

Chapter 2

Design of Planar Linkages with Specified Positions

This chapter proposes a uniform design equation for planar four-bar linkages with n prescribed positions. The coordinates of circle point at the first position are used as the design variables to build the distance constraint equations for the following successive positions through matrix transformation. Expanding the quadratic equations and canceling the quadratic items of the coordinates of the center point, a set of linear equations are obtained. The augmented coefficient matrix consisting of the coordinates of circle point at the first position is obtained to construct a uniform expression for the design of planar four-bar linkages.

2.1 Analysis of Planar Four-Bar Linkages with Multispecified Positions

For planar four-bar linkage synthesis, Burmester problem as a classic issue has been extensively discussed in the previous literatures [1–5]. Hunt [1], Suh and Radcliffe [2], Bottema and Roth [3], Angeles [4], and McCarthy [5] investigated the planar Burmester problem with a focus on five-pose synthesis by intersecting two center/circle-point curves of four-position combinations for the specified five positions.

The objective of planar four-bar motion generation is to calculate the mechanism parameters required to achieve a set of specified coupler poses which includes the positions and orientations of a link [6]. To satisfy this requirement, the crank center must be located within certain area of the fixed plane which can be designed in advance [7]. This mechanism design target is particularly useful when the coupler must achieve a specific displacement sequence for effective operation [6]. Burmester circle-point and center-point curves could give the driving cranks of crank-rocker or slider-crank linkages when combined with a prescribed driven crank [8–12]. For the

centenary of Burmester's classical textbook, Koetsier [13] gave some biographical data on Burmester and briefly commented on the importance of the book. Historical surveys [14–16] trace the early events relating to mechanical inventions and their applications that were instrumental in, but preceded, the eventual development of the science of linkage mechanism as originally expounded by Reuleaux. Analytical studies of the properties of coupler curves commenced in 1875 [15], when Samuel Roberts published the first account of the algebraic properties of the planar four-bar curve. Planar synthesis for orderly progression through four specified positions can be solved with Burmester theory [17, 18]. As to the comment on the Burmester theory [19], Koetsier addressed the importance of Burmester's classical textbook to the development of planar linkage synthesis [13].

There are three problems that have to be considered in the synthesis: branch, order, and crank problem [17]. Different kinds of methods have been proposed to solve the Burmester problem. To provide parameters of value in selecting linkage dimensions to meet design requirements, Davies et al. [20] described transformations which map the dimensions of planar 4-bar linkages onto finite regions of three-dimensional space. Waldron [8] developed a method to locate regions of the Burmester circle-point curve, which gives the driving cranks of crank-rocker or slider-crank linkages when combined with a specified driven crank. Waldron and Strong [9] presented a method to resolve the ambiguity in Waldron's solution to the order problem of planar Burmester synthesis by inspection of an image pole circle. They also proposed another method to allow the mapping of permissible segments of the circle-point curve in Waldron's solution to the branch problem. Prentis [21] discussed the classical problem of choosing a four-bar linkage to move its coupler through four prescribed positions. To satisfy this requirement, the crank center must be located within certain areas of a fixed plane. The result shows that these areas can be determined in advance. Simple extensions of the analysis offer a detailed picture of the movements of both driving crank and followers. Dyads method modeled by complex numbers [12] is presented in several different forms for three prescribed positions of either motion, path generation with the prescribed timing, or function generation. Complex number method [22] is applied to the dimensional synthesis of planar triads, for six exact positions, and movement generation with the prescribed timing. For rigid-body guidance mechanism synthesis reaching four specified task positions, Burmester curve equations are fit for being presented by a displacement matrix method [23].

Kim and Kwak [24] proposed a selective accurate synthesis technique by adopting the displacement matrix approach. For more than five prescribed poses, a four-bar linkage can be generally synthesized to access the poses approximately [18]. Although adjustable driving crank for 4-bar linkages could allow different motions [11], the requirement of an additional control of the driving crank increases the complexity and expense of the system. Therefore, this chapter focuses on the planar four-bar linkage and discusses the uniform expression of multiposition design.

2.2 Uniform Equation of Planar Four-Bar Linkages with n Specified Positions

For a planar four-bar linkage, any point on the perpendicular bisector of the two corresponding position points can be selected as the fixed joints on the base when two positions are specified. Therefore, there are numerous solutions for two-position synthesis of a planar four-bar linkage. The fixed joints can also be obtained by finding the crosspoint of perpendicular bisector of every two corresponding position points when three positions shown in Fig. 2.1a are prescribed. Figure 2.1b shows the process to find the fixed joint corresponding to the 3-position points P_1 , P_2 , and P_3 . Connecting P_1 , P_2 , and P_3 in turn and sketching the perpendicular bisectors of P_1P_2 and P_2P_3 , one can find a crosspoint, denoted by A , which is the pivot corresponding to position points P_1 , P_2 , and P_3 . Similarly, the pivot D , corresponding to position points Q_1 , Q_2 , and Q_3 , could be obtained, the process of which is shown in Fig. 2.1c. Figure 2.1d shows the case that the planar four-bar linkage is in the first position.

However, the solution obtained through the above process is based on the assumption that the joints are just the position points on the link. There would be other solutions if the joints were chosen differently. As a result, the number of solutions for 3-position synthesis of a planar four-bar linkage is numerous. What is discussed above is obtained through a geometric method. Because there are too many geometry lines to obtain the solution of four or more position problems, what follows will concentrate on the analytical means to get the solutions of multiposition synthesis of planar four-bar linkages.

2.2.1 General Requirements for Multiposition Design of a Planar Four-Bar Linkage

Suppose that the link of a planar four-bar linkage shown in Fig. 2.2 is required to pass a number of specified positions, $P_1Q_1, P_2Q_2, \dots, P_nQ_n$ ($n \geq 3$). The target of the synthesis is to find the pivots on the fixed base and the link as well.

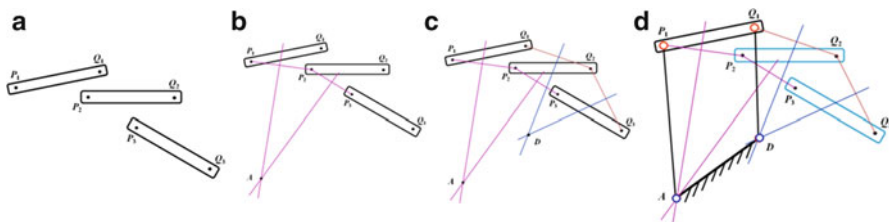


Fig. 2.1 3-position synthesis for a planar four-bar linkage. (a) Prescribed 3 positions (b) Left center point (c) Right center point and (d) Four-bar linkage

Fig. 2.2 General multiposition synthesis of planar four-bar linkage

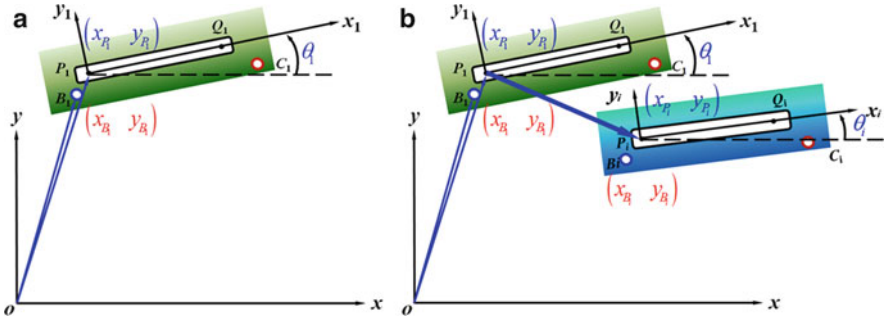
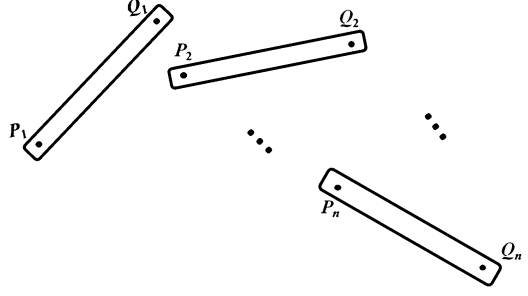


Fig. 2.3 Transformation between two positions. (a) The first prescribed position and (b) The second prescribed position

Figure 2.3a shows the case that link P_1Q_1 is in the first position. In the absolute coordinate system, the position is represented by the absolute coordinates of joint P_1 and the rotational angle θ_1 with respect to the absolute coordinate system about the z -axis which is perpendicular to the xoy -plane and therefore is omitted here. Assume that the pivots on the link corresponding to the first position are denoted by B_1 and C_1 . When the i th ($i \leq n$) position, P_iQ_i , shown in Fig. 2.3b, is specified, the transformation from the first coordinate system, $x_1B_1y_1$, to the i th one, $x_iB_iy_i$, can be uniquely determined:

$$\mathbf{r}_{B_i} = \mathbf{r}_{P_1} + \mathbf{r}_{P_1P_i} + \mathbf{R}(\theta_{1i}) \mathbf{r}_{P_1B_1} \quad (2.1)$$

where $\mathbf{r}_{B_i}$ denotes the absolute vector coordinates of joint B_i , $\mathbf{r}_{P_1}$ denotes the absolute vector coordinates of joint P_1 , $\mathbf{r}_{P_1P_i}$ ($i = 1, 2, \dots, n$) denotes the absolute vector coordinates of P_1P_i , and

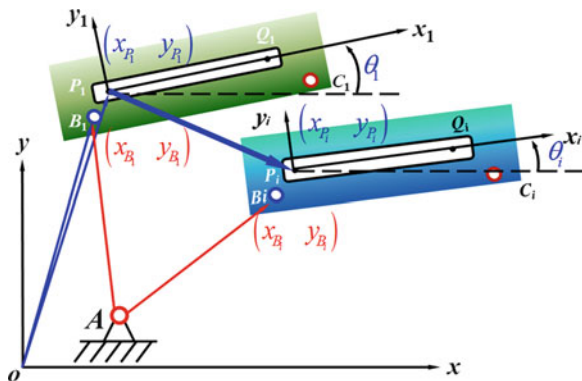
$$\mathbf{R}(\theta_{1i}) = \begin{bmatrix} \cos \theta_{1i} & -\sin \theta_{1i} \\ \sin \theta_{1i} & \cos \theta_{1i} \end{bmatrix} \quad (2.2)$$

where $\theta_{1i} = \theta_i - \theta_1$.

Because

$$\mathbf{r}_{P_i} = \mathbf{r}_{P_1} + \mathbf{r}_{P_1P_i} \quad (2.3)$$

Fig. 2.4 Circle point and center point



and $\mathbf{r}_{P_i}$ is prescribed, Eq. (2.1) can also be simplified as

$$\mathbf{r}_{B_i} = \mathbf{r}_{P_i} + \mathbf{R}(\theta_{1i}) \mathbf{r}_{P_1 B_1} \quad (2.4)$$

For the joints $B_1, B_2, \dots$, and B_i are all circle points, they must locate on a circle whose center is the fixed pivot on the base and which is denoted by A in Fig. 2.4.

Immediately, the following equation is obtained:

$$\|\mathbf{r}_{AB_i}\|^2 = \|\mathbf{r}_{AB_1}\|^2 \quad (i = 2, 3, \dots, n) \quad (2.5)$$

where $\|\cdot\|$ represents the 2-norm of vector “.”. Expanding Eq. (2.5) presents

$$c_{i1}x_A + c_{i2}y_A + c_{i3} = 0 \quad (2.6)$$

where $i = 2, 3, \dots, n$, x_A , and y_A denote the absolute coordinates of center point A and

$$\begin{cases} c_{i1} = 2[x_{B_1}(1 - \cos \theta_{1i}) + y_{B_1} \sin \theta_{1i} + x_{P_1} \cos \theta_{1i} - y_{P_1} \sin \theta_{1i} - x_{P_i}] \\ c_{i2} = 2[y_{B_1}(1 - \cos \theta_{1i}) - x_{B_1} \sin \theta_{1i} + y_{P_1} \cos \theta_{1i} + x_{P_1} \sin \theta_{1i} - y_{P_i}] \\ c_{i3} = 2(-x_{P_i}y_{B_1} - y_{P_i}x_{P_1} + x_{P_i}y_{P_1} + y_{P_i}x_{B_1}) \sin \theta_{1i} \\ \quad + 2(x_{P_i}x_{B_1} - x_{P_i}x_{P_1} - y_{P_i}y_{P_1} + y_{P_i}y_{B_1}) \cos \theta_{1i} \\ \quad - 2y_{B_1}y_{P_1} - 2x_{B_1}x_{P_1} + y_{P_1}^2 + x_{P_1}^2 + y_{P_1}^2 + x_{P_1}^2 \end{cases} \quad (2.7)$$

When three specified positions of the link are known, which are illustrated in Fig. 2.1a, the following equation set should hold:

$$\begin{cases} c_{21}x_A + c_{22}y_A + c_{23} = 0 \\ c_{31}x_A + c_{32}y_A + c_{33} = 0 \end{cases} \quad (2.8)$$

Solving Eq. (2.8) yields

$$\begin{cases} x_A = -\frac{\begin{vmatrix} c_{23} & c_{22} \\ c_{33} & c_{32} \end{vmatrix}}{\begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix}} \\ y_A = -\frac{\begin{vmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{vmatrix}}{\begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix}} \end{cases} \quad (2.9)$$

Equation (2.7) indicates that the coefficients of the linear equation set (2.8) are the function of the coordinates of joint B_1 ; therefore, there is one solution set when joint B_1 is prescribed as point P_1 . There will be other solutions when joint B_1 is changed to another one within the plane determined by link P_1Q_1 . When four specified positions of the link are provided, the following equation set holds:

$$\begin{cases} c_{21}x_A + c_{22}y_A + c_{23} = 0 \\ c_{31}x_A + c_{32}y_A + c_{33} = 0 \\ c_{41}x_A + c_{42}y_A + c_{43} = 0 \end{cases} \quad (2.10)$$

Equation set (2.10) has two variables, x_A and y_A , but there are three equations. Therefore, only two independent equations in equation set (2.10) are necessary to determine the coordinates of joint A . In other words, the sufficient and necessary condition that Eq. (2.10) has at least one solution is

$$R(\mathbf{C}_{3 \times 3}) \leq 2 \quad (2.11)$$

where $R(\cdot)$ denotes the rank of matrix “.” and $\mathbf{C}_{3 \times 3} = \begin{bmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix}$.

Therefore, the conditions that there are solutions for the coordinates of center point A can be alternatively expressed as

$$|\mathbf{C}_{3 \times 3}| = 0 \quad (2.12)$$

where $|\cdot|$ represents the determinant of matrix “.”. Equation (2.7) indicates that every element of the determinant (2.12) is a linear function of the coordinates of circle point B_1 . Equation (2.12) represents a 3-order curve of the coordinates of point B_1 [23].

When five specified positions of the link are known, the following equation set should hold:

$$\begin{cases} c_{21}x_A + c_{22}y_A + c_{23} = 0 \\ c_{31}x_A + c_{32}y_A + c_{33} = 0 \\ c_{41}x_A + c_{42}y_A + c_{43} = 0 \\ c_{51}x_A + c_{52}y_A + c_{53} = 0 \end{cases} \quad (2.13)$$

Equation set (2.13) also has two variables, x_A and y_A , but there are four equations. Only two independent equations in equation set (2.13) are needed to decide the coordinates of joint A . In other words, the sufficient and necessary condition that Eq. (2.13) has one solution at least is

$$R(\mathbf{C}_{4 \times 3}) < 3 \quad (2.14)$$

where $\mathbf{C}_{4 \times 3} = \begin{bmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \end{bmatrix}$. There should be $C_4^3 = 4$ 3-order determinants if Eq.

(2.14) is replaced by the zero-determinant expressions [2–5, 25], namely,

$$\begin{vmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{vmatrix} = 0, \quad \begin{vmatrix} c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \end{vmatrix} = 0, \quad \begin{vmatrix} c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \\ c_{21} & c_{22} & c_{23} \end{vmatrix} = 0, \quad \begin{vmatrix} c_{51} & c_{52} & c_{53} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 0 \quad (2.15)$$

The circle point B_1 must be the intersection of these four 3-order curves represented by Eq. (2.15). Similarly, when n specified positions of the link are known, the following equation set should hold when there is one solution at least:

$$\begin{cases} c_{21}x_A + c_{22}y_A + c_{23} = 0 \\ c_{31}x_A + c_{32}y_A + c_{33} = 0 \\ \vdots \\ c_{n1}x_A + c_{n2}y_A + c_{n3} = 0 \end{cases} \quad (2.16)$$

where $n = 3, 4, \dots$

Equation set (2.16) also has two variables only, namely, x_A and y_A , but there are $n - 1$ constraint equations. Therefore, only two independent equations in equation set (2.16) are required to determine the coordinates of joint A . Hence, the sufficient and necessary conditions that Eq. (2.16) has a solution could be expressed as

$$R(\mathbf{C}_{(n-1) \times 3}) < 3 \quad (2.17)$$

where

$$\mathbf{C}_{(n-1) \times 3} = \begin{bmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & c_{n3} \end{bmatrix} \quad (2.18)$$

If Eq. (2.17) is equivalently represented as that the determinants of 3×3 matrices are zeros simultaneously, there should be $C_{n-1}^3 = \frac{(n-1)(n-2)(n-3)}{6}$ 3-order determinants. It is a tricky job to search the intersections of so many 3-order curves.

2.2.2 Uniform Equation for the Design of a Planar Four-Bar Linkage with n Specified Positions

When n specified positions of the link are known, the condition that there is one solution at least is that the rank of the augmented coefficient matrix should always be less than 3. The solution can be expressed as the determinant of the square matrix is zero if a 3×3 matrix could be conceived with the augmented coefficient matrix in Eq. (2.18). Suppose

$$\mathbf{M} = \mathbf{C}_{(n-1) \times 3}^T \mathbf{C}_{(n-1) \times 3} \quad (2.19)$$

Therefore, $\mathbf{M}$ represented in Eq. (2.19) is a 3×3 matrix. There is the following theorem:

The circle points satisfying the n -position requirements must meet

$$|(\mathbf{M})_{3 \times 3}| = 0 \quad (2.20)$$

Proof.

1. Necessity

Because $R(\mathbf{C}_{(n-1) \times 3}^T \mathbf{C}_{(n-1) \times 3}) \leq R(\mathbf{C}_{(n-1) \times 3})$ and $R(\mathbf{C}_{(n-1) \times 3}) < 3$, the following equation should hold:

$$R(\mathbf{C}_{(n-1) \times 3}^T \mathbf{C}_{(n-1) \times 3}) < 3 \quad (2.21)$$

As a result, there must be

$$|(\mathbf{M})_{3 \times 3}| = |\mathbf{C}_{(n-1) \times 3}^T \mathbf{C}_{(n-1) \times 3}| = 0 \quad (2.22)$$

2. Sufficiency

Because $|(\mathbf{M})_{3 \times 3}| = 0$, the following equation should hold:

$$R(\mathbf{M}) < 3 \quad (2.23)$$

which means

$$R(\mathbf{C}_{(n-1) \times 3}^T \mathbf{C}_{(n-1) \times 3}) < 3 \quad (2.24)$$

Therefore, there is at least one 3-dimensional nonzero vector $\mathbf{x}_3$ such that the following equation holds:

$$\left(\mathbf{C}_{(n-1)3}^T \mathbf{C}_{(n-1)3} \right) \mathbf{x}_3 = 0 \quad (2.25)$$

Left multiplying $\mathbf{x}_3^T$ at both sides of Eq. (2.25) yields

$$\mathbf{x}_3^T \left(\mathbf{C}_{(n-1)3}^T \mathbf{C}_{(n-1)3} \right) \mathbf{x}_3 = 0 \quad (2.26)$$

Rearranging Eq. (2.26) presents

$$\left(\mathbf{C}_{(n-1) \times 3} \mathbf{x}_3 \right)^T \left(\mathbf{C}_{(n-1) \times 3} \mathbf{x}_3 \right) = 0 \quad (2.27)$$

which can be rewritten as

$$\left\| \mathbf{C}_{(n-1) \times 3} \mathbf{x}_3 \right\|^2 = 0 \quad (2.28)$$

where $\|\cdot\|$ represents the 2-normal of vector “ $\cdot$ ”. Immediately, Eq. (2.28) leads to

$$\mathbf{C}_{(n-1) \times 3} \mathbf{x}_3 = 0 \quad (2.29)$$

Because $\mathbf{x}_3$ is a 3-dimensional nonzero vector, there must be

$$R \left(\mathbf{C}_{(n-1) \times 3} \right) < 3 \quad (2.30)$$

The necessary and sufficient condition expressed by Eq. (2.20) provides a theoretical uniform equation for the circle point of planar four-bar linkages with n prescribed positions.

However, Eq. (2.19) indicates that each element of matrix $\mathbf{M}$ is a quadratic function of x_{B_1} and y_{B_1} and it is positive-semidefinite; the determinant of $\mathbf{M}$ denoted by Eq. (2.20) is larger than or equal to zero. Suppose that

$$f(x_{B_1}, y_{B_1}) = |\mathbf{M}| \quad (2.31)$$

The optimum solutions for x_{B_1} and y_{B_1} of the above problem must occur when $f(x_{B_1}, y_{B_1})$ has the minimum value. Let

$$\begin{cases} \frac{\partial f(x_{B_1}, y_{B_1})}{\partial x_{B_1}} = 0 \\ \frac{\partial f(x_{B_1}, y_{B_1})}{\partial y_{B_1}} = 0 \end{cases} \quad (2.32)$$

where $\frac{\partial f(x_{B_1}, y_{B_1})}{\partial x_{B_1}}$ indicates the partial derivative of $f(x_{B_1}, y_{B_1})$ with respect to x_{B_1} and $\frac{\partial f(x_{B_1}, y_{B_1})}{\partial y_{B_1}}$ denotes the partial derivative of $f(x_{B_1}, y_{B_1})$ with respect to y_{B_1} .

Solving partial derivative equation set (2.32) yields the stationary points. Assume that

$$\begin{cases} a = \frac{\partial^2 f(x_{B_1}, y_{B_1})}{\partial x_{B_1}^2} \\ b = \frac{\partial^2 f(x_{B_1}, y_{B_1})}{\partial x_{B_1} \partial y_{B_1}} \\ c = \frac{\partial^2 f(x_{B_1}, y_{B_1})}{\partial y_{B_1}^2} \end{cases} \quad (2.33)$$

According to calculus, the extreme values of $f(x_{B_1}, y_{B_1})$ occur at $a > 0$ and $ac - b^2 > 0$. The minimum value point can be found when these solutions are substituted into Eq. (2.31). Therefore, the circle point (x_{B_1}, y_{B_1}) is obtained. It will be an accurate solution when the multipositions specified meet the requirements of a circle point; otherwise it is an approximate solution. Substituting the coordinates of the circle point into Eq. (2.16), the coefficients of the equation are all known. Rearranging equation set (2.16) presents

$$\mathbf{K}\mathbf{x}_2 = \mathbf{h} \quad (2.34)$$

where $\mathbf{K} = \begin{bmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \\ \vdots & \vdots \\ c_{n1} & c_{n2} \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$, and $\mathbf{h} = -\begin{bmatrix} c_{23} \\ c_{33} \\ \vdots \\ c_{n3} \end{bmatrix}$. Because $R(\mathbf{K}) = 2$

always holds for different position problems, the general solution of $\mathbf{x}_2$ can be obtained by the least-squares method:

$$\mathbf{x}_2 = (\mathbf{K}^T \mathbf{K})^{-1} \mathbf{K}^T \mathbf{h} \quad (2.35)$$

where $(\mathbf{K}^T \mathbf{K})^{-1} \mathbf{K}^T$ denotes the pseudo inverse of $\mathbf{K}$ and $(\mathbf{K}^T \mathbf{K})^{-1} \mathbf{K}^T = \mathbf{K}^{-1}$ when $\mathbf{K}$ is a 2×2 full rank matrix.

The results expressed by Eq. (2.35) will be the precise solution if the specified positions satisfy the circle-point requirements; otherwise, the results will be the best approximate solutions for the proscribed multiposition problem. After obtaining the solutions, Grashof conditions should be considered in the final selection and mechanism design which has been discussed in Sect. 1.2.

2.3 Application Examples and Discussions

Because the augmented coefficient matrices, $\mathbf{C}$, for 2- and 3-position problems are 1×3 and 2×3 , respectively, Eq. (2.20) will therefore always hold. So the solutions for the circle point B_1 can be any point within the plane that the link locates, for

Table 2.1 Four specified positions

Items	1st position	2nd position	3rd position	4th position
x-coordinate (mm)	0	4	1	3
y-coordinate (mm)	−5	3	6	5
Pose angle (°)	90	45	−60	30

2- and 3-position problems. In this section, examples will concentrate on the 4-, 5-, and multiposition designs.

2.3.1 Design of Planar Four-Bar Linkages with Four Accurate Positions

The coordinates of circle point and its center point satisfy equation set (2.10) when four specified positions of the link are known. The augmented coefficient matrix of the constraint equation set is

$$\mathbf{C}_{3 \times 3} = \begin{bmatrix} c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix} \quad (2.36)$$

Because $\mathbf{C}_{3 \times 3}$ is a square matrix, Eq. (2.22) renders that

$$|(\mathbf{M})_{3 \times 3}| = |\mathbf{C}_{3 \times 3}^T \mathbf{C}_{3 \times 3}| = |\mathbf{C}_{3 \times 3}^T| \cdot |\mathbf{C}_{3 \times 3}| = |\mathbf{C}_{3 \times 3}|^2 = 0$$

which can be simplified as

$$|\mathbf{C}_{3 \times 3}| = 0 \quad (2.37)$$

This proves Eq. (2.12) again. Suppose that there are four different positions of the link which are listed in Table 2.1.

Substituting these parameters into Eq. (2.37), the precise circle-point curve can be drawn. Figure 2.5 shows the circle-point loci that satisfy the four positions in Table 2.1.

Substituting the coordinates of a point from the circle-point curve in Fig. 2.5 into any two equations of Eq. (2.10) or (2.35), the exact center-point coordinates can be obtained. For example, the center point is expressed with Eq. (2.9) when the first two equations of equation set (2.10) are selected. As previous literature [12, 23, 24, 26] investigated this problem repeatedly, this chapter does not discuss it again.

Fig. 2.5 Circle-point curve of four-position design

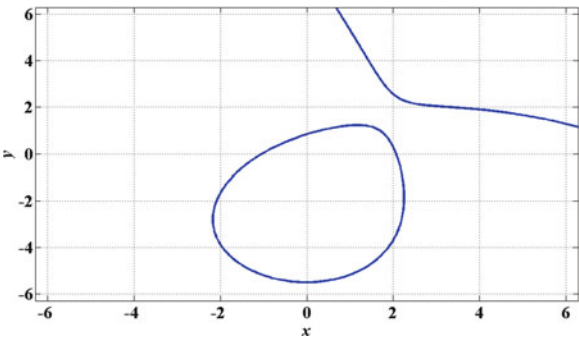


Table 2.2 Five specified positions

Items	1st position	2nd position	3rd position	4th position	5th position
x -coordinate (mm)	10	−6	−13	−17	6
y -coordinate (mm)	−5	−22	−11	10	5
Pose angle ($^{\circ}$)	0	−26	−40	−48	−15

When the parameters are prescribed differently from those listed in Table 2.1, the center-point and circle-point curves might be two-branch, thumb-shape, or knot curves [25]. For practical applications, users might prescribe a lot of design requirements, such as link position limit, dimension interval, and avoiding a certain specific defect. To solve these problems, Prentis [7], Waldron [8], Waldron and Strong [9], Chen and Fu [17], and Chen and Angeles [18] discussed different methods; Han and Qian [26] recently proposed a method of feasible mechanism solution regions where the solutions satisfy all required constraints above. This can be directly cooperated with the method proposed in this chapter to get an optimum solution within specified region. Detailed process for optimization design can be found from previous literatures [6, 14, 15, 26].

2.3.2 Design of Planar Four-Bar Linkages with Five Positions

When five positions for the link are specified, the circle points can be solved with the process discussed in Sect. 2.2. The results obtained here will be accurate if the 5-position problem has exact solutions; otherwise, the results will be the best approximate ones for the 5-position problem. Table 2.2 lists the five positions a planar four-bar linkage should trace.

Substituting the parameters of Table 2.2 into Eq. (2.19) yields

$$\mathbf{M} = \begin{bmatrix} (0.702454x_{B_1}^2 - 3.59562x_{B_1}y_{B_1} + 96.5332x_{B_1} \\ + 4.89839y_{B_1}^2 - 258.246y_{B_1} + 3490.95) \\ (1.79781x_{B_1}^2 - 4.19593x_{B_1}y_{B_1} + 105.939x_{B_1} \\ - 1.79781y_{B_1}^2 + 97.9667y_{B_1} - 1054.19) \\ (-53.2624x_{B_1}^2 + 116.772x_{B_1}y_{B_1} - 2776.09x_{B_1} \\ + 51.8288y_{B_1}^2 - 3684.6y_{B_1} + 53181.1) \\ (1.79781x_{B_1}^2 - 4.19593x_{B_1}y_{B_1} + 105.939x_{B_1} \\ - 1.79781y_{B_1}^2 + 97.9667y_{B_1} - 1054.19) \\ (4.89839x_{B_1}^2 + 3.59562x_{B_1}y_{B_1} - 99.4003x_{B_1} \\ + 0.702454y_{B_1}^2 - 46.3680y_{B_1} + 3034.54) \\ (-134.539x_{B_1}^2 - 105.091x_{B_1}y_{B_1} + 4331.14x_{B_1} \\ - 17.7675y_{B_1}^2 - 664.276y_{B_1} - 29219.3) \\ (-53.2624x_{B_1}^2 + 116.772x_{B_1}y_{B_1} - 2776.09x_{B_1} \\ + 51.8288y_{B_1}^2 - 3684.60y_{B_1} + 53181.1) \\ (-134.539x_{B_1}^2 - 105.091x_{B_1}y_{B_1} + 4331.14x_{B_1} \\ - 17.7675y_{B_1}^2 - 664.276y_{B_1} - 29219.3) \\ (4236.81x_{B_1}^2 + 1731.64x_{B_1}y_{B_1} - 122679.x_{B_1} \\ + 2288.69y_{B_1}^2 - 33612.8y_{B_1} + 921452.) \end{bmatrix} \quad (2.38)$$

Substituting Eq. (2.38) into (2.20) presents

$$\begin{aligned} |\mathbf{M}| &= 39.1785x_{B_1}^6 - 185.13x_{B_1}^5y_{B_1} + 4608.05x_{B_1}^5 + 433.967x_{B_1}^4y_{B_1}^2 \\ &\quad - 13925.2x_{B_1}^4y_{B_1} - 320.867x_{B_1}^4 - 370.259x_{B_1}^3y_{B_1}^3 + 1955.7x_{B_1}^3y_{B_1}^2 \\ &\quad + 902479.0x_{B_1}^3y_{B_1} - 1.94144e7x_{B_1}^3 + 750.399x_{B_1}^2y_{B_1}^4 - 83039.4x_{B_1}^2y_{B_1}^3 \\ &\quad + 3460755.0x_{B_1}^2y_{B_1}^2 - 6.40205e7x_{B_1}^2y_{B_1} + 4.9395e8x_{B_1}^2 - 185.13x_{B_1}y_{B_1}^5 \\ &\quad - 2652.35x_{B_1}y_{B_1}^4 + 2919555.0x_{B_1}y_{B_1}^3 - 1.74407e8x_{B_1}y_{B_1}^2 \\ &\quad + 3.29746e9x_{B_1}y_{B_1} - 1.48565e10x_{B_1} + 355.61y_{B_1}^6 - 69114.3y_{B_1}^5 \\ &\quad + 5022288.0y_{B_1}^4 - 1.76564e8y_{B_1}^3 + 3.57256e9y_{B_1}^2 - 5.12775e10y_{B_1} \\ &\quad + 4.50738 \times 10^{11} \\ &= 0 \end{aligned} \quad (2.39)$$

Table 2.3 Four different solutions

x_{B_1} (mm)	y_{B_1} (mm)	x_A (mm)	y_A (mm)
-30.568446	20.960137	191.467702	-64.838556
26.545161	32.179730	-44.146967	-17.273592
-0.818763	64.804165	-13.399540	-60.318106
38.477471	3.216875	-25.700624	9.351623

There are four different real solutions for the circle point $B_1 (x_{B_1}, y_{B_1})$ for equation (2.39) which are shown in Table 2.3. Substituting these solutions into Eq. (2.34) allows one to obtain the coordinates of the center point. For example, substituting the first solution of point B_1 into Eq. (2.34) presents

$$\begin{bmatrix} 1.028114 & 3.686569 \\ -6.356135 & -28.006752 \\ -11.429996 & -73.117632 \\ -8.202628 & -39.230632 \end{bmatrix} \begin{bmatrix} x_A \\ y_A \end{bmatrix} = - \begin{bmatrix} -42.181186 \\ 598.922840 \\ 2552.366548 \\ 973.119227 \end{bmatrix} \quad (2.40)$$

Solving Eq. (2.40) with equation (2.35) yields

$$\begin{cases} x_A = 191.467702 \\ y_A = -64.838556 \end{cases}$$

Similarly, the other center points corresponding to the circle points can be gained. The 4 different solutions for the specified 5-position problem are listed in Table 2.3.

According to Burmester theory [19], planar revolute-jointed four-bar linkage could exactly trace five arbitrary positions with the specified poses at most. But the method proposed in this chapter could also present the best approximate result when the 5-position design has no precise solution. The same is true for more position problems which will be discussed next.

2.3.3 Design of Planar Four-Bar Linkages with More Approximate Positions

Table 2.4 lists six positions for a planar four-bar linkage. The solutions for the circle point B_1 and its center point A and the circle point C and its center point D can be obtained with the process discussed in Sect. 2.2.

Generally, there are only approximate solutions for circle points and center points when more than five positions for the link are specified. Fortunately, the 6-position design prescribed in Table 2.4 has accurate solutions which could be directly gained with Eq. (2.20) or (2.31) through (2.35). The solutions for the center points and circle points are listed in Table 2.5 which demonstrates the effectiveness of the

Table 2.4 Six specified positions

Positions	x_P -coordinate	y_P -coordinate	x_Q -coordinate	y_Q -coordinate
1	1.046800	1.382700	2.577900	1.777300
2	1.127700	1.812200	2.708300	1.855700
3	0.684600	1.987400	2.264400	1.921500
4	0.119700	1.786300	1.700700	1.764300
5	-0.256500	1.325300	1.315600	1.494400
6	-0.316000	0.867500	1.189100	1.352100

Table 2.5 Exact solutions for the circle point and centre point of six prescribed positions

Circle point B_1	Circle point C_1	Center point A	Center point D
(0.749678, -0.000165)	(3.017368, 1.466613)	(-0.000396, -0.000156)	(2.700123, -0.001168)

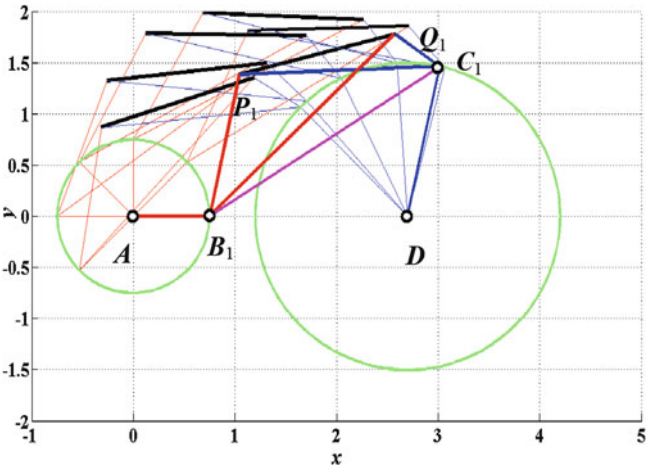


Fig. 2.6 A mechanism exactly meets six different positions

method proposed in this chapter. The mechanism exactly satisfying the six positions prescribed in Table 2.4 is illustrated by Fig. 2.6. The quadrangle $B_1C_1Q_1P_1$ shown in Fig. 2.6 represents link B_1C_1 .

There are only approximate solutions when these six positions are changed a little. Suppose the changed 6 positions are listed in Table 2.6 where only the second and fourth positions are altered near their original places.

The approximate circle point and center point corresponding to the positions in Table 2.6 are represented in Table 2.7, and the mechanism approximately satisfying the positions prescribed in Table 2.6 is shown in Fig. 2.7. The light bold lines represent the accurate positions specified by Table 2.6, and the dark bold lines are the real positions that the planar linkage could approximately trace.

Table 2.6 Six arbitrary positions

Positions	x_P -coordinate	y_P -coordinate	x_Q -coordinate	y_Q -coordinate
1	1.046800	1.382700	2.577900	1.777300
2	1.227700	1.752200	2.808300	1.795700
3	0.684600	1.987400	2.264400	1.921500
4	−0.080300	1.886300	1.500700	1.864300
5	−0.256500	1.325300	1.315600	1.494400
6	−0.316000	0.867500	1.189100	1.352100

Table 2.7 Approximate solutions for the circle point and centre point of six arbitrary positions

Circle point B_1	Circle point C_1	Center point A	Center point D
(0.819348, 0.055679)	(3.001470, 1.756439)	(0.064983, 0.073748)	(2.621502, 0.539307)

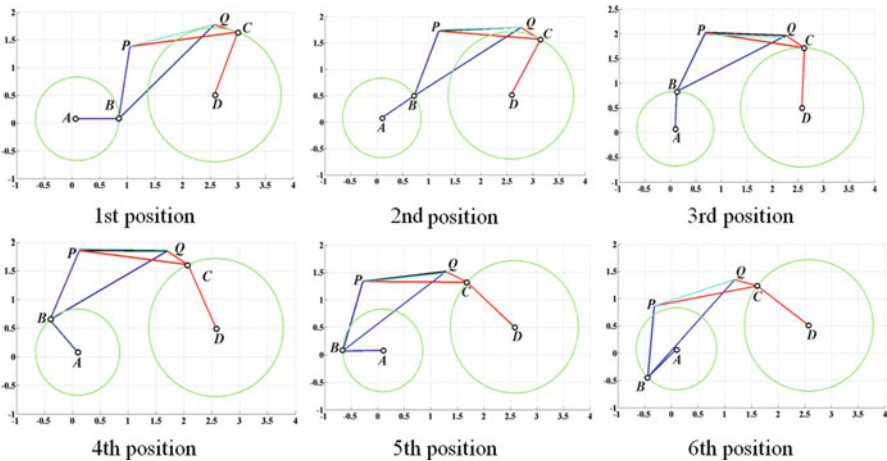


Fig. 2.7 A mechanism approximately meets six different positions

When seven positions are specified, the solutions for the circle point B_1 and its center point A and the circle point C and its center point D can be obtained with the similar process. Figure 2.8 illustrates the planar four-bar linkage that exactly satisfies the seven positions prescribed in Table 2.8. Figure 2.8 presents the case when the mechanism is in the second position for the specified seven positions. The solutions for the circle point and center point are listed in Table 2.9.

There are only approximate solutions when seven arbitrary positions are specified. Suppose the seven positions are listed in Table 2.10 where the second, fourth, and seventh positions are changed compared with those listed in Table 2.8.

The approximate circle point and center point corresponding to the positions in Table 2.10 are represented in Table 2.11, and the mechanism approximately satisfying the positions prescribed in Table 2.10 is shown in Fig. 2.9. The light bold lines represent the accurate positions specified by Table 2.10, and the dark bold lines are the real positions that the planar mechanism could approximately trace.

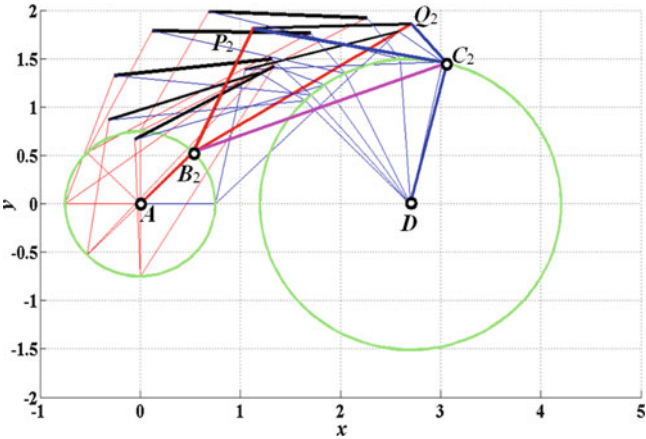


Fig. 2.8 The mechanism in the second position

Table 2.8 Seven specified positions

Positions	x_P -coordinate	y_P -coordinate	x_Q -coordinate	y_Q -coordinate
1	1.046800	1.382700	2.577900	1.777300
2	1.127700	1.812200	2.708300	1.855700
3	0.684600	1.987400	2.264400	1.921500
4	0.119700	1.786300	1.700700	1.764300
5	-0.256500	1.325300	1.315600	1.494400
6	-0.316000	0.867500	1.189100	1.352100
7	-0.051700	0.663300	1.335700	1.421600

Table 2.9 Precise solutions for the circle point and centre point of seven prescribed positions

Circle point B_1	Circle point C_1	Center point A	Center point D
(0.749685, -0.000162)	(3.017259, 1.466166)	(-0.000386, -0.000153)	(2.6996, -0.004914)

Table 2.10 Seven arbitrary positions

Positions	x_P -coordinate	y_P -coordinate	x_Q -coordinate	y_Q -coordinate
1	1.046800	1.382700	2.577900	1.777300
2	1.227700	1.752200	2.808300	1.795700
3	0.684600	1.987400	2.264400	1.921500
4	-0.080300	1.886300	1.500700	1.864300
5	-0.256500	1.325300	1.315600	1.494400
6	-0.316000	0.867500	1.189100	1.352100
7	-0.001700	0.663300	1.385700	1.421600

Table 2.11 Approximate solutions for the circle point and centre point of seven arbitrary positions

Circle point B_1	Circle point C_1	Center point A	Center point D
(0.819348, 0.055679)	(3.001470, 1.756439)	(0.064983, 0.073748)	(2.621502, 0.539307)

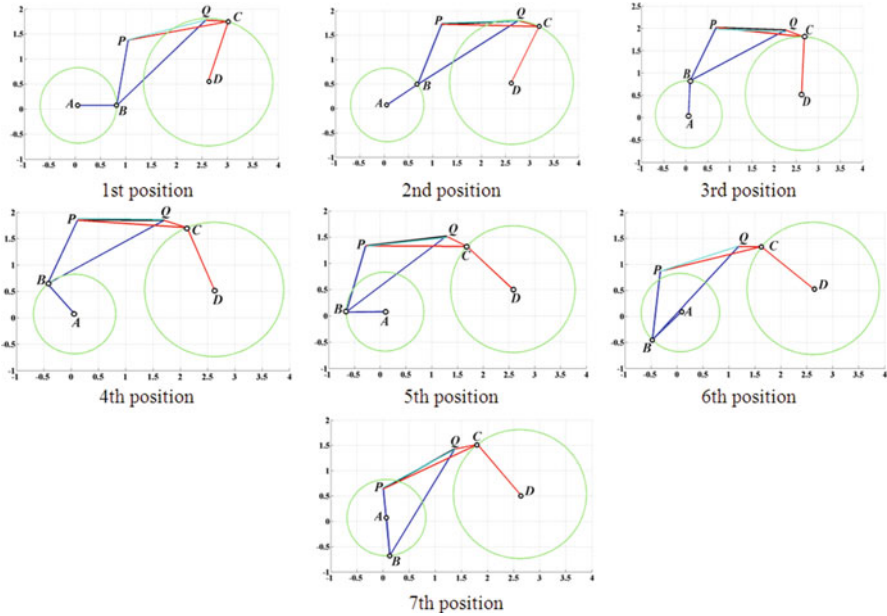


Fig. 2.9 A planar four-bar linkage approximately meets seven different positions

Table 2.12 Eight specified positions

Positions	x_P -coordinate	y_P -coordinate	x_Q -coordinate	y_Q -coordinate
1	1.046800	1.382700	2.577900	1.777300
2	1.127700	1.812200	2.708300	1.855700
3	0.684600	1.987400	2.264400	1.921500
4	0.119700	1.786300	1.700700	1.764300
5	−0.256500	1.325300	1.315600	1.494400
6	−0.316000	0.867500	1.189100	1.352100
7	−0.051700	0.663300	1.335700	1.421600
8	0.469100	0.882600	1.851300	1.650300

When eight accurate positions are prescribed in Table 2.12, the solutions for the circle point B_1 and its center point A and the circle point C_1 and its center point D can be obtained with the same process.

The precise solutions for the specified eight positions shown in Table 2.12 can be immediately obtained with the process proposed in Sect. 2.2. The solutions are represented by Table 2.13.

Table 2.13 Precise solutions for the circle point and centre point of eight prescribed positions

Circle point B_1	Circle point C_1	Center point A	Center point D
(0.749755, -0.000061)	(3.017255, 1.466168)	(-0.000271, -0.000030)	(2.699542, -0.005128)

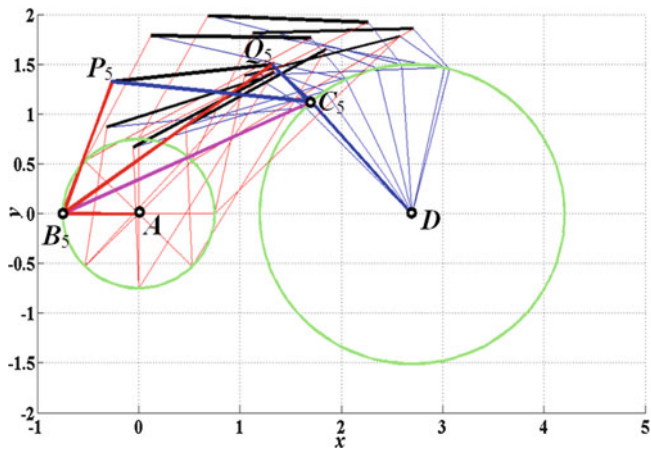


Fig. 2.10 The mechanism in the fifth position

Table 2.14 Eight arbitrary positions

Positions	x_P -coordinate	y_P -coordinate	x_Q -coordinate	y_Q -coordinate
1	1.046800	1.382700	2.577900	1.777300
2	1.227700	1.752200	2.808300	1.795700
3	0.684600	1.987400	2.264400	1.921500
4	-0.080300	1.886300	1.500700	1.864300
5	-0.256500	1.325300	1.315600	1.494400
6	-0.316000	0.867500	1.189100	1.352100
7	-0.001700	0.663300	1.385700	1.421600
8	0.489100	0.852600	2.808300	1.795700

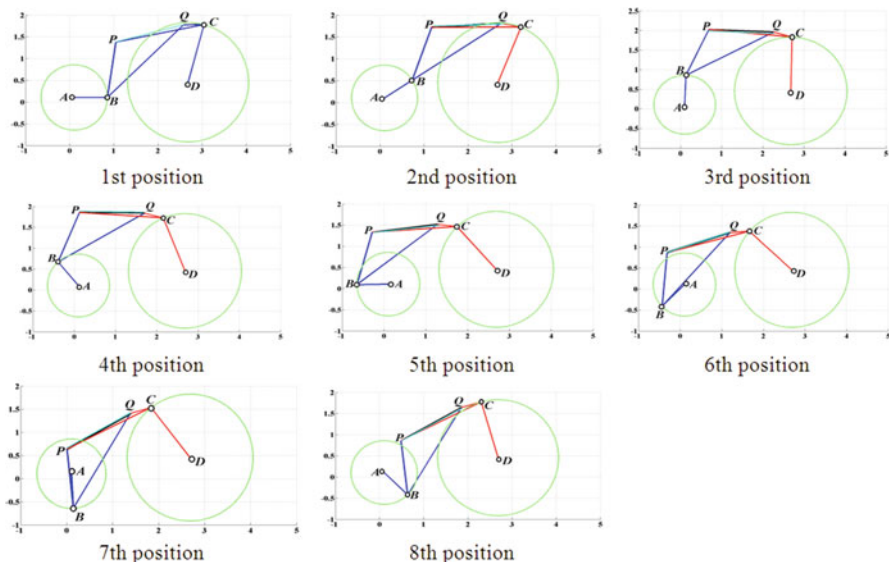
Figure 2.10 shows the case when the mechanism moves to the fifth position for the specified 8-position problem.

There are also only approximate solutions when these eight arbitrary positions are specified. Table 2.14 shows the eight positions of a planar four-bar linkage where the second, fourth, seventh, and eighth positions are different from those listed in Table 2.12.

The approximate circle point and center point corresponding to the positions in Table 2.14 are represented in Table 2.15, and the mechanism approximately satisfying the positions prescribed in Table 2.14 is shown in Fig. 2.11. The light bold lines represent the accurate positions specified by Table 2.12, and the dark bold lines are the real positions that the planar mechanism could approximately trace.

Table 2.15 Approximate solutions for the circle point and centre point of eight arbitrary positions

Circle point B_1	Circle point C_1	Center point A	Center point D
(0.852813, 0.092069)	(3.032621, 1.789466)	(0.102542, 0.108504)	(2.683970, 0.458268)

**Fig. 2.11** A mechanism approximately meets eight different positions

The method can also be directly used when more positions are required. The examples illustrate that the uniform equation is very convenient in engineering applications for the design of planar four-bar linkages with n specified positions. It can search the accurate results when there are such solutions for the specified n precise positions; otherwise, the best approximate solutions will be found. The merits of the method proposed in this chapter is that the engineers could directly obtain the planar four-bar linkages when the specified positions are known but need not intersect any two center/circle-point curves of four-position by combining the positions.

2.4 Conclusions

This chapter has proposed a uniform equation for the design of planar four-bar linkages with n specified positions. The coordinates of circle point at the first position are used as the design variables to develop the distance constraint equations for the following successive positions through matrix transformation. Expanding the quadratic equations and canceling the quadratic items of the coordinates of the

center point, a set of linear equations are obtained. The augmented coefficient matrix consisting the coordinates of circle point at the first position is then utilized to form a 3×3 -matrix, $\mathbf{M}$. Solutions of the circle points could be represented as $|\mathbf{M}| = 0$ which provides a uniform expression for the design of planar four-bar linkages. The least-squares method is used to get the best approximate solution for the problem of more than four positions that might not have any precise solution.

References

1. Hunt KH (1978) Kinematic geometry of mechanisms. Oxford University Press, New York
2. Suh CH, Radcliffe CW (1978) Kinematics and mechanisms design. Wiley, New York
3. Bottema O, Roth B (1979) Theoretical kinematics. North-Holland, New York
4. Angeles J (1982) Spatial kinematic chains: analysis, synthesis, and optimisation. Springer, Berlin
5. McCarthy JM, Soh GS (2011) Geometric design of linkages. Springer, New York
6. Shen Q, Al-Smadi YM, Martin PJ, Russell K, Sodhi RS (2009) An extension of mechanism design optimization for motion generation. *Mech Mach Theory* 44(9):1759–1767
7. Prentis JM (1991) The pole triangle, burmester theory and order and branching problems-I the order problem. *Mech Mach Theory* 26(1):19–30
8. Waldron KJ (1978) Location of burmester synthesis solutions with fully rotatable cranks. *Mech Mach Theory* 13(2):125–137
9. Waldron KJ, Strong RT (1978) Improved solutions of the branch and order problems of burmester linkage synthesis. *Mech Mach Theory* 13(2):199–207
10. Rao AC (1979) Synthesis of 4-bar function-generators using geometric programming. *Mech Mach Theory* 14(2):141–149
11. Ahmad A, Waldron KJ (1979) Synthesis of adjustable planar 4-bar mechanisms. *Mech Mach Theory* 14(6):405–411
12. Erdman AG (1981) Three and four precision point kinematic synthesis of planar linkages. *Mech Mach Theory* 16(3):227–245
13. Koetsier T (1989) The centenary of ludwig Burmester's "Lehrbuch der kinematik". *Mech Mach Theory* 24(1):37–38
14. Nolle H (1974) Linkage coupler curve synthesis: a historical review – I. Developments up to 1875. *Mech Mach Theory* 9(2):147–168
15. Nolle H (1974) Linkage coupler curve synthesis: a historical review – II. Developments after 1875. *Mech Mach Theory* 9(3–4):325–348
16. Nolle H (1975) Linkage coupler curve synthesis: a historical review – III. Spatial synthesis and optimization. *Mech Mach Theory* 10(1):41–55
17. Chen Y, Fu J (1999) A computational approach for determining location of burmester solutions with fully rotatable cranks. *Mech Mach Theory* 34(4):549–558
18. Chen C, Angeles J (2008) A novel family of linkages for advanced motion synthesis. *Mech Mach Theory* 43(7):882–890
19. Burmester L (1888) Lehrbuch der kinematik. Arthur Felix Verlag, Leipzig
20. Davies TH, Baker JE, Thompson AGR (1979) A finite 3-dimensional atlas of 4-bar linkages. *Mech Mach Theory* 14(6):389–403
21. Chen J-S, Chiang CH (1992) Coordinations of coupler-line positions with input-link rotations by a planar four-bar linkage. *Mech Mach Theory* 27(5):555–561
22. Lin C-S, Erdman AG (1987) Dimensional synthesis of planar triads: motion generation with prescribed timing for six precision positions. *Mech Mach Theory* 22(5):411–419
23. Yang T, Han J, Yin L (2011) A unified synthesis method based on solution regions for four finitely separated and mixed "point-order" positions. *Mech Mach Theory* 46(11):1719–1731

24. Kim K-Y, Kwak B-M (1995) A modified selective precision synthesis technique for planar four-Bar rigid-body guidance. *Proc Inst Mech Eng Part C J Mech Eng Sci* 209(1):39–47
25. Han J (2004) *Advanced mechanism design: analysis and synthesis*. China Machine Press, Beijing, In Chinese
26. Han J, Qian W (2009) On the solution of region-based planar four-bar motion generation. *Mech Mach Theory* 44(2):457–465

Design of Special Planar Linkages

Zhao, J.-S.; Feng, Z.; Ma, N.; Chu, F.

2014, XII, 294 p. 330 illus.,

ISBN: 978-3-642-38448-6