F_n(t_j). \quad (2.3)$$

Let us subtract the first inequality (2.3) from the second:

$$F_n(t_{j-1}) - F(t_j) \leq F_n(t) - F(t) \leq F_n(t_j) - F(t_{j-1}), \quad (2.4)$$

Let us continue with the right hand side using (2.1) and (2.2):

$$\begin{aligned} F_n(t) - F(t) &\leq F_n(t_j) - F(t_{j-1}) \\ &= \underbrace{\{F_n(t_j) - F(t_j)\}}_{\leq \varepsilon} + \underbrace{\{F(t_j) - F(t_{j-1})\}}_{=\varepsilon} \leq 2\varepsilon, \end{aligned}$$

In the same way (considering the left part of (2.4)), one can prove that

$$F_n(t) - F(t) \geq -2\varepsilon$$

So,

$$|F_n(t) - F(t)| \leq 2\varepsilon. \quad (2.5)$$

Thus for all $\varepsilon > 0$ there exists constant $n(\varepsilon) > 0$ such that for every $n > n(\varepsilon)$ the inequality (2.5) holds for all $t \in \mathbb{R}$.

2. By $T = \{t_m\}_{m=1}^{+\infty}$ we denote points of discontinuity of function $F(x)$. Of course, these points are also points of discontinuity of function $F_n(t)$ (for any n).

Let us fix some $\varepsilon > 0$ and let us construct some finite set $S(\varepsilon)$. We include in $S(\varepsilon)$ the following points:

- (a) Points such that at least one inequality fulfills:

$$F(t_m) - F(t_{m-1}) > \varepsilon \quad \text{or} \quad F(t_{m+1}) - F(t_m) > \varepsilon$$

(b) Continuous set of points such that

$$F(t_m) - F(t_{m-1}) < \varepsilon$$

Denote amount of elements in $S(\varepsilon)$ by M .

We know that $F_n(t) \rightarrow F(t)$ almost sure. In particular

$$F_n(t_m) \xrightarrow{a.s.} F(t_m), \quad \forall m \in S(\varepsilon).$$

By definition

$$\exists n_m(\varepsilon) \in \mathbb{N} : \quad \forall n > n_m(\varepsilon) \quad |F_n(t_m) - F(t_m)| < \varepsilon$$

Define $n(\varepsilon) \stackrel{\text{def}}{=} \max\{n_1(\varepsilon), \dots, n_M(\varepsilon)\}$. Then for all $t_m \in S(\varepsilon)$

$$\forall n > n(\varepsilon) \quad |F_n(t_m) - F(t_m)| < \varepsilon.$$

Let us prove that the inequality

$$\forall n > n(\varepsilon) \quad |F_n(t_m) - F(t_m)| < 2\varepsilon. \quad (2.6)$$

is also true for all points $t_m \notin S(\varepsilon)$. Fix some $t_m \notin S(\varepsilon)$ and find index s such that

$$F(t_{s-1}) \leq F(t_m) \leq F(t_s), \quad F_n(t_{s-1}) \leq F_n(t_m) \leq F_n(t_s).$$

Consider

$$\begin{aligned} F_n(t_m) - F(t_m) &\leq F_n(t_s) - F(t_{s-1}) \\ &= \underbrace{\{F_n(t_s) - F(t_s)\}}_{< \varepsilon} + \underbrace{\{F(t_s) - F(t_{s-1})\}}_{\leq \varepsilon} \leq 2\varepsilon, \end{aligned}$$

Similarly, one can prove that

$$F_n(t_m) - F(t_m) \geq -2\varepsilon$$

This means that

$$|F_n(t_m) - F(t_m)| \leq 2\varepsilon$$

So, (2.6) is true for all $t_m \in T$.

For all t there exists some point $t_m \in T$ such that

$$F_n(t) = F_n(t_m) \quad \text{and} \quad F(t) = F(t_m).$$

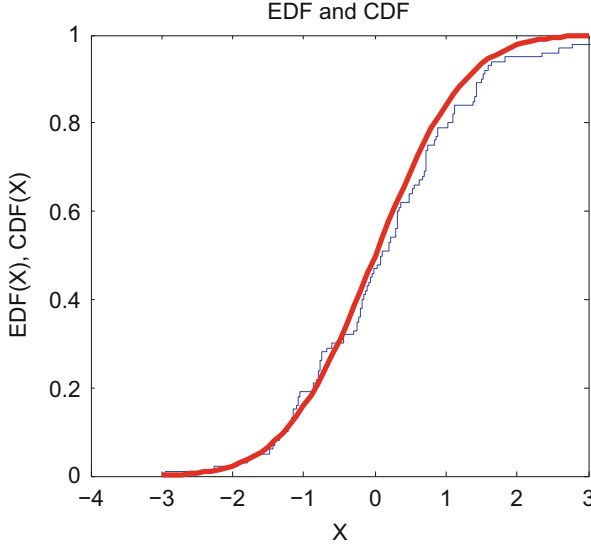


Fig. 2.1 The standard normal cdf (*thick line*) and the empirical distribution function (*thin line*) for $n = 100$.  MSEdnormal

Thus

$$\forall n > n(\varepsilon) \quad |F_n(t) - F(t)| < \varepsilon.$$

This observation completes the proof.

For an illustration of the asymptotic property, we draw $\{X_i\}_{i=1}^n$ i.i.d. samples from the standard normal distribution. Figure 2.1 shows the case of $n = 100$ and Fig. 2.2 shows the case of $n = 1,000$. The empirical cdf and theoretical cdf are close in the limit as n becomes larger.

Exercise 2.2 (Illustration of the Glivenko-Cantelli theorem). Denote by F the cdf of

1. Standard normal law,
2. Exponential law with parameter $\lambda = 1$.

Consider the sample $\{X_i\}_{i=1}^n$. Draw the plot of the empirical distribution function F_n and cumulative distribution function F . Find the index $i^* \in \{1, \dots, n\}$ such that

$$|F_n(X_{i^*}) - F(X_{i^*})| = \sup_i |F_n(X_i) - F(X_i)|.$$

The examples for the code can be found in the Quantnet. The readers are suggested to change the sample size n to compare the results (Figs. 2.3 and 2.4).

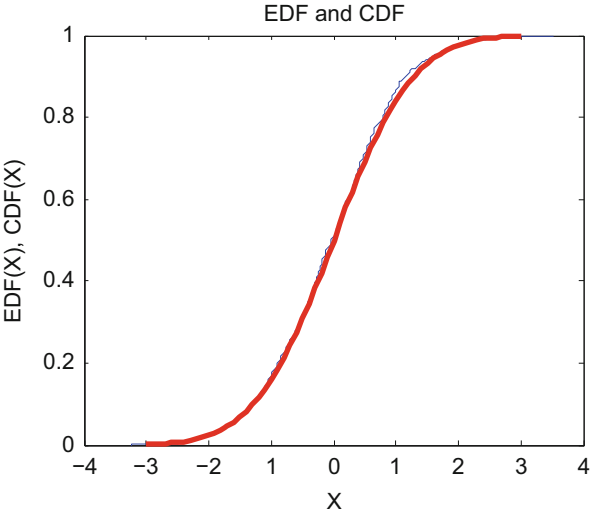


Fig. 2.2 The standard normal cdf (*thick line*) and the empirical distribution function (*thin line*) for $n = 1,000$.  MSEdnormal

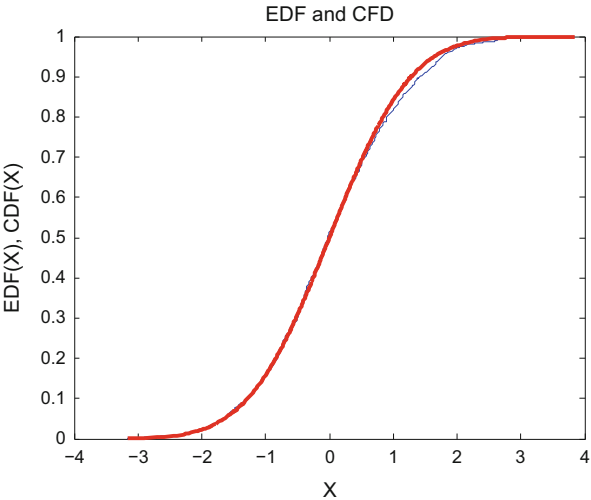


Fig. 2.3 The standard normal cdf (*thick line*) and the empirical distribution function (*thin line*) for $n = 1,000$. The maximal distance in this case occurs at $X_{i^*} = 1.0646$ where $i^* = 830$.  MSEGChmnorm

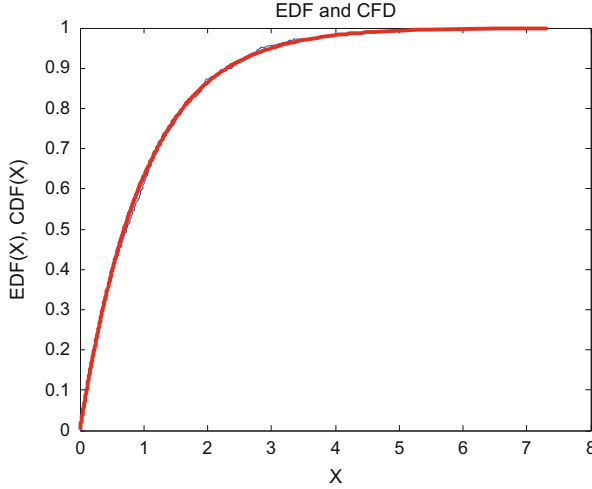


Fig. 2.4 The exponential ($\lambda = 1$) cdf (thick line) and the empirical distribution function (thin line) for $n = 1,000$. The maximal distance in this case occurs at $X_{i*} = 0.9184$ where $i* = 577$.
 MSEdnormal

Exercise 2.3. Compute the estimate of method of moments for the following parametric models:

1. Multinomial model:

$$\mathbb{P}_\theta(X = k) = \binom{m}{k} \theta^k (1 - \theta)^{m-k}, \quad k = 0, \dots, m.$$

2. Exponential model

$$\mathbb{P}_\theta(X > x) = e^{-x/\theta}.$$

In both cases one can follow the algorithm consisting of two steps:

- Calculate mathematical expectation $m(\theta) = \mathbb{E}_\theta X$;
- Solve the equation $m(\tilde{\theta}) = n^{-1} \sum_{i=1}^n X_i$; the solution is the required estimate.

Let us apply this:

1. Multinomial model, we first calculate expectation:

$$\begin{aligned} m(\theta) &= n^{-1} \sum_{i=1}^n X_i = \sum_{k=0}^m k \binom{m}{k} \theta^k (1 - \theta)^{m-k} \\ &= m \sum_{k=1}^m \binom{m-1}{k-1} \theta^k (1 - \theta)^{m-k} = m\theta. \end{aligned}$$

Secondly we solve the equation

$$m(\tilde{\theta}) = \frac{1}{n} \sum_{i=1}^n X_i;$$

which gives the solution:

$$\tilde{\theta} = \frac{1}{nm} \sum_{i=1}^n X_i.$$

2. Exponential family. Both items are trivial: $m(\theta) = \frac{1}{\theta}$ and $\tilde{\theta} = n \left(\sum_{i=1}^n X_i \right)^{-1}$.

Exercise 2.4. Let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from a distribution with Lebesgue density

$$f_{\theta}(x) = \frac{1}{2} (1 + \theta x) I_{[-1,1]}(x)$$

1. Find an estimator via the method of moments;
2. Find a consistent estimator.

Let us begin with calculation of the mathematical expectation:

$$\mathbb{E}_{\theta} X_1 = \frac{1}{2} \int_{-1}^1 (1 + \theta x) x \, dx = \frac{1}{3} \theta$$

Both items of the exercise follow immediately:

1. The estimator of method of moments is a solution of the equality

$$\mathbb{E}_{\tilde{\theta}} X_1 = n^{-1} \sum_{i=1}^n X_i$$

$$\text{So, } \tilde{\theta} = 3n^{-1} \sum_{i=1}^n X_i$$

2. By the law of large numbers,

$$n^{-1} \sum_{i=1}^n X_i \xrightarrow{a.s.} \mathbb{E} X_i = \frac{1}{3} \theta, \quad n \rightarrow \infty.$$

This means that

$$3n^{-1} \sum_{i=1}^n X_i \xrightarrow{a.s.} \theta, \quad n \rightarrow \infty,$$

hence the estimator $\hat{\theta} = 3n^{-1} \sum_{i=1}^n X_i$ is consistent.

Exercise 2.5. Consider the model

$$X_i = \theta^* + \varepsilon_i,$$

where θ^* is the parameter of interest and ε_i are independent normal errors $\mathcal{N}(0, \sigma_i^2)$.

Compute the MLE $\tilde{\theta}$ of the parameter θ^* and prove that this estimate has the following properties:

- (a) The estimate $\tilde{\theta}$ is unbiased: $\mathbb{E}_{\theta^*} \tilde{\theta} = \theta^*$.
(b) The quadratic risk of $\tilde{\theta}$ is equal to

$$\mathcal{R}(\tilde{\theta}, \theta^*) \stackrel{\text{def}}{=} \mathbb{E}_{\theta^*} |\tilde{\theta} - \theta^*|^2 = \left(\sum_{i=1}^n \sigma_i^2 \right)^{-1}.$$

The corresponding log-likelihood reads

$$L(\theta) = -\frac{1}{2} \sum_{i=1}^n \left\{ \log(2\pi\sigma_i^2) + \frac{(X_i - \theta)^2}{\sigma_i^2} \right\}.$$

The first derivative is equal to

$$\frac{\partial L(\theta)}{\partial \theta} = \sum_{i=1}^n \frac{X_i - \theta}{\sigma_i^2} = \sum_{i=1}^n \frac{X_i}{\sigma_i^2} - \theta \sum_{i=1}^n \frac{1}{\sigma_i^2}.$$

Then the MLE $\tilde{\theta}$ equals

$$\tilde{\theta} \stackrel{\text{def}}{=} \operatorname{argmax}_{\theta} L(\theta) = \frac{1}{N} \sum \frac{X_i}{\sigma_i^2},$$

where $N = \sum \sigma_i^{-2}$.

(a)

$$\mathbb{E}_{\theta^*} \tilde{\theta} = \frac{1}{N} \sum \frac{\mathbb{E} X_i}{\sigma_i^2} = \frac{1}{N} \sum \frac{\theta^*}{\sigma_i^2} = \frac{\theta^*}{N} \sum \frac{1}{\sigma_i^2} = \theta^*.$$

(b) The quadratic risk of $\tilde{\theta}$ is equal to the variance $\operatorname{Var}(\tilde{\theta})$:

$$\begin{aligned} \mathcal{R}(\tilde{\theta}, \theta^*) &\stackrel{\text{def}}{=} \mathbb{E}_{\theta^*} |\tilde{\theta} - \theta^*|^2 = \mathbb{E}_{\theta^*} \left| \frac{1}{N} \sum \frac{X_i}{\sigma_i^2} - \theta^* \right|^2 \\ &= \mathbb{E}_{\theta^*} \left| \frac{1}{N} \sum \frac{X_i}{\sigma_i^2} - \theta^* \frac{1}{N} \sum \frac{1}{\sigma_i^2} \right|^2 \\ &= \mathbb{E}_{\theta^*} \left| \frac{1}{N} \sum \frac{X_i - \theta^*}{\sigma_i^2} \right|^2 = \frac{1}{N^2} \sum \mathbb{E}_{\theta^*} \left| \frac{X_i - \theta^*}{\sigma_i^2} \right|^2. \end{aligned}$$

Note that random value $X_i - \theta^*$ has a normal distribution with zero mean and variance σ_i^2 . Then $(X_i - \theta^*)/\sigma_i^2 \sim \mathcal{N}(0, \sigma_i^{-2})$ and

$$\mathcal{R}(\tilde{\theta}^\circ, \theta^*) = \frac{1}{N^2} \sum \sigma_i^{-2} = \frac{1}{N}.$$

Exercise 2.6. Let $\{X_i\}_{i=1}^n$ be an i.i.d. sample with distribution that depends on some parameter θ . Let $\hat{\theta}_n$ be an estimate of parameter θ .

Assume that this estimate is root- n normal, i.e. there exists a function $\sigma(\theta)$ such that

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma(\theta)^2), \quad n \rightarrow \infty.$$

Prove that $\hat{\theta}_n$ is consistent,

$$\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$$

This fact can be briefly formulated as “root- n normality implies consistency”. We need Slutsky’s Theorem:

1. Let a_n (sequence of real numbers) be convergent in probability,

$$a_n \xrightarrow{\mathbb{P}} a, \quad n \rightarrow \infty$$

Let η_n (sequence of random variables) be convergent in distribution,

$$\eta_n \xrightarrow{\mathcal{L}} \text{Law}(\eta), \quad n \rightarrow \infty$$

Then

$$a_n \eta_n \xrightarrow{\mathcal{L}} \text{Law}(a\eta), \quad n \rightarrow \infty$$

2. Let ξ_n be a sequence of random variables that converges in law to the distribution that is degenerated in some point c (we denote this degenerated distribution by $\text{Law}(c)$). Then ξ_n also tends to c in probability.

Let us apply these observations to our situation. We use the first part of Slutsky’s Theorem with $a_n = \frac{1}{\sqrt{n}}$ and $\xi_n = \sqrt{n}(\hat{\theta}_n - \theta)$.

The sequence a_n tends to zero and the sequence ξ_n tends in probability to a normal distribution. So,

$$a_n \xi_n \xrightarrow{\mathcal{L}} \text{Law}(0)$$

According to the second part, this sequence also tends to zero in probability. Thus,

$$a_n \xi_n = \hat{\theta}_n - \theta \xrightarrow{\mathbb{P}} 0.$$

Remark 2.1. In fact our proof is true for any estimate that has an asymptotic distribution (not necessarily normal).

Exercise 2.7. Let F be the distribution function of a random variable X and let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from F . Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that

$$\sigma_g^2 \stackrel{\text{def}}{=} \text{Var}\{g(X)\} < \infty$$

Denote

$$s \stackrel{\text{def}}{=} \mathbb{E}g(X), \quad S_n \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n g(X_i)$$

1. Prove that

- (a) $S_n \xrightarrow{\mathbb{P}} s, \quad n \rightarrow \infty$
- (b) $\sqrt{n}(S_n - s) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_g^2), \quad n \rightarrow \infty.$

2. Let $h(z)$ be a twice continuously differentiable function on the real line such that $h'(s) \neq 0$ and $h''(s)$ is bounded in some neighborhood of s . Prove that

- (a) $h(S_n) \xrightarrow{\mathbb{P}} h(s)$
- (b) $\sqrt{n}\{h(S_n) - h(s)\} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_h^2), \quad n \rightarrow \infty,$
where $\sigma_h^2 \stackrel{\text{def}}{=} |h'(s)|^2 \sigma_g^2.$

1. (a) Note that $\{g(X_i)\}_{i=1}^n$ is a sample from the distribution with expectation equal to $\mathbb{E}g(X)$.

One can apply the law of large numbers for the sequence $\{g(X_i)\}_{i=1}^n$:

$$n^{-1} \sum_{i=1}^n g(X_i) \xrightarrow{\mathbb{P}} \mathbb{E}g(X) \quad n \rightarrow \infty.$$

(b) This statement directly follows by the CLT for i.i.d. random variables:

$$\frac{n^{-1} \sum_{i=1}^n g(X_i) - \mathbb{E}g(X)}{\sqrt{\frac{1}{n} \text{Var}\{g(X)\}}} \sim \mathcal{N}(0, 1)$$

In other words,

$$\sqrt{n} \left\{ n^{-1} \sum_{i=1}^n g(X_i) - \mathbb{E}g(X) \right\} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_g^2), \quad n \rightarrow \infty.$$

2. (a) We know:

$$S_n \xrightarrow{\mathbb{P}} s, \quad n \rightarrow \infty$$

Then for any continuous function g :

$$g(S_n) \xrightarrow{\mathbb{P}} g(s), \quad n \rightarrow \infty$$

(b) One can find a neighborhood U of the point s such that

- (i) S_n belongs with high probability to U ;
- (ii) $h''(s)$ is bounded in U .

Applying the Taylor expansion to h in this neighborhood U :

$$\sqrt{n} \{h(S_n) - h(s)\} = \sqrt{n} h'(s) (S_n - s) + \frac{\sqrt{n}}{2} h''(\tilde{s}) (S_n - s)^2, \quad (2.7)$$

where $\tilde{s}$ is some point between s and S_n . The right hand side of (2.7) is a sum of two random variables. First random variable $\sqrt{n} h'(s) (S_n - s)$ tends to $\mathcal{N}(0, |h'(s)|^2 \sigma_g^2)$ in distribution.

Let us show that the second component tends to zero in probability. Actually,

$$\left| \frac{\sqrt{n}}{2} h''(\tilde{s}) (S_n - s)^2 \right| \leq \frac{U}{2} \frac{1}{\sqrt{n}} \{ \sqrt{n} (S_n - s) \}^2,$$

where U is an upper bound for $h''(s)$ in the considering neighborhood. Expression in the right hand side is a product of the sequence $\frac{1}{\sqrt{n}}$, which tends to zero, and sequence $\{ \sqrt{n} (S_n - s) \}^2$, which converges in distribution. Then

$$\left| \frac{\sqrt{n}}{2} h''(\tilde{s}) (S_n - s)^2 \right| \xrightarrow{\mathbb{P}} 0$$

Thus, the right hand side in (2.7) (and left hand side also) tends to $\mathcal{N}(0, |h'(s)|^2 \sigma_g^2)$ in distribution.

Exercise 2.8. (Analogue of the Exercise 2.7 for multi-dimensional case) Let $\mathbf{g}(\cdot) = (g_1(\cdot), \dots, g_m(\cdot))^\top : \mathbb{R} \rightarrow \mathbb{R}^m$ be a function such that

$$\Sigma_{jk} \stackrel{\text{def}}{=} \mathbb{E}[g_j(X)g_k(X)] < \infty, \text{ for } j, k \leq m.$$

Denote

$$\begin{aligned} \mathbf{s} &= \mathbb{E}\mathbf{g}(X) = (\mathbb{E}g_1(X), \dots, \mathbb{E}g_m(X))^\top, \\ \mathbf{S}_n &= \frac{1}{n} \sum_{i=1}^n \mathbf{g}(X_i) = \left(\frac{1}{n} \sum_{i=1}^n g_1(X_i), \dots, \frac{1}{n} \sum_{i=1}^n g_m(X_i) \right)^\top. \end{aligned}$$

1. Prove that

$$\begin{aligned} (a) \quad \mathbf{S}_n &\xrightarrow{\mathbb{P}} \mathbf{s}, \quad n \rightarrow \infty \\ (b) \quad \sqrt{n}(\mathbf{S}_n - \mathbf{s}) &\xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma), \quad n \rightarrow \infty, \\ &\text{where } \Sigma = (\Sigma_{jk})_{j,k=1,\dots,m} \end{aligned}$$

2. Let $H(z) : \mathbb{R}^m \rightarrow \mathbb{R}$ be a twice continuously differentiable function such that $\nabla H(z)$ and $\|\nabla^2 H(z)\|$ is bounded in some neighborhood of $\mathbf{s}$. Prove that

$$\begin{aligned} (a) \quad H(\mathbf{S}_n) &\xrightarrow{\mathbb{P}} H(\mathbf{s}) \\ (b) \quad \sqrt{n}\{H(\mathbf{S}_n) - H(\mathbf{s})\} &\xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_H^2), \quad n \rightarrow \infty, \\ &\text{where } \sigma_H^2 \stackrel{\text{def}}{=} \nabla H(\mathbf{s})^\top \Sigma \nabla H(\mathbf{s}). \end{aligned}$$

First note that items 1a and 2a follow from items 1b and 2b correspondingly. Let us check items 1b and 2b.

Consider for every $\mathbf{v} = (v_1, \dots, v_m)^\top \in \mathbb{R}^m$ the scalar products $\mathbf{v}^\top \mathbf{g}(\cdot)$, $\mathbf{v}^\top \mathbf{s}$, $\mathbf{v}^\top \mathbf{S}_n$. For the statement 1b, it suffices to show that

$$\sqrt{n}\mathbf{v}^\top (\mathbf{S}_n - \mathbf{s}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \mathbf{v}^\top \Sigma \mathbf{v}), \quad n \rightarrow \infty.$$

Actually

$$\begin{aligned} \sqrt{n}\mathbf{v}^\top (\mathbf{S}_n - \mathbf{s}) &= \sqrt{n} \sum_j v_j \left\{ \frac{1}{n} \sum_i g_j(X_i) - \mathbb{E}g_j(X) \right\} \\ &= \sqrt{n} \left[\frac{1}{n} \sum_i \left\{ \sum_j v_j g_j(X_i) \right\} - \mathbb{E} \left\{ \sum_j v_j g_j(X) \right\} \right] \\ &= \sqrt{n} \left\{ \frac{1}{n} \sum_i G(X_i) - \mathbb{E}G(X) \right\}, \end{aligned}$$

where $G(\cdot) = \sum_j v_j g_j(\cdot) = \mathbf{v}^\top \mathbf{g}(\cdot)$.

Now one can apply result of the Exercise 2.7 (item 1b) for the function $G(\cdot)$ and obtain the required statement.

For the statement 2b, consider the Taylor expansion

$$\sqrt{n} \{H(\mathbf{S}_n) - H(s)\} = \sqrt{n} \nabla H(s)^\top (\mathbf{S}_n - s) + \frac{\sqrt{n}}{2} (\mathbf{S}_n - s)^\top \nabla^2 H(\tilde{s}) (\mathbf{S}_n - s).$$

This formula is an analogue of (2.7). One can continue the line of reasoning in the same way as in the proof of (2.7) (item 2b).

In fact,

$$\sqrt{n} \nabla H(s)^\top (\mathbf{S}_n - s) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nabla H(s)^\top \Sigma \nabla H(s)),$$

and

$$\left| \frac{\sqrt{n}}{2} (\mathbf{S}_n - s)^\top \nabla^2 H(\tilde{s}) (\mathbf{S}_n - s) \right| \leq \frac{1}{2\sqrt{n}} \|\sqrt{n} (\mathbf{S}_n - s)\|^2 \max_s \|\nabla^2 H(s)\|$$

$$\xrightarrow{\mathbb{P}} 0$$

These two observations conclude the proof.

Exercise 2.9.

1. Consider a sample $\{X_i\}_{i=1}^n$ from a distribution $P_{\theta^*} \in (P_\theta, \theta \in \Theta \in \mathbb{R})$. Let $\tilde{\theta}$ be an estimator of θ such that the bias

$$b(\tilde{\theta}, \theta^*) \stackrel{\text{def}}{=} \mathbb{E}_{\theta^*} \tilde{\theta} - \theta^*$$

and the variance $\text{Var}_{\theta^*}(\tilde{\theta})$ tend to zero as $n \rightarrow \infty$. Prove that $\tilde{\theta}$ is consistent.

2. Let $\{X_i\}_{i=1}^n$ be a sample from the uniform distribution on $[0, \theta]$. Using the first item of this exercise, prove that the estimator

$$\tilde{\theta}_1 = \max \{X_1, \dots, X_n\}$$

is consistent.

1. Applying the so called bias-variance decomposition, which is true for any estimate $\tilde{\theta}$:

$$\mathbb{E}_{\theta^*} (\tilde{\theta} - \theta^*)^2 = \text{Var}_{\theta^*}(\tilde{\theta}) + b^2(\tilde{\theta}, \theta^*). \quad (2.8)$$

Let us prove (2.8):

$$\begin{aligned}
 \mathbb{E}_{\theta^*} \left(\tilde{\theta} - \theta^* \right)^2 &= \mathbb{E}_{\theta^*} \left\{ \tilde{\theta} - \mathbb{E}(\tilde{\theta}) + \mathbb{E}(\tilde{\theta}) - \theta^* \right\}^2 \\
 &= \mathbb{E}_{\theta^*} \left\{ \tilde{\theta} - \mathbb{E}(\tilde{\theta}) + b(\tilde{\theta}, \theta^*) \right\}^2 \\
 &= \text{Var}_{\theta^*}(\tilde{\theta}) + 2b(\tilde{\theta}, \theta^*)\mathbb{E}_{\theta^*} \left(\tilde{\theta} - \mathbb{E}\tilde{\theta} \right) + b^2(\tilde{\theta}, \theta^*) \\
 &= \text{Var}_{\theta^*}(\tilde{\theta}) + b^2(\tilde{\theta}, \theta^*)
 \end{aligned}$$

If bias and variance tend to zero as $n \rightarrow \infty$, then

$$\mathbb{E}_{\theta^*} \left(\tilde{\theta} - \theta^* \right)^2 \rightarrow 0, \quad n \rightarrow \infty$$

This means that $\tilde{\theta}$ tends to θ^* in L_2 sense. Then $\tilde{\theta}$ also tends to θ^* in probability, i.e. $\tilde{\theta}$ is a consistent estimator.

2. First of all, let us calculate the cdf of $\tilde{\theta}_1$.

$$\begin{aligned}
 \mathbb{P}_{\theta^*} \left(\tilde{\theta}_1 \leq x \right) &= \mathbb{P}_{\theta^*} (X_1 \leq x, \dots, X_n \leq x) \\
 &= \{ \mathbb{P}_{\theta^*} (X_1 \leq x) \}^n = \left(\frac{x}{\theta^*} \right)^n, \quad x \in [0, \theta^*]
 \end{aligned}$$

Afterwards we can take the derivative and obtain the density function

$$p(x) = n(\theta^*)^{-n} x^{n-1} \mathbf{1}(0 \leq x \leq \theta^*)$$

For applying the first item, one has to calculate expectation and variance of $\tilde{\theta}_1$:

$$\mathbb{E}\tilde{\theta}_1 = \frac{n}{n+1}\theta^*, \quad \text{Var}(\tilde{\theta}_1) = \frac{n}{(n+1)^2(n+2)}\theta^{*2}$$

Now we are ready for applying the first item:

$$b(\tilde{\theta}_1, \theta^*) = \frac{n}{n+1}\theta^* - \theta^* = -\frac{1}{n+1}\theta^* \rightarrow 0, \quad n \rightarrow \infty.$$

$$\text{Var}_{\theta^*}(\tilde{\theta}) = \frac{n}{(n+1)^2(n+2)}\theta^{*2} \rightarrow 0, \quad n \rightarrow \infty.$$

So, assumptions are fulfilled. This concludes the proof.

Exercise 2.10. Check that the i.i.d. experiment from the uniform distribution on the interval $[0, \theta]$ with unknown θ is not regular.

First condition from the definition of the regular family is the following one: the sets $A(\theta) \stackrel{\text{def}}{=} \{y : p(y, \theta) = 0\}$ are the same for all $\theta \in \Theta$.

The uniform distribution on the interval $[0, \theta]$ doesn't satisfy this condition,

$$A(\theta) = (-\infty, 0) \cup (\theta, +\infty).$$

This exercise gives a local approximation of the Kullback-Leibler divergence.

Exercise 2.11. *Let (P_θ) be a regular family.*

1. *Show that the KL-divergence $\mathcal{K}(\theta, \theta')$ satisfies for any θ, θ' :*

(a)

$$\mathcal{K}(\theta, \theta') \Big|_{\theta'=\theta} = 0;$$

(b)

$$\frac{d}{d\theta'} \mathcal{K}(\theta, \theta') \Big|_{\theta'=\theta} = 0;$$

(c)

$$\frac{d^2}{d\theta'^2} \mathcal{K}(\theta, \theta') \Big|_{\theta'=\theta} = I(\theta).$$

2. *Show that in a small neighborhood of θ , the KL-divergence can be approximated by*

$$\mathcal{K}(\theta, \theta') \approx I(\theta) |\theta' - \theta|^2 / 2.$$

1. Note that

$$\mathcal{K}(\theta, \theta') = \mathbb{E}_\theta \log p(x, \theta) - \mathbb{E}_\theta \log p(x, \theta')$$

(a) First item is trivial.

(b)

$$\begin{aligned} \frac{d}{d\theta'} \mathcal{K}(\theta, \theta') &= -\frac{d}{d\theta'} \mathbb{E}_\theta \log p(x, \theta') \\ &= -\frac{d}{d\theta'} \int \log p(x, \theta') p(x, \theta) dx \\ &= -\int \frac{p'_{\theta'}(x, \theta')}{p(x, \theta')} p(x, \theta) dx, \end{aligned}$$

where $p'_{\theta'}(x, \theta') \stackrel{\text{def}}{=} \frac{d}{d\theta'} p(x, \theta')$. Substitution $\theta' = \theta$ gives

$$\begin{aligned} \frac{d}{d\theta'} \mathcal{K}(\theta, \theta') \Big|_{\theta'=\theta} &= - \int \frac{d}{d\theta'} \{p(x, \theta')\} dx \Big|_{\theta'=\theta} \\ &= - \frac{d}{d\theta'} \int p(x, \theta') dx \Big|_{\theta'=\theta} = 0. \end{aligned}$$

(c)

$$\begin{aligned} \frac{d^2}{d\theta'^2} \mathcal{K}(\theta, \theta') &= - \int \frac{d}{d\theta'} \left\{ \frac{p'_{\theta'}(x, \theta')}{p(x, \theta')} \right\} p(x, \theta) dx \\ &= - \int \left[\frac{p''_{\theta'}(x, \theta') p(x, \theta') - \{p'_{\theta'}(x, \theta')\}^2}{\{p(x, \theta')\}^2} \right] p(x, \theta) dx. \end{aligned}$$

Substitution $\theta' = \theta$ yields

$$\begin{aligned} \frac{d^2}{d\theta'^2} \mathcal{K}(\theta, \theta') \Big|_{\theta'=\theta} &= \underbrace{\int p''_{\theta'}(x, \theta') dx \Big|_{\theta'=\theta}}_{\frac{d^2}{d\theta'^2} \int p(x, \theta') dx \Big|_{\theta'=\theta} = 0} + \underbrace{\int \frac{\{p'_{\theta'}(x, \theta')\}^2}{p(x, \theta')} dx}_{=I(\theta)} = I(\theta). \end{aligned}$$

2. The required representation directly follows from the Taylor expansion at the point $\theta' = \theta$.

The following exercise

1. Illustrates two methods for checking the R-efficiency;
2. Shows that the Fisher information can depend on the parameter (for some parametric families), but can be a constant (for other parametric families).

Exercise 2.12. Consider two families:

- (a) the Gaussian shift (b) the Poisson family

1. Compute the Fisher Information for these families.
2. Check that the Cramér-Rao inequality for the empirical mean estimate $\tilde{\theta} = n^{-1} \sum_{i=1}^n X_i$ is in fact an equality, i.e.

$$\text{Var}_{\theta}(\tilde{\theta}) = n^{-1} I^{-1}(\theta).$$

3. Check R-efficiency of $\tilde{\theta}$

- (i) Using only the definition;
- (ii) Using the Theorem 2.6.3. of [Spokoiny and Dickhaus \(2014\)](#)

1. (a) Recall

$$p(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(x - \theta)^2}{2} \right\}.$$

Then

$$\begin{aligned} I(\theta) &= \mathbb{E}_\theta \left| \frac{\partial \log p(X, \theta)}{\partial \theta} \right|^2 = \mathbb{E}_\theta \left| \frac{\partial}{\partial \theta} \left\{ -\frac{(X - \theta)^2}{2} \right\} \right|^2 \\ &= \mathbb{E}_\theta |X - \theta|^2 \\ &= \mathbb{E}_\theta |X - \mathbb{E}_\theta X|^2 \\ &= \text{Var}(X) = 1. \end{aligned}$$

Therefore, the Fisher information is equal to 1 for any values of the parameter θ .

(b)

$$p(x, \theta) = \frac{\theta^x}{x!} e^{-\theta}, \quad x = 1, 2, \dots$$

$$\begin{aligned} I(\theta) &= \mathbb{E}_\theta \left| \frac{\partial \log p(X, \theta)}{\partial \theta} \right|^2 = \mathbb{E}_\theta \left| \frac{\partial}{\partial \theta} (X \log \theta - \log X! - \theta) \right|^2 \\ &= \mathbb{E}_\theta \left| \frac{X}{\theta} - 1 \right|^2 = \frac{1}{\theta^2} \mathbb{E}_\theta |X - \theta|^2 \\ &= \frac{1}{\theta^2} \mathbb{E}_\theta |X - \mathbb{E}_\theta X|^2 = \frac{1}{\theta^2} \text{Var}_\theta(X) = \frac{1}{\theta}. \end{aligned}$$

So, in the case of the Poisson family, the Fisher information depends on θ .

2. Estimator $\tilde{\theta}$ is unbiased for both cases. Then the Cramér-Rao inequality stands that

$$\text{Var}_\theta(\tilde{\theta}) = \text{Var}_\theta \left(\frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n} \text{Var}_\theta(X_1) \geq n^{-1} I^{-1}(\theta).$$

So, the aim is to check that

$$\text{Var}_\theta(X_1) I(\theta) = 1. \quad (2.9)$$

(a) For the Gaussian shift $\text{Var}_\theta(X_1) = 1$ and $I(\theta) = 1$. Hence, (2.9) is fulfilled.

(b) For the Poisson family, $\text{Var}_\theta(X_1) = \theta$ and $I(\theta) = 1/\theta$. Hence, (2.9) is also fulfilled.

3. (i) The definition says that R-efficient estimators are exactly the estimators that give the equality in the Cramér-Rao inequality. So, this item is already proved.
(ii) The estimate $\tilde{\theta}$ can be represented as

$$\tilde{\theta} = n^{-1} \sum U(Y_i)$$

with $U(x) = x$. The aim is to show that the log-density $\ell(y, \theta)$ of P_θ can be represented as

$$\ell(x, \theta) = C(\theta)x - B(\theta) + \ell(x), \quad (2.10)$$

for some functions $C(\cdot)$ and $B(\cdot)$ on Θ and a function $\ell(\cdot)$ on $\mathbb{R}$.

(a)

$$\ell(x, \theta) = \theta x - \theta^2/2 + \left(-\frac{x^2}{2} + \log \frac{1}{\sqrt{2\pi}} \right),$$

and (2.10) follows with $C(\theta) = \theta$, $B(\theta) = \theta^2/2$, and $\ell(x) = -x^2/2 + \log 1/\sqrt{2\pi}$.

(b)

$$\ell(x, \theta) = \log(\theta)x - \theta + \log(x!), \quad (2.11)$$

and (2.11) follows with $C(\theta) = \log \theta$, $B(\theta) = \theta$, and $\ell(x) = \log(x!)$.

Exercise 2.13. Let X be a random variable with a distribution from $(P_\theta, \theta \in \Theta \subset \mathbb{R})$. Let also a function $\psi^\circ : \mathbb{R} \times \Theta \rightarrow \mathbb{R}$ be such that

$$\psi^\circ(x, \theta) = a(x - \theta)^2 + b(x - \theta) + c,$$

where $a, b, c \in \mathbb{R}$.

1. Find a condition on the constants a, b, c and the family (P_θ) such that the function $\psi^\circ(x, \theta)$ is a contrast.
2. Find a condition on the constants a, b, c such that the function $\psi^\circ(x, \theta)$ is a contrast for the model of the Gaussian shift $\mathcal{N}(\theta, 1)$.
1. By definition, the function ψ° is a contrast if and only if

$$\operatorname{argmin}_{\theta'} \mathbb{E}_\theta \psi^\circ(X, \theta') = \theta, \quad \forall \theta.$$

Introduce a function

$$\begin{aligned} f(\theta, \theta') &\stackrel{\text{def}}{=} \mathbb{E}_\theta [\psi^\circ(X, \theta')] \\ &= (a \mathbb{E}_\theta X^2 + b \mathbb{E}_\theta X + c) - (2a\mathbb{E}_\theta X + b) \theta' + a\theta'^2. \end{aligned} \quad (2.12)$$

The aim is to find a condition on the constants a, b, c and the family (P_θ) such that

$$\operatorname{argmin}_{\theta'} f(\theta, \theta') = \theta, \quad \forall \theta. \quad (2.13)$$

Take the derivative of the function $f(\theta, \theta')$ with respect to θ' and solve the equation $\partial f(\theta, \theta')/\partial \theta' = 0$:

$$\frac{\partial f(\theta, \theta')}{\partial \theta'} = -(2a\mathbb{E}_\theta X + b) + 2a\theta' = 0$$

This means that

$$\operatorname{argmin}_{\theta'} f(\theta, \theta') = \mathbb{E}_\theta X + \frac{b}{2a}.$$

Together with (2.13), this yields the required condition on the constants a, b and the family (P_θ) :

$$\theta = \mathbb{E}_\theta X + \frac{b}{2a}, \quad \forall \theta. \quad (2.14)$$

Constant c can be chosen arbitrary.

2. For the model of the Gaussian shift $\mathcal{N}(\theta, 1)$,

$$\mathbb{E}_\theta X = \theta, \quad \forall \theta.$$

Condition (2.14) in this case yields $b = 0$. This means, that any function $\psi^\circ(x, \theta)$ with $b = 0$ and any constants a and c is a contrast for the Gaussian shift.

Exercise 2.14. Let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from a distribution $P_{\theta^*} \in (P_\theta, \theta \in \Theta \subset \mathbb{R})$.

1. Let also $g(x)$ satisfy $\int g(x)dP_{\theta^*}(x) = \theta^*$, leading to the moment estimate

$$\tilde{\theta} \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n g(X_i).$$

Show that this estimate can be obtained as the M -estimate for a properly selected function $\psi(\cdot)$.

2. Let $\int g(x)dP_{\theta^*}(x) = m(\theta^*)$ for the given functions $g(\cdot)$ and strictly monotonic and continuously differentiable $m(\cdot)$. Show that the moment estimate $\tilde{\theta} = m^{-1}\{\sum g(X_i)/n\}$ can be obtained as the M -estimate for a properly selected function $\psi(\cdot)$.

1. It is the worth mentioning that

$$\tilde{\theta} \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n g(X_i) = \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^n \{g(X_i) - \theta\}^2. \quad (2.15)$$

This observation helps us to find an appropriate function ψ . Fix

$$\psi(x, \theta) \stackrel{\text{def}}{=} \{g(x) - \theta\}^2$$

and prove that

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \mathbb{E}_{\theta^*} \psi(X, \theta), \quad (2.16)$$

where X is a variable that has the distribution P_{θ^*} .

The proof of (2.16) is straightforward:

$$\mathbb{E}_{\theta^*} \psi(X, \theta) = \mathbb{E}_{\theta^*} \{g(X) - \theta\}^2 = \mathbb{E}_{\theta^*} g^2(X) - 2\theta \mathbb{E}_{\theta^*} g(X) + \theta^2$$

Minimizing the right hand side expression by θ yields

$$\theta_{\min} = \mathbb{E}_{\theta^*} g(X) = \theta^*.$$

This concludes the proof.

2. The proof follows the same lines as the proof of the first statement. Note that

$$\tilde{\theta} \stackrel{\text{def}}{=} m^{-1} \left\{ \frac{1}{n} \sum_{i=1}^n g(X_i) \right\} = \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^n \{g(X_i) - m(\theta)\}^2.$$

Function

$$\psi(x, \theta) \stackrel{\text{def}}{=} \{g(x) - m(\theta)\}^2$$

is appropriate because of

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \mathbb{E}_{\theta^*} \{g(X) - m(\theta)\}^2 \quad (2.17)$$

In fact, fix some $\theta \in \Theta$ and find a minimum value of the function

$$f(\theta) = \mathbb{E}_{\theta^*} \{g(X) - m(\theta)\}^2$$

In order to minimize this function, solve the equation $f'(\theta) = 0$:

$$\frac{df(\theta)}{d\theta} = \frac{df(\theta)}{dm(\theta)} \frac{dm(\theta)}{d\theta} = 2\mathbb{E}_{\theta}^* \{g(X) - m(\theta)\} \frac{dm(\theta)}{d\theta} = 0$$

The first derivative of the function $m(\theta)$ doesn't change the sign because of monotonicity. This means that the minimum value of function f satisfies the following equation

$$m(\theta_{\min}) = \mathbb{E}_{\theta^*} g(X).$$

Then (2.17) fulfills. This completes the proof.

Exercise 2.15. Let $\{X_i\}_{i=1}^{n_1}$ be a sample from the distribution with the pdf

$$p(x, \theta) = \frac{2x}{\theta^2}, \quad x \in [0, \theta].$$

Find the MLE of the median of the distribution.

First let us find a relation between θ and the median m . By the definition of the median,

$$\int_{-\infty}^m p(x, \theta) dx = \int_0^m \frac{2x}{\theta^2} dx = 1/2,$$

i.e. $\theta = \sqrt{2}m$. Then the likelihood function

$$L(m) = \prod_{i=1}^n p(X_i, \sqrt{2}m) = \prod_{i=1}^n \frac{X_i}{m^2} \mathbf{1}(X_i \in [0, \sqrt{2}m])$$

has a maximum at the point $\hat{m} = \max_i X_i / \sqrt{2}$.

Exercise 2.16. Let $\{X_i^{(1)}\}_{i=1}^{n_1}$ and $\{X_i^{(2)}\}_{i=1}^{n_2}$ be two independent samples from the Poisson distributions with unknown parameters μ_1 and $\mu_2 = \mu_1 + \mu$ correspondingly. Find the maximum likelihood estimator for the parameter μ .

Hint: Is it possible to find separately $\hat{\mu}_1$ (the MLE for μ_1) from the first sample, $\hat{\mu}_2$ (the MLE for μ_2) from the second sample, and then obtain the MLE estimator for μ as the difference $\hat{\mu} = \hat{\mu}_1 - \hat{\mu}_2$?

Denote by $L_1(\mu_1)$ and $L_2(\mu_2)$ the log-likelihood functions for the first and the second samples correspondingly.

The MLE estimate for the parameter μ is determined as

$$(\hat{\mu}_1, \hat{\mu}) = \operatorname{argmax}_{\mu_1, \mu} \{L_1(\mu_1) + L_2(\mu_1 + \mu)\}.$$

Hence, $\hat{\mu} = \operatorname{argmax}_{\mu} L_2(\hat{\mu}_1 + \mu)$. The maximal value of the function L_2 is achieved at the point $\hat{\mu}_2$. This yields

$$\max_{\mu} L_2(\hat{\mu}_1 + \mu) = L_2(\hat{\mu}_2) = L_2\{\hat{\mu}_1 + (\hat{\mu}_2 - \hat{\mu}_1)\}.$$

So, $\hat{\mu} = \hat{\mu}_2 - \hat{\mu}_1$.

In the case of the Poisson distribution,

$$L_1(\mu_1) = \sum_{i=1}^{n_1} \log \left(e^{-\mu_1} \frac{\mu_1^{X_i^{(1)}}}{X_i^{(1)}!} \right) \quad \text{and} \quad L_2(\mu_2) = \sum_{i=1}^{n_2} \log \left(e^{-\mu_2} \frac{\mu_2^{X_i^{(2)}}}{X_i^{(2)}!} \right),$$

and the MLE of the parameter is the mean value, i.e. $\hat{\mu}_j = n_j^{-1} \sum_{i=1}^{n_j} X_i^{(j)} \stackrel{\text{def}}{=} \overline{X}^{(j)}$, $j = 1, 2$. Thus, we conclude that

$$\hat{\mu} = \overline{X}^{(2)} - \overline{X}^{(1)}.$$

Exercise 2.17. Let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from a distribution with the Lebesgue density

$$p(x, \theta) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} I_{(0, \beta)}(x),$$

where $\alpha, \beta > 0$ and $\theta \stackrel{\text{def}}{=} (\alpha, \beta)$. Find estimators for the multivariate parameter θ using the following approaches:

1. Maximum likelihood approach;
2. Method of moments.

1. The likelihood function in this case

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n p(X_i, \theta) = \frac{\alpha^n}{\beta^{\alpha n}} \prod_{i=1}^n X_i^{\alpha-1} I_{(0, \beta)}(X_i) \\ &= \frac{\alpha^n}{\beta^{\alpha n}} I_{(0, \beta)}(X_{(n)}) \prod_{i=1}^n X_i^{\alpha-1} \end{aligned}$$

is equal to zero if $\beta < X_{(n)}$ and decreases for $\beta \geq X_{(n)}$. Therefore the maximum likelihood estimator for the parameter β is $\tilde{\beta} = X_{(n)}$. In order to find MLE for the parameter α , one should maximize the function

$$f(\alpha) = C_1 \alpha^n C_2^{\alpha-1},$$

where $C_1 = I_{(0, \tilde{\beta})}(X_{(n)})\tilde{\beta}^{-n}$, $C_2 = \prod_{i=1}^n X_i / \tilde{\beta} = \prod_{i=1}^n X_i / X_{(n)}$. The equation $f'(\alpha) = 0$ gives the MLE of the parameter α :

$$\tilde{\alpha} = \frac{n}{-\log C_2} = \frac{n}{\sum_{i=1}^n \log \frac{X_{(n)}}{X_i}}.$$

So, the MLE is

$$\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta}) = \left(\frac{n}{\sum_{i=1}^n \log \frac{X_{(n)}}{X_i}}, X_{(n)} \right).$$

2. Firstly we compute the first and the second moments:

$$\begin{aligned} m_1(\theta) &= \mathbb{E}_{\theta} X_1 = \int x p(x, \theta) dx = \int_0^{\beta} \frac{\alpha}{\beta^{\alpha}} x^{\alpha} dx = \frac{\alpha\beta}{\alpha+1} \\ m_2(\theta) &= \mathbb{E}_{\theta} X_1^2 = \int x^2 p(x, \theta) dx = \int_0^{\beta} \frac{\alpha}{\beta^{\alpha}} x^{\alpha+1} dx = \frac{\alpha\beta^2}{\alpha+2} \end{aligned}$$

The empirical counterparts are

$$M_1 = \frac{1}{n} \sum_{i=1}^n X_i, \quad M_2 = \frac{1}{n} \sum_{i=1}^n X_i^2. \quad (2.18)$$

The required estimators are the solutions of the system of equations

$$\begin{cases} M_1 = \alpha\beta/(\alpha+1) \\ M_2 = \alpha\beta^2/(\alpha+2) \end{cases} \quad (2.19)$$

Raise both parts of the first equation to the second power and divide it to the second equation:

$$\frac{M_1}{M_2} = \frac{\alpha(\alpha+2)}{(\alpha+1)^2}.$$

This yields the following quadratic equation w.r.t α :

$$\alpha^2 + 2\alpha + \frac{M_1}{M_1 - M_2} = 0. \quad (2.20)$$

If $\frac{M_1}{M_1 - M_2} < 0$ (or equivalently $M_1 < M_2$) then (2.20) has one positive solution

$$\hat{\alpha} = -1 + \sqrt{1 - \frac{M_1}{M_1 - M_2}}.$$

The first equation of system (2.19) gives

$$\hat{\beta} = \frac{\hat{\alpha} + 1}{\hat{\alpha}} M_1.$$

So, the estimate by the method of moments is

$$(\hat{\alpha}, \hat{\beta}) = \left(-1 + \sqrt{1 - \frac{M_1}{M_1 - M_2}}, \quad \frac{\sqrt{1 - \frac{M_1}{M_1 - M_2}}}{-1 + \sqrt{1 - \frac{M_1}{M_1 - M_2}}} M_1 \right),$$

where M_1 and M_2 are given by (2.18).

Exercise 2.18. Let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from a distribution with the Lebesgue density that depends on the parameter $\theta \in \mathbb{R}$ (σ is a fixed positive number):

$$p(x, \theta) = (2\sigma)^{-1} e^{-|x-\theta|/\sigma}.$$

Compute the maximum likelihood estimate for the parameter θ .

This model is known as a shift of a Laplace law.

The maximum likelihood approach leads to maximizing the sum

$$L(\theta) = -n \log(2\sigma) - \sum_{i=1}^n |X_i - \theta|/\sigma,$$

or equivalently to minimizing the sum $\sum_{i=1}^n |X_i - \theta|$:

$$\tilde{\theta} = \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^n |X_i - \theta|.$$

Order the observations $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ and consider two cases.

1. Suppose that n is even. Denote $k = n/2 \in \mathbb{N}$. It is worth mentioning that

$$|X_{(1)} - \theta| + |X_{(n)} - \theta| \geq |X_{(n)} - X_{(1)}|, \quad (2.21)$$

where equality takes place if and only if $\theta \in [X_{(1)}, X_{(n)}]$ Analogously,

$$|X_{(2)} - \theta| + |X_{(n-1)} - \theta| \geq |X_{(n-1)} - X_{(2)}| \quad (2.22)$$

...

$$|X_{(k)} - \theta| + |X_{(k+1)} - \theta| \geq |X_{(k+1)} - X_{(k)}| \quad (2.23)$$

This yields that

$$\begin{aligned} \sum_{i=1}^n |X_i - \theta| &= \sum_{i=1}^n |X_{(i)} - \theta| = \sum_{j=1}^k (|X_{(j)} - \theta| + |X_{(n-j+1)} - \theta|) \\ &\geq \sum_{j=1}^k |X_{(n-j+1)} - X_{(j)}|. \end{aligned} \quad (2.24)$$

Equality in (2.24) takes place if and only if all the inequalities (2.21)–(2.23) are in fact equalities. This means that $\operatorname{argmin} \sum |X_i - \theta|$ is minimized by any $\theta \in [X_{(k)}, X_{(k+1)}]$, in particular by

$$\tilde{\theta} = \operatorname{med} X_i = \frac{X_{(k)} + X_{(k+1)}}{2}.$$

2. Suppose that n is odd. Denote $k = (n - 1)/2 \in \mathbb{N}$. Equalities (2.21)–(2.23) are still true. This yields the analogue for (2.24):

$$\begin{aligned} \sum_{i=1}^n |X_i - \theta| &= \sum_{i=1}^n |X_{(i)} - \theta| = \underbrace{|X_{(k+1)} - \theta|}_{\geq 0} + \sum_{j=1}^k \underbrace{|X_{(j)} - \theta| + |X_{(n-j+1)} - \theta|}_{\geq |X_{(n-j+1)} - X_{(j)}|} \\ &\geq \sum_{j=1}^k |X_{(n-j+1)} - X_{(j)}|. \end{aligned} \quad (2.25)$$

Note that the following two equalities take place only in the case of $\tilde{\theta} = \operatorname{med} X_i = X_{(k+1)}$:

$$\begin{aligned} |X_{(k+1)} - \theta| &= 0 \\ \sum_{j=1}^k [|X_{(j)} - \theta| + |X_{(n-j+1)} - \theta|] &= \sum_{j=1}^k |X_{(n-j+1)} - X_{(j)}| \end{aligned}$$

This completes the proof.

Exercise 2.19. Consider the volatility model with parameter θ :

$$Y = \xi^2, \quad \xi \sim \mathcal{N}(0, \theta).$$

1. Prove that θ is a natural parameter.
2. Find a canonical parameter for this model.
3. Compute the Fisher information for this model with canonical parameter.

1. The proof is straightforward:

$$\mathbb{E}Y = \mathbb{E}\xi^2 = \underbrace{\text{Var}\xi}_{=\theta} + \underbrace{(\mathbb{E}\xi)^2}_{=0} = \theta.$$

2. Denote by $p_\xi(x)$ the pdf of ξ :

$$p_\xi(x) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{x^2}{2\theta}\right).$$

The density function of Y can be derived from $p_\xi(x)$:

$$\begin{aligned} p_Y(y, \theta) &= \frac{1}{2\sqrt{y}} p_\xi(\sqrt{y}) = \frac{1}{2\sqrt{2\pi\theta y}} \exp\left(-\frac{y}{2\theta}\right) \\ &= \frac{1}{2\sqrt{2\pi y}} \exp\left(-\frac{y}{2\theta} - \frac{1}{2} \log \theta\right). \end{aligned} \quad (2.26)$$

This density representation means that $C(\theta) = -(2\theta)^{-1}$. The canonical parameter is determined by the equality $v \stackrel{\text{def}}{=} C(\theta)$, i.e. $v = -(2\theta)^{-1}$. This yields

$$p_Y(y, v) = \frac{1}{2\sqrt{2\pi y}} \exp\{yv - d(v)\},$$

where $d(v) = 1/2 \log\{-1/(2v)\}$.

3. According to the general theory,

$$I(v) = d''(v) = \frac{1}{2v^2}.$$

Exercise 2.20. Let (P_v) be a Gaussian shift experiment, that is $P_v = \mathcal{N}(v, 1)$, $v \in \mathbb{R}$. Let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from a distribution P_{v^*} .

1. Is the parameter v a natural parameter? Is it a canonical parameter?
2. Check that

$$\mathcal{K}(v_1, v_2) = (v_1 - v_2)^2 / 2.$$

3. Check that for any v_0 and any $C > 0$, the equation

$$\mathcal{K}(v_0 + u, v_0) = C \quad (2.27)$$

has only one positive (u^+) and only one negative (u^-) solution.

4. Compute the maximum likelihood estimator $\tilde{v}$ and check that

$$L(\tilde{v}, v) = (\tilde{v} - v)^2 n / 2.$$

5. Fix some $\zeta > 0$. Consider the equation (2.27) with $v_0 = v^*$ and $C = \zeta/n$. According to item 3, this equation has two solutions: denote the positive solution by u^+ , and the negative solution by u^- . Denote also $v^+ = v^* + u^+$, and $v^- = v^* + u^-$.

- (a) Compute the sets $\{L(\tilde{v}, v^*) \geq \zeta\}$, $\{L(v^+, v^*) \geq \zeta\}$, $\{L(v^-, v^*) \geq \zeta\}$.
 (b) Check that

$$\{L(\tilde{v}, v^*) \geq \zeta\} \subseteq \{L(v^+, v^*) \geq \zeta\} \cup \{L(v^-, v^*) \geq \zeta\}.$$

Note that the last item is fulfilled for any v^* (not necessary the true value).

1. Parameter v is a natural parameter, because the expected value of a r.v. with distribution $\mathcal{N}(v, 1)$ is equal to v . The parameter v is also a canonical parameter, because the density function can be represented in the following way

$$p(x, v) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(x - v)^2}{2} \right\} = p(x) \exp \{xv - d(v)\},$$

where

$$p(x) = \varphi(x), \quad d(v) = \frac{v^2}{2}.$$

2. According to the formula for the canonical parametrization,

$$\mathcal{K}(v_1, v_2) = d'(v_1)(v_1 - v_2) - \{d(v_1) - d(v_2)\}. \quad (2.28)$$

In the case of a Gaussian shift, (2.28) yields

$$\mathcal{K}(v_1, v_2) = v_1(v_1 - v_2) - \frac{v_1^2 - v_2^2}{2} = (v_1 - v_2) \left(v_1 - \frac{v_1 + v_2}{2} \right) = \frac{(v_1 - v_2)^2}{2}.$$

3. The statement is a straightforward corollary from the previous item:

$$\mathcal{K}(u, v_0) = \frac{(v_0 + u - v_0)^2}{2} = \frac{u^2}{2} = C.$$

This equation has two solutions: one positive $u^+ = \sqrt{2C}$ and one negative $u^- = -\sqrt{2C}$.

4. The maximum likelihood approach leads to maximizing the sum

$$L(v) = n \log \frac{1}{2\pi} - \sum_{i=1}^n \frac{(X_i - v)^2}{2}.$$

Then the maximum likelihood estimator is equal to $\tilde{v} = \sum_i X_i / n$. Consider the difference between $L(\tilde{v})$ and $L(v)$:

$$\begin{aligned} L(\tilde{v}, v) &= L(\tilde{v}) - L(v) = - \sum_i \frac{(X_i - \tilde{v})^2}{2} + \sum_i \frac{(X_i - v)^2}{2} \\ &= \frac{1}{2} \sum_i \left\{ (X_i - v)^2 - (X_i - \tilde{v})^2 \right\} = \frac{1}{2} \sum_i (2X_i - \tilde{v} - v) (\tilde{v} - v) \\ &= \frac{1}{2} \left(\underbrace{2 \sum_i X_i}_{2n\tilde{v}} - n\tilde{v} - nv \right) (\tilde{v} - v) = \frac{(\tilde{v} - v)^2 n}{2}. \quad (2.29) \end{aligned}$$

5. (a) Formula (2.29) yields

$$\begin{aligned} \{L(\tilde{v}, v^*) \geq \zeta\} &= \left\{ \frac{(\tilde{v} - v^*)^2 n}{2} \geq \zeta \right\} \\ &= \left\{ \tilde{v} \geq \sqrt{\frac{2\zeta}{n}} + v^* \right\} \cup \left\{ \tilde{v} \leq -\sqrt{\frac{2\zeta}{n}} + v^* \right\} \end{aligned}$$

From (2.27) (in item 3) we know that $v^+ = v^* + \sqrt{2\zeta/n}$ and $v^- = v^* - \sqrt{2\zeta/n}$. Then

$$\begin{aligned} \{L(v^+, v^*) \geq \zeta\} &= \left\{ \frac{n}{2} (2\tilde{v} - v^+ - v^*) (\tilde{v} - v^*) \geq \zeta \right\} \\ &= \left\{ \frac{n}{2} (2\tilde{v} - 2v^* - \sqrt{\frac{2\zeta}{n}}) \sqrt{\frac{2\zeta}{n}} \geq \zeta \right\} \\ &= \left\{ \tilde{v} \geq v^* + \sqrt{\frac{2\zeta}{n}} \right\}. \end{aligned}$$

Analogously,

$$\{L(v^-, v^*) \geq \zeta\} \supset \left\{ \tilde{v} \leq v^* - \sqrt{\frac{2\zeta}{n}} \right\}.$$

(b) The required embedding is trivial.

Natural parametrization has some “nice” properties:

1.

$$\tilde{\theta} = \frac{1}{n} \sum_{i=1}^n Y_i.$$

2.

$$L(\tilde{\theta}, \theta) = n\mathcal{K}(\mathbb{P}_{\tilde{\theta}}, \mathbb{P}_{\theta}).$$

The following exercise shows, that the choice of parametrization is crucial for the first property, but the second one is fulfilled for any parametrization.

Exercise 2.21. *Let (P_{θ}) be an exponential family (θ – **any** parameter). Let $\{X_i\}_{i=1}^n$ be an i.i.d. sample from distribution that belongs to (P_{θ}) , and X be a random variable with the same distribution.*

Show that the maximum likelihood estimator $\tilde{\theta}$ has the following properties:

1.

$$\mathbb{E}_{\tilde{\theta}} X = \frac{1}{n} \sum_{i=1}^n X_i.$$

2.

$$L(\tilde{\theta}, \theta) = n\mathcal{K}(\mathbb{P}_{\tilde{\theta}}, \mathbb{P}_{\theta}).$$

1. $\tilde{\theta}$ is a point of maximum of the function

$$L(\theta) = \sum_{i=1}^n \log p(X_i, \theta) = C(\theta) \sum_{i=1}^n X_i - nB(\theta).$$

Differentiating w.r.t θ yields the equation for $\tilde{\theta}$:

$$C'(\tilde{\theta}) \sum_{i=1}^n X_i - nB'(\tilde{\theta}) = 0. \quad (2.30)$$

On the other hand, differentiating both sides of the equality

$$\int p(x, \theta) dx = 1$$

w.r.t. θ yields

$$\begin{aligned}
 0 &= \int \frac{\partial}{\partial \theta} \{p(x, \theta)\} dx = \int \frac{\partial}{\partial \theta} \{\log p(x, \theta)\} p(x, \theta) dx \\
 &= \int \{x C'(\theta) - B'(\theta)\} p(x, \theta) dx \\
 &= C'(\theta) \underbrace{\int x p(x, \theta) dx}_{=\mathbb{E}_\theta X} - B'(\theta) \underbrace{\int p(x, \theta) dx}_{=1}.
 \end{aligned}$$

This means that the equality

$$C'(\theta) \mathbb{E}_\theta X - B'(\theta) = 0$$

holds for any parameter θ , in particular for $\theta = \tilde{\theta}$:

$$C'(\tilde{\theta}) \mathbb{E}_{\tilde{\theta}} X - B'(\tilde{\theta}) = 0. \quad (2.31)$$

Comparison of the equations (2.30) and (2.31) (using positivity of the first derivative of function $C(\theta)$) completes the proof.

2. Transformation of the left-hand side yields:

$$\begin{aligned}
 L(\tilde{\theta}, \theta) &= \sum_{i=1}^n \left\{ \log p(X_i, \tilde{\theta}) - \log p(X_i, \theta) \right\} \\
 &= \{C(\tilde{\theta}) - C(\theta)\} \sum_{i=1}^n X_i - n\{B(\tilde{\theta}) - B(\theta)\}. \quad (2.32)
 \end{aligned}$$

The Kullback-Leibler divergence in the right-hand side can be transformed in the following way:

$$\begin{aligned}
 \mathcal{K}(\mathbb{P}_{\tilde{\theta}}, \mathbb{P}_\theta) &= \int \log \left\{ \frac{p(x, \tilde{\theta})}{p(x, \theta)} \right\} P_{\tilde{\theta}}(dx) \\
 &= \{C(\tilde{\theta}) - C(\theta)\} \int x P_{\tilde{\theta}}(dx) - \{B(\tilde{\theta}) - B(\theta)\} \\
 &= \{C(\tilde{\theta}) - C(\theta)\} \mathbb{E}_{\tilde{\theta}} X - \{B(\tilde{\theta}) - B(\theta)\}. \quad (2.33)
 \end{aligned}$$

Comparison of the equalities (2.32) and (2.33) using the first item completes the proof.

Exercise 2.22 (Suhov and Kelbert 2005). *There is widespread agreement amongst the managers of the Reliable Motor Company that the number x of faulty cars produced in a month has a binomial distribution*

$$\mathbb{P}(x = s) = \binom{n}{s} p^s (1 - p)^{n-s}, s = 0, 1, \dots, n; 0 \leq p \leq 1.$$

There is, however, some dispute about the parameter p . The general manager has a prior distribution for p which is uniform (i.e. with the pdf $f_p(x) = \mathbf{1}(0 \leq x \leq 1)$), while the more pessimistic production manager has a prior distribution with density $f_p(x) = 2x\mathbf{1}(0 \leq x \leq 1)$. Both pdfs are concentrated on $(0, 1)$.

- (i) In a particular month, s faulty cars are produced. Show that if the general manager's loss function is $(\hat{p} - p)^2$, where $\hat{p}$ is her estimate and p is the true value, then her best estimate of p is

$$\hat{p} = \frac{s + 1}{n + 2}$$

- (ii) The production manager has responsibilities different from those of the general manager, and a different loss function given by $(1 - p)(\hat{p} - p)^2$. Find his best estimator of p and show that it is greater than that of the general manager unless $s \geq n/2$.

You may assume that, for non-negative integers α, β ,

$$\int_0^1 p^\alpha (1 - p)^\beta dp \approx \frac{\alpha! \beta!}{(\alpha + \beta + 1)!}$$

As $\mathbb{P}_p(X = s) = \alpha p^s (1 - p)^{n-s}$, $s = 0, 1, \dots, n$, the posterior for the general manager (GM) is

$$\pi^{GM}(p | s) = \alpha p^s (1 - p)^{n-s} \mathbf{1}(0 < p < 1),$$

and for the production manager (PM)

$$\pi^{PM}(p | s) = \alpha p p^s (1 - p)^{n-s} \mathbf{1}(0 < p < 1).$$

Then the expected loss for the GM is minimized at the posterior mean:

$$\begin{aligned} \hat{p}^{GM} &= \frac{\int_0^1 p p^s (1 - p)^{n-s} dp}{\int_0^1 p^s (1 - p)^{n-s} dp} \\ &= \frac{(s + 1)!(n - s)!}{(n - s + s + 2)!} \frac{(n - s + s + 1)!}{s!(n - s)!} = \frac{s + 1}{n + 2}. \end{aligned}$$

For the PM, the expected loss

$$\int_0^1 (1 - p) (p - a)^2 \pi^{PM}(p | s) dp$$

is minimized at

$$a = \frac{\int_0^1 p(1-p)\pi^{PM}(p|s) dp}{\int_0^1 (1-p)\pi^{PM}(p|s) dp},$$

which yields

$$\begin{aligned}\hat{p}^{PM} &= \frac{\int_0^1 p(1-p)pp^s(1-p)^{n-s} dp}{\int_0^1 p(1-p)pp^s(1-p)^{n-s} dp} \\ &= \frac{(s+2)!(n-s+1)!}{(n-s+s+4)!} \frac{(n-s+s+3)!}{(s+1)!(n-s+1)!} = \frac{s+2}{n+4}.\end{aligned}$$

We see that $(s+2)/(n+4) > (s+1)/(n+2)$, i.e., $s < n/2$.

Exercise 2.23. Denote the number of incoming telecom signals between $[0, t]$ as $C(0, t)$. Assume that $C(0, t)$ satisfies

- (a) The number of arrivals in disjoint time intervals are independent;
- (b) The distribution of $C(s, t)$ depends on $t - s$;
- (c) For $h > 0$ small, $P\{C(0, h) = 1\} = \lambda h + o(h)$, where $\lambda > 0$ is a constant;
- (d) $P\{C(0, h) \geq 2\} = o(h)$.

Please answer the following questions:

1. Prove that $C(0, t)$ follows a Poisson distribution with mean λt .
2. Find the function $p(y)$, $C(\theta)$ and $B(\theta)$ of the natural parametrization

$$p(y, \theta) \stackrel{\text{def}}{=} p(y)e^{yC(\theta)-B(\theta)}$$

and function $d(\theta)$ of the canonical parametrization

$$p(y, \theta) \stackrel{\text{def}}{=} e^{y\theta-d(\theta)}$$

for this Poisson distribution with mean λt .

3. Find an estimator for constant λ .

1. Let $X_m^n = C\{(m-1)t/n, mt/n\}$, $1 \leq m \leq n$, X_m^n are i.i.d. by assumption (a). Define Y_m^n be i.i.d. Bernoulli random variable such that $Y_m^n = 1$ with probability $/n$, $1 \leq m \leq n$. Define

$$S_n = X_1^n + \dots + X_n^n$$

and

$$T_n = Y_1^n + \dots + Y_n^n.$$

Suppose $P\{C(0, h) = 1\} = \lambda h + g_1(h)$ and $P\{C(0, h) \geq 2\} = g_2(h)$ where $g_1(h)$ and $g_2(h)$ are of order $\mathcal{O}(h)$. We claim the following lemma:

Lemma 2.1. *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be complex numbers with modulus $\leq c$, then*

$$\left| \prod_{m=1}^n a_m - \prod_{m=1}^n b_m \right| \leq c^{n-1} \sum_{m=1}^n |a_m - b_m|.$$

The proof of this simple lemma is left to the reader (hint: use induction). The modulus of $\varphi_Y(\xi) = \exp(\mathbf{i}Y_m^n \xi)$ and $\varphi_{X_m^n}(\xi) = \exp(\mathbf{i}X_m^n \xi)$ are less than 1, $|\varphi_{X_m^n}(\xi) - \varphi_Y(\xi)| \leq 2g_1(t/n) + 2g_2(t/n)$ (verify!). By the lemma,

$$\begin{aligned} & |\mathbb{E} \exp(\mathbf{i}T_n \xi) - \mathbb{E} \exp(\mathbf{i}S_n \xi)| \\ &= \left| \prod_{m=1}^n \varphi_{X_m^n}(\xi) - \prod_{m=1}^n \varphi_{Y_m}(\xi) \right| \\ &\leq \sum_{m=1}^n |\varphi_{X_m^n}(\xi) - \varphi_Y(\xi)| \\ &\leq \sum_{m=1}^n 2 \left\{ \left| g_1\left(\frac{t}{n}\right) \right| + \left| g_2\left(\frac{t}{n}\right) \right| \right\} \\ &\rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Now we show that $\mathbb{E} \exp(\mathbf{i}T_n \xi) \rightarrow \exp\{\lambda t(\exp(\mathbf{i}\xi) - 1)\}$, the characteristic function of the Poisson distribution with mean λt and finish the proof. Observe that $|\mathbb{E} \exp(\mathbf{i}Y_m^n \xi)| = (1 - \lambda t/n) + (\lambda t/n) \exp(\mathbf{i}\xi) = 1 + (\lambda t/n)\{\exp(\mathbf{i}\xi) - 1\}$ and $|\exp(\mathbf{i}\xi) - 1| \leq 2$. When n large, $\lambda t/n \leq 1/2$. Using the lemma again,

$$\begin{aligned} & \left| \exp(\lambda t\{\exp(\mathbf{i}\xi) - 1\}) - \prod_{m=1}^n [1 + (\lambda t/n)\{\exp(\mathbf{i}\xi) - 1\}] \right| \\ &\leq \sum_{m=1}^n \left| \exp\left[\frac{\lambda t}{n}\{\exp(\mathbf{i}\xi) - 1\}\right] - \left[1 + \frac{\lambda t}{n}\{\exp(\mathbf{i}\xi) - 1\}\right] \right| \\ &\leq \sum_{m=1}^n \left(\frac{\lambda t}{n}\right)^2 |\exp(\mathbf{i}\xi) - 1|^2 \\ &\leq 4 \left(\frac{\lambda t}{n}\right) \lambda t \\ &\rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. This finishes the proof.

2. The Poisson density with mean λt is

$$p(y, \lambda t) = \exp(-\lambda t)(\lambda t)^y / y!.$$

The $p(y)$, $C(\lambda)$, $B(\lambda)$ of the natural parametrization is

$$\begin{aligned} p(y) &= \frac{1}{y!}; \\ C(\lambda t) &= \log(\lambda t); \\ B(\lambda t) &= \lambda t. \end{aligned}$$

The $d(\lambda t)$ for the canonical parametrization is

$$d(\lambda t) = -\lambda t(y + 1) + y \log(\lambda t) - \log y!.$$

3. Suppose we have an observation of the number of signal y between time 0 and t .

The maximizer for the log natural parametrization is $\hat{\lambda} = y/t$.

Exercise 2.24. Let Y be an i.i.d. sample from $P_{\theta^*} \in (P_{\theta})$, where (P_{θ}) is a regular parametric family. The fundamental exponential bound for the maximum likelihood is given by the fact that for any $0 < \varrho < 1$, $0 < s < 1$, $\mu > 0$, the log-likelihood process $L(\theta, \theta^*)$ fulfills for a fixed constant $\varOmega(\varrho, s)$

$$\mathbb{E} \exp \left[\varrho \sup_{\theta \in \Theta} \{ \mu L(\theta, \theta^*) + s \mathcal{M}(\mu, \theta, \theta^*) \} \right] \leq \varOmega(\varrho, s), \quad (2.34)$$

see [Spokoiny and Dickhaus \(2014\)](#). Denote the set $\mathcal{A}(\mathfrak{z}, \theta^*) = \{ \theta : \mathcal{M}(\mu, \theta, \theta^*) \leq \mathfrak{z} \}$, where $\mathfrak{z}$ is positive, and $\mathcal{M}(\mu, \theta, \theta^*)$ is the rate function defined for $\mu > 0$ by

$$\mathcal{M}(\mu, \theta, \theta^*) \stackrel{\text{def}}{=} -\log \mathbb{E}_{\theta^*} \exp \{ \mu L(\theta, \theta^*) \}.$$

Using (2.34), prove that for any $\varrho' < \varrho$,

1.

$$\mathbb{E} \left[\exp \left\{ \varrho' s \mathcal{M}(\mu, \tilde{\theta}, \theta^*) \right\} \mathbf{1} \left\{ \tilde{\theta} \notin \mathcal{A}(\mathfrak{z}, \theta^*) \right\} \right] \leq \varOmega(\varrho, s) \exp \{ -(\varrho - \varrho') s \mathfrak{z} \};$$

in particular,

$$\mathbb{P} \{ \tilde{\theta} \notin \mathcal{A}(\mathfrak{z}, \theta^*) \} \leq \varOmega(\varrho, s) \exp(-\varrho s \mathfrak{z}).$$

2.

$$\mathbb{E} \left[\mathcal{M}(\mu, \tilde{\theta}, \theta^*) \mathbf{1}(\tilde{\theta} \notin \mathcal{A}(\mathfrak{z}, \theta^*)) \right] \leq \frac{1}{\varrho' s} \varOmega(\varrho, s) \exp \{ -(\varrho - \varrho') s \mathfrak{z} \}.$$

1. The inequalities $L(\tilde{\theta}, \theta^*) \geq 0$ and $\mathcal{M}(\mu, \tilde{\theta}, \theta^*) > \mathfrak{z}$ for $\tilde{\theta} \notin \mathcal{A}(\mathfrak{z}, \theta^*)$ imply

$$\begin{aligned}
 & \mathbb{E} \left[\exp\{(\varrho - \varrho')s\mathfrak{z}\} \exp\{\varrho's \mathcal{M}(\mu, \tilde{\theta}, \theta^*)\} \mathbf{1}\{\tilde{\theta} \notin \mathcal{A}(\mathfrak{z}, \theta^*)\} \right] \\
 & \leq \mathbb{E} \left[\exp\{\varrho s \mathcal{M}(\mu, \tilde{\theta}, \theta^*)\} \mathbf{1}\{\tilde{\theta} \notin \mathcal{A}(\mathfrak{z}, \theta^*)\} \right] \\
 & \leq \mathbb{E} \left[\exp\{\varrho s \mathcal{M}(\mu, \tilde{\theta}, \theta^*)\} \right] \\
 & \leq \mathbb{E} \left[\exp\{\varrho \mu L(\tilde{\theta}, \theta^*) + \varrho s \mathcal{M}(\mu, \tilde{\theta}, \theta^*)\} \right] \\
 & \leq \mathfrak{Q}(\varrho, s),
 \end{aligned}$$

and the assertion follows.

2. The second item directly follows from the first one, because $x < e^x$ for any positive x .

Exercise 2.25. Consider a multivariate normal rv $\mathbf{Y} \sim \mathcal{N}(\theta^*, \Sigma)$, where $\Sigma = (nD^2)^{-1}$ for some matrix D . In other words, $\mathbf{Y} = \theta^* + \xi$ with $\xi \sim \mathcal{N}\{0, (nD^2)^{-1}\}$.

1. Check that the log-likelihood ratio computed on one observation of $\mathbf{Y}$ is equal to

$$L(\theta, \theta^*) = n(\theta - \theta^*)^\top D^2 \xi - n\|D(\theta - \theta^*)\|^2/2. \quad (2.35)$$

2. Prove that the r.v. ξ is equal to

$$\xi = (nD^2)^{-1} \nabla L(\theta^*).$$

1. The log-likelihood is equal to

$$\begin{aligned}
 & L(\theta, \theta^*) \\
 & = L(\theta) - L(\theta^*) \\
 & = -\frac{1}{2}(\mathbf{Y} - \theta)^\top \Sigma^{-1}(\mathbf{Y} - \theta) + \frac{1}{2}(\mathbf{Y} - \theta^*)^\top \Sigma^{-1}(\mathbf{Y} - \theta^*) \\
 & = -\frac{1}{2}(\mathbf{Y} - \theta^* + \theta^* - \theta)^\top \Sigma^{-1}(\mathbf{Y} - \theta^* + \theta^* - \theta) \\
 & \quad + \frac{1}{2}(\mathbf{Y} - \theta^*)^\top \Sigma^{-1}(\mathbf{Y} - \theta^*) \\
 & = -(\theta^* - \theta)^\top \Sigma^{-1}(\mathbf{Y} - \theta^*) - \frac{1}{2}(\theta^* - \theta)^\top \Sigma^{-1}(\theta^* - \theta).
 \end{aligned}$$

To conclude the proof, it is sufficient to note that

$$\Sigma^{-1}(\mathbf{Y} - \theta^*) = nD^2 \xi,$$

and

$$\begin{aligned} \frac{1}{2}(\boldsymbol{\theta}^* - \boldsymbol{\theta})^\top \Sigma^{-1}(\boldsymbol{\theta}^* - \boldsymbol{\theta}) &= \frac{1}{2}(\boldsymbol{\theta}^* - \boldsymbol{\theta})^\top nD^2(\boldsymbol{\theta}^* - \boldsymbol{\theta}) \\ &= n\|D(\boldsymbol{\theta} - \boldsymbol{\theta}^*)\|^2/2. \end{aligned}$$

2. The proof is straightforward:

$$\begin{aligned} \nabla L(\boldsymbol{\theta}^*) &= \nabla \left\{ -\frac{n}{2} \log |2\pi \Sigma| - \frac{1}{2}(\mathbf{Y} - \boldsymbol{\theta}^*)^\top \Sigma^{-1}(\mathbf{Y} - \boldsymbol{\theta}^*) \right\} \\ &= \Sigma^{-1}(\mathbf{Y} - \boldsymbol{\theta}^*) = nD^2 \boldsymbol{\xi}. \end{aligned}$$

Exercise 2.26. Consider the model from the previous exercise, $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\theta}^*, \Sigma)$ with $\Sigma = (nD^2)^{-1}$ for some matrix D .

Using the formula (2.35), simulate the log-likelihood ratio for $D^2 = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.

$$\boldsymbol{\theta}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \text{ and } \boldsymbol{\theta} = \boldsymbol{\theta}_1 \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{\theta} = \boldsymbol{\theta}_2 \stackrel{\text{def}}{=} \begin{pmatrix} 1.2 \\ 1 \end{pmatrix}.$$

Draw a plot for $L(\boldsymbol{\theta}, \boldsymbol{\theta}^*)$ as a function of $\boldsymbol{\xi}$ and a plot for an estimator of the density function of $L(\boldsymbol{\theta}, \boldsymbol{\theta}^*)$.

Define $\mu \stackrel{\text{def}}{=} -n\|D(\boldsymbol{\theta} - \boldsymbol{\theta}^*)\|^2/2$, and note that $nD^2 \boldsymbol{\xi} \sim \mathcal{N}(0, nD^2)$. Therefore, by formula (2.35), the rv $L(\boldsymbol{\theta}, \boldsymbol{\theta}^*)$ has the distribution

$$L(\boldsymbol{\theta}, \boldsymbol{\theta}^*) \sim \mathcal{N}\{\mu, (\boldsymbol{\theta} - \boldsymbol{\theta}^*)^\top (nD^2)(\boldsymbol{\theta} - \boldsymbol{\theta}^*)\}.$$

The square root of D^2 can be found via the Jordan decomposition $D^2 = \Gamma \Lambda \Gamma^\top$, where Γ is the eigenvector matrix and Λ is the diagonal matrix of eigenvalues of D^2 . In our case, the diagonal entries of the matrix Λ are $\lambda_1 = (5 + \sqrt{5})/2$ and $\lambda_2 = (5 - \sqrt{5})/2$.

Figure 2.5 describes the simulation of the r.v. $\boldsymbol{\xi}$ for $n = 1,000$, $\boldsymbol{\theta} = \boldsymbol{\theta}_1$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_2$.

Exercise 2.27 (Shao 2005). Let $(X_1, \dots, X_n)$ be a random sample from a distribution on $\mathbb{R}$ with the Lebesgue density $\frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$, where $f(x) > 0$ is a known Lebesgue density and $f'(x)$ exists for all $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $\sigma > 0$. Let $\boldsymbol{\theta} = (\mu, \sigma)$. Show that the Fisher information about $\boldsymbol{\theta}$ contained in $X_1, \dots, X_n$ is

$$I(\boldsymbol{\theta}) = \frac{n}{\sigma^2} \begin{pmatrix} \int \frac{\{f'(x)\}^2}{f(x)} dx & \int \frac{f'(x)\{xf'(x)+f(x)\}}{f(x)} dx \\ \int \frac{f'(x)\{xf'(x)+f(x)\}}{f(x)} dx & \int \frac{\{xf'(x)+f(x)\}^2}{f(x)} dx \end{pmatrix},$$

assuming that all integrals are finite.

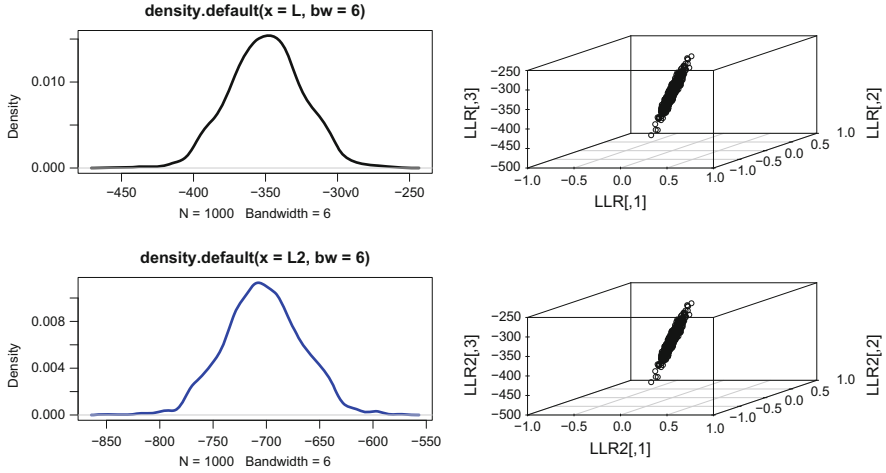


Fig. 2.5 Plots of density estimator and log-likelihood ratio function.  MSEloglikelihood

Denote $g(\mu, \sigma, x) \stackrel{\text{def}}{=} \log \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$. Then

$$\frac{\partial}{\partial \mu} g(\mu, \sigma, x) = -\frac{f'\left(\frac{x-\mu}{\sigma}\right)}{\sigma f\left(\frac{x-\mu}{\sigma}\right)}$$

$$\frac{\partial}{\partial \sigma} g(\mu, \sigma, x) = -\frac{(x-\mu) f'\left(\frac{x-\mu}{\sigma}\right)}{\sigma f\left(\frac{x-\mu}{\sigma}\right)} - \frac{1}{\sigma}.$$

By the direct computation,

$$\begin{aligned} \mathbb{E} \left\{ \frac{\partial}{\partial \mu} g(\mu, \sigma, X_1) \right\}^2 &= \frac{1}{\sigma^2} \int \left\{ \frac{f'\left(\frac{x-\mu}{\sigma}\right)}{f\left(\frac{x-\mu}{\sigma}\right)} \right\}^2 \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx \\ &= \frac{1}{\sigma^2} \int \frac{\{f'\left(\frac{x-\mu}{\sigma}\right)\}^2}{f\left(\frac{x-\mu}{\sigma}\right)} d\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma^2} \int \frac{\{f'(x)\}^2}{f(x)} dx, \end{aligned}$$

$$\mathbb{E} \left\{ \frac{\partial}{\partial \sigma} g(\mu, \sigma, X_1) \right\}^2 = \frac{1}{\sigma^2} \int \left\{ \frac{x-\mu}{\sigma} \frac{f'\left(\frac{x-\mu}{\sigma}\right)}{f\left(\frac{x-\mu}{\sigma}\right)} + 1 \right\}^2 \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx$$

$$\begin{aligned}
&= \frac{1}{\sigma^2} \int \left\{ x \frac{f'(x)}{f(x)} + 1 \right\}^2 f(x) dx \\
&= \frac{1}{\sigma^2} \int \frac{\{x f'(x) + f(x)\}^2}{f(x)} dx,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E} \left\{ \frac{\partial}{\partial \mu} g(\mu, \sigma, x) \frac{\partial}{\partial \sigma} g(\mu, \sigma, x) \right\} \\
&= \frac{1}{\sigma^2} \int \frac{f' \left(\frac{x-\mu}{\sigma} \right)}{f \left(\frac{x-\mu}{\sigma} \right)} \left\{ \frac{x-\mu}{\sigma} \frac{f' \left(\frac{x-\mu}{\sigma} \right)}{f \left(\frac{x-\mu}{\sigma} \right)} + 1 \right\} \frac{1}{\sigma} f \left(\frac{x-\mu}{\sigma} \right) dx \\
&= \frac{1}{\sigma^2} \int \frac{f'(x) \{x f'(x) + f(x)\}}{f(x)} dx.
\end{aligned}$$

The result follows since

$$I(\theta) = n \mathbb{E} \left\{ \frac{\partial}{\partial \theta} \log \frac{1}{\sigma} f \left(\frac{X_1 - \mu}{\sigma} \right) \right\} \left\{ \frac{\partial}{\partial \theta} \log \frac{1}{\sigma} f \left(\frac{X_1 - \mu}{\sigma} \right) \right\}^\top.$$

Exercise 2.28 (Shao 2005). Let X be a random variable having a cumulative distribution function F . Show that if $\mathbb{E}X$ exists, then

$$\mathbb{E}X = \int_0^\infty \{1 - F(x)\} dx - \int_{-\infty}^0 F(x) dx.$$

By Fubini's theorem,

$$\begin{aligned}
\int_0^\infty \{1 - F(x)\} dx &= \int_0^\infty \int_{(x, \infty)} dF(y) dx \\
&= \int_0^\infty \int_{(0, y)} dx dF(y) \\
&= \int_0^\infty y dF(y).
\end{aligned}$$

Similarly,

$$\int_{-\infty}^0 F(x) dx = \int_{-\infty}^0 \int_{(-\infty, x]} dF(y) dx = - \int_{-\infty}^0 y dF(y).$$

If $\mathbb{E}X$ exists, then at least one of $\int_0^\infty y \, dF(y)$ and $\int_{-\infty}^0 y \, dF(y)$ is finite and

$$\mathbb{E}X = \int_{-\infty}^\infty yF(y) = \int_0^\infty \{1 - F(x)\} \, dx - \int_{-\infty}^0 F(x) \, dx.$$

Exercise 2.29 (Shao 2005). Let $(X_1, \dots, X_n)$ be a random sample from the exponential distribution on (a, ∞) with scale parameter 1, where $a \in \mathbb{R}$ is unknown.

1. Construct $(1 - \alpha)$ – confidence interval for a using the cumulative distribution function of the smallest order statistic $X_{(1)}$.
 2. Show that the confidence interval in (i) can also be obtained using a pivotal quantity.
1. The cumulative distribution function of $X_{(1)}$ is

$$F_a(t) = \begin{cases} 0 & t \leq a \\ 1 - \exp^{-n(t-a)} & t > a, \end{cases}$$

which is decreasing in a for fixed $t > a$. A $(1 - \alpha)$ – confidence interval for a has upper limit equal to the unique solution of $F_a(T) = \alpha_1$ and lower limit equal to the unique solution of $F_a(T) = 1 - \alpha_2$, where $\alpha_1 + \alpha_2 = \alpha$. Then, $[T + n^{-1} \log(\alpha_2), T + n^{-1} \log(1 - \alpha_1)]$ is the resulting confidence interval.

2. Note that $W(a) = n(X_{(1)} - a)$ has the exponential distribution on $(0, \infty)$ with scale parameter 1. Therefore the distribution of $W(a)$ doesn't depend on the parameter and, hence, $W(a)$ is a pivotal quantity. The $1 - \alpha$ confidence interval for a constructed this random variable is the same as that derived in item (i).

Exercise 2.30 (Shao 2005). Let F_n be the edf based on a random sample of size n from cdf F on $\mathbb{R}$ having Lebesgue density f . Let $\varphi_n(t)$ be the Lebesgue density of the p th sample quantile $F_n^{-1}(p)$.

Denote by m_p the integer part of np . Introduce also the quantity ℓ_p , which is equal to m_p if np is an integer and is equal to $m_p + 1$ if np is not an integer. Prove that

$$\varphi_n(t) = n \binom{n-1}{\ell_p-1} \{F(t)\}^{\ell_p-1} \{1 - F(t)\}^{n-\ell_p} f(t),$$

1. Using the fact that $nF_n(t)$ has a binomial distribution;
2. Using the Lebesgue density of the j -th order statistic.

1. Since $nF_n(t)$ has the binomial distribution with size n and probability $F(t)$, for any $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}\{F_n^{-1}(p) \leq t\} &= \mathbb{P}\{F_n(t) \geq p\} \\ &= \sum_{i=\ell_p}^n \binom{n}{i} \{F(t)\}^i \{1 - F(t)\}^{n-i}. \end{aligned}$$

Differentiating term by term leads to

$$\begin{aligned}
 \varphi_n(t) &= \sum_{i=l_p}^n \binom{n}{i} i \{F(t)\}^{i-1} \{1-F(t)\}^{n-i} f(t) \\
 &\quad - \sum_{i=l_p}^n \binom{n}{i} (n-i) \{F(t)\}^i \{1-F(t)\}^{n-i-1} f(t) \\
 &= \binom{n}{l_p} l_p \{F(t)\}^{l_p-1} \{1-F(t)\}^{n-l_p} f(t) \\
 &\quad + n \sum_{i=l_p+1}^n \binom{n-1}{i-1} \{F(t)\}^{i-1} \{1-F(t)\}^{n-i} f(t) \\
 &\quad - n \sum_{i=l_p}^{n-1} \binom{n-1}{i} \{F(t)\}^i \{1-F(t)\}^{n-i-1} f(t) \\
 &= n \binom{n-1}{l_p-1} \{F(t)\}^{l_p-1} \{1-F(t)\}^{n-l_p} f(t).
 \end{aligned}$$

2. The Lebesgue density of the j -th order statistic is

$$n \binom{n-1}{j-1} \{F(t)\}^{j-1} \{1-F(t)\}^{n-j} f(t).$$

Then, the result follows from the fact that

$$F_n^{-1}(p) = \begin{cases} X_{(m_p)} & \text{if } np \text{ is an integer,} \\ X_{(m_p+1)} & \text{if } np \text{ is not an integer.} \end{cases}$$

Exercise 2.31. Consider samples $\{Y_i\}_{i=1}^n$, where Y_i are i.i.d. with distribution function $F_Y(y)$. We want to estimate the τ th quantile of the distribution function $F_Y^{-1}(\tau)$:

$$F_Y^{-1}(\tau) \stackrel{\text{def}}{=} \inf \{y \in \mathbb{R} : \tau \leq F_Y(y)\}.$$

This problem can be seen as in a location model:

$$Y_i = \theta^* + \varepsilon_i, \quad \varepsilon_i \sim \text{ALD}(\tau),$$

where $F_{\varepsilon}^{-1}(\tau) = 0$ and ε_i 's are i.i.d. The QMLE estimation follows the framework with ALD likelihood, where ALD stands for "Asymmetric Laplace Distribution", and has probability density function

$$f(u|\tau) = \tau(1 - \tau)\exp\{-\rho_{\tau}(u)\},$$

with $\rho_{\tau}(u) = u\{\tau\mathbf{1}(u \geq 0) - (1 - \tau)\mathbf{1}(u < 0)\}$.

1. Prove that

$$\operatorname{argmin}_{\theta} \mathbb{E}\rho_{\tau}(Y_i - \theta) = F_Y^{-1}(\tau) = \theta^*. \quad (2.37)$$

2. Please write the empirical loss function for the estimation of $F_Y^{-1}(\tau)$.

1. To prove (2.37),

$$\begin{aligned} & \frac{\partial \mathbb{E}\rho_{\tau}(Y_i - \theta)}{\partial \theta} \\ &= \frac{\partial \int \{\tau(Y_i - \theta)\mathbf{1}(Y_i - \theta > 0) + (1 - \tau)\mathbf{1}(Y_i - \theta \leq 0)\} dF_Y(u)}{\partial \theta} \\ &= -\tau \theta f_Y(\theta) - \tau \{1 - F_Y(\theta)\} + \tau \theta f(\theta) - (1 - \tau) f_Y(\theta) + (1 - \tau)(F_Y(\theta) + \theta f_Y(\theta)) \\ &= (1 - \tau)F_Y(\theta) - \tau \{1 - F_Y(\theta)\} \\ &= F_Y(\theta) - \tau \end{aligned}$$

Solve

$$\frac{\partial \mathbb{E}\rho_{\tau}(Y_i - \theta)}{\partial \theta} = 0,$$

we get

$$F(\theta^*) = \tau.$$

Thus, $\theta^* = F_Y^{-1}(\tau)$.

2. An estimator of θ^* would be

$$\operatorname{argmin}_{\theta} \sum_{i=1}^n \{\tau \mathbf{1}(Y_i > \theta) - (1 - \tau) \mathbf{1}(Y_i < \theta)\}.$$

Exercise 2.32. Consider samples $\{(X_i, Y_i)\}_{i=1}^n$ i.i.d., in a regression framework, we now want to estimate the conditional τ th quantile of the conditional distribution function $F_{Y|X}^{-1}(\tau)$. If we believe in the following linear model:

$$Y_i = X_i^\top \theta^* + \epsilon_i, \quad \epsilon_i \sim \text{ALD}(\tau),$$

where $F_{\epsilon|X}^{-1}(\tau) = 0$ and ϵ_i s are i.i.d. Similarly we take a QMLE in an ALD likelihood.

1. Prove that

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \mathbb{E}_{Y|X} \rho_\tau(Y_i - X_i^\top \theta) \quad (2.38)$$

$$F_{Y|X_i}^{-1}(\tau) = X_i^\top \theta^* \quad (2.40)$$

2. Suppose now $\{(X_i, Y_i)\}_{i=1}^n$ is a bivariate i.i.d. sequence from a joint normal distribution $N(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

Please write down the theoretical form of $F_{Y|X}^{-1}(\tau)$. (Hint: Observe that the conditional distribution is again normally distributed, with $\mu_{Y|X=x} = \mu_1 + \sigma_{12}\sigma_{22}^{-1}(x - \mu_2)$ and $\sigma_{Y|X} = \sigma_{11} - \sigma_{12}^2/\sigma_{22}$.)

1. To prove (2.40),

$$\begin{aligned} & \frac{\partial \mathbb{E} \rho_\tau(Y_i - X_i^\top \theta)}{\partial \theta_j} \\ &= \frac{\partial \int \{\tau(Y_i - X_i^\top \theta) \mathbf{1}(Y_i - X_i^\top \theta > 0)\} dF_{Y|X}(u)}{\partial \theta_j} \\ & \quad - \frac{(1 - \tau) \int \{(Y_i - X_i^\top \theta) \mathbf{1}(Y_i - X_i^\top \theta \leq 0)\} dF_{Y|X}(u)}{\partial \theta_j} \\ &= -\tau X_{ij} X_i^\top \theta f_Y(X_i^\top \theta) - X_{ij} \tau \{1 - F_Y(X_i^\top \theta)\} + X_{ij} \tau X_i^\top \theta f_Y(X_i^\top \theta) \\ & \quad - (1 - \tau) X_{ij} f_Y(X_i^\top \theta) + (1 - \tau) X_{ij} (F_Y(X_i^\top \theta) + X_i^\top \theta f_Y(X_i^\top \theta)) \\ &= (1 - \tau) X_{ij} F_{Y|X}(X_i^\top \theta) - \tau X_{ij} \{1 - F_{Y|X}(X_i^\top \theta)\} \\ &= X_{ij} F_{Y|X}(X_i^\top \theta) - \tau X_{ij} \end{aligned}$$

Solve

$$\frac{\partial \mathbb{E}_{Y|X} \rho_\tau(Y_i - X_i^\top \theta)}{\partial \theta_j} = 0, \forall j \in 1, \dots, d$$

we get

$$F_{Y|X}(X_i^\top \theta^*) = \tau, \forall i, 1, \dots, n$$

Thus, $F_{Y|X_i}^{-1}(\tau) = X_i^\top \theta^*$.

2. Use the hint, we have the normal conditional distribution. Given $X = x$, $(Y_i - u_{Y|X=x})/\sigma_{Y|X} \sim \mathbf{N}(0, 1)$. Denote $\Phi^{-1}(\tau)$ as the τ th quantile of a standard normal distribution. Then we have,

$$F_{Y|X=x}^{-1}(\tau) = \sigma_{Y|X} \Phi^{-1}(\tau) + u_{Y|X=x}$$

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