Chapter 2
Governing Equation of Motions

Although all dynamic modelings are essentially "lies," in this book I try to present those that may prove useful about a dynamic system in the real world.

The first step for performing a dynamic analysis is to set up the equations of motions. We start with Newton's second law, which is followed by the virtual work principle (D'Alembert's principle), Hamilton's principles, and Lagrange's equations. We shall also see the important role of energy in the study of dynamic analysis. It is noted that each of the formulations basically represents the same dynamic equilibrium but in a unique form of expression.

2.1 Dynamic Equilibrium

We start with Newton's second law of motions from *Philosophiæ Naturalis Principia Mathematica* (Fig. 2.1), which is the most powerful of Newton's three laws of motions. It states that "the rate of change of momentum of a mass equals the force acting on it." This is illustrated in Fig. 2.2 and expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t} m \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right) = F(t) \tag{2.1}$$

where F, m and a are the external force on a body and the mass and acceleration of the body, respectively, and t is the time.

Here, we describe the position of mass in a Cartesian coordinate, which is referred to as an inertia frame in three dimensions X, Y, and Z (in the equation above, only one dimension X is used). Essentially, no inertia frames exist—a conclusion derived from the debates going back to the late nineteenth. However, it is usually convenient to find a frame for the purpose of a particular situation so that the dynamic analysis agrees with the observations. In this sense, the Earth is normally taken to be an inertial frame even if it rotates forever. However, the Earth is not an appropriate inertia frame for large scale motions such as those of the atmosphere and oceans [36]. For example, it is common knowledge that winds

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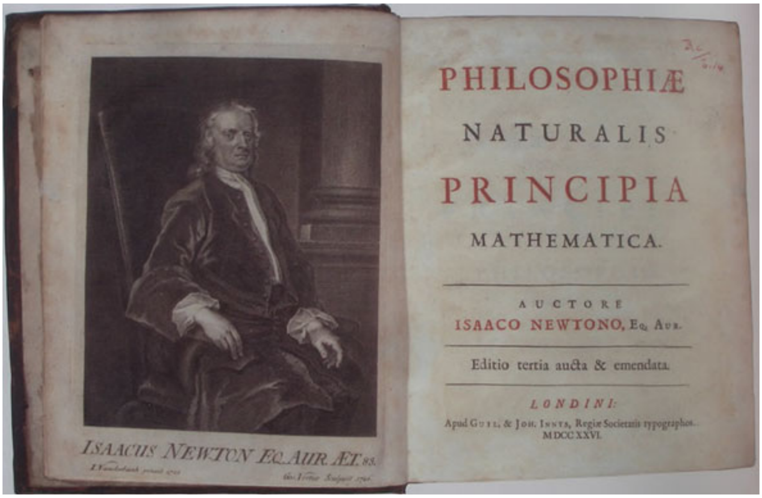

Fig. 2.1 *Philosophiæ Naturalis Principia Mathematica*, which laid the foundation for dynamic equilibrium

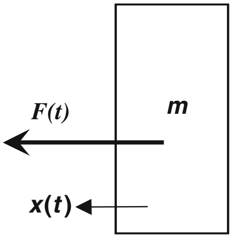

Fig. 2.2 A single mass under external force $F(t)$

blow more frequently along an east–west direction, but they would blow north–south if the Earth was not rotating.

Newton's second law of motion was a breakthrough in the understanding of dynamics, and it confirms that force only causes a change of velocity, correcting the previous view, proposed by Aristotle (384–322 BC), that force maintains the velocity.

If one attaches a spring and a damper to the mass, as shown in Fig. 2.3, by assuming that the spring obeys Hooke's law and the damping is of viscous type (the damping force is proportional to the velocity of the mass), the equilibrium is formulated by adding the terms of spring- and damping-induced forces:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{2.2}$$

where the dot over the symbol represents differentiation with respect to time t.

In the equilibrium equation above, all the motion parameters (displacement, velocity and acceleration) are lower-order differentiations of displacement with

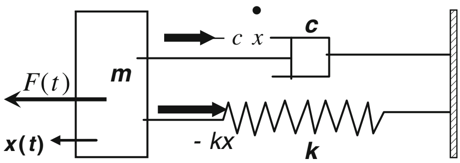

Fig. 2.3 A SDOF spring-mass-damper system

respect to time, and are sensible by human beings. Higher-order differentiations do not appear in the equilibrium equation because scientists do not find their link to equilibrium of force, but they are already used in many engineering applications. For example, the time-derivative of acceleration is called jerk/jolt, i.e., the change of acceleration with time. The jerk can be illustrated by the example of a person gradually pressing the surface of a wall: one's hand can feel that the force is increasing (a change in the force) until it reaches a constant force pressure. The jerk can also be sensed when, moving quickly on a bicycle, one suddenly brakes hard. If the bicycle were to slide off a paved track onto wet grass, even though the friction between the bicycle tire and grass is still present due to sliding, it will decrease and the bicycle will slow down less rapidly, thus undergoing a positive jerk. The rider would feel pressure on his/her hip, as if the bicycle were speeding up, even if it does not really go any faster. As its name suggests, jerk is important when evaluating the destructive effect of motions on a mechanism or the discomfort caused to passengers onboard vehicles. Movement-sensitive instruments need to be kept within specified limits of jerk as well as acceleration to avoid damage. For passenger comfort, a train in operation will typically be required to keep jerk below 2 m/s^3. In the aerospace industry, a type of instrument called a jerk-meter is used for measuring jerk.

A system that has a small number of degrees-of-freedoms can be evaluated efficiently by directly using the equation of the motions above. However, when the degrees-of-freedom become too large for an analyst to handle using this simplified direct method, other methods for formulating and solving the equations of motions have to be used instead. Examples of the former type are Hamilton's principle or the finite element method, while the latter type include the modal superposition method (Sect. 13.4), and direct time integration method (Sect. 13.5) etc. In addition, for continuous systems with non-uniform mass or stiffness distribution, the Rayleigh energy method (Chap. 5) can also be used to obtain a quick but approximate characterization of the dynamic characteristics.

2.2 Principle of Virtual Displacements

Newton's second law can be more practically expressed as D'Alembert's principle: the condition for dynamic equilibrium is that the total force is in equilibrium with the inertia forces.

For a complex structure or a system, it is not practical to describe the forces acting on each mass point as a vectorial addition of all those forces. In this situation, the principle of virtual displacement is particularly appealing. It solves the dynamic equilibrium by indirectly formulating the equations of motions, which is essentially an energy approach, i.e., the total virtual work done by effective forces applied on a system through virtual displacements, which is compatible with the system constraints, will be zero. By saying “effective forces,” they contain both the normal and inertia forces.

2.3 Hamilton’s Principle Through Lagrange’s Equations

Note that even if D’Alembert’s principle eliminates the problem of force addition in a vectorial context, the virtual work itself is still a product of force vector and virtual displacement vector, and the equations of motions are formulated in terms of position coordinates that may not all be independent. This problem can be solved by the more powerful Hamilton’s principle through Lagrange’s equations.

Many principles describe a system by minimizing certain physical quantities, as does Hamilton’s principle, which states that (as shown in Fig. 2.4), for a conservative system, of all the possible paths along which a dynamical system may travel from one point to another within a specified time interval (consistent with any constraints), the actual path (called the true, Newtonian or dynamical path) followed is that which minimizes the time integral of the difference between the kinetic and potential energy:

$$\delta I = \int_{t_1}^{t_2} \delta(T - V)dt = \int_{t_1}^{t_2} \delta L dt = 0 \tag{2.3}$$

where δ is the first variation and L is called the Lagrangian function, which represents the difference between kinetic (T) and potential (V) energy of the system; the former is a function of particle velocity, the latter is a function of position. Hamilton’s principle represents the most condensed description of motion for a single system. For a single-degree-of-freedom system, as shown in Fig. 2.3, it can be expressed as:

$$T = \frac{1}{2} m \dot{x}(t)^2 \tag{2.4}$$

$$V = \frac{1}{2} k x(t)^2 \tag{2.5}$$

Surprisingly, in certain senses, Hamilton’s principle coincides with the statements of the Chinese philosopher Laozi (Fig. 2.5), who expressed in his *Daodejing* (No 48, 《道德经》) around 500 BC that “non-action is all action” (“无为而无不为”).

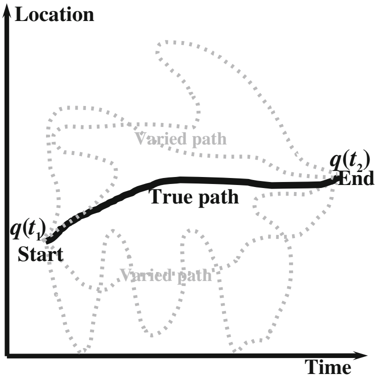

Fig. 2.4 Many paths, among which the true path (*solid line*) is that which follows Newton's law (minimize the action); varied paths are not possible [38]

Fig. 2.5 Chinese philosopher Laozi (老子) (painting by Jing Yu)

The convenience of Hamilton's principle lies in the fact that all the system differential equations of motions can be derived from two scalar functions, the kinetic energy and the potential energy, with the virtual work corresponding to

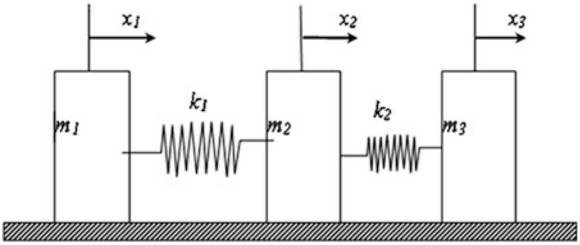

Fig. 2.6 *Three* masses resting on the horizontal ground are free to move horizontally, no friction is assumed between the mass and the ground surface

non-conservative forces [37]. Describing a system by applying Hamilton's principle allows people to determine the equations of motions for a system for which we would not be able to derive these equations easily from Newton's laws. However, one nevertheless needs to bear in mind that Hamilton's principle is not a new law, but simply provides a new description of Newton's laws.

The extension from Newton's second law to Hamilton's principle also makes it possible to handle dynamic problems for deformable bodies by using continuum mechanics. This paves the way for the development of finite element discretization of deformable bodies [24].

In a more general case with a system comprising non-conservative forces (such as the ones caused by frictions), one may express Hamilton's principle via Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = Q_j, j = 1, 2, \ldots, n \tag{2.6}$$

where D is the dissipation function. For a single degree-of-freedom system shown in Fig. 2.3, $D = \frac{1}{2} c \dot{x}(t)^2 \cdot Q_j$ represents the non-conservative forces and q_j is the generalized degree-of-freedom (coordinate or path), which is not unique and related to the physical coordinate. The dot over the symbols represents differentiation with respect to time. It is noted that the forces are not direct knowns; instead, their information is contained in the kinetic and potential energy terms.

Since $\frac{\partial T}{\partial q_j}$ is zero, the equation above is finally written as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial D}{\partial \dot{q}_j} + \frac{\partial V}{\partial q_j} = Q_j, j = 1, 2, \ldots, n \tag{2.7}$$

Each item in this equation exactly corresponds to the equations of motions (Eq. (2.2)).

Example [39]: Consider a system with three masses connected by springs as shown in Fig. 2.6. Establish the equations of motions using Lagrange's equations and summarize them in a matrix form.
Solution: The Lagrange's equations are written as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}+\frac{\partial V}{\partial q_j}+\frac{\partial D}{\partial \dot{q}_j}= Q_j, j=1,2,\ldots,n$$

Since there is neither friction nor external forces, $D = 0$ and $Q_j = 0$.
Let q_1, q_2 and q_3 be the generalized degree-of-freedom. We have

$$T=\frac{1}{2}m_1\dot{q}_1^2+\frac{1}{2}m_2\dot{q}_2^2+\frac{1}{2}m_3\dot{q}_3^2$$

$$\begin{aligned} V &=\frac{1}{2}k_1(q_2-q_1)^2+\frac{1}{2}k_2(q_3-q_2)^2 \\ &=\frac{1}{2}k_1\left(q_2^2-2q_1q_2+q_1^2\right)+\frac{1}{2}k_2\left(q_3^2-2q_3q_2+q_2^2\right) \end{aligned}$$

With regard to q_1:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right)-\frac{d}{dt}\left(\frac{\partial T}{\partial q_1}\right)=\frac{d}{dt}\left(m_1\ \dot{q}_1\right)-0=m_1\ \ddot{q}_1$$

$$\frac{\partial V}{\partial q_1}=\frac{1}{2}k_1(-2q_2+2q_1)=-k_1q_2+k_1q_1$$

With regard to q_2:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right)-\frac{d}{dt}\left(\frac{\partial T}{\partial q_2}\right)=\frac{d}{dt}\left(m_2\ \dot{q}_2\right)-0=m_2\ \ddot{q}_2$$

$$\frac{\partial V}{\partial q_2}=k_1q_2-k_1q_1-k_2q_3+k_2q_2$$

With regard to q_3:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right)-\frac{d}{dt}\left(\frac{\partial T}{\partial q_3}\right)=\frac{d}{dt}\left(m_3\ \dot{q}_3\right)-0=m_3\ \ddot{q}_3$$

$$\frac{\partial V}{\partial q_3}=k_2q_3-k_2q_2$$

Insert the equations above into the Lagrange's equations:

$$m_1\ddot{q}_1=k_1q_2-k_1q_1$$

$$m_2\ddot{q}_2=-k_1q_2+k_1q_1+k_2q_3-k_2q_2$$

$$m_3 \ddot{q}_3 = -k_2 q_3 + k_2 q_2$$

Sum up the three equations above in a matrix form:ą

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = 0$$

2.4 Momentum Equilibrium

Once again, we recall Newton's second law: if there are no forces acting on the mass, the momentum will be constant. For systems comprising more than one body mass, each one has an individual momentum, but their sum will be constant if there are no external forces acting on the system. The sum of the momentum is expressed as:

$$Momentum\, sum = m_1 v_1 + m_2 v_2 + \ldots + m_n v_n \tag{2.8}$$

where the lower indices serve to identify the mass.

The momentum equilibrium has abundant applications in the engineering world. One example of its application is the calculation of the speed of two cars before and after their collision. In structural engineering, we can also find their applications in the design of various types of dynamic absorbers [23].

In civil engineering, this momentum equilibrium can be utilized in designing, for example, a tuned mass damper (TMD) or an impact damper, both of which comprise a secondary mass attached to (in the case of TMD) or constrained by (in the case of impact damper) a vibrating structure (main structure). This mass has dynamic characteristics that relate closely to that of the main structure. By varying the ratio of the mass to the primary body (main structure), the frequency ratio between the two masses, and the damping ratio associated to the secondary mass, the momentum exchange can control the maximum responses of the main structure.

Figure 2.7 shows a single TMD fitted to the underside of a concrete deck at the Infinity Bridge in the north-east of England. The installation of more such TMDs is planned for when the issue of maintenance arises. Today, many TMDs are installed on high-rise buildings and bridge structures to mitigate the dynamic responses due to dynamic loadings induced by wind, earthquake, impact and mechanical vibrations [23]. Representative examples are the Taipei World Financial Center (Fig. 1.23), Washington National Airport Tower, Sydney Tower, Citicorp Center (New York), the John Hancock Building (Boston), and the Crystal Tower (Osaka, Japan).

Figure 2.8 shows an impact damper, which comprises a small rigid mass placed inside a container mounted on the side of the structure. There is a small optimal

Fig. 2.7 TMDs installed under the bridge deck of the Infinity Bridge (photo by John Yeadon)

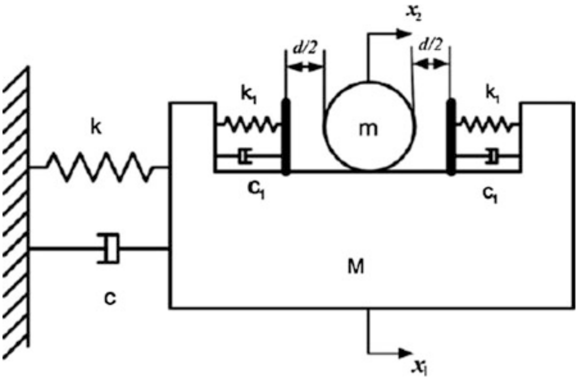

Fig. 2.8 An impact damper model [40]

clearance between the small mass and the container wall, thus allowing collisions between the mass and the container wall to occur when the displacement along the clearance direction exceeds the optimal clearance. The collision achieves both momentum exchange and energy dissipation, the latter of which is mainly produced on the contacting surface between the mass and the wall. A schematic diagram of an impact damper is shown in Fig. 2.8.

2.5 Validity of Classical Dynamics

Before ending this chapter, this author would like to quote a famous conversation that is reported to have taken place between Napoleon Bonaparte, Laplace and Lagrange (Fig. 2.9) [41]:

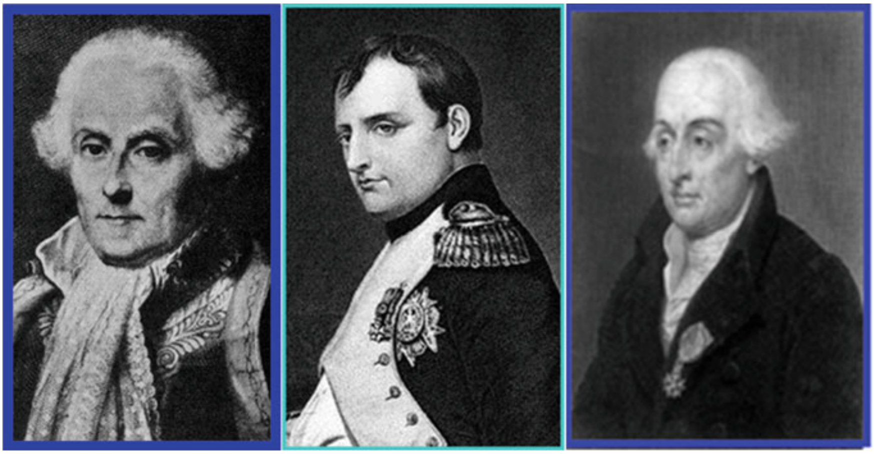

Fig. 2.9 Laplace (*left*), Napoleon (*middle*) and Lagrange (*right*)

Napoleon: How is it that, although you say so much about the Universe in this huge book, you say nothing about its Creator?

Laplace: No, Sire, I had no need of that hypothesis. [This is an example of his arrogance at that time].

Lagrange: Ah, but it is such a good hypothesis, it explains so many things!

Laplace: Indeed, Sire, Monsieur Lagrange has, with his usual sagacity, put his finger on the precise difficulty with the hypothesis, it explains everything, but predicts nothing.

Laplace was confident that he could predict the motions of everything, but now we know that the equations of motions introduced in this chapter are only valid for the mechanical universe [38] and do not apply for particles at rather small distances or with extremely high velocities. Nevertheless, they remain valid for the scientific research in the fields of civil and mechanical engineering.

Arrogant and humble attitudes are both necessary for scientific research. In the author's opinion, however, it is most important to be humble when investigating problems that are beyond our knowledge.

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