Chapter 2
The Lévy–Itô Decomposition and Path Structure

The main aim of this chapter is to establish a rigorous understanding of the structure of the paths of Lévy processes. The way we shall do this is to prove the assertion in Theorem 1.6 that, given any characteristic exponent, Ψ, belonging to an infinitely divisible distribution, there exists a Lévy process with the same characteristic exponent. This will be done by establishing the so-called Lévy–Itô decomposition, which describes the structure of a general Lévy process in terms of three independent auxiliary Lévy processes, each with a different type of path behaviour. In doing so it will be necessary to digress temporarily into the theory of Poisson random measures and associated square-integrable martingales. Understanding the Lévy–Itô decomposition will allow us to distinguish a number of important, but nonetheless general, subclasses of Lévy processes according to their path type. The chapter is concluded with a discussion of the interpretation of the Lévy–Itô decomposition in the context of some of the applied probability models mentioned in Chap. 1.

2.1 The Lévy–Itô Decomposition

According to Theorem 1.3, any characteristic exponent Ψ belonging to an infinitely divisible distribution can be written, after some simple reorganisation, in the form

$$\begin{aligned}\Psi(\theta) &= \left\{\mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2\right\} \\ &\quad + \left\{\Pi\big(\mathbb{R}\backslash(-1,1)\big)\int_{|x|\geq 1}\big(1-\mathrm{e}^{\mathrm{i}\theta x}\big)\frac{\Pi(\mathrm{d}x)}{\Pi(\mathbb{R}\backslash(-1,1))}\right\} \\ &\quad + \left\{\int_{0<|x|<1}\big(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x\big)\Pi(\mathrm{d}x)\right\}, \qquad (2.1)\end{aligned}$$

for all $\theta \in \mathbb{R}$, where $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1\wedge x^2)\Pi(\mathrm{d}x) < \infty$. Note that this condition on Π implies that $\Pi(A) < \infty$ for

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all Borel A such that 0 is in the interior of A^c and, in particular, that $\Pi(\mathbb{R}\backslash(-1,1)) \in [0,\infty)$. In the case that $\Pi(\mathbb{R}\backslash(-1,1)) = 0$, one should think of the second set of curly brackets in (2.1) as absent. Call the contents of the three sets of curly brackets in (2.1) $\Psi^{(1)}(\theta)$, $\Psi^{(2)}(\theta)$ and $\Psi^{(3)}(\theta)$. The essence of the Lévy–Itô decomposition revolves around showing that $\Psi^{(1)}(\theta)$, $\Psi^{(2)}(\theta)$ and $\Psi^{(3)}(\theta)$ correspond to the characteristic exponents of three different types of Lévy processes. Therefore, Ψ may be considered as the characteristic exponent of the independent sum of these three Lévy processes, which is again a Lévy process (cf. Exercise 1.1). Indeed, as we have already seen in Chap. 1, $\Psi^{(1)}$ and $\Psi^{(2)}$ correspond, respectively, to a linear Brownian motion, say, $X^{(1)} = \{X_t^{(1)} : t \geq 0\}$, where

$$X_t^{(1)} = \sigma B_t - at, \quad t \geq 0, \tag{2.2}$$

and an independent compound Poisson process, say $X^{(2)} = \{X_t^{(2)} : t \geq 0\}$, where,

$$X_t^{(2)} = \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0, \tag{2.3}$$

$\{N_t : t \geq 0\}$ is a Poisson process with rate $\Pi(\mathbb{R}\backslash(-1,1))$ and $\{\xi_i : i \geq 1\}$ are independent and identically distributed with common distribution $\Pi(\mathrm{d}x)/\Pi(\mathbb{R}\backslash(-1,1))$ concentrated on $\{x : |x| \geq 1\}$ (unless $\Pi(\mathbb{R}\backslash(-1,1)) = 0$ in which case $X^{(2)}$ is the process which is identically zero).

The proof of existence of a Lévy process with characteristic exponent given by (2.1) thus boils down to showing the existence of a Lévy process, $X^{(3)}$, whose characteristic exponent is given by $\Psi^{(3)}$. Note that

$$\begin{aligned}\int_{0<|x|<1} &\left(1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x\right)\Pi(\mathrm{d}x) \\ &= \sum_{n\geq 0}\left\{\lambda_n \int_{2^{-(n+1)}\leq|x|<2^{-n}} \left(1 - \mathrm{e}^{\mathrm{i}\theta x}\right)F_n(\mathrm{d}x)\right. \\ &\quad \left. + \mathrm{i}\theta\lambda_n\left(\int_{2^{-(n+1)}\leq|x|<2^{-n}} xF_n(\mathrm{d}x)\right)\right\},\end{aligned} \tag{2.4}$$

where $\lambda_n = \Pi(\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\})$ and $F_n(\mathrm{d}x) = \Pi(\mathrm{d}x)/\lambda_n$, restricted to $\{x : 2^{-(n+1)} \leq |x| < 2^{-n}\}$ (again with the understanding that the n-th integral is absent if $\lambda_n = 0$). It would appear from (2.4) that the process $X^{(3)}$ consists of the superposition of (at most) a countable number of independent compound Poisson processes with different arrival rates and additional linear drift. To understand the mathematical sense of this superposition, we shall need to establish some facts concerning Poisson random measures and related martingales. This is done in Sects. 2.2 and 2.3. The precise construction of $X^{(3)}$ is given in Sect. 2.5.

The identification of a Lévy process, X, as the independent sum of processes $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ is attributed to Lévy (1954) and Itô (1942) and is thus known as the *Lévy–Itô decomposition*. Formally speaking, and in a little more detail, we quote the Lévy–Itô decomposition in the form of a theorem.

Theorem 2.1 (Lévy–Itô decomposition) *Given any $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and measure Π concentrated on $\mathbb{R}\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}} \big(1 \wedge x^2\big) \Pi(\mathrm{d}x) < \infty,$$

there exists a probability space on which three independent Lévy processes exist, $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$, where $X^{(1)}$ is a linear Brownian motion given by (2.2), *$X^{(2)}$ is a compound Poisson process given by* (2.3) *and $X^{(3)}$ is a square-integrable martingale with an almost surely countable number of path discontinuities* (*or jumps*) *on each finite time interval, which are of magnitude less than unity, and with characteristic exponent given by $\Psi^{(3)}$. Moreover, by taking $X = X^{(1)} + X^{(2)} + X^{(3)}$, the conclusion of Theorem* 1.6 *holds, namely that there exists a probability space on which a Lévy process is defined with characteristic exponent*

$$\Psi(\theta) = a\mathrm{i}\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \big(1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbf{1}_{(|x|<1)}\big) \Pi(\mathrm{d}x), \tag{2.5}$$

for $\theta \in \mathbb{R}$.

2.2 Poisson Random Measures

Poisson random measures turn out to be the right mathematical mechanism to describe the jump structure embedded in any Lévy process. Before engaging in an abstract study of Poisson random measures, we give a rough idea of how they are related to the jump structure of Lévy processes by considering the less complicated case of a compound Poisson process.

Suppose that $X = \{X_t : t \geq 0\}$ is a compound Poisson process with a drift taking the form

$$X_t = \delta t + \sum_{i=1}^{N_t} \xi_i, \qquad t \geq 0,$$

where $\delta \in \mathbb{R}$ and, as usual, $\{\xi_i : i \geq 1\}$ are independent and identically distributed random variables with common distribution function F. Further, let $\{T_i : i \geq 1\}$ be the times of arrival of the Poisson process $N = \{N_t : t \geq 0\}$ with rate $\lambda > 0$. See Fig. 2.1.

Suppose now that we pick any set in $A \in \mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R}\backslash\{0\})$. Define

$$N(A) = \#\big\{i \geq 1 : (T_i, \xi_i) \in A\big\} = \sum_{i=1}^{\infty} \mathbf{1}_{((T_i,\xi_i)\in A)}. \tag{2.6}$$

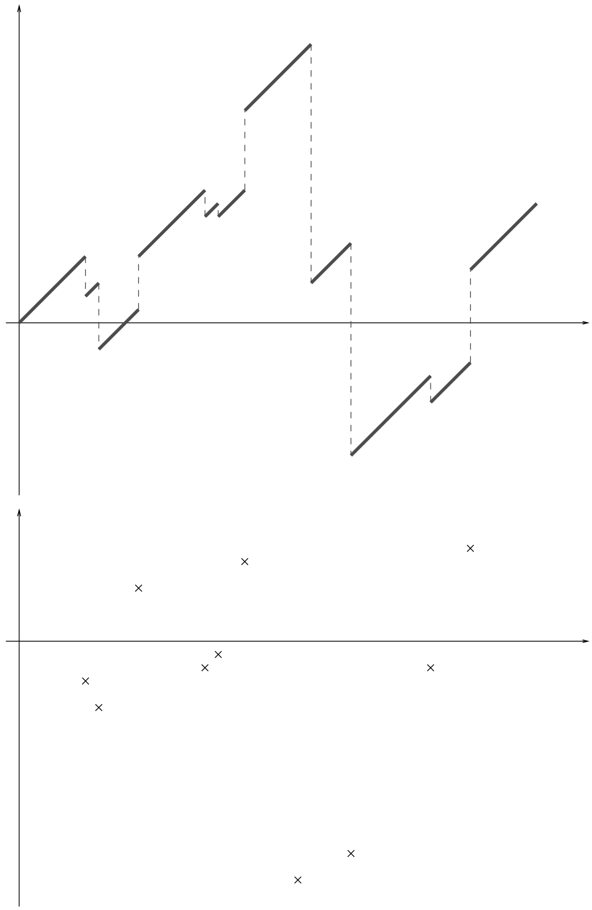

Fig. 2.1 The initial period of a sample path of a compound Poisson process with drift $\{X_t : t \geq 0\}$ and the field of points it generates.

Clearly, since X experiences an almost surely finite number of jumps over a finite period of time, it follows that $N(A) < \infty$ almost surely when $t \geq 0$ and $A \subseteq \mathcal{B}[0,t) \times \mathcal{B}(\mathbb{R}\backslash\{0\})$.

Lemma 2.2 *Choose $k \geq 1$. If $A_1, \ldots, A_k$ are disjoint sets in $\mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R}\backslash\{0\})$, then $N(A_1), \ldots, N(A_k)$ are mutually independent and Poisson distributed*[1] *with*

[1]We understand a Poisson random variable whose parameter is infinite to be infinite valued with probability 1.

parameters $\lambda_i := \lambda \int_{A_i} \mathrm{d}t \times F(\mathrm{d}x)$, *respectively. Further, for* $\mathbb{P}$*-almost every realisation of* X, $N : \mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R}\backslash\{0\}) \to \{0,1,2,\dots\} \cup \{\infty\}$ *is a measure.*[2]

Proof First recall a classic result concerning the Poisson process $\{N_t : t \geq 0\}$. That is, when $t > 0$, the law of $\{T_1, \dots, T_n\}$ conditional on the event $\{N_t = n\}$ is the same as the law of an ordered independent sample of size n from the uniform distribution on $[0,t]$. (See Exercise 2.2.) This, together with the fact that the variables $\{\xi_i : i = 1, \dots, n\}$ are independent and identically distributed with common law F, implies that, conditional on $\{N_t = n\}$, the joint law of the pairs $\{(T_i, \xi_i) : i = 1, \dots, n\}$ is that of n independent bivariate random variables, with common distribution $t^{-1}\mathrm{d}s \times F(\mathrm{d}x)$ on $[0,t] \times (\mathbb{R}\backslash\{0\})$, ordered in time. In particular, for any $A \in \mathcal{B}[0,t] \times \mathcal{B}(\mathbb{R}\backslash\{0\})$, the random variable $N(A)$ conditional on the event $\{N_t = n\}$ is a binomial random variable with probability of success given by $\int_A t^{-1}\mathrm{d}s \times F(\mathrm{d}x)$. A generalisation of this statement for the k-tuple $(N(A_1), \dots, N(A_k))$, where $A_1, \dots, A_k$ are mutually disjoint and chosen from $\mathcal{B}[0,t] \times \mathcal{B}(\mathbb{R}\backslash\{0\})$, is the following. Suppose that $A_0 = \{[0,t] \times \mathbb{R}\}\backslash\{A_1 \cup \cdots \cup A_k\}$, $\sum_{i=1}^k n_i \leq n$, $n_0 = n - \sum_{i=1}^k n_i$ and $\lambda_0 = \int_{A_0} \lambda \mathrm{d}s \times F(\mathrm{d}x) = \lambda t - \lambda_1 - \cdots - \lambda_k$, then $(N(A_1), \dots, N(A_k))$ has the following multinomial law,

$$\begin{aligned}
&\mathbb{P}\big(N(A_1) = n_1, \dots, N(A_k) = n_k | N_t = n\big) \\
&\quad = \frac{n!}{n_0! n_1! \cdots n_k!} \prod_{i=0}^k \left(\frac{\lambda_i}{\lambda t}\right)^{n_i}.
\end{aligned}$$

Summing out the conditioning on N_t, it follows that

$$\begin{aligned}
&\mathbb{P}\big(N(A_1) = n_1, \dots, N(A_k) = n_k\big) \\
&\quad = \sum_{n \geq \sum_{i=1}^k n_i} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{n!}{n_0! n_1! \cdots n_k!} \prod_{i=0}^k \left(\frac{\lambda_i}{\lambda t}\right)^{n_i} \\
&\quad = \sum_{n \geq \sum_{i=1}^k n_i} \mathrm{e}^{-\lambda_0} \frac{\lambda_0^{(n - \sum_{i=1}^k n_i)}}{(n - \sum_{i=1}^k n_i)!} \left(\prod_{i=1}^k \mathrm{e}^{-\lambda_i} \frac{\lambda_i^{n_i}}{n_i!}\right) \\
&\quad = \prod_{i=1}^k \mathrm{e}^{-\lambda_i} \frac{\lambda_i^{n_i}}{n_i!},
\end{aligned}$$

[2] Specifically, $\mathbb{P}$-almost surely, $N(\emptyset) = 0$ and for disjoint $A_1, A_2, \dots$ in $\mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R}\backslash\{0\})$, we have

$$N\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} N(A_i).$$

showing that $N(A_1), \ldots, N(A_k)$ are independent and Poisson distributed, as claimed.

To complete the proof for arbitrary disjoint $A_1, \ldots, A_k$, for each $i = 1, \ldots, k$, write A_i as a countable union of disjoint sets, each of which belongs to $\mathcal{B}[0, t') \times \mathcal{B}(\mathbb{R}\backslash\{0\})$ for some $t' > 0$. Recall that the sum of an independent sequence of Poisson random variables is Poisson distributed with the sum of their rates. If we agree that a Poisson random variable with infinite rate is infinite with probability one (see Exercise 2.1), then the proof is complete.

Finally the fact that N is a measure $\mathbb{P}$-almost surely follows immediately from its definition. □

Lemma 2.2 shows that $N : \mathcal{B}[0, \infty) \times \mathcal{B}(\mathbb{R}\backslash\{0\}) \to \{0, 1, \ldots\} \cup \{\infty\}$ fulfils the following definition of a Poisson random measure.

Definition 2.3 (Poisson random measure) Let $(S, \mathcal{S}, \eta)$ be an arbitrary sigma-finite measure space and $(\Omega, \mathcal{F}, P)$ a probability space. Let $N : \Omega \times \mathcal{S} \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ in such a way that the family $\{N(\cdot, A) : A \in \mathcal{S}\}$ are random variables defined on $(\Omega, \mathcal{F}, P)$. Henceforth, for convenience, we shall suppress the dependency of N on ω. Then N is called a Poisson random measure on $(S, \mathcal{S}, \eta)$ (or sometimes a Poisson random measure on S with intensity η) if

(i) for mutually disjoint $A_1, \ldots, A_n$ in $\mathcal{S}$, the variables $N(A_1), \ldots, N(A_n)$ are independent,
(ii) for each $A \in \mathcal{S}$, $N(A)$ is Poisson distributed with parameter $\eta(A)$ (here we allow $0 \leq \eta(A) \leq \infty$),
(iii) $N(\cdot)$ is a measure P-almost surely.

In the second condition, we note that, if $\eta(A) = 0$, then it is understood that $N(A) = 0$ with probability one and if $\eta(A) = \infty$ then $N(A)$ is infinite with probability one.

In the case of (2.6), we have $S = [0, \infty) \times (\mathbb{R}\backslash\{0\})$ and $\mathrm{d}\eta = \lambda \mathrm{d}t \times \mathrm{d}F$. Note also that, by construction of the compound Poisson process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for each $A \in \mathcal{B}[0, \infty) \times \mathcal{B}(\mathbb{R}\backslash\{0\})$, the random variable $\mathbf{1}_{((T_i, \xi_i) \in A)}$ is $\mathcal{F}$-measurable, and hence so is the variable $N(A)$.

We complete this section by proving that a Poisson random measure, as defined above, exists. This is done in Theorem 2.4 below, the proof of which has many similarities to the proof of Lemma 2.2.

Theorem 2.4 *There exists a Poisson random measure $N(\cdot)$ as in Definition 2.3.*

Proof First suppose that S is such that $0 < \eta(S) < \infty$. There exists a standard construction of an infinite product space, say $(\Omega, \mathcal{F}, P)$, defined on which are the independent random variables

$$\mathtt{N} \quad \text{and} \quad \{\upsilon_1, \upsilon_2, \ldots\},$$

such that $\mathtt{N}$ has a Poisson distribution with parameter $\eta(S)$ and each of the variables υ_i has distribution $\eta(\mathrm{d}x)/\eta(S)$ on S. Define for each $A \in \mathcal{S}$,

$$N(A) = \sum_{i=1}^{\mathtt{N}} \mathbf{1}_{(\upsilon_i \in A)}, \tag{2.7}$$

so that $\mathtt{N} = N(S)$. For each $A \in \mathcal{S}$ and $i \geq 1$, the random variables $\mathbf{1}_{(\upsilon_i \in A)}$ and $\mathtt{N}$ are $\mathcal{F}$-measurable, hence so are the random variables $N(A)$.

When presented with mutually disjoint sets of $\mathcal{S}$, say $A_1, \dots, A_k$, a calculation identical to the one given in the proof of Lemma 2.2 shows, again, that

$$P\big(N(A_1) = n_1, \dots, N(A_k) = n_k\big) = \prod_{i=1}^{k} \mathrm{e}^{-\eta(A_i)} \frac{\eta(A_i)^{n_i}}{n_i!},$$

for non-negative integers $n_1, n_2, \dots, n_k$. Returning to Definition 2.3, it is now clear from the previous calculation that conditions (i)–(iii) are met by $N(\cdot)$. In particular, similar to the case dealt with in Lemma 2.2, the third condition is automatic as $N(\cdot)$ is a counting measure by definition.

Next, we turn to the case that $(S, \mathcal{S}, \eta)$ is a sigma-finite measure space. The sigma-finite assumption means that there exists a countable disjoint exhaustive sequence of sets $B_1, B_2, \dots$ in S such that $0 < \eta(B_i) < \infty$ for each $i \geq 1$. Define, for each $i \geq 1$, the measures $\eta_i(\cdot) = \eta(\cdot \cap B_i)$. The first part of this proof shows that, for each $i \geq 1$, there exists some probability space, say $(\Omega_i, \mathcal{F}_i, P_i)$, on which we can define a Poisson random measure, say $N_i(\cdot)$, in $(B_i, \mathcal{S} \cap B_i, \eta_i)$, where $\mathcal{S} \cap B_i = \{A \cap B_i : A \in \mathcal{S}\}$ (the reader should verify easily that $\mathcal{S} \cap B_i$ is indeed a sigma-algebra on B_i). The idea is now to show that

$$N(\cdot) = \sum_{i \geq 1} N_i(\cdot \cap B_i)$$

is a Poisson random measure on S, with intensity η, defined on the product space

$$(\Omega, \mathcal{F}, P) := \prod_{i \geq 1} (\Omega_i, \mathcal{F}_i, P_i).$$

First note, again from its definition, that $N(\cdot)$ is P-almost surely a measure. In particular with the help of Fubini's Theorem, for disjoint $A_1, A_2, \dots$, we have

$$\begin{aligned} N\bigg(\bigcup_{j \geq 1} A_j\bigg) &= \sum_{i \geq 1} N_i\bigg(\bigcup_{j \geq 1} A_j \cap B_i\bigg) = \sum_{i \geq 1} \sum_{j \geq 1} N(A_j \cap B_i) \\ &= \sum_{j \geq 1} \sum_{i \geq 1} N(A_j \cap B_i) \\ &= \sum_{j \geq 1} N(A_j). \end{aligned}$$

Next, for each $i \geq 1$, we have that $N_i(A \cap B_i)$ is Poisson distributed with parameter $\eta_i(A)$; Exercise 2.1 tells us that under P the random variable $N(A)$ is Poisson distributed with parameter $\eta(A)$. The proof is complete once we show that, for disjoint $A_1, \ldots, A_k$ in $\mathcal{S}$, the variables $N(A_1), \ldots, N(A_k)$ are all independent under P. However this is obvious since the double array of variables,

$$\left\{N_i(A_j \cap B_i) : i = 1, 2, \ldots \text{ and } j = 1, \ldots, k\right\},$$

is also an independent sequence of variables. □

From the construction of the Poisson random measure, the following two corollaries should be clear.

Corollary 2.5 *Suppose that $N(\cdot)$ is a Poisson random measure on $(S, \mathcal{S}, \eta)$. Then for each $A \in \mathcal{S}$, $N(\cdot \cap A)$ is a Poisson random measure on $(S \cap A, \mathcal{S} \cap A, \eta(\cdot \cap A))$. Further, if $A, B \in \mathcal{S}$ and $A \cap B = \emptyset$, then $N(\cdot \cap A)$ and $N(\cdot \cap B)$ are independent.*

Corollary 2.6 *Suppose that $N(\cdot)$ is a Poisson random measure on $(S, \mathcal{S}, \eta)$, then the support of $N(\cdot)$ is P-almost surely countable. If, in addition, η is a finite measure, then the support is P-almost surely finite.*

Finally, note that, if η is a measure with an atom at, say, the singleton $s \in S$ and $\{s\} \in \mathcal{S}$, then it follows from the definition of $N(\cdot)$ in the proof of Theorem 2.4 that $P(N(\{s\}) \geq 1) > 0$. Conversely, if η has no atoms then $P(N(\{s\}) = 0) = 1$ for all singletons $s \in S$ such that $\{s\} \in \mathcal{S}$. For further discussion on this point, the reader is referred to Kingman (1993).

2.3 Functionals of Poisson Random Measures

Suppose as in Sect. 2.2 that $N(\cdot)$ is a Poisson random measure on the measure space $(S, \mathcal{S}, \eta)$. As $N(\cdot)$ is P-almost surely a measure, classical measure theory now allows us to talk of

$$\int_S f(x) N(\mathrm{d}x) \tag{2.8}$$

as a well-defined $[0, \infty]$-valued random variable, for measurable functions $f : S \to [0, \infty]$. Further, (2.8) is still well defined and $[-\infty, \infty]$ valued for signed measurable f provided at most one of the integrals of $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ is infinite. Note however, from the construction of the Poisson random measure in the proof of Theorem 2.4, the integral in (2.8) may be interpreted as equal to

$$\sum_{\upsilon \in \Upsilon} f(\upsilon) m_\upsilon,$$

where Υ is the support of $N(\cdot)$, which, from Corollary 2.6, is countable, and m_υ is the multiplicity of points at υ. Recalling the remarks following Corollary 2.6, if η has no atoms then $m_\upsilon = 1$ for all $\upsilon \in \Upsilon$.

We move to the main theorem of this section for which the reader is referred to Sect. 9.8 of Moran (1968), Kingman (1967) and the earlier work of Campbell (1909, 1910).

Theorem 2.7 *Suppose that N is a Poisson random measure on $(S, \mathcal{S}, \eta)$. Let $f : S \to \mathbb{R}$ be a measurable function.*

(i) *Then*

$$X = \int_S f(x) N(\mathrm{d}x)$$

is almost surely absolutely convergent if and only if

$$\int_S \big(1 \wedge |f(x)|\big) \eta(\mathrm{d}x) < \infty. \tag{2.9}$$

(ii) *When condition* (2.9) *holds, then* (*with E as expectation with respect to P*)

$$E\big(\mathrm{e}^{\mathrm{i}\beta X}\big) = \exp\left\{-\int_S \big(1 - \mathrm{e}^{\mathrm{i}\beta f(x)}\big)\eta(\mathrm{d}x)\right\} \tag{2.10}$$

for any $\beta \in \mathbb{R}$.

(iii) *Further*

$$E(X) = \int_S f(x)\eta(\mathrm{d}x) \quad \textit{when} \quad \int_S |f(x)|\eta(\mathrm{d}x) < \infty \tag{2.11}$$

and

$$E\big(X^2\big) = \int_S f(x)^2 \eta(\mathrm{d}x) + \left(\int_S f(x)\eta(\mathrm{d}x)\right)^2$$

when

$$\int_S f(x)^2 \eta(\mathrm{d}x) < \infty \quad \textit{and} \quad \int_S |f(x)|\eta(\mathrm{d}x) < \infty. \tag{2.12}$$

Proof (i) We begin by defining simple functions to be those of the form

$$f(x) = \sum_{i=1}^{n} f_i \mathbf{1}_{A_i}(x),$$

where f_i is constant and $\{A_i : i = 1, \ldots, n\}$ are disjoint sets in $\mathcal{S}$ and further $\eta(A_1 \cup \cdots \cup A_n) < \infty$.

For such functions, we have

$$X = \sum_{i=1}^{n} f_i N(A_i),$$

which is clearly finite with probability one since each $N(A_i)$ has a Poisson distribution with parameter $\eta(A_i) < \infty$. Recall the well-known fact that the moment-generating function of a Poisson distribution with parameter $\lambda > 0$ is $\exp\{-\lambda(1 - \mathrm{e}^{-\theta})\}$, for $\theta \geq 0$. For the same range of θ, we have

$$\begin{aligned} E\left(\mathrm{e}^{-\theta X}\right) &= \prod_{i=1}^{n} E\left(\mathrm{e}^{-\theta f_i N(A_i)}\right) \\ &= \prod_{i=1}^{n} \exp\{-\left(1 - \mathrm{e}^{-\theta f_i}\right)\eta(A_i)\} \\ &= \exp\left\{-\sum_{i=1}^{n}\left(1 - \mathrm{e}^{-\theta f_i}\right)\eta(A_i)\right\}. \end{aligned}$$

Since $1 - \mathrm{e}^{-\theta f(x)} = 0$ on $S \backslash (A_1 \cup \cdots \cup A_n)$, we may thus conclude that

$$E\left(\mathrm{e}^{-\theta X}\right) = \exp\left\{-\int_S \left(1 - \mathrm{e}^{-\theta f(x)}\right)\eta(\mathrm{d}x)\right\}.$$

Next we establish the above equality for a general positive measurable f. For this class of f, there exists a pointwise increasing sequence of positive simple functions, $\{f_n : n \geq 0\}$, such that $\lim_{n\uparrow\infty} f_n = f$, where the limit is also understood in the pointwise sense. Since N is an almost surely sigma-finite measure, we have that

$$\lim_{n\uparrow\infty} \int_S f_n(x) N(\mathrm{d}x) = \int_S f(x) N(\mathrm{d}x) = X$$

almost surely. An application of bounded convergence followed by an application of monotone convergence tells us that, for any $\theta > 0$,

$$\begin{aligned} E\left(\mathrm{e}^{-\theta X}\right) &= E\left(\exp\left\{-\theta \int f(x) N(\mathrm{d}x)\right\}\right) \\ &= \lim_{n\uparrow\infty} E\left(\exp\left\{-\theta \int f_n(x) N(\mathrm{d}x)\right\}\right) \\ &= \lim_{n\uparrow\infty} \exp\left\{-\int_S \left(1 - \mathrm{e}^{-\theta f_n(x)}\right)\eta(\mathrm{d}x)\right\} \\ &= \exp\left\{-\int_S \left(1 - \mathrm{e}^{-\theta f(x)}\right)\eta(\mathrm{d}x)\right\}. \end{aligned} \tag{2.13}$$

Note that the integral on the right-hand side of (2.13) is either infinite, for all $\theta > 0$, or finite, for all $\theta > 0$, accordingly as $X = \infty$ with probability one or $X = \infty$ with probability less than one, respectively. If $\int_S (1 - \mathrm{e}^{-\theta f(x)})\eta(\mathrm{d}x) < \infty$ for all $\theta > 0$ then as, for each $x \in S$, $(1 - \mathrm{e}^{-\theta f(x)}) \le (1 - \mathrm{e}^{-f(x)})$, for all $0 < \theta < 1$, dominated convergence implies that

$$\lim_{\theta \downarrow 0} \int_S \big(1 - \mathrm{e}^{-\theta f(x)}\big)\eta(\mathrm{d}x) = 0,$$

and hence dominated convergence, as $\theta \downarrow 0$, applied again in (2.13) tells us that $P(X = \infty) = 0$.

In conclusion, we have that $X < \infty$ almost surely if and only if $\int_S (1 - \mathrm{e}^{-\theta f(x)})\eta(\mathrm{d}x) < \infty$, for all $\theta > 0$. Moreover, it can be checked (see Exercise 2.3) that this happens if and only if

$$\int_S \big(1 \wedge f(x)\big)\eta(\mathrm{d}x) < \infty.$$

Note that both sides of (2.13) may be analytically continued by replacing θ by $\theta - \mathrm{i}\beta$ for $\beta \in \mathbb{R}$. Then taking limits on both sides as $\theta \downarrow 0$, we deduce (2.10).

Now we shall remove the restriction that f is positive. Henceforth assume, as in the statement of the theorem, that f is a measurable function. We may write $f = f^+ - f^-$ where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are both measurable. The sum X can be written $X_+ - X_-$ where

$$X_+ = \int_S f(x) N_+(\mathrm{d}x) \quad \textit{and} \quad X_- = \int_S f(x) N_-(\mathrm{d}x)$$

and $N_+ = N(\cdot \cap \{x \in S : f(x) \ge 0\})$ and $N_- = N(\cdot \cap \{x \in S : f(x) < 0\})$. From Corollary 2.5, we know that N_+ and N_- are both Poisson random measures with respective intensities $\eta(\cdot \cap \{f \ge 0\})$ and $\eta(\cdot \cap \{f < 0\})$. Further, they are independent and hence the same is true of X_+ and X_-. It is now clear that, almost surely, X converges absolutely if and only if X_+ and X_- are convergent. The analysis of the case when f is positive applied to the sums X_+ and X_- now tells us that absolute convergence of X occurs if and only if

$$\int_S \big(1 \wedge |f(x)|\big)\eta(\mathrm{d}x) < \infty, \tag{2.14}$$

and the proof of (i) is complete.

To complete the proof of (ii), assume that (2.14) holds. Using the independence of X_+ and X_-, as well as the conclusion of part (i), we have that, for any $\beta \in \mathbb{R}$,

$$
\begin{aligned}
E\left(\mathrm{e}^{\mathrm{i}\beta X}\right) &= E\left(\mathrm{e}^{\mathrm{i}\beta X_+}\right)E\left(\mathrm{e}^{-\mathrm{i}\beta X_-}\right)\\
&= \exp\left\{-\int_{\{f\geq 0\}}\left(1-\mathrm{e}^{\mathrm{i}\beta f^+(x)}\right)\eta(\mathrm{d}x)\right\}\\
&\quad\times \exp\left\{-\int_{\{f<0\}}\left(1-\mathrm{e}^{-\mathrm{i}\beta f^-(x)}\right)\eta(\mathrm{d}x)\right\}\\
&= \exp\left\{-\int_S\left(1-\mathrm{e}^{\mathrm{i}\beta f(x)}\right)\eta(\mathrm{d}x)\right\},
\end{aligned}
$$

and the proof of (ii) is complete.

Part (iii) is dealt with similarly as in the above treatment. That is, first consider positive, simple f, then extend to positive measurable f and then to a general measurable f by treating its positive and negative parts separately.

Alternatively one may take the identity (2.10) and differentiate in β, once for $E(X)$ and twice for $E(X^2)$, and then set $\beta = 0$. The integrability conditions in (2.11) and (2.12) are used in applying the Dominated Convergence Theorem to differentiate through the integral on the right-hand side of (2.10). The details are left to the reader. □

2.4 Square-Integrable Martingales

We shall predominantly use the identities in Theorem 2.7 for a Poisson random measure, $N(\cdot)$, on $([0,\infty)\times\mathbb{R}, \mathcal{B}[0,\infty)\times\mathcal{B}(\mathbb{R}), \mathrm{d}t\times\Pi(\mathrm{d}x))$, where Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$. We shall be interested in integrals of the form

$$
\int_{[0,t]}\int_B xN(\mathrm{d}s\times\mathrm{d}x), \tag{2.15}
$$

where $B\in\mathcal{B}(\mathbb{R})$. The relevant integrals appearing in (2.9)–(2.12), with $f(x)=x$, for the above Poisson random measure, can now be checked to take the form

$$
t\int_B\left(1\wedge|x|\right)\Pi(\mathrm{d}x),\qquad t\int_B\left(1-\mathrm{e}^{\mathrm{i}\beta x}\right)\Pi(\mathrm{d}x),
$$
$$
t\int_B|x|\Pi(\mathrm{d}x),\quad \text{and}\quad t\int_B x^2\Pi(\mathrm{d}x),
$$

with the appearance of the factor t in front of each of the integrals being a consequence of the involvement of Lebesgue measure in the intensity of N. The following two lemmas capture the context in which we use sums of the form (2.15). The first may be considered as a converse to Lemma 2.2 and the second shows the relationship with martingales.

Lemma 2.8 *Suppose that $N(\cdot)$ is a Poisson random measure on* $([0,\infty)\times\mathbb{R}, \mathcal{B}[0,\infty)\times\mathcal{B}(\mathbb{R}), \mathrm{d}t\times\Pi(\mathrm{d}x))$, *where Π is a measure concentrated on* $\mathbb{R}\backslash\{0\}$

and $B \in \mathcal{B}(\mathbb{R})$ such that $0 < \Pi(B) < \infty$. Then

$$X_t := \int_{[0,t]} \int_B x N(\mathrm{d}u \times \mathrm{d}x), \quad t \geq 0,$$

is a compound Poisson process with arrival rate $\Pi(B)$ and jump distribution $\Pi(B)^{-1}\Pi(\mathrm{d}x)|_B$.

Proof First note that since it is assumed $\Pi(B) < \infty$, from Corollary 2.6, we know that, for each $t > 0$, X_t may be written as an almost surely finite sum. This explains why $X = \{X_t : t \geq 0\}$ is right-continuous with left limits. (One may also see finiteness of X_t from Theorem 2.7 (i).) Next note that, for all $0 \leq s < t < \infty$,

$$X_t - X_s = \int_{(s,t]} \int_B x N(\mathrm{d}u \times \mathrm{d}x),$$

which is independent of $\{X_u : u \leq s\}$ as $N(\cdot)$ has independent counts over disjoint regions. From the construction of $N(\cdot)$, see for example (2.7), and the fact that its intensity measure takes the specific form $\mathrm{d}t \times \Pi(\mathrm{d}x)$, it also follows that $X_t - X_s$ has the same distribution as X_{t-s}. Further, according to Theorem 2.7 (ii), we have that, for all $\theta \in \mathbb{R}$ and $t \geq 0$,

$$E\left(\mathrm{e}^{\mathrm{i}\theta X_t}\right) = \exp\left\{-t \int_B \left(1 - \mathrm{e}^{\mathrm{i}\theta x}\right)\Pi(\mathrm{d}x)\right\}. \tag{2.16}$$

The Lévy–Khintchine exponent in (2.16) is that of a compound Poisson process with jump distribution and arrival rate given by $\Pi(B)^{-1}\Pi(\mathrm{d}x)|_B$ and $\Pi(B)$, respectively. □

Just as in the discussion following Definition 1.1, we assume that $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is the filtration generated by X satisfying the conditions of natural enlargement.

Lemma 2.9 *Suppose that N is the same as in the previous lemma and B is such that $\int_B |x|\Pi(\mathrm{d}x) < \infty$.*

(i) *The compound Poisson process with drift*

$$M_t := \int_{[0,t]} \int_B x N(\mathrm{d}s \times \mathrm{d}x) - t \int_B x \Pi(\mathrm{d}x), \quad t \geq 0,$$

is a P-martingale with respect to the filtration $\mathbb{F}$.

(ii) *If further, $\int_B x^2 \Pi(\mathrm{d}x) < \infty$ then it is a square-integrable martingale.*

Proof (i) First note that the process $M = \{M_t : t \geq 0\}$ is adapted to the filtration $\mathbb{F}$. Next note that, for each $t > 0$,

$$E\left(|M_t|\right) \leq E\left(\int_{[0,t]} \int_B |x| N(\mathrm{d}s \times \mathrm{d}x) + t \int_B |x|\Pi(\mathrm{d}x)\right),$$

which, from Theorem 2.7 (iii), is finite because $\int_B |x|\Pi(\mathrm{d}x)$ is. Next use the fact that M has stationary independent increments to deduce that, for $0 \le s \le t < \infty$,

$$\begin{aligned}
&E(M_t - M_s|\mathcal{F}_s)\\
&\quad = E(M_{t-s})\\
&\quad = E\left(\int_{(s,t]}\int_B xN(\mathrm{d}u \times \mathrm{d}x)\right) - (t-s)\int_B x\Pi(\mathrm{d}x)\\
&\quad = 0,
\end{aligned}$$

where in the final equality we have used Theorem 2.7 (iii) again.

(ii) To see that M is square-integrable, we may yet again appeal to Theorem 2.7 (iii), together with the assumption that $\int_B x^2\Pi(\mathrm{d}x) < \infty$, to deduce that

$$E\left(\left\{M_t + t\int_B x\Pi(\mathrm{d}x)\right\}^2\right) = t\int_B x^2\Pi(\mathrm{d}x) + t^2\left(\int_B x\Pi(\mathrm{d}x)\right)^2.$$

Recalling from the martingale property that $E(M_t) = 0$, it follows by developing the left-hand side in the previous display that

$$E\left(M_t^2\right) = t\int_B x^2\Pi(\mathrm{d}x) < \infty,$$

as required. □

The conditions in both Lemmas 2.8 and 2.9 mean that we may consider sets, for example, of the form $B_\varepsilon := (-1, -\varepsilon) \cup (\varepsilon, 1)$ for any $\varepsilon \in (0, 1)$. However, it is not necessarily the case that we may consider sets of the form $B = (-1, 0) \cup (0, 1)$. Consider for example the case that $\Pi(\mathrm{d}x) = \mathbf{1}_{(x>0)}x^{-(1+\alpha)}\mathrm{d}x + \mathbf{1}_{(x<0)}|x|^{-(1+\alpha)}\mathrm{d}x$ for $\alpha \in (1, 2)$. In this case, we have that $\int_B |x|\Pi(\mathrm{d}x) = \infty$ whereas $\int_B x^2\Pi(\mathrm{d}x) < \infty$. It will turn out to be quite important in the proof of the Lévy–Itô decomposition to understand the limit of the martingale in Lemma 2.8 for sets of the form B_ε as $\varepsilon \downarrow 0$. For this reason, let us now state and prove the following theorem.

Theorem 2.10 *Suppose that $N(\cdot)$ is as in Lemma 2.8 and $\int_{(-1,1)} x^2\Pi(\mathrm{d}x) < \infty$. For each $\varepsilon \in (0, 1)$ define the martingale*

$$M_t^\varepsilon = \int_{[0,t]}\int_{B_\varepsilon} xN(\mathrm{d}s \times \mathrm{d}x) - t\int_{B_\varepsilon} x\Pi(\mathrm{d}x), \quad t \ge 0.$$

Then there exists a martingale $M = \{M_t : t \geq 0\}$ *with the following properties*:

(i) *for each* $T > 0$, *there exists a deterministic subsequence* $\{\varepsilon_n^T : n = 1, 2, \ldots\}$ *with* $\varepsilon_n^T \downarrow 0$ *along which*

$$P\Big(\lim_{n\uparrow\infty} \sup_{0\leq s\leq T} \big(M_s^{\varepsilon_n^T} - M_s\big)^2 = 0\Big) = 1,$$

(ii) *it is adapted to the filtration* $\mathbb{F}$,
(iii) *it has right-continuous paths with left limits almost surely*,
(iv) *it has*, *at most*, *a countable number of discontinuities on* $[0, T]$ *almost surely and*
(v) *it has stationary and independent increments*.

In short, *there exists a Lévy process*, *which is also a martingale with a countable number of jumps to which*, *for any fixed* $T > 0$, *the sequence of martingales* $\{M_t^\varepsilon : t \leq T\}$ *converges uniformly on* $[0, T]$ *with probability one along a subsequence in* ε *which may depend on* T.

Before proving Theorem 2.10, we need to remind ourselves of some general facts concerning square-integrable martingales. In our account, we shall recall a number of well-established facts coming from straightforward L^2 theory, measure theory and continuous-time martingale theory. The reader is referred to Sects. 2.4, 2.5 and 9.6 of Ash and Doléans-Dade (2000) for a clear account of the necessary background.

Fix a time horizon $T > 0$. Let us assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, P)$ is a filtered probability space in which the filtration $\{\mathcal{F}_t : t \geq 0\}$ satisfies the conditions of natural enlargement.

Definition 2.11 Fix $T > 0$. Define $\mathcal{M}_T^2 = \mathcal{M}_T^2(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, P)$ to be the space of real-valued, zero mean, almost surely right-continuous, square-integrable P-martingales with respect to the given filtration over the finite time period $[0, T]$.

One luxury that follows from the assumptions on $\{\mathcal{F}_t : t \geq 0\}$ is that any zero mean square-integrable martingale with respect to this filtration has a right-continuous modification[3] which is also a member of $\mathcal{M}_T^2$.

If we quotient out the equivalent classes of versions[4] of each martingale, it is straightforward to deduce that $\mathcal{M}_T^2$ is a vector space over the real numbers with

[3] Recall that $M' = \{M'_t : t \in [0, T]\}$ is a modification of M if, for every $t \geq 0$, we have $P(M'_t = M_t) = 1$.

[4] Recall that $M' = \{M'_t : t \in [0, T]\}$ is a version of M if it is defined on the same probability space and $\{\exists t \in [0, T] : M'_t \neq M_t\}$ is measurable with zero probability. Note that, if M' is a modification of M, then it is not necessarily a version of M. However, it is obviously the case that, if M' is a version of M, then it also fulfils the requirement of being a modification.

zero element $M_t = 0$ for all $t \in [0, T]$ and all $\omega \in \Omega$. In fact, as we shall shortly see, $\mathcal{M}_T^2$ is a Hilbert space[5] with respect to the inner product

$$\langle M, N \rangle = E(M_T N_T),$$

where $M, N \in \mathcal{M}_T^2$. It is left to the reader to verify the fact that $\langle\cdot,\cdot\rangle$ forms an inner product. The only mild technical difficulty in this verification is showing that, for $M \in \mathcal{M}_T^2$, $\langle M, M\rangle = 0$ implies that $M = 0$, the zero element. Note that, if $\langle M, M\rangle = 0$, then by Doob's Maximal Inequality, which says that for $M \in \mathcal{M}_T^2$,

$$E\Big(\sup_{0\le s\le T} M_s^2\Big) \le 4E\big(M_T^2\big),$$

we have that $\sup_{0\le t\le T}|M_t| = 0$ almost surely. It follows necessarily that $M_t = 0$ for all $t \in [0, T]$ with probability one. This corresponds to the zero element in the quotient space.

As alluded to above, we can show without too much difficulty that $\mathcal{M}_T^2$ is a Hilbert space. To do that, we are required to show that, if $\{M^{(n)} : n = 1, 2, \ldots\}$ is a Cauchy sequence of martingales taken from $\mathcal{M}_T^2$, then there exists an $M \in \mathcal{M}_T^2$ such that

$$\big\|M^{(n)} - M\big\| \to 0,$$

as $n \uparrow \infty$, where $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$. To this end let us assume that the sequence of processes $\{M^{(n)} : n = 1, 2, \ldots\}$ is a Cauchy sequence, in other words,

$$E\big[\big(M_T^{(m)} - M_T^{(n)}\big)^2\big]^{1/2} \to 0 \quad \text{as } m, n \uparrow \infty.$$

Necessarily the sequence of *random variables* $\{M_T^{(k)} : k \ge 1\}$ is a Cauchy sequence in the Hilbert space of zero mean, square-integrable random variables defined on $(\Omega, \mathcal{F}_T, P)$, say $L^2(\Omega, \mathcal{F}_T, P)$, endowed with the inner product $\langle M, N\rangle = E(MN)$. Hence, there exists a limiting variable, say M_T in $L^2(\Omega, \mathcal{F}_T, P)$, satisfying

$$E\big[\big(M_T^{(n)} - M_T\big)^2\big]^{1/2} \to 0,$$

as $n \uparrow \infty$. Define the martingale M to be the right-continuous version[6] of

$$E(M_T|\mathcal{F}_t) \text{ for } t \in [0, T]$$

[5]Recall that $\langle\cdot,\cdot\rangle$: $L \times L \to \mathbb{R}$ is an inner product on a vector space L over the reals if it satisfies the following properties, for $f, g \in L$ and $a, b \in \mathbb{R}$; (i) $\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$ for all $h \in L$, (ii) $\langle f, g\rangle = \langle g, f\rangle$, (iii) $\langle f, f\rangle \ge 0$ and (iv) $\langle f, f\rangle = 0$ if and only if $f = 0$.

For each $f \in L$, let $\|f\| = \langle f, f\rangle^{1/2}$. The pair $(L, \langle\cdot,\cdot\rangle)$ are said to form a Hilbert space if all sequences, $\{f_n : n = 1, 2, \ldots\}$ in L that satisfy $\|f_n - f_m\| \to 0$ as $m, n \to \infty$, i.e. so-called Cauchy sequences, have a limit in L.

[6]Here, we use the fact that $\{\mathcal{F}_t : t \in [0, T]\}$ satisfies the conditions of natural enlargement.

and note that, by definition,

$$\left\| M^{(n)} - M \right\| \to 0,$$

as n tends to infinity. Clearly it is an $\mathcal{F}_t$-adapted process and by Jensen's inequality

$$\begin{aligned} E\left(M_t^2\right) &= E\left(E(M_T|\mathcal{F}_t)^2\right) \\ &\le E\left(E\left(M_T^2|\mathcal{F}_t\right)\right) \\ &= E\left(M_T^2\right), \end{aligned}$$

which is finite. Hence Cauchy sequences converge in $\mathcal{M}_T^2$ and we see that $\mathcal{M}_T^2$ is indeed a Hilbert space.

We are now ready to return to Theorem 2.10.

Proof of Theorem 2.10 (i) Choose $0 < \eta < \varepsilon < 1$, fix $T > 0$ and define $M^\varepsilon = \{M_t^\varepsilon : t \in [0, T]\}$. A calculation similar to the one in Lemma 2.9 (ii) gives

$$\begin{aligned} & E\left(\left(M_T^\varepsilon - M_T^\eta\right)^2\right) \\ &\quad = E\left(\left\{\int_{[0,T]} \int_{\eta \le |x| < \varepsilon} x N(\mathrm{d}s \times \mathrm{d}x) - T \int_{\eta < |x| < \epsilon} x \Pi(\mathrm{d}x)\right\}^2\right) \\ &\quad = T \int_{\eta \le |x| < \varepsilon} x^2 \Pi(\mathrm{d}x). \end{aligned}$$

Note, however, that the left-hand side above is also equal to $\|M^\varepsilon - M^\eta\|^2$ (where as in the previous discussion, $\|\cdot\|$ is the norm induced by the inner product on $\mathcal{M}_T^2$).

Thanks to the assumption that $\int_{(-1,1)} x^2 \Pi(\mathrm{d}x) < \infty$, we now have that $\lim_{\varepsilon,\eta \downarrow 0} \|M^\varepsilon - M^\eta\| = 0$ and hence that $\{M^\varepsilon : 0 < \varepsilon < 1\}$ is a Cauchy sequence in $\mathcal{M}_T^2$. As $\mathcal{M}_T^2$ is a Hilbert space, we know that there exists a right-continuous martingale $M = \{M_s : s \in [0, T]\} \in \mathcal{M}_T^2$ such that

$$\lim_{\varepsilon \downarrow 0} \left\| M - M^\varepsilon \right\| = 0.$$

An application of Doob's Maximal Inequality tells us that, in fact,

$$\lim_{\varepsilon \downarrow 0} E\left[\sup_{0 \le s \le T} \left(M_s - M_s^\varepsilon\right)^2\right] \le 4 \lim_{\varepsilon \downarrow 0} \left\| M - M^\varepsilon \right\| = 0. \tag{2.17}$$

From this, one may deduce that the limit $\{M_s : s \in [0, T]\}$ does not depend on T. Indeed, suppose it did and we adjust our notation accordingly so that $\{M_{s,T} : s \le T\}$ represents the limit. Then from (2.17), we see that, for any $0 < T' < T$,

$$\lim_{\varepsilon \downarrow 0} E\left[\sup_{0 \le s \le T'} \left(M_s^\varepsilon - M_{s,T'}\right)^2\right] = 0$$

as well as

$$\lim_{\varepsilon\downarrow 0} E\Big[\sup_{0\le s\le T'}\big(M_s^\varepsilon - M_{s,T}\big)^2\Big] \le \lim_{\varepsilon\downarrow 0} E\Big[\sup_{0\le s\le T}\big(M_s^\varepsilon - M_{s,T}\big)^2\Big] = 0,$$

where the inequality is the result of a trivial upper bound. Hence, using that, for any two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, $\sup_n a_n^2 = (\sup_n |a_n|)^2$ and $\sup_n |a_n + b_n| \le \sup_n |a_n| + \sup_n |b_n|$, we have, together with an application of Minkowski's inequality, that

$$\begin{aligned} E\Big[\sup_{0\le s\le T'}(M_{s,T'} - M_{s,T})^2\Big]^{1/2} &\le \lim_{\varepsilon\downarrow 0} E\Big[\sup_{0\le s\le T'}\big(M_s^\varepsilon - M_{s,T'}\big)^2\Big]^{1/2} \\ &\quad + \lim_{\varepsilon\downarrow 0} E\Big[\sup_{0\le s\le T'}\big(M_s^\varepsilon - M_{s,T}\big)^2\Big]^{1/2} \\ &= 0, \end{aligned}$$

thus showing that the processes $M_{\cdot,T}$ and $M_{\cdot,T'}$ are almost surely uniformly equal on $[0,T']$. Since T' and T may be arbitrarily chosen, we may now speak of a well-defined limiting martingale, $M = \{M_t : t \ge 0\}$.

From the limit (2.17), we may also deduce that there exists a deterministic subsequence $\{\varepsilon_n^T : n \ge 0\}$ along which

$$\lim_{\varepsilon_n^T\downarrow 0}\sup_{0\le s\le T}\big(M_s^{\varepsilon_n^T} - M_s\big)^2 = 0$$

P-almost surely. This follows from the well-established fact that L^2 convergence of a sequence of random variables implies almost sure convergence on a deterministic subsequence.

(ii) and (iii) Since, for each $T < \infty$, $\{M_s : s \in [0,T]\} \in \mathcal{M}_T^2$, it is automatic from the definition of this space of martingales that M is $\mathbb{F}$-adapted with right-continuous paths. It remains to show that the paths of M have left limits. To this end, note that the paths of M^ε are right-continuous with left limits. Hence, almost sure uniform convergence (along a subsequence) on finite time intervals implies that the limiting process, M, also has paths which are right-continuous with, in particular, left limits. We are using here the fact that, if $D[0,1]$ is the space of functions $f : [0,1] \to \mathbb{R}$ which are right-continuous with left limits, then $D[0,1]$ contains all its limit points under the metric $d(f,g) = \sup_{t\in[0,1]}|f(t) - g(t)|$ for $f, g \in D[0,1]$. See Exercise 2.4.

(iv) According to Corollary 2.6, there are at most an almost surely countable number of points in the support of N. Further, recalling the discussion after Corollary 2.6, as the measure $\mathrm{d}t \times \Pi(\mathrm{d}x)$ has no atoms, the random measure $N(\cdot)$ is necessarily $\{0,1\}$-valued on time-space singletons. Hence every discontinuity in $\{M_s : s \ge 0\}$ corresponds to a unique point in the support of $N(\cdot)$. It follows that M has at most a countable number of discontinuities. Another way to see that there are, at most, a countable number of discontinuities is simply to note that the same is true of functions in the space $D[0,1]$; see Exercise 2.4.

(v) For any $n \in \mathbb{N}$, $0 \le s_1 \le t_1 \le \cdots \le s_n \le t_n \le T < \infty$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$, dominated convergence and almost sure uniform convergence along the subsequence $\{\epsilon_n^T : t \ge 0\}$ gives

$$
\begin{aligned}
E\left[\prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}\theta_j (M_{t_j} - M_{s_j})}\right] &= \lim_{n\uparrow\infty} E\left[\prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}\theta_j (M_{t_j}^{\varepsilon_n^T} - M_{s_j}^{\varepsilon_n^T})}\right] \\
&= \lim_{n\uparrow\infty} \prod_{j=1}^{n} E\big[\mathrm{e}^{\mathrm{i}\theta_j M_{t_j - s_j}^{\varepsilon_n^T}}\big] \\
&= \prod_{j=1}^{n} E\big[\mathrm{e}^{\mathrm{i}\theta_j M_{t_j - s_j}}\big],
\end{aligned}
$$

which, thanks to Exercise 1.1, is sufficient to deduce that M has stationary and independent increments. This concludes the proof. □

2.5 Proof of the Lévy–Itô Decomposition

As previously indicated in Sect. 2.1, we will take $X^{(1)}$ to be the linear Brownian motion (2.2), now defined on some probability space $(\Omega^{\#}, \mathcal{F}^{\#}, P^{\#})$.

Given Π in the statement of Theorem 2.1, we know from Theorem 2.4 that there exists a probability space, say $(\Omega^*, \mathcal{F}^*, P^*)$, on which we may construct a Poisson random measure, N, on $([0,\infty) \times \mathbb{R}, \mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R}), \mathrm{d}t \times \Pi(\mathrm{d}x))$. We may think of the points in the support of N as having a time and space coordinate, or alternatively, as points in $\mathbb{R}\backslash\{0\}$ arriving in time.

Now define

$$
X_t^{(2)} = \int_{[0,t]} \int_{|x|\ge 1} x N(\mathrm{d}s \times \mathrm{d}x), \qquad t \ge 0,
$$

and note from Lemma 2.8 that, since $\Pi(\mathbb{R}\backslash(-1,1)) < \infty$, it is a compound Poisson process with rate $\Pi(\mathbb{R}\backslash(-1,1))$ and jump distribution

$$
\Pi\big(\mathbb{R}\backslash(-1,1)\big)^{-1} \Pi(\mathrm{d}x)|_{\mathbb{R}\backslash(-1,1)}.
$$

(We can assume without loss of generality that $\Pi(\mathbb{R}\backslash(-1,1)) > 0$ as otherwise, we may take the process $X^{(2)}$ as the process which is identically zero.)

Next, we construct a Lévy process having only small jumps. For each $1 > \varepsilon > 0$, define similarly the compound Poisson process with drift,

$$
X_t^{(3,\varepsilon)} = \int_{[0,t]} \int_{\varepsilon \le |x| < 1} x N(\mathrm{d}s \times \mathrm{d}x) - t \int_{\varepsilon \le |x| < 1} x \Pi(\mathrm{d}x), \qquad t \ge 0. \tag{2.18}
$$

(As in the definition of $X^{(2)}$, we shall assume without loss of generality $\Pi(\{x : |x| < 1\}) > 0$, otherwise the process $X^{(3)}$ may be taken as the process which is identically zero.) Using Theorem 2.7 (ii), we can compute its characteristic exponent,

$$\Psi^{(3,\varepsilon)}(\theta) := \int_{\varepsilon \le |x| < 1} \big(1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x\big)\Pi(\mathrm{d}x).$$

According to Theorem 2.10, there exists a Lévy process, which is also a square-integrable martingale, defined on $(\Omega^*, \mathcal{F}^*, P^*)$, to which $X^{(3,\varepsilon)}$ converges uniformly on $[0, T]$ along an appropriate deterministic subsequence in ε. Note that it is precisely at this point that we use the assumption that $\int_{(-1,1)} x^2 \Pi(\mathrm{d}x) < \infty$. It is clear that the characteristic exponent of the aforementioned Lévy process is equal to

$$\Psi^{(3)}(\theta) = \int_{|x|<1} \big(1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x\big)\Pi(\mathrm{d}x).$$

From Corollary 2.5, we know that, for each $t > 0$, N has independent counts over the two domains $[0, t] \times \{\mathbb{R}\backslash(-1, 1)\}$ and $[0, t] \times (-1, 1)$. It follows that $X^{(2)}$ and $X^{(3)}$ are independent.

To conclude the proof of the Lévy–Itô decomposition in line with the statement of Theorem 2.1, define the process

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)}, \quad t \ge 0. \tag{2.19}$$

This process is defined on the product space

$$(\Omega, \mathcal{F}, \mathbb{P}) = \big(\Omega^{\#}, \mathcal{F}^{\#}, P^{\#}\big) \times \big(\Omega^*, \mathcal{F}^*, P^*\big),$$

has stationary independent increments, has paths that are right-continuous with left limits and has characteristic exponent

$$\begin{aligned}\Psi(\theta) &= \Psi^{(1)}(\theta) + \Psi^{(2)}(\theta) + \Psi^{(3)}(\theta)\\ &= \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \big(1 - \mathrm{e}^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbf{1}_{(|x|<1)}\big)\Pi(\mathrm{d}x),\end{aligned}$$

as required. □

Let us conclude this section with some additional remarks on the Lévy–Itô decomposition.

Recall from (2.4) that the exponent $\Psi^{(3)}$ appears to have the form of the infinite sum of characteristic exponents belonging to compound Poisson processes with drift. This suggests that $X^{(3)}$ may be taken as the superposition of such processes. We now see from the above proof that this is exactly the case. Indeed, moving ε to zero through the sequence $\{2^{-k} : k \ge 0\}$ shows us that in the appropriate sense of L^2 convergence

$$
\begin{aligned}
\lim_{k\uparrow\infty} X_t^{(3,2^{-k})} &= \lim_{k\uparrow\infty} \int_{[0,t]} \int_{2^{-k}<|x|<1} xN(\mathrm{d}s\times\mathrm{d}x) - t\int_{2^{-k}<|x|<1} x\Pi(\mathrm{d}x) \\
&= \lim_{k\uparrow\infty} \sum_{i=0}^{k-1} \left\{ \int_{[0,t]} \int_{2^{-(i+1)}<|x|<2^{-i}} xN(\mathrm{d}s\times\mathrm{d}x) \right. \\
&\qquad \left. - t\int_{2^{-(i+1)}<|x|<2^{-i}} x\Pi(\mathrm{d}x) \right\}.
\end{aligned}
$$

It is also worth remarking that the definition of $X^{(2)}$ and $X^{(3)}$ in the proof of the Lévy–Itô decomposition, corresponding to the partition of $\mathbb{R}\backslash\{0\}$ into $\mathbb{R}\backslash(-1,1)$ and $(-1,1)\backslash\{0\}$, is to some extent arbitrary. The point is that one needs to deal differently with the contributions to the path from N which come from a neighbourhood of the origin, and which come from its complement. In this respect one could have redrafted the proof replacing $(-1,1)$ by (α,β), for any $\alpha<0$ and $\beta>0$. In which case, one would need to choose a different value of a in the definition of $X^{(1)}$ in order to make terms add up precisely to the expression given in the Lévy–Khintchine exponent. To be more precise, if for example $\alpha<-1$ and $\beta>1$, then one should take $X_t^{(1)}=a't+\sigma B_t$ where

$$
a' = a - \int_{\alpha<|x|\le -1} x\Pi(\mathrm{d}x) - \int_{1\le|x|<\beta} x\Pi(\mathrm{d}x).
$$

This also shows that the Lévy–Khintchine formula (2.1) is not a unique representation and, indeed, the indicator $\mathbf{1}_{(|x|<1)}$ in (2.1) may be replaced by $\mathbf{1}_{(\alpha<x<\beta)}$ with an appropriate adjustment in the constant a.

Taking a much deeper view of things, the Lévy–Itô decomposition illustrates one of many examples where a Markov process can be decomposed according to an endogenous Poisson point process. This approach was pursued by K. Itô. See for example Itô (2004, 1970). Later on, in Chap. 6, we shall see another path-decomposition of Lévy processes in this spirit. In that case, the path is decomposed according to a Poisson point process of excursions of the Lévy process from its maximum.

2.6 Lévy Processes Distinguished by Their Path Type

As is clear from the proof of the Lévy–Itô decomposition, we should think of the measure Π given in the Lévy–Khintchine formula as characterising a Poisson random measure which encodes the rate at which the jumps of the associated Lévy process occur. In this section we shall re-examine elements of the proof of the Lévy–Itô decomposition and show that, with additional assumptions on Π, we may further identify special classes of Lévy processes embedded within the general class.

2.6.1 Path Variation

It is clear from the Lévy–Itô decomposition that the presence of the linear Brownian motion $X^{(1)}$ would imply that paths of the Lévy process have unbounded variation. On the other hand, should it be the case that $\sigma = 0$, then the Lévy process may or may not have unbounded variation. The term $X^{(2)}$, being a compound Poisson process, has only bounded variation. Hence, in the case $\sigma = 0$, understanding whether the Lévy process has unbounded variation is an issue determined by the limiting process $X^{(3)}$; that is to say, the process of compensated small jumps.

Reconsidering the definition of $X^{(3)}$, it is natural to ask under what circumstances

$$\lim_{\varepsilon \downarrow 0} \int_{[0,t]} \int_{\varepsilon \le |x| < 1} x N(\mathrm{d}s \times \mathrm{d}x)$$

exists almost surely without the need for compensation by its mean as in (2.18). Once again, the answer is given by Theorem 2.7 (i). Here we are told that

$$\int_{[0,t]} \int_{|x|<1} |x| N(\mathrm{d}s \times \mathrm{d}x) < \infty$$

if and only if $\int_{|x|<1} |x| \Pi(\mathrm{d}x) < \infty$. In that case, we may identify $X^{(3)}$ directly via

$$X_t^{(3)} = \int_{[0,t]} \int_{|x|<1} x N(\mathrm{d}s \times \mathrm{d}x) - t \int_{|x|<1} x \Pi(\mathrm{d}x), \quad t \ge 0.$$

This also tells us that $X^{(3)}$ will be of bounded variation if and only if $\int_{|x|<1} |x| \Pi(\mathrm{d}x) < \infty$. Note that this is a stronger integrability condition than the general integrability condition $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(\mathrm{d}x) < \infty$. We get the following lemma.

Lemma 2.12 *A Lévy process with Lévy–Khintchine exponent corresponding to the triple* (a, σ, Π) *has paths of bounded variation if and only if*

$$\sigma = 0 \quad \textit{and} \quad \int_{\mathbb{R}} (1 \wedge |x|) \Pi(\mathrm{d}x) < \infty. \tag{2.20}$$

Note that the finiteness of the integral in (2.20) also allows for the Lévy–Khintchine exponent of any such bounded variation process to be rewritten as

$$\Psi(\theta) = -\mathrm{i}\delta\theta + \int_{\mathbb{R}} \left(1 - \mathrm{e}^{\mathrm{i}\theta x}\right) \Pi(\mathrm{d}x), \tag{2.21}$$

where the constant $\delta \in \mathbb{R}$ relates to the constant a and Π via

$$\delta = -\left(a + \int_{|x|<1} x \Pi(\mathrm{d}x)\right).$$

In this case, we may write the Lévy process in the form

$$X_t = \delta t + \int_{[0,t]} \int_{\mathbb{R}} x N(\mathrm{d}s \times \mathrm{d}x), \quad t \geq 0. \tag{2.22}$$

In view of the decomposition of the Lévy–Khintchine formula for a process of bounded variation and the corresponding representation (2.22), the term δ is often referred to as the *drift*. Strictly speaking, one should not talk of drift in the case of a Lévy process whose jump part is a process of unbounded variation. If drift is to be understood in terms of a purely deterministic trend, then it is ambiguous on account of the "infinite limiting compensation" that one sees in $X^{(3)}$ coming from the second term on the right-hand side of (2.18).

From the expression given in (1.4) of Chap. 1, we see that, if X is a compound Poisson process with drift, then its characteristic exponent takes the form of (2.21) with $\Pi(\mathbb{R}) < \infty$. Conversely, if $\sigma = 0$ and Π has finite total mass, then we know from Lemma 2.8 that (2.22) is a compound Poisson process with drift δ. In conclusion, we have the following lemma.

Lemma 2.13 *A Lévy process is a compound Poisson process with drift if and only if* $\sigma = 0$ *and* $\Pi(\mathbb{R}) < \infty$.

2.6.2 One-Sided Jumps

Suppose now that $\Pi(-\infty, 0) = 0$. From the proof of the Lévy–Itô decomposition, we see that the corresponding Lévy process has no negative jumps. If further we have that $\int_{(0,\infty)} (1 \wedge x)\Pi(\mathrm{d}x) < \infty$, $\sigma = 0$ and, in the representation (2.21) of the characteristic exponent, $\delta \geq 0$, then from the representation (2.22) it becomes clear that the Lévy process has non-decreasing paths. Conversely, if a Lévy process has non-decreasing paths, then necessarily it has bounded variation. Hence $\int_{(0,\infty)} (1 \wedge x) \Pi(\mathrm{d}x) < \infty$, $\sigma = 0$ and then it is easy to see that in the representation (2.21) of the characteristic exponent, we necessarily have $\delta \geq 0$. Examples of such a process were given in Chap. 1 (the gamma process and the inverse Gaussian process) and were named *subordinators*. Summarising, we have the following.

Lemma 2.14 *A Lévy process is a subordinator if and only if* $\Pi(-\infty, 0) = 0$, $\int_{(0,\infty)} (1 \wedge x)\Pi(\mathrm{d}x) < \infty$, $\sigma = 0$ *and* $\delta = -(a + \int_{(0,1)} x\Pi(\mathrm{d}x)) \geq 0$.

For the sake of clarity, we note that, when X is a subordinator, further to (2.21), its Lévy–Khintchine formula may be written as

$$\Psi(\theta) = -i\delta\theta + \int_{(0,\infty)} \left(1 - \mathrm{e}^{\mathrm{i}\theta x}\right)\Pi(\mathrm{d}x). \tag{2.23}$$

If $\Pi(-\infty, 0) = 0$ and X does not have monotone paths, that is to say, it is not a subordinator and it is not a pure negative linear drift, then it is referred to in general as a *spectrally positive* Lévy process. A Lévy process, X, will then be referred to as a *spectrally negative Lévy process* if $-X$ is spectrally positive. Together, these two classes of processes are called *spectrally one-sided*. Spectrally one-sided Lévy processes may be of bounded or unbounded variation and, in the latter case, may or may not possess a Gaussian component. Note in particular that when $\sigma = 0$, it is still possible to have paths of unbounded variation. If a spectrally positive Lévy process has bounded variation, then it must take the form

$$X_t = -\delta t + S_t, \quad t \geq 0,$$

where $\{S_t : t \geq 0\}$ is a pure jump subordinator and, necessarily, $\delta > 0$. Note that if $\delta \leq 0$, then X would conform to the definition of a subordinator. Note that the above decomposition implies that if $\mathbb{E}(X_1) \leq 0$, then $\mathbb{E}(S_1) < \infty$, as opposed to the case that $\mathbb{E}(X_1) > 0$, in which case it is possible that $\mathbb{E}(S_1) = \infty$.

A special feature of spectrally positive processes is that, if $\tau_x^- = \inf\{t > 0 : X_t < x\}$, where $x < 0$, then $\mathbb{P}(\tau_x^- < \infty) > 0$. Hence, as there are no downwards jumps,

$$\mathbb{P}\left(X_{\tau_x^-} = x \,|\, \tau_x^- < \infty\right) = 1, \tag{2.24}$$

with a similar property for first passage upwards being true for spectrally negative processes. A rigorous proof of the first of the above two facts will be given in Corollary 3.13, at the end of Sect. 3.3. It turns out that (2.24) plays a very important role in the simplification of a number of theorems we shall encounter later on in this text, which concern the fluctuations of general Lévy processes.

2.7 Interpretations of the Lévy–Itô Decomposition

Let us return to some of the models considered in Chap. 1 and consider how our understanding of the Lévy–Itô decomposition helps to justify working with more general classes of Lévy processes.

2.7.1 The Structure of Insurance Claims

Recall from Sect. 1.3.1 that the Cramér–Lundberg model corresponds to a Lévy process with characteristic exponent given by

$$\Psi(\theta) = -\mathrm{i}c\theta + \lambda \int_{(-\infty,0)} \left(1 - \mathrm{e}^{\mathrm{i}\theta x}\right) F(\mathrm{d}x),$$

for $\theta \in \mathbb{R}$. In other words, a compound Poisson process with arrival rate $\lambda > 0$ and negative jumps, corresponds to claims having common distribution F, as well as a drift $c > 0$ corresponding to a steady income due to premiums. Suppose instead we work with a general spectrally negative Lévy process, that is a process for which $\Pi(0,\infty)=0$ (but without monotone paths). In this case, the Lévy–Itô decomposition offers an interpretation for large-scale insurance companies as follows. The Lévy–Khintchine exponent may be written in the form

$$\Psi(\theta)=\left\{\frac{1}{2}\sigma^2\theta^2\right\}+\left\{-\mathrm{i}\theta c+\int_{(-\infty,-1]}\left(1-\mathrm{e}^{\mathrm{i}\theta x}\right)\Pi(\mathrm{d}x)\right\} \\ +\left\{\int_{(-1,0)}\left(1-\mathrm{e}^{\mathrm{i}\theta x}+\mathrm{i}\theta x\right)\Pi(\mathrm{d}x)\right\} \qquad (2.25)$$

for $\theta\in\mathbb{R}$. Assume that $\Pi(-\infty,0)=\infty$, and so Ψ is genuinely different from the characteristic of a Cramér–Lundberg model. We may understand the third bracket in (2.25) as a Lévy process representing a countably infinite number of arbitrarily small claims, compensated by a deterministic positive drift (which may be infinite in the case that $\int_{(-1,0)}|x|\Pi(\mathrm{d}x)=\infty$), corresponding to the accumulation of premiums over an infinite number of contracts. Roughly speaking, the way in which claims occur is such that, in any arbitrarily small period of time $\mathrm{d}t$, a claim of size $|x|$ (for $x<0$) is made independently with probability $\Pi(\mathrm{d}x)\mathrm{d}t+o(\mathrm{d}t)$. The insurance company thus counterbalances such claims by ensuring that it collects premiums in such a way that in any $\mathrm{d}t$, $|x|\Pi(\mathrm{d}x)\mathrm{d}t$ of its income is devoted to the compensation of claims of size $|x|$. The second bracket in (2.25) can be understood as coming from large claims, which occur occasionally and are compensated for by a steady income at rate $c>0$, just as in the Cramér–Lundberg model. Here "large" is taken to mean claims of size one or more and $c=-a$, in the terminology of the Lévy–Khintchine formula given in Theorem 1.6. Finally, the first bracket in (2.25) may be seen as a stochastic perturbation of the system of claims and premium income.

Since the contents of the first and third set of curly brackets in (2.25) correspond to martingales, the company may guarantee that its surplus drifts to infinity over an infinite time horizon by assuming that such behaviour applies to the compensated process of large claims corresponding to the second bracket in (2.25).

2.7.2 General Storage Models

The workload of the $M/G/1$ queue was presented in Sect. 1.3.2 as a spectrally negative compound Poisson process with rate $\lambda>0$ and jump distribution F with positive unit drift, reflected in its supremum. In other words, the underlying Lévy process has characteristic exponent

$$\Psi(\theta)=-\mathrm{i}\theta+\lambda\int_{(-\infty,0)}\left(1-\mathrm{e}^{\mathrm{i}\theta x}\right)F(\mathrm{d}x),$$

for all $\theta\in\mathbb{R}$. A general storage model, described for example in the classic books of Prabhu (1998) and Takács (1966), consists of working with a Lévy process, X, which is the difference of a positive drift and a subordinator and then reflected in its supremum. Its Lévy–Khintchine exponent thus takes the form

$$\Psi(\theta) = -\mathrm{i}\delta\theta + \int_{(-\infty,0)} \left(1 - \mathrm{e}^{\mathrm{i}\theta x}\right)\Pi(\mathrm{d}x),$$

where $\delta > 0$ and $\int_{(-\infty,0)}(1 \wedge |x|)\Pi(\mathrm{d}x) < \infty$. As with the case of the $M/G/1$ queue, the reflected process

$$W_t = (w \vee \overline{X}_t) - X_t, \quad t \geq 0,$$

may be thought of as the stored volume or workload of some system, where $\overline{X}$ is the running supremum and w is the initial volume in the system. The Lévy–Itô decomposition tells us that, during the periods of time that X is away from its supremum, there is a natural "drainage" of volume or "processing" of workload, corresponding to the downward movement of W in a linear fashion with rate δ. At the same time new "volume for storage" or equivalently new "jobs" arrive independently so that in each $\mathrm{d}t$, one arrives of size $|x|$ (where $x < 0$) with probability $\Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t)$ (thus giving similar interpretation to the occurrence of jumps in the insurance risk model described above). When $\Pi(-\infty, 0) = \infty$, the number of jumps are countably infinite over any finite time interval, thus indicating that our model is processing with "infinite frequency" in comparison to the finite activity of the workload of the $M/G/1$ process.

Of course one may also envisage working with a jump measure which has some mass on the positive half-line. This would correspond to negative jumps in the process W. This, in turn, can be interpreted as follows. Over and above the natural drainage or processing at rate δ, in each $\mathrm{d}t$ there is independent removal of a "volume" or "processing time of job" of size $y > 0$ with probability $\Pi(\mathrm{d}y)\mathrm{d}t + o(\mathrm{d}t)$. One may also consider moving to models of unbounded variation. However, in this case, the interpretation of drift is lost.

2.7.3 Financial Models

Financial mathematics has become a field of applied probability which has also embraced the use of Lévy processes, in particular, for the purpose of modelling the evolution of risky assets. We shall not attempt to give anything like a comprehensive exposure of this topic here, nor elsewhere in this book, especially since textbooks of Boyarchenko and Levendorskii (2002b), Schoutens (2003) and Cont and Tankov (2004) already offer a clear and up-to-date overview between them. It is worth mentioning briefly some of the connections between path properties of Lévy processes seen above and modern perspectives within financial modelling.

One may say that financial mathematics proper begins with the thesis of Louis Bachelier who proposed the use of linear Brownian motion to model the value of a risky asset, say the value of a stock. See Bachelier (1900, 1901). However, the *classical* model for the evolution of a risky asset, proposed by Samuelson (1965), is generally accepted to be that of an exponential linear Brownian motion with drift;

$$S_t = s\exp\{\sigma B_t + \mu t\}, \quad t \geq 0, \tag{2.26}$$

where $s > 0$ is the initial value of the asset, $B = \{B_t : t \geq 0\}$ is a standard Brownian motion, $\sigma > 0$ and $\mu \in \mathbb{R}$. This choice of model offers the feature that asset values have multiplicative stationarity and independence in the sense that for any $0 \leq u < t < \infty$,

$$S_t = S_u \times \widetilde{S}_{t-u}, \tag{2.27}$$

where $\widetilde{S}_{t-u}$ is independent of $\{S_v : v \leq u\}$ and has the same distribution as S_{t-u}. Whether or not this is a realistic assumption in terms of temporal correlations in financial markets is open to debate. Nonetheless, for the purpose of a theoretical framework in which one may examine certain economic mechanisms, such as risk-neutrality, hedging and arbitrage, as well as giving sense to the value of certain financial products such as option contracts, exponential Brownian motion has proved to be a successful model in capturing the imagination of mathematicians, economists and financial practitioners alike. Indeed, what makes (2.26) "classical" is that Black and Scholes (1973) and Merton (1973) demonstrated how one may construct rational arguments leading to the pricing of a call option on a risky asset driven by exponential Brownian motion.

Two particular points, of the many, where the above model of a risky asset can be shown to be inadequate, concern the continuity of the paths and the distribution of log-returns of the value of a risky asset. Clearly (2.26) has continuous paths and therefore cannot accommodate for jumps which arguably are present in observed historical data of certain risky assets due to shocks in the market. The feature (2.27) suggests that for a fixed period of time Δ, for each $n \geq 1$, the innovations $\log(S_{(n+1)\Delta}/S_{n\Delta})$ are independent and normally distributed with mean $\mu\Delta$ and standard deviation $\sigma\sqrt{\Delta}$. Empirical data suggests that the tails of the distribution of the log-returns are asymmetric as well as having heavier tails than those of normal distributions. Note that the tails of normal distributions are particularly light as they decay like $\exp\{-x^2\}$ for large values of $|x|$. See for example the discussion in Schoutens (2003).

Recent literature suggests that a possible remedy is to work with

$$S_t = s\mathrm{e}^{X_t}, \quad t \geq 0,$$

instead of (2.26), where again $s > 0$ is the initial value of the risky asset and $X = \{X_t : t \geq 0\}$ is a Lévy process. This preserves multiplicative stationary and independent increments, as well as allowing for jumps, distributional asymmetry and the possibility of heavier tails than the normal distribution can offer. A simple example of how this may happen is simply to take for X a compound Poisson process whose jump distribution is asymmetric and heavy tailed. A more sophisticated example, and indeed quite a popular model in the research literature, is the so-called *variance-gamma* process, introduced by Madan and Seneta (1990). This Lévy process is pure jump, that is to say $\sigma = 0$, and has Lévy measure given by

$$\Pi(\mathrm{d}x) = \mathbf{1}_{(x<0)} \frac{C}{|x|} \mathrm{e}^{Gx} \mathrm{d}x + \mathbf{1}_{(x>0)} \frac{C}{x} \mathrm{e}^{-Mx} \mathrm{d}x,$$

where $C, G, M > 0$. It is easily seen by computing explicitly the integral $\int_{\mathbb{R}} (1 \wedge |x|) \Pi(\mathrm{d}x)$ and the total mass $\Pi(\mathbb{R})$ that the variance-gamma process has paths of bounded variation and further is not a compound Poisson process. It turns out that the exponential weighting in the Lévy measure ensures that the distribution of the variance-gamma process at a fixed time t has exponentially decaying tails (as opposed to the much lighter tails of the Gaussian distribution).

Working with pure jump processes implies that there is no diffusive nature to the evolution of risky assets. Diffusive behaviour is often found attractive for modelling purposes as it has the taste of a physical interpretation in which increments in infinitesimal periods of time are explained through the Central Limit Theorem as the aggregate effect of many simultaneous conflicting external forces.[7] Geman et al. (2001) argue the case for modelling the value of risky assets with Lévy processes which have paths of bounded variation which are not compound Poisson processes. In their reasoning, such processes have a countable number of jumps over finite periods of time, which correspond to the countable, but nonetheless infinite number of purchases and sales of the asset which collectively dictate its value as a net effect. In particular, being of bounded variation means the Lévy process can be written as the difference to two independent subordinators (see Exercise 2.8). These two subordinators should be thought of the total prevailing price buy orders and total prevailing price sell orders on the logarithmic price scale.

Despite the fundamental difference between modelling with bounded variation Lévy processes and Brownian motion, Geman et al. (2001) also provide an interesting link to the classical model (2.26) via time change. The basis of their ideas lies with the following lemma.

Lemma 2.15 *Suppose that $X = \{X_t : t \geq 0\}$ is a Lévy process with characteristic exponent Ψ and $\tau = \{\tau_s : s \geq 0\}$ is an independent subordinator with characteristic exponent $\Xi(\theta)$. Then $Y = \{X_{\tau_s} : s \geq 0\}$ is again a Lévy process with characteristic exponent $\Xi(\mathrm{i}\Psi(\theta))$.*

Proof First let us make some remarks about Ξ. We already know that the formula

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta\tau_s}\right) = \mathrm{e}^{-\Xi(\theta)s}$$

[7] See for example the second volume of Lucretius (ca. 99 BC–ca. 55 BC) and the formalisation in Einstein (1905).

holds for all $\theta \in \mathbb{R}$. However, since τ is a non-negative valued process, via analytical continuation, we may claim that the previous equality is still valid for[8] $\theta \in \{z \in \mathbb{C} : \Im z \geq 0\}$. Note in particular that, since

$$\Re \Psi(u) = \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \big(1 - \cos(ux)\big) \Pi(\mathrm{d}x) \geq 0,$$

for all $u \in \mathbb{R}$, the equalities

$$\mathbb{E}\big(\mathrm{e}^{\mathrm{i}uX_{\tau_s}}\big) = \mathbb{E}\big(\mathrm{e}^{-\Psi(u)\tau_s}\big) = \mathbb{E}\big(\mathrm{e}^{\mathrm{i}(\mathrm{i}\Psi(u))\tau_s}\big) = \mathrm{e}^{-\Xi(\mathrm{i}\Psi(u))s} \tag{2.28}$$

hold.

Since X and τ have right-continuous paths, then so does Y. Next consider $n \in \mathbb{N}$, $0 \leq s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n < \infty$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$. Then, by first conditioning on τ and noting that $0 \leq \tau_{s_1} \leq \tau_{t_1} \leq \cdots \leq \tau_{s_n} \leq \tau_{t_n} < \infty$, we have

$$\begin{aligned}
\mathbb{E}\left(\prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}\theta_j (Y_{t_j} - Y_{s_j})}\right) &= \mathbb{E}\left(\prod_{j=1}^{n} \mathrm{e}^{-\Psi(\theta_j)(\tau_{t_j} - \tau_{s_j})}\right) \\
&= \mathbb{E}\left(\prod_{j=1}^{n} \mathrm{e}^{-\Psi(\theta_j)\tau_{t_j - s_j}}\right) \\
&= \prod_{j=1}^{n} \mathrm{e}^{-\Xi(\mathrm{i}\Psi(\theta_j))(t_j - s_j)},
\end{aligned}$$

where in the final equality, we have used the fact that τ has stationary independent increments together with (2.28). Exercise 1.1 now allows us to conclude that Y has stationary and independent increments. $\square$

Suppose in the above lemma, we take for X a linear Brownian motion with drift as in the exponent of (2.26). By sampling this continuous path process along the range of an independent subordinator, one recovers another Lévy process. Geman et al. (2001) suggest that one may consider the value of a risky asset to evolve as the process (2.26) on an abstract time scale suitable to the rate of business transactions, called *business time*. The link between business time and real time is given by the subordinator τ. That is to say, one assumes that the value of a given risky asset follows the process $Y = X \circ \tau$ because, at real time $s > 0$, τ_s units of business time have passed and hence the value of the risky asset is positioned at X_{τ_s}.

Returning to the example of the variance-gamma process given above, it turns out that one may recover it from a linear Brownian motion by applying a time change using a gamma subordinator. See Exercise 2.9 for more details on the facts mentioned here concerning the variance-gamma process as well as Exercise 2.12 for more examples of Lévy processes which may be written in terms of a subordinated Brownian motion with drift.

[8] The notation $\Im z$ refers to the imaginary part of z.

Exercises

2.1 The objective of this exercise is to give a reminder of the additive property of Poisson distributions (which is also the reason why they belong to the class of infinite divisible distributions). Suppose that $\{N_i : i = 1, 2, \ldots\}$ is an independent sequence of random variables defined on $(\Omega, \mathcal{F}, P)$ which are Poisson distributed with parameters λ_i, for $i = 1, 2, \ldots$, respectively. Let $S = \sum_{i\geq 1} N_i$. Show that

(i) if $\sum_{i\geq 1} \lambda_i < \infty$ then S is Poisson distributed with parameter $\sum_{i\geq 1} \lambda_i$ and hence in particular $P(S < \infty) = 1$,
(ii) if $\sum_{i\geq 1} \lambda_i = \infty$ then $P(S = \infty) = 1$.

2.2 Denote by $\{T_i : i \geq 1\}$ the arrival times in the Poisson process $N = \{N_t : t \geq 0\}$ with parameter λ.

(i) By recalling that inter-arrival times are independent and exponentially distributed, show that, for any $A \in \mathcal{B}([0,\infty)^n)$,

$$P\big((T_1, \ldots, T_n) \in A | N_t = n\big) = \int_A \frac{n!}{t^n} \mathbf{1}_{(0\leq t_1 \leq \cdots \leq t_n \leq t)} \mathrm{d}t_1 \times \cdots \times \mathrm{d}t_n.$$

(ii) Deduce that the distribution of $(T_1, \ldots, T_n)$, conditional on $N_t = n$, has the same law as the distribution of an ordered independent sample of size n taken from the uniform distribution on $[0, t]$.

2.3 If η is a measure on $(S, \mathcal{S})$ and $f : S \to [0, \infty)$ is measurable then show that $\int_S (1 - \mathrm{e}^{-\phi f(x)})\eta(\mathrm{d}x) < \infty$ for all $\phi > 0$ if and only if $\int_S (1 \wedge f(x))\eta(\mathrm{d}x) < \infty$.

2.4 Recall that $D[0, 1]$ is the space of functions $f : [0, 1] \to \mathbb{R}$ which are right-continuous with left limits.

(i) Define the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Use the triangle inequality to deduce that, if $\{f_n : n \geq 1\}$ is a sequence in $D[0, 1]$ and $f : [0, 1] \to \mathbb{R}$ such that $\lim_{n\uparrow\infty} \|f_n - f\| = 0$, then $f \in D[0, 1]$.
(ii) Suppose that $f \in D[0, 1]$ and let $\Delta = \{t \in [0, 1] : |f(t) - f(t-)| \neq 0\}$ (the set of discontinuity points). Show that Δ is countable if Δ_c is countable, for all $c > 0$, where $\Delta_c = \{t \in [0, 1] : |f(t) - f(t-)| > c\}$. Next fix $c > 0$. Suppose for contradiction that Δ_c has an accumulation point, say x. Show that the existence of either a left or right limit at x leads to the conclusion that there is no left or right limit of f at x. Deduce that Δ_c, and hence Δ, is countable.

2.5 The explicit construction of a Lévy process given in the Lévy–Itô decomposition begs the question as to whether one may construct examples of *deterministic* functions which have similar properties to those of the paths of Lévy processes. The objective of this exercise is to do precisely that. The reader is warned, however, that this is purely an analytical exercise and one should not necessarily think of the paths of Lévy processes as being entirely similar to the functions constructed below in all respects.

(i) Let us recall the definition of the Cantor function, which we shall use to construct a deterministic function that has bounded variation, that is right-continuous with left limits and whose points of discontinuity are dense in its domain. Take the interval $C_0 := [0,1]$ and perform the following iteration. For $n \geq 0$ define C_n as the union of intervals which remain when removing the middle third of each of the intervals which make up C_{n-1}. The Cantor set C is the limiting object, $\bigcap_{n\geq 0} C_n$ and can be described by

$$C = \left\{ x \in [0,1] : x = \sum_{k\geq 1} \frac{\alpha_k}{3^k} \text{ such that } \alpha_k \in \{0,2\} \text{ for each } k \geq 1 \right\}.$$

One sees that the Cantor set is simply the remaining points in $[0,1]$ after omitting numbers whose tertiary expansion contains the digit 1. To describe the Cantor function, for each $x \in [0,1]$, let $j(x)$ be the smallest j for which $\alpha_j = 1$ in the tertiary expansion of $\sum_{k\geq 1} \alpha_k/3^k$ of x. If $x \in C$, then $j(x) = \infty$ and otherwise, if $x \in [0,1]\backslash C$, then $1 \leq j(x) < \infty$. The Cantor function is defined as follows

$$f(x) = \frac{1}{2^{j(x)}} + \sum_{i=1}^{j(x)-1} \frac{\alpha_i}{2^{i+1}} \quad \text{for } x \in [0,1].$$

Now consider the function $g : [0,1] \to \mathbb{R}$, given by $g(x) = f^{-1}(x) - ax$ for $a \in \mathbb{R}$. Here, we understand $f^{-1}(x) = \inf\{\theta : f(\theta) > x\}$. Note that g is monotone if and only if $a \leq 0$. Show that g has only positive jumps and the values of x for which g jumps form a dense set in $[0,1]$. Show further that g has bounded variation on $[0,1]$.

(ii) Now let us construct an example of a deterministic function which has unbounded variation and that is right-continuous with left limits. Denote by $\mathbb{Q}_2$ the dyadic rationals. Consider a function $J : [0,\infty) \to \mathbb{R}$ as follows. For all $x \geq 0$ which are not in $\mathbb{Q}_2$, set $J(x) = 0$. It remains to assign a value to J for each $x = a/2^n$ where $a = 1,3,5,\ldots$ (even values of a cancel). Let

$$J\left(a/2^n\right) = \begin{cases} 2^{-n} & \text{if } a = 1,5,9,\ldots \\ -2^{-n} & \text{if } a = 3,7,11,\ldots \end{cases}$$

and define

$$f(x) = \sum_{s\in[0,x]\cap\mathbb{Q}_2} J(s).$$

Show that f is uniformly bounded on $[0,1]$, is right-continuous with left limits and has unbounded variation over $[0,1]$.

2.6 Suppose that X is a Lévy process with Lévy measure Π.

(i) For each $n \geq 2$ show that for each $t > 0$,

$$\mathbb{E}\left[\int_{[0,t]}\int_{\mathbb{R}} |x|^n N(\mathrm{d}s \times \mathrm{d}x)\right] < \infty$$

almost surely if and only if

$$\int_{|x|\geq 1} |x|^n \Pi(\mathrm{d}x) < \infty.$$

(ii) Suppose now that Π satisfies $\int_{|x|\geq 1} |x|^n \Pi(\mathrm{d}x) < \infty$ for $n \geq 2$. Show that

$$\int_{[0,t]}\int_{\mathbb{R}} x^n N(\mathrm{d}s \times \mathrm{d}x) - t\int_{\mathbb{R}} x^n \Pi(\mathrm{d}x), \quad t \geq 0,$$

is a martingale.

2.7 Let X be a Lévy process with Lévy measure Π. Denote by N the Poisson random measure associated with its jumps.

(i) Show that

$$\mathbb{P}\left(\sup_{0<s\leq t} |X_s - X_{s-}| \geq a\right) = 1 - \mathrm{e}^{-t\Pi(\mathbb{R}\setminus(-a,a))},$$

for $a > 0$.

(ii) Show that the paths of X are continuous if and only if $\Pi = 0$.

(iii) Show that the paths of X are piecewise linear if and only if it is a compound Poisson process with drift if and only if $\sigma = 0$ and $\Pi(\mathbb{R}) < \infty$. (Recall that a function $f : [0,\infty) \to \mathbb{R}$ is right-continuous and piecewise linear if there exist sequence of times $0 = t_0 < t_1 < \cdots < t_n < \cdots$ with $\lim_{n\uparrow\infty} t_n = \infty$ such that on $[t_{j-1}, t_j)$ the function f is linear.)

(iv) Now suppose that $\Pi(\mathbb{R}) = \infty$. Argue by contradiction that, for each positive rational $q \in \mathbb{Q}$, there exists a decreasing sequence of jump times for X, say $\{T_n(\omega) : n \geq 0\}$, such that $\lim_{n\uparrow\infty} T_n = q$. Hence deduce that the set of jump times are dense in $[0,\infty)$.

2.8 Show that any Lévy process of bounded variation may be written as the difference of two independent subordinators.

2.9 This exercise gives another explicit example of a Lévy process, the variance-gamma process, introduced by Madan and Seneta (1990) to model financial data.

(i) Suppose that $\Gamma = \{\Gamma_t : t \geq 0\}$ is a gamma subordinator with parameters α, β and that $B = \{B_t : t \geq 0\}$ is an independent standard Brownian motion. Show that, for $c \in \mathbb{R}$ and $\sigma > 0$, the variance-gamma process

$$X_t := c\Gamma_t + \sigma B_{\Gamma_t}, \quad t \geq 0,$$

is a Lévy process with characteristic exponent

$$\Psi(\theta) = \beta \log\left(1 - \mathrm{i}\frac{\theta c}{\alpha} + \frac{\sigma^2\theta^2}{2\alpha}\right), \quad \theta \in \mathbb{R}.$$

(ii) Show that the variance-gamma process is equal in law to the Lévy process

$$\Gamma^{(1)} - \Gamma^{(2)} = \left\{\Gamma_t^{(1)} - \Gamma_t^{(2)} : t \geq 0\right\},$$

where $\Gamma^{(1)}$ is a gamma subordinator with parameters

$$\alpha^{(1)} = \left(\sqrt{\frac{1}{4}\frac{c^2}{\alpha^2} + \frac{1}{2}\frac{\sigma^2}{\alpha}} + \frac{1}{2}\frac{c}{\alpha}\right)^{-1} \quad \text{and} \quad \beta^{(1)} = \beta$$

and $\Gamma^{(2)}$ is a gamma subordinator, independent of $\Gamma^{(1)}$, with parameters

$$\alpha^{(2)} = \left(\sqrt{\frac{1}{4}\frac{c^2}{\alpha^2} + \frac{1}{2}\frac{\sigma^2}{\alpha}} - \frac{1}{2}\frac{c}{\alpha}\right)^{-1} \quad \text{and} \quad \beta^{(2)} = \beta.$$

2.10 Suppose that d is an integer greater than one. Choose $\mathbf{a} \in \mathbb{R}^d$ and let Π be a measure concentrated on $\mathbb{R}^d \backslash \{0\}$ satisfying

$$\int_{\mathbb{R}^d} \left(1 \wedge |\mathbf{x}|^2\right) \Pi(\mathrm{d}\mathbf{x}) < \infty,$$

where $|\cdot|$ is the standard Euclidean norm. Show that it is possible to construct a d-dimensional process $\mathbf{X} = \{\mathbf{X}_t : t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having the following properties.

(i) The paths of $\mathbf{X}$ are right-continuous with left limits $\mathbb{P}$-almost surely in the sense that, for each $t \geq 0$,

$$\mathbb{P}\left(\lim_{s \downarrow t} \mathbf{X}_s = \mathbf{X}_t\right) = 1 \quad \text{and} \quad \mathbb{P}\left(\lim_{s \uparrow t} \mathbf{X}_s \text{ exists}\right) = 1.$$

(ii) $\mathbb{P}(\mathbf{X}_0 = \mathbf{0}) = 1$, the zero vector in $\mathbb{R}^d$.
(iii) For $0 \leq s \leq t$, $\mathbf{X}_t - \mathbf{X}_s$ is independent of $\{\mathbf{X}_u : u \leq s\}$.
(iv) For $0 \leq s \leq t$, $\mathbf{X}_t - \mathbf{X}_s$ is equal in distribution to $\mathbf{X}_{t-s}$.
(v) For any $t \geq 0$ and $\theta \in \mathbb{R}^d$,

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\theta\cdot\mathbf{X}_t}\right) = \mathrm{e}^{-\Psi(\theta)t}$$

and

$$\Psi(\theta) = \mathrm{i}\mathbf{a}\cdot\theta + \frac{1}{2}\theta\cdot\mathbf{A}\theta + \int_{\mathbb{R}^d}\left(1 - \mathrm{e}^{\mathrm{i}\theta\cdot\mathbf{x}} + \mathrm{i}(\theta\cdot\mathbf{x})\mathbf{1}_{(|\mathbf{x}|<1)}\right)\Pi(\mathrm{d}\mathbf{x}), \qquad (2.29)$$

where for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^d$, $\mathbf{x}\cdot\mathbf{y}$ is the usual inner product and $\mathbf{A}$ is a $d \times d$ Gaussian covariance matrix.

2.11 Suppose that X is a subordinator.

(i) Show that it has a Laplace exponent given by

$$-\log \mathbb{E}\big(\mathrm{e}^{-qX_1}\big) =: \Phi(q) = \delta q + \int_{(0,\infty)} \big(1-\mathrm{e}^{-qx}\big)\Pi(\mathrm{d}x),$$

for $q \geq 0$, where $\delta \geq 0$ and $\int_{(0,\infty)}(1 \wedge x)\Pi(\mathrm{d}x) < \infty$.

(ii) Show using integration by parts that

$$\Phi(q) = \delta q + q\int_0^\infty \mathrm{e}^{-qx}\,\Pi(x,\infty)\mathrm{d}x$$

and hence that the drift term δ may be recovered from the limit

$$\lim_{q\uparrow\infty}\frac{\Phi(q)}{q} = \delta.$$

(iii) Show that

$$\lim_{q\downarrow 0}\frac{\Phi(q)}{q} = \mathbb{E}(X_1) = \delta + \int_{(0,\infty)} x\Pi(\mathrm{d}x) \in (0,\infty].$$

(iv) Finally, prove that $\Phi(\infty) < \infty$ if and only if X is a compound Poisson subordinator. That is to say, $\delta = 0$ and $\Pi(0,\infty) < \infty$, in which case $\Phi(\infty) = \delta + \Pi(0,\infty)$.

2.12 Here are some more examples of Lévy processes which may be written as a subordinated Brownian motion.

(i) Let $\alpha \in (0,2)$. Show that a Brownian motion subordinated by a stable process of index $\alpha/2$ is a symmetric stable process of index α.

(ii) Suppose that $X = \{X_t : t \geq 0\}$ is a compound Poisson process with Lévy measure given by

$$\Pi(\mathrm{d}x) = \big\{\mathbf{1}_{(x<0)}\mathrm{e}^{-a|x|} + \mathbf{1}_{(x>0)}\mathrm{e}^{-ax}\big\}\mathrm{d}x,$$

for $a > 0$. Now let $\tau = \{\tau_s : s \geq 0\}$ be a pure jump subordinator with Lévy measure

$$\pi(\mathrm{d}x) = \mathbf{1}_{(x>0)}2a\mathrm{e}^{-a^2x}\mathrm{d}x.$$

Show that $\{\sqrt{2}B_{\tau_s} : s \geq 0\}$ has the same law as X, where $B = \{B_t : t \geq 0\}$ is a standard Brownian motion independent of τ.

(iii) Suppose now that $X = \{X_t : t \geq 0\}$ is a compound Poisson process with Lévy measure given by

$$\Pi(\mathrm{d}x) = \frac{\lambda\sqrt{2}}{\sigma\sqrt{\pi}}\mathrm{e}^{-x^2/2\sigma^2}\mathrm{d}x,$$

for $x \in \mathbb{R}$. Show that $\{\sigma B_{N_t} : t \geq 0\}$ has the same law as X, where B is as in part (ii) and $\{N_s : s \geq 0\}$ is a Poisson process with rate 2λ independent of B.

The final part of this question gives a simple example of Lévy processes which may be written as a subordinated Lévy process.

(iv) Suppose that X is a symmetric stable process of index $\alpha \in (0, 2)$. Show that X can be written as a symmetric stable process of index α/β subordinated by an independent stable subordinator of index $\beta \in (0, 1)$.

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