Chapter 1
Passage of Charged Particles Through Matter

1.1 Various Types of Processes

When charged particles pass through matter, the following processes may take place:

(1) Inelastic collisions with the bound electrons of the atoms of the medium, in which case the particle energy is spent in the excitation or ionization of atoms and molecules. The energy losses of this kind of collisions are called ionization losses (collision losses) to distinguish them from radiation losses that are concerned with the generation of bremsstrahlung.
(2) Inelastic collisions with nuclei, leading to the production of bremsstrahlung quanta, to the excitation of nuclear levels, or to the nuclear reactions.
(3) Elastic collisions with nuclei, in which part of the kinetic energy of the incident particle is transferred to the recoil nuclei. However, the total kinetic energy of the colliding particles remains unchanged. A particular type of elastic scattering is the Rutherford scattering which results from the interaction of a charged particle with the Coulomb field of the target nucleus in single encounters. When thick materials are used, cumulative single scatterings give rise to the phenomenon of multiple scattering.
(4) Elastic collisions with bound electrons.
(5) Cerenkov effect, i.e. emission of light by charged particles passing through matter with a velocity exceeding the velocity of light waves in the given medium.

1.2 Kinematics

1.2.1 Laboratory (Lab) System (LS) and Centre of Mass System (CM)

In order to describe the motion of particles in the collision problem one must choose a definite frame of reference (co-ordinate system). Two frames of reference are im-

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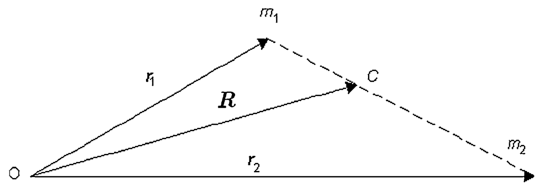


Fig. 1.1 The position vectors m_1 and m_2 and their centre of mass is shown

portant, one is the lab system (LS) and the other one is centre of mass system (CMS). In the lab system, the observer who is at rest in the lab views the collision process. In the CM system the centre of mass is at rest initially and always. Observations are usually made in the lab system but theoretical calculations are made in the CM system. It is of great interest to find out how various quantities like velocity, angle of scattering, etc. are related in these two systems. It is easier to perform calculations in the CM system rather than in the lab system. For, the great merit of CM system is that the total linear momentum of particles is always zero so that in the two-body process particles move directly towards each other before the collision and they recede in the opposite direction after the collision.

The collision process in the CM system may be visualized as the one in which a particle of reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ moving with initial velocity u_1 collides with a fixed scattering centre. Here, u_1 is the initial velocity of m_1 moving towards the target particle of mass m_2 at rest.

1.2.2 Total Linear Momentum in the CM System Is Zero

In Fig. 1.1, the position of the centre of mass of two particles m_1 and m_2 is shown by C. The position of masses m_1 and m_2 are indicated by the position vectors r_1 and r_2 and that of the centre of mass by R. By definition

$$\boldsymbol{R} = \frac{m_1 r_1 + m_2 r_2}{M} \quad \text{or}$$

$$M\boldsymbol{R} = m_1 r_1 + m_2 r_2$$

Differentiating with respect to time

$$M\dot{R} = m_1 \dot{r} + m_2 \dot{r} \quad \text{or}$$

$$M v_c = m_1 u_1 + m_2 u_2$$

where u_1 and u_2 are the initial velocities of particles 1 and 2, respectively and v_c is the CM velocity. Since m_2 is initially at rest, $u_2 = 0$, and the centre of mass which is located at $M = m_1 + m_2$, must move in the lab system towards m_2 with velocity

$$v_c = \frac{m_1 u_1}{m_1 + m_2} \tag{1.1}$$

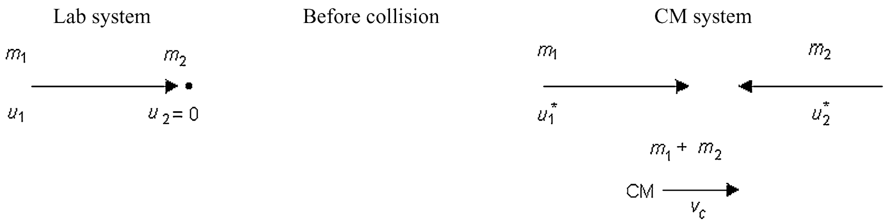


Fig. 1.2 Collision in the LS and CMS are shown

In the lab system, let m_1 move from left to right with initial velocity u_1, m_2 being initially at rest as in Fig. 1.2. As m_2 is initially at rest, its initial velocity in the CMS must be just equal to v_c in magnitude but oppositely directed. Denoting the velocities in the CMS by asterisk (*) we get

$$u_2^* = v_c = \frac{m_1 u_1}{m_1 + m_2} \tag{1.2}$$

$$u_2^* = -v_c \tag{1.3}$$

The initial velocity of m_1 in the CMS is reduced by an amount equal to v_c

$$u_1^* = u_1 - v_c$$

$$u_1^* = u_1 - \frac{m_1 u_1}{m_1 + m_2} = \frac{m_2 u_1}{m_1 + m_2}$$

where we have used (1.2). Total initial linear momentum of m_1 and m_2 in the CMS is

$$P^* = P_1^* + P_2^* = m_1 u_1^* + m_1 u_2^* = \frac{m_1 m_2 u_1^*}{m_1 + m_2} - \frac{m_2 m_1 u_1^*}{m_1 + m_2} = 0 \tag{1.4}$$

where we have used (1.2), (1.3) and (1.4). Thus total linear momentum of particles in the CMS is zero before the collision and by conservation of momentum, this must be so after the collision.

1.2.3 Relation Between Velocities in the LS and CMS

Lab system	CM system	
$m_1 : u_1,$	$u_1^* = \dfrac{m_2 u_1}{m_1 + m_2}$	(1.5)
$m_2 : u_2 = 0,$	$u_2^* = \dfrac{m_1 u_1}{m_1 + m_2}$	(1.6)

For elastic collisions, both momentum and kinetic energy must be conserved. This implies that the respective velocities of the particles before and after the collisions in the CMS must be equal

$$u_1^* = v_1^*; \qquad u_2^* = v_2^* \tag{1.7}$$

$$v_1^* = \frac{m_2 u_1}{m_1 + m_2} \tag{1.8}$$

Observe that in both the LS and CMS, the relative velocity of the two particles is equal to u_1. We know

$$u(\text{rel}) = u_1^* + u_2^* = \frac{m_2 u_1}{m_1 + m_2} + \frac{m_1 u_1}{m_1 + m_2} = u_1$$

Using (1.2) and (1.8),

$$\frac{v_c}{v_1^*} = \frac{m_1}{m_2} = \gamma. \tag{1.9}$$

It is seen that if $m_1 < m_2$, then $v_c < v_1^*$ and if $m_1 > m_2$, $v_c > v_1^*$.

1.2.4 Relation Between the Angles in LS and CMS

Figure 1.3 shows the scattering and recoil angles in the LS and CMS.

The lab velocity v_1 of m_1 after the collision is obtained by combining vectorially its velocity v_1^* in the CMS and the CM velocity v_c (Fig. 1.4)

$$v_1 = v_1^* + v_c$$

Let m_1 be scattered at an angle θ as seen in the LS, its corresponding angle in the CMS being θ^*. In the velocity triangle (Fig. 1.4) resolving the velocities along the x-axis and y-axis, we get

$$v_1 \sin\theta = v_1^* \sin\theta^* \tag{1.10}$$

$$v_1 \cos\theta = v_1^* \cos\theta^* + v_c \tag{1.11}$$

Dividing (1.10) by (1.11)

$$\tan\theta = \frac{v_1^* \sin\theta^*}{v_1^* \cos\theta^* + v_c} = \frac{\sin\theta^*}{\cos\theta^* + v_c/v_1^*} = \frac{\sin\theta^*}{\cos\theta^* + m_1/m_2} \tag{1.12}$$

where we have used (1.9).

Special cases

(i) $m_1 \ll m_2$; $\theta \simeq \theta^*$. Here $v_c \to 0$ and the CMS is reduced to the LS.
Example: α-gold nucleus scattering.

(ii) $m_1 \gg m_2$; $\theta \simeq 0°$.
Example: nucleus-electron scattering.

(iii) $m_1 = m_2$; $\tan\theta = \frac{\sin\theta^*}{\cos\theta^* + 1} = \tan\frac{1}{2}\theta^*$ so that $\theta = \frac{1}{2}\theta^*$.
Example: proton-proton scattering.

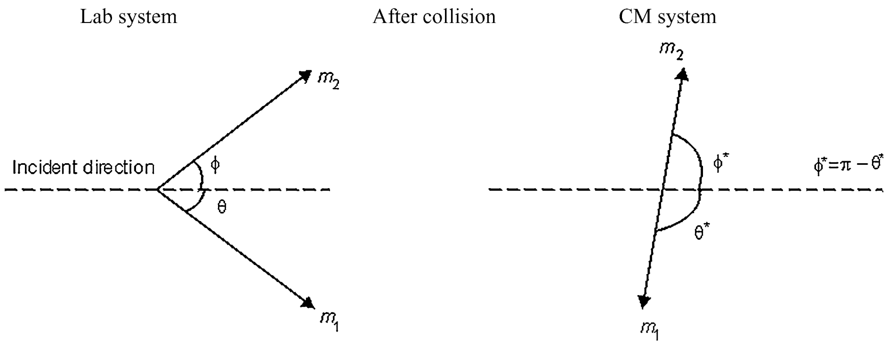
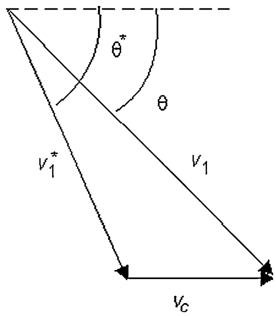


Fig. 1.3 Relation between the angles in LS and CMS

Fig. 1.4 Velocity triangle for the scattered particle



1.2.5 Recoil Angle

Let m_2 recoil with velocity v_2 at an angle ϕ with the incident direction in the LS. Let its velocity be v_2^* at angle ϕ^* in the CMS. From the velocity triangle in Fig. 1.5, we get

$$v_2 \sin\phi = v_2^* \sin\phi^* \tag{1.13}$$

$$v_2 \cos\phi = v_2^* \cos\phi^* + v_c \tag{1.14}$$

Dividing (1.13) by (1.14)

$$\tan\phi = \frac{v_2^* \sin\phi^*}{v_2^* \cos\phi^* + v_c}$$

but by (1.1), (1.6) and (1.7), $v_2^* = v_c$

$$\therefore \quad \tan\phi = \frac{\sin\phi^*}{\cos\phi^* + 1} = \tan\frac{\phi^*}{2} \quad \text{or}$$

$$\phi = \phi^*/2 \quad \text{(regardless of the ratio } m_1/m_2\text{)} \tag{1.15}$$

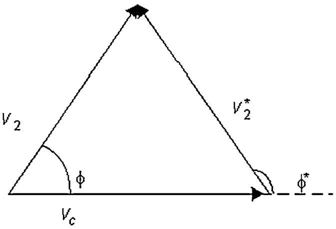


Fig. 1.5 Velocity triangle for the recoil particle

1.2.6 Limits on the Scattering Angle θ

Case (i) $m_2 > m_1$, or $\gamma < 1$; i.e. $v_1^* > v_c$.

In Fig. 1.6, the circle is drawn with O as the centre and radius $OP = v_1^*$. A is a point within the circle such that $AO = v_c$, and the line AOB represents the incident direction. As before, the lab velocity of m_1 is v_1 which is obtained by compounding v_1^* and v_c vectorially. The lab angle $\theta =$ angle PAO and the CM angle $\theta^* =$ angle POB. As the point P moves counterclockwise on the circumference, θ^* increases and so does θ. When P approaches P', $\theta^* = \theta = \pi$. Thus, θ increases monotonically from 0 to π, and in this case there is no restriction on the scattering angle in the LS. In other words, m_1 can be scattered in completely backward direction.

Case (ii) $m_2 = m_1$, or $\gamma = 1$; i.e. $v_1^* = v_c$.

Here A lies on the circumference of the circle (Fig. 1.7). As θ^* increases, θ also increases. But when P approaches A, PA becomes tangential at A and so $\theta \to \frac{\pi}{2}$. θ varies from 0 to π. Thus, in this case $m_1$can be scattered up to a maximum angle of $\pi/2$ but not beyond. In other words, backward scattering in the LS is not permissible.

Case (iii) $m_2 < m_1$, or $\gamma > 1$, i.e. $v_1^* < v_c$.

Here A lies outside the circle (Fig. 1.8). There are two positions P and P^l for which the same scattering angle θ is obtained for two different values of θ^*. As P moves back on the circumference, θ increases. The maximum angle θ_m is reached when AP becomes tangent to the circle (Fig. 1.9). In that case

$$\sin\theta_m = \frac{OP}{AO} = \frac{v_1^*}{v_c} = \frac{m_2}{m_1} \quad \text{or}$$

$$\theta_m = \sin^{-1}(m_2/m_1)$$

Thus, there is a limitation on the scattering angle when $m_2 < m_1$. θ first increases from 0 to a maximum value $\sin^{-1}(1/\gamma)$ which is less than $\pi/2$, as θ^* increases from 0 to $\cos^{-1}(-1/\gamma)$. θ then decreases to 0 as θ^* further increases to π. At a given angle θ between 0 and $\sin^{-1}(1/\gamma)$, there will be two groups of particles associated with different velocities corresponding to the two values of θ^*.

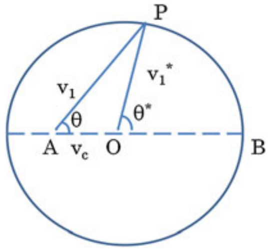


Fig. 1.6 Limits on the scattering angle for $m_2 > m_1$, θ max $= \pi$

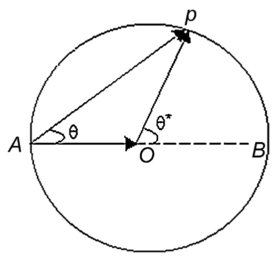


Fig. 1.7 Limits on the scattering angle θ for $m_1 = m_2$, θ max $= \pi/2$

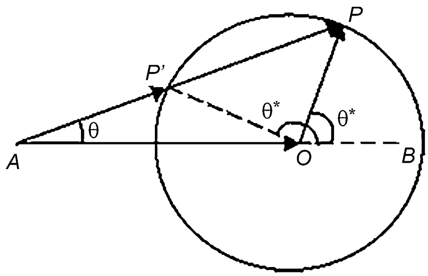


Fig. 1.8 Limits on the scattering angle θ. For $m_2 < m_1$, $\theta(\max) = \sin^{-1}(\frac{m_2}{m_1})$

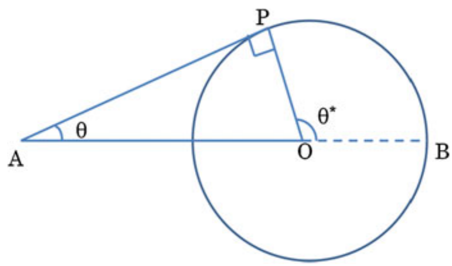


Fig. 1.9 $\theta_{\max}$ is reached when AP is a tangent to the circle

1.2.7 Limits on the Recoil Angle ϕ

Since $v_2^* = v_c$ (always), $\phi = \frac{1}{2}\phi^*$ by (1.15). Since the maximum angle of ϕ^* is π, the maximum angle ϕ_m is $\frac{1}{2}\pi$. In other words, the target particle cannot recoil in the backward hemisphere.

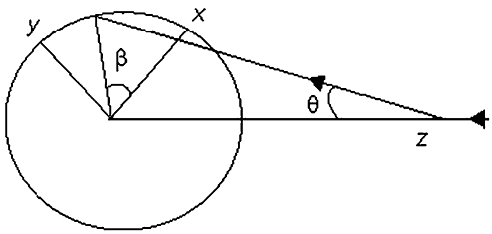


Fig. 1.10 Azimuth angle β is measured with respect to the positive axis in the xy plane $\perp$ to the direction of incidence

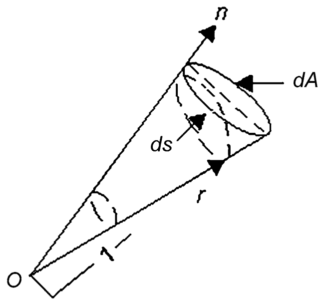


Fig. 1.11 Element of solid angle $d\Omega = ds/r^2$

1.2.8 Scattering in Three Dimensions

Since scattering is described under central forces, a particle which is incident on a target particle and initially moves in a certain plane would be necessarily confined to this plane after the scattering because of the conservation of angular momentum. Thus, a single scattering event is completely described in two dimensions. However, in practice one is concerned with a flux of particles incident say along the z-axis on a target material. Since various particles proceed in different planes, the scattering on the whole will be in three dimensions. In order to fix the orientation of the plane of scattering, we need to introduce the azimuth angle β which is measured with respect to the positive x-axis in the xy plane, Fig. 1.10. We must also consider the element of solid angle into which the particles are scattered. This is illustrated in Fig. 1.11.

Let dA denote an element of surface area and connect all points on the boundary of dA to O so as to form a cone. Let ds be the area of that portion of a sphere with O as the centre and radius r which is cut out by this cone. The solid angle subtended by dA at O is defined as $d\Omega = ds/r^2$ and is numerically equal to the area cut out by a sphere with centre O and unit radius. From Fig. 1.12, it is seen that $ds = r^2 \sin\theta d\theta d\beta$ so that $d\Omega = \sin\theta d\theta d\beta = 2\pi \sin\theta d\theta$, where we have integrated over $d\beta$. When the scattering is independent of the azimuth angle then the area subtended at O is due to the entire circular strip, $ds = 2\pi r^2 \sin\theta d\theta$ as in Fig. 1.13, so that the element of solid angle $d\Omega = 2\pi \sin\theta d\theta$. Observe that the maximum solid angle is 4π since it is given by the entire surface area of a sphere $(4\pi r^2)$ divided by r^2.

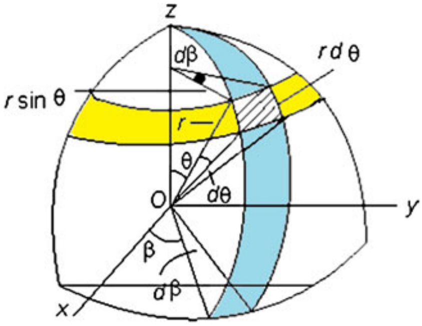


Fig. 1.12 Elements of solid angle for a general case

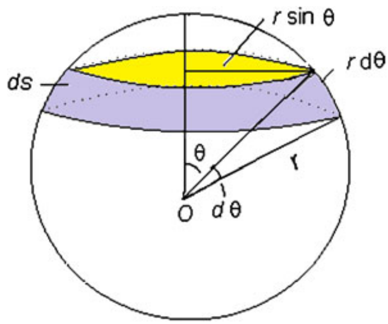


Fig. 1.13 Element of solid angle for an azimuthal symmetry

1.2.9 Scattering Cross-Section

In order to describe the angular distribution of particles scattered by target particles which are initially stationary, the concept of cross-section is introduced. Let a uniform parallel flux of N_0 particles be incident per unit area normal to the direction per unit time on a group of n scattering centres. Let N particles be scattered per unit time into a small solid angle $d\Omega$ centred towards a direction which has polar angle θ and azimuth angle β with respect to the incident direction as polar axis. N will be proportional to N_0, n and $d\Omega$ provided the flux is small enough to ensure that the incident particles do not interfere with one another, that there is no appreciable decrease in the number of scattering centres on account of their being knocked out due to collisions and that the incident particles are far enough apart so that each collision is made only by one of them.

The number of incident particles that emerge per unit time in $d\Omega$ can be written as:

$$N = nN_0\sigma(\theta, \beta)d\Omega \tag{1.16}$$

where the proportionality factor $\sigma(\theta, \beta)$ is called the differential scattering cross-section. The quantity $\sigma(\theta, \beta)$ is a measure of the probability of scattering in a given direction (θ, β), per unit solid angle from the given nucleus.

The integral of $\sigma(\theta, \beta)$ over the sphere is called the total scattering cross-section

$$\sigma = \int \sigma(\theta, \beta) d\Omega \tag{1.17}$$

σ has the dimension of area. The unit of σ is a Barn (b). 1 b $= 10^{-24}$ cm^2. The unit of $\sigma(\theta, \beta)$ is Barn/Steradian, where Steradian (sr) is the unit of solid angle. $\sigma(\theta, \beta)$ is also written as $d\sigma(\theta, \beta)/d\Omega$.

Compared to the scattering in two dimensions the only additional parameter which has been introduced to describe scattering in three dimensions is known as the azimuth angle β.

1.2.10 Relation Between Differential Scattering Cross-Sections

The relation between the differential cross-section in the laboratory and the centre-of-mass co-ordinate systems can be obtained from their definition which implies that the number of particles scattered into the element of solid angle $d\Omega$ about θ, β is the same as are scattered into $d\Omega^*$ about θ^*, β^*. In polar co-ordinates $d\Omega = \sin\theta d\theta d\beta$ and $d\Omega^* = \sin^* d\theta^* d\beta^*$

$$\sigma(\theta, \beta) \sin\theta d\theta d\beta = \sigma\left(\theta^*, \beta^*\right) \sin\theta^* d\theta^* d\beta^*$$

but

$$\beta = \beta^*$$

$$\therefore \quad \sigma(\theta) = \sigma\left(\theta^*\right) \frac{\sin^* d\theta^*}{\sin\theta d\theta} \tag{1.18}$$

Differentiating (1.12)

$$\sec^2\theta d\theta = \frac{|1 + \gamma\cos\theta^*| d\theta^*}{(\cos\theta^* + \gamma)^2} \tag{1.19}$$

Using (1.12) and the identity, $\sec^2\theta = 1 + \tan^2\theta$ (1.19) is easily reduced to the form:

$$\frac{d\theta^*}{d\theta} = \frac{[1 + 2\gamma\cos\theta^* + \gamma^2]}{|1 + \gamma\cos\theta^*|} \tag{1.20}$$

Also

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta^*}{\cos\theta^* + \gamma} \tag{1.12}$$

whence

$$\frac{\sin\theta^*}{\sin\theta} = \frac{\cos\theta^* + \gamma}{\cos\theta} \tag{1.21}$$

Also

$$\frac{1}{\cos\theta} = \sec\theta = \sqrt{1+\tan^2\theta} \tag{1.22}$$

Using (1.12) in (1.22) and re-arranging them we get

$$\frac{\cos\theta^* + \gamma}{\cos\theta} = \sqrt{1 + 2\gamma\cos\theta^* + \gamma^2} = \frac{\sin\theta^*}{\sin\theta} \tag{1.23}$$

Using (1.20) and (1.23) in (1.18)

$$\sigma(\theta) = \frac{(1+\gamma^2+2\gamma\cos\theta^*)^{3/2}\sigma(\theta^*)}{|1+\gamma\cos\theta^*|} \tag{1.24}$$

It must be pointed out that the total cross-section is the same for both lab and CM systems, since the occurrence of total number of collisions is independent of the mode of description of the process.

1.2.11 Kinematics of Elastic Collisions

We have to obtain an expression for velocity v_1 as a function of scattering angle θ. From the velocity triangle (Fig. 1.14)

$$v_1^{*2} = v_c^2 - 2v_1 v_c \cos\theta + v_1^2 \tag{1.25}$$

Substituting for v_c and v_1^* from (1.1) and (1.8), (1.25) becomes

$$v_1^2 - \frac{2m_1 u_1 \cos\theta\, v_1}{m_1+m_2} + u_1^2 \frac{(m_1-m_2)}{m_1+m_2} = 0$$

This is a quadratic equation in v_1 whose solutions are found to be

$$v_1 = \frac{m_1 u_1}{m_1+m_2}\left[\cos\theta \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\theta}\right] \tag{1.26}$$

For the special case, $m_1 = m_2$, (1.26) simplifies to:

$$v_1 = u_1\cos\theta$$

so that the ratio of kinetic energy T_1 and T_0 of the scattered and incident particle becomes

$$\frac{T_1}{T_0} = \frac{v_1^2}{u_1^2} = \cos^2\theta$$

with the restriction, $\theta \le 90^\circ$, as pointed out earlier.

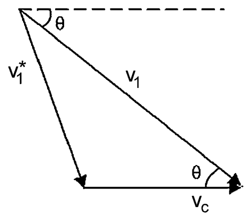


Fig. 1.14 Velocity triangle for the scattered particle

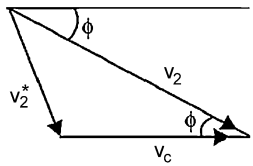


Fig. 1.15 Velocity triangle for recoil particle

1.2.12 To Derive an Expression for the Recoil Velocity v_2 as a Function of ϕ

From the velocity triangle (Fig. 1.15)

$$v_2^{*2} = v_c^2 + v_2^2 - 2v_c v_2 \cos\phi$$

Since

$$v_2^* = v_c$$

$$v_2 = 2v_c \cos\phi = \frac{2m_1 u_1}{m_1 + m_2} \cos\phi \tag{1.27}$$

where we have used (1.1).

The ratio of kinetic energy of the recoil particle and original kinetic energy of the incident particle is:

$$\frac{T_2}{T_0} = \frac{m_2 v_2^2}{m_1 u_1^2} = \frac{4m_1 m_2}{(m_1 + m_2)^2} \cos^2\phi \tag{1.28}$$

For the special case $m_1 = m_2$

$$\frac{T_2}{T_0} = \cos^2\phi \tag{1.29}$$

1.2.13 Available Energy in the Lab System and CM System

Assuming that the target particle is at rest before the collision, total kinetic energy in the lab system is

$$T = T_0, \quad \text{with } T_0 = \frac{1}{2} m_1 u_1^2 \tag{1.30}$$

In the CM system, m_1 has kinetic energy

$$T_1^* = \frac{1}{2}m_1\left(u_1^*\right)^2 = \frac{1}{2}m_1\left[\frac{m_2 u_1}{m_1+m_2}\right]^2$$

where we have used (1.4).

In the CM system, m_2 has kinetic energy:

$$T_2^* = \frac{1}{2}m_2\left(u_2^*\right)^2 = \frac{1}{2}m_2\left[\frac{m_1 u_1}{m_1+m_2}\right]^2$$

where we have used (1.2).

Total kinetic energy available in the CM system is:

$$T^* = T_1^* + T_2^* = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}u_1^2 = \frac{1}{2}\mu u_1^2 \tag{1.31}$$

where μ is the reduced mass.

Formula (1.31) shows that the two-body problem is reduced to a one-body problem by imagining that a particle of mass $\mu = m_1 m_2/(m_1+m_2)$ is directed towards a scattering centre, with the velocity u_1. Using (1.30) in (1.31)

$$T^* = \frac{m_2 T_0}{m_1+m_2}$$

where $T^* < T_0$.

Thus less energy of motion is available in the CM system. It can easily be shown that the difference in energy in the lab and CM systems is associated with the motion of CM system

$$\Delta T = T_0 - T^* = \frac{1}{2}m_1 u_1^2 - \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}u_1^2 = \frac{1}{2}\frac{m_1^2 u_1^2}{(m_1+m_2)} = \frac{1}{2}(m_1+m_2)v_c^2 \tag{1.32}$$

where we have used (1.1).

Formula (1.32) shows that the difference of energy goes into the motion of CM of mass (m_1+m_2) with velocity v_c. We conclude that in the CM system energy that is available is always less than that in the lab system, for some energy must go into the motion of CM system.

For the special case $m_1 = m_2$

$$T^* = \frac{1}{4}m_1 u_1^2 = \frac{1}{2}T_0$$

This fact has a bearing on production thresholds, i.e. minimum energy that is to be provided in order to produce particles. Consider, for example, the case of pion production in proton-proton collisions. The rest mass of pion is only 140 MeV/c^2. However, this much energy must be available in the CM system. This means that in the lab system, the incident proton must have double this energy viz, 280 MeV in order to produce a pion. Relativistic calculations actually give a value of 290 MeV.

These considerations are also important in the invention of a new class of high energy accelerators in recent years, in which colliding beams of particles are used; i.e. one beam travels in one direction and is intercepted by another beam of similar or dissimilar particles of the same energy moving in the opposite direction. In this case, the CM system is realized in the laboratory itself and lot of energy is made available.

Example 1.1 If a particle of mass m collides elastically with one of mass M at rest, and if the former is scattered at an angle θ and the latter recoils at an angle ϕ with respect to the line of motion of the incident particle, then show that

$$\tan\theta = \frac{\sin 2\phi}{\frac{m}{M} - \cos 2\phi}$$

Hence, show that

$$\frac{m}{M} = \frac{\sin(2\phi + \theta)}{\sin\theta}$$

Solution

$$\tan\theta = \frac{\sin\theta^*}{\cos\theta^* + m/M} \quad \text{but } \theta^* = \pi - \phi^* = \pi - 2\phi$$

$$\therefore \quad \sin\theta^* = \sin(\pi - 2\phi) = \sin 2\phi$$

$$\cos\theta^* = \cos(\pi - 2\phi) = -\cos 2\phi$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin 2\phi}{m/M - \cos 2\phi}$$

Re-arranging the above we get

$$\frac{m}{M}\sin\theta = \sin\theta\cos 2\phi + \cos\theta\sin 2\phi = \sin(\theta + 2\phi)$$

$$m/M = \frac{\sin(2\phi + \theta)}{\sin\theta}$$

Example 1.2 A particle makes an elastic collision with another particle of identical mass, initially at rest. Prove that after scattering, the lab angle between the outgoing particles is 90°.

Solution

First Method

We use the lab system. Let the particle of mass m, momentum P and kinetic energy T move along the x-axis. After collision the particles have momenta P_1 and P_2 at

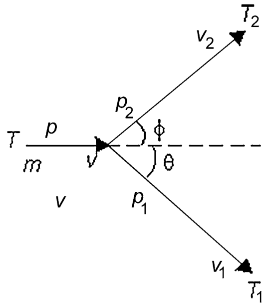


Fig. 1.16 Elastic collision in LS for $m_1 = m_2$

angles θ and ϕ as in Fig. 1.16. Conservation of momentum along the direction of incidence (x-axis) gives

$$P = P_1 \cos\theta + P_2 \cos\phi \tag{1}$$

Conservation of momentum along the perpendicular direction (y-axis) yields

$$P_2 \sin\phi - P_1 \sin\theta = 0 \quad \text{or}$$

$$0 = P_1 \sin\theta - P_2 \sin\phi \tag{2}$$

Squaring and adding (1) and (2) and simplifying we get

$$P^2 = P_1^2 + P_2^2 + 2P_1 P_2 \cos(\theta + \phi) \tag{3}$$

Energy conservation gives:

$$\frac{P^2}{2m} = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} \quad \text{or} \tag{4}$$

$$P^2 = P_1^2 + P_2^2 \tag{5}$$

Using (5) in (3):

$$2P_1 P_2 \cos(\theta + \phi) = 0$$

Since $P_1 \neq 0$; $P_2 \neq 0$; $\cos(\theta + \phi) = 0$ or $\theta + \phi = 90°$.

Second Method (Vector Method)

Momentum of conservation demands that (Fig. 1.17)

$$\boldsymbol{P_1} + \boldsymbol{P_2} = \boldsymbol{P}$$

Taking the scalar product

$$\boldsymbol{P} \cdot \boldsymbol{P} = (\boldsymbol{P_1} + \boldsymbol{P_2}) \cdot (\boldsymbol{P_1} + \boldsymbol{P_2})$$

$$\boldsymbol{P} \cdot \boldsymbol{P} = \boldsymbol{P_1} \cdot \boldsymbol{P_1} + \boldsymbol{P_2} \cdot \boldsymbol{P_2} + \boldsymbol{P_1} \cdot \boldsymbol{P_2} + \boldsymbol{P_2} \cdot \boldsymbol{P_1}$$

$$\boldsymbol{P^2} = \boldsymbol{P_1^2} + \boldsymbol{P_2^2} + \boldsymbol{2P_1} \cdot \boldsymbol{P_2} \tag{6}$$

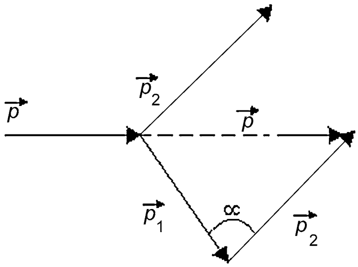


Fig. 1.17 Momentum triangle for the elastic collision

since the scalar product of a vector by itself is the square of the magnitude of the vectors and the order of scalar product is immaterial.
In view of energy conservation, i.e. with the aid of (5), we find

$$2P_1P_2 = 0 \quad \text{or} \quad 2P_1P_2\cos\alpha = 0$$

where α is the angle between the vectors P_1 and P_2

$$\therefore \quad \alpha = 90^\circ$$

Third Method
Because of the conservation of momentum, P_1, P_2, and P form a closed triangle. Their magnitudes are indicated in Fig. 1.17. Because of energy conservation we further have the relation:

$$P_1^2 + P_2^2 = P^2$$

i.e. the triangle must be a right angle triangle. Hence, $\alpha = 90^\circ$.
Fourth Method
We use the following formula for transformation of angles between LS and the CMS. Set $\frac{m}{M} = 1$ in (1.12)

$$\tan\theta = \frac{\sin\theta^*}{1+\cos\theta^*} = \tan\frac{\theta^*}{2}$$

$$\therefore \quad \theta = \frac{\theta^*}{2}$$

But $\theta^* = \pi - \phi^*$ and $\phi^* = 2\phi$, always

$$\phi^* = 2\theta = \pi - 2\phi \quad \text{whence } \theta + \phi = \frac{\pi}{2}$$

Example 1.3 Show that if a particle of mass m is scattered by a particle of mass M initially at rest, then the angle between the final directions of motion in the lab system is:

$$\frac{\pi}{2} + \frac{1}{2}\theta - \frac{1}{2}\sin^{-1}\left(\frac{m}{M}\sin\theta\right)$$

Hence, show that for particles of equal masses, the angle between final directions of motion is always 90°.

Solution From Example 1.1 we get

$$\frac{m}{M} = \frac{\sin(2\phi + \theta)}{\sin\theta} \quad \text{or}$$

$$\frac{m}{M}\sin\theta = \sin\big[\pi - (2\phi + \theta)\big]$$

$$\sin^{-1}\frac{m}{M}\sin\theta = \big[\pi - (2\phi + \theta)\big]$$

$$\phi + \frac{\theta}{2} = \frac{\pi}{2} - \frac{1}{2}\sin^{-1}\frac{m}{M}\sin\theta$$

The angle between the final directions of motion is

$$\alpha = \phi + \theta = \frac{\pi}{2} + \frac{1}{2}\theta - \frac{1}{2}\sin^{-1}\frac{m}{M}\sin\theta$$

For $m/M = 1$, α reduces to $\frac{1}{2}\pi$.

Example 1.4 At low energies, neutron-proton scattering is isotropic in the C-system. If K is neutron lab energy and σ the total cross section, show that in the lab, the proton energy distribution is

$$d\sigma_p/dK_p = \text{const} = \sigma/K_0$$

Solution In Fig. 1.18, ABC is the momentum triangle. Since the angle between the scattered neutron and recoil proton must be a right angle

$$P_P = P_0\cos\phi$$

$$K_P = P_P^2/2m_P \quad \text{and} \quad K_0 = P_0^2/2m_n$$

but, $m_p \simeq m_n = m$

$$K_P/K_0 = P_P^2/P_0^2 = \cos^2\phi \quad \text{or} \quad K_P = K_0\cos^2\phi$$

$$dK_p = -2K_0\cos\phi\sin\phi d\phi = -K_0\sin 2\phi d\phi$$

but, $\phi = \phi^*/2$ and $d\phi = d\phi^*/2$

$$dK_p = -\frac{1}{2}K_0\sin\phi^* d\phi^*$$

Isotropy requires that $d\sigma_p/d\Omega^* = \sigma/4\pi$

$$\frac{d\sigma_p}{dK_p} = -\frac{d\sigma_p}{d\Omega^*}\frac{d\Omega^*}{dK_p} = \frac{\sigma}{4\pi}\frac{2\pi\sin\phi^* d\phi^*}{\frac{1}{2}K_0\sin\phi^* d\phi^*} = \frac{\sigma}{K_0} = \text{const}$$

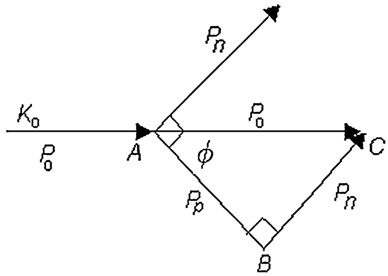


Fig. 1.18 Momentum triangle for n–p scattering

Negative sign is introduced in the last equation because as ϕ^* increases K_p decreases.

Example 1.5 A beam of particles of mass m is elastically scattered by target particles of mass M initially at rest. If the angular distribution is spherically symmetrical in the centre of mass system, what is it for M in the lab system?

Solution

$$\sigma(\phi^*) = \frac{\sigma}{4\pi} = \text{const}$$

$$\sigma(\phi) = \frac{\sin\phi^* d\phi^*}{\sin\phi d\phi}\sigma\left(\phi^*\right)$$

but

$$\phi^* = 2\phi \quad \text{and} \quad d\phi^* = 2d\phi$$

$$\sigma(\phi) = \frac{\sin 2\phi 2d\phi}{\sin\phi d\phi}\frac{\sigma}{4\pi} = \frac{\sigma}{\pi}\cos\phi$$

It may be recalled that ϕ is limited to 90°, i.e. the target particles can recoil only in the forward hemisphere in the lab system. It is instructive to note that

$$\int \sigma(\phi) d\Omega = \int_0^{\frac{1}{2}\pi} \frac{\sigma}{\pi}\cos\phi 2\pi \sin\phi d\phi$$

$$= 2\sigma \int_0^1 \sin\phi d(\sin\phi) = \sigma \quad \text{(as it should)}$$

Example 1.6 Small balls of negligible radii are projected against an infinitely heavy sphere of radius R. Assuming the balls are elastically scattered and bounce off in such a way that the angle of reflection (r) is equal to the angle of incidence (i). Prove that the scattering is isotropic, i.e. $\sigma(\phi)$ is independent of θ and that the total cross-section is equal to πR^2.

Solution Let b be the impact parameter (perpendicular distance of the line of flight from the central axis). The angle of incidence and reflection are measured with

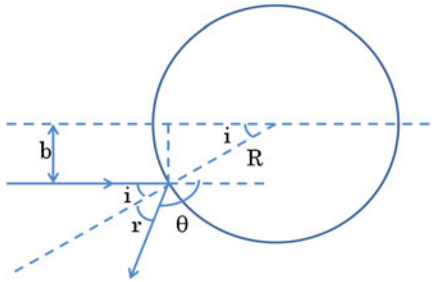


Fig. 1.19 Scattering of a small ball off a heavy sphere

respect to the normal at the point of scattering (Fig. 1.19)

$$\sigma(\theta) = -\frac{bdb}{\sin\theta d\theta} \tag{1}$$

See Eq. (1.56).

From the geometry of Fig. 1.19

$$\theta = \pi - (i + r) = \pi - 2i \tag{2}$$

$$\sin\theta = \sin(\pi - 2i) = \sin 2i = 2\sin i \cos i \tag{3}$$

since

$$r = i, \quad \text{and} \quad d\theta = -2di \tag{4}$$

but

$$b = R\sin i \tag{5}$$

Hence

$$db = R\cos i di \tag{6}$$

using (2), (4), (5) and (6) in (1) and cancelling various factors; we get $\sigma(\theta) = R^2/4$.

The right hand side of $\sigma(\theta)$ is independent of θ, the scattering angle. Hence, the scattering is isotropic, i.e. equally in all directions.

The total cross-section is given by

$$\sigma = \int \sigma(\theta) d\Omega = \int_0^\pi \frac{R^2}{4} 2\pi \sin\theta d\theta = \pi R^2$$

Observe that the total cross-section σ has the dimension of area, and in the above example it is equal to the projected area of the sphere. It is, therefore, called geometrical cross-section. Formula (5) shows that if $b = O$ (head-on collision), $i = 0$ and from (2), $\theta = 180°$. Thus, in this case the ball bounces in the opposite direction. Again, when $b = R$, $i = 90°$ and $\theta = 0°$ (glancing collision). Thus, the ball having hit the edge of the sphere, does not suffer any deviation and continues its flight in the incident direction. Of course, if $b > R$, the ball goes undeviated and there is no scattering. The above example shows the concept of $\sigma(\theta)$ and σ.

1.3 Rutherford Scattering

1.3.1 Derivation of Scattering Formula

Here we are concerned with the scattering (deflection) of point charged particles by a massive centre of electric force. The force is assumed to be central, i.e. directed along the line joining the centres of the colliding particles. Rutherford supposed that all the positive charge and hence practically all the mass of the atom is concentrated in a core or nucleus whose volume is very much less than that of the atom. Outside the nucleus is a relatively empty space only occupied by a few electrons. Suppose, an alpha particle is fired against the atom, then it is permitted to penetrate close to the nucleus and owing to the electrical interaction with the nucleus it may suffer a large angle deflection and recede from the nucleus and the effect of widely dispersed electrons can be neglected.

A particle of charge $+ze$ (for alpha particle, $z = 2$) at a distance r from the nucleus of charge $+Ze$ (Z being the atomic number) experiences a repulsive force zZe^2/r^2 (Coulomb's inverse square law) and the corresponding potential energy will be zZe^2/r. When the incident (incoming) particle is at a very large distance, the potential energy will be zero, and the energy is entirely kinetic due to the motion of the particle.

Let the particle of charge $+ze$ and mass m be incident from a very large distance from the nucleus (for example at a point A, Fig. 1.20), with velocity v_0. In the absence of forces between the nucleus (henceforth called target nucleus) at F and the incident particle, the particle would have continued to move along the straight line AOB. Let FQ be perpendicular on AOB. Then $b = FQ$ is called impact parameter. Since the target nucleus is considered infinitely heavy, it does not move during the encounter. The analysis will therefore be made in the lab system. The force is repulsive and central. We shall prove that under the influence of Coulomb's force, the trajectory is a hyperbola with the external focus F at the nucleus.

It is convenient to introduce the polar co-ordinates r, θ. The radial distance r is measured from the focus F and the angle θ with the x-axis, which is arbitrarily chosen. When the particle is near the nucleus, it will be deviated from the rectilinear trajectory under the action of electrical forces and its typical position at some instant would be at some point P with co-ordinates (r, θ) and velocity v. Since the force is repulsive, $v < v_0$. After the complete encounter, the particle is deflected through angle θ_0 and would recede to a remote distance beyond which it would continue along the straight path OD, and at a distant point like D. It would again have the original speed v_0, as the potential energy again approaches zero.

If the original path of the incident particle lies in a plane (here plane of paper) then because angular momentum is conserved, the particle would continue its path in the same plane throughout.

The conservation of energy gives:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + zZe^2/r \tag{1.33}$$

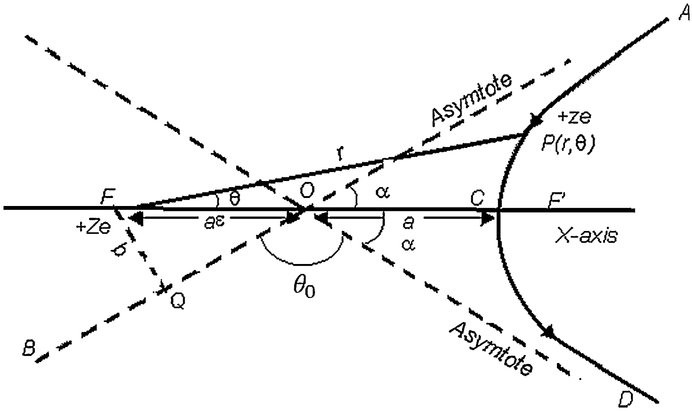


Fig. 1.20 Geometry of Rutherford scattering

In polar co-ordinates, the components of velocity along and perpendicular to r are $\dot{r}$ and $r\dot{\theta}$, respectively, so that

$$v^2 = (\dot{r})^2 + (r\dot{\theta})^2 \tag{1.34}$$

where dot means differentiation with respect to time. Eliminating v between (1.33) and (1.34) and re-arranging we get:

$$1 = \frac{2zZe^2}{mv_0^2 r} + \frac{1}{v_0^2}\left[(\dot{r})^2 + (r\dot{\theta})^2\right] \tag{1.35}$$

Also in the absence of external forces, angular momentum must be conserved. Take the angular momentum about an axis passing through the nucleus and perpendicular to the plane. Initially the momentum mv_0 is in the direction AB and the perpendicular distance FQ is b. Therefore, the initial angular momentum $= (mv_0)b$. At the point P, the component of velocity perpendicular to FP is $r\dot{\theta}$, and the distance $FP = r$. Hence, the angular momentum at P is $m(r\dot{\theta})r$.

Conservation of angular momentum gives,

$$mv_0 b = mr^2\dot{\theta} \quad \text{or} \tag{1.36}$$

$$\dot{\theta} = v_0 b / r^2 \tag{1.37a}$$

Also

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{v_0 b}{r^2}\frac{dr}{d\theta} \tag{1.37b}$$

Using (1.37a) and (1.37b) in (1.35) and dividing by b^2, we find:

$$\frac{1}{b^2} = \frac{2zZe^2}{mv_0^2 b^2 r} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} \tag{1.38}$$

It is desirable to have a change of variable, i.e.

$$u = \frac{1}{r} \tag{1.39}$$

$$\frac{dr}{d\theta} = \frac{dr}{du}\frac{du}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta} \tag{1.40}$$

Also, it is convenient to call

$$R_0 = \frac{2zZe^2}{mv_0^2} = \frac{zZe^2}{T_0} \tag{1.41}$$

Here T_0 is the initial kinetic energy of the incident particle. R_0 is the distance of closest approach for the head-on collision ($b = 0$). At the distance R_0, the particle momentarily comes to rest ($v = 0$) before it makes a sharp U-turn. Using (1.39), (1.40) and (1.41) in (1.38) and re-arranging, we get:

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{b^2} - u^2 - \frac{R_0}{b^2}u \tag{1.42}$$

Above differential equation can be solved easily if it is differentiated once with respect to θ, and bearing in mind that b is a constant for a given encounter,

$$2\frac{d^2u}{d\theta^2}\frac{du}{d\theta} = -2u\frac{du}{d\theta} - \frac{R_0}{b^2}\frac{du}{d\theta}$$

Cancelling the common factor $du/d\theta$ and re-arranging we get:

$$\frac{d^2u}{d\theta^2} + u + \frac{R_0}{2b^2} = 0 \tag{1.43}$$

This has the obvious solution

$$u = A\cos(\theta - \delta) - \frac{R_0}{2b^2} \tag{1.44}$$

where A and δ are the constants of integration. We may choose $\delta = 0$ to make the trajectory symmetrical about the x-axis. Call

$$g = 2b^2/R_0 \tag{1.45}$$

with $\delta = 0$, we find from (1.44)

$$\frac{1}{u} = r = \frac{g}{gA\cos\theta - 1} \tag{1.46}$$

This may be compared with the equation for a conic

$$r = \frac{a(\varepsilon^2 - 1)}{\varepsilon\cos\theta - 1} \tag{1.47}$$

where ε is the eccentricity and 'a' is the semi-major axis. We therefore, identify

$$g = a\left(\varepsilon^2 - 1\right) \tag{1.48}$$

$$\varepsilon = gA \tag{1.49}$$

Using (1.44) and (1.45) in (1.42) and simplifying

$$A^2 = \frac{1}{g^2} + \frac{1}{b^2} \tag{1.50}$$

Eliminating A between (1.49) and (1.50) and using (1.48) we can find ε

$$\varepsilon = \sqrt{1 + \frac{4b^2}{R_0^2}} = \sqrt{1 + \frac{4b^2 T_0^2}{z^2 Z^2 e^4}} \tag{1.51}$$

where we have used (1.41).

It is seen from the above formula that $\varepsilon > 1$ even if the charge is negative and the eccentricity is same in both the cases. The orbit is always a hyperbola and never an ellipse. For a repulsive Coulomb force (positively charged incident particle) the orbit is a hyperbola with the target nucleus at the external focus F, whereas for attractive Coulomb force (negatively charged incident particle) the orbit is a hyperbola with the target nucleus at the inner focus F'.

As $r \to \infty$, the denominator of the right hand side of (1.47) becomes zero, and the limiting angle α is given by:

$$\begin{aligned} \cos\alpha &= \frac{1}{\varepsilon} \quad \text{or} \\ \cot\alpha &= \frac{1}{\sqrt{\varepsilon^2 - 1}} \end{aligned} \tag{1.52}$$

Observe that α is very nearly equal to half of the angle subtended between the asymtotes, since the angle contained between the radius vector r and the x-axis is almost equal to α when $r \to \infty$. The scattering angle θ_0 is equal to angle BOD and is given by $\theta_0 = \pi - 2\alpha$ or, $\alpha = \frac{\pi}{2} - \frac{\theta_0}{2}$. Hence

$$\tan\frac{\theta_0}{2} = \cot\alpha = \frac{1}{\sqrt{\varepsilon^2 - 1}} = \frac{R_0}{2b} \tag{1.53}$$

where we have used (1.52) and (1.51).

Formula (1.53) can be derived by a shorter method by assuming that the trajectory is a hyperbola and by considering the velocity v at the point C which is at distance 'a' from the centre of the nucleus (Fig. 1.20); v being perpendicular to 'a'. Energy conservation gives:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{zZe^2}{a}$$

which yields

$$\left(\frac{v}{v_0}\right)^2 = 1 - \frac{R_0}{a} \tag{i}$$

Angular momentum conservation gives

$$mv_0 b = mva \quad \text{or}$$

$$\frac{v}{v_0} = \frac{b}{a} \tag{ii}$$

Using (ii) in (i), we find

$$b^2 = a(a - R_0) \tag{iii}$$

From the properties of hyperbola, we know

$$a = b\cot\frac{\alpha}{2} \tag{iv}$$

Eliminating 'a' between (iii) and (iv)

$$\frac{R_0}{b} = \frac{\cot^2\frac{\alpha}{2} - 1}{\cot\frac{\alpha}{2}} = 2\cot\alpha$$

$$\cot\alpha = \tan\frac{\theta_0}{2} = \frac{R_0}{2b}$$

Equation (1.53) shows that smaller the impact parameter b, larger is the scattering angle θ_0, and vice versa. Physically a larger value of b implies a weaker force and so a smaller deflection is to be expected.

In particular, $b = \infty$, implies $\theta_0 = 0$ and $b = 0$ implies $\theta_0, = \pi$. Figure 1.21 shows three typical scattering events. They are:

(a) with a large b,
(b) with a moderate value of b, and
(c) for a very small value of b.

Eliminating g between (1.45) and (1.48)

$$a = \frac{2b^2}{R_0(\varepsilon^2 - 1)} = \frac{R_0}{2} \tag{1.54}$$

where we have used (1.51).

For a particular value of b, the closest distance of approach will be FC which is given by putting $\theta = 0$ in (1.47)

$$r(\min) = a(\varepsilon + 1) = \frac{R_0}{2}\left[1 + \sqrt{1 + \frac{4b^2}{R_0^2}}\right] \tag{1.55}$$

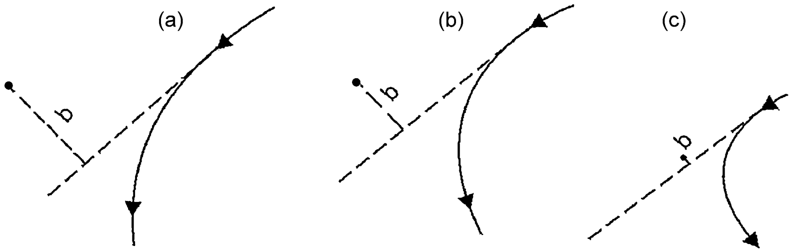


Fig. 1.21 Rutherford scattering for three different parameters b

Considering various scattering events with different b, $r(\text{min})$ will take on the least value for $b = 0$, i.e. for the head-on collision and in this case $r(\text{min}) = R_0$. Thus the significance of R_0 given by (1.41) is that it represents the least distance of the closest approach. It is also called Collision diameter. This result also follows from very simple considerations. As the positively charged particle approaches the target nucleus, due to the Coulomb's repulsion, it loses kinetic energy and when it is closest to the nucleus in a head-on collision, it would lose all its kinetic energy. Putting $v = 0$ in (1.33) we get

$$r(\text{min}) = \frac{2zZe^2}{mv_0^2} = R_0$$

We can find the numerical value of R_0 in the scattering of 5 MeV alpha particles (typical alpha energy from the radioactive sources) from a gold foil

$$T_0 = 5\ \text{MeV} = 5 \times 1.6 \times 10^{-13}\ \text{J} = S \times 10^{-13}\ \text{J}$$
$$e = 1.6 \times 10^{-19}\ \text{Coul}$$
$$z = 2, \qquad Z = 79$$

For S.I. units, formula (1.41) becomes: $R_0 = \frac{zZe^2}{4\pi\varepsilon_0 T_0}$.

For numerical calculations, $R_0\ (\text{fm}) = \frac{1.44 zZ}{T_0\ (\text{MeV})} = \frac{1.44 \times 2 \times 79}{5} = 45.5$ fm.

This value may be compared with the radius of gold nucleus which is 8 fm, a value which is smaller than the minimum distance of closest approach. This then ensures that the alpha particle of 5 MeV stays well outside the gold nucleus in any type of encounter including the head-on collision and that the inverse square law would be valid for all the orbits. For much greater energy, in close encounters, alpha particles may be able to penetrate the target nucleus itself in which case the inverse square law would no longer be valid, and other complications would be introduced into which we shall not enter at the moment.

From (1.53) it is obvious that given the impact parameter b, the scattering angle θ_0 can be determined. But, in practice, it is impossible to know the value of b. However, we can compute the expected angular distribution from the entire range of b's. Consider a uniform beam of particles fired against the target material. The beam intensity I is defined as the number of particles crossing unit area normal to the beam

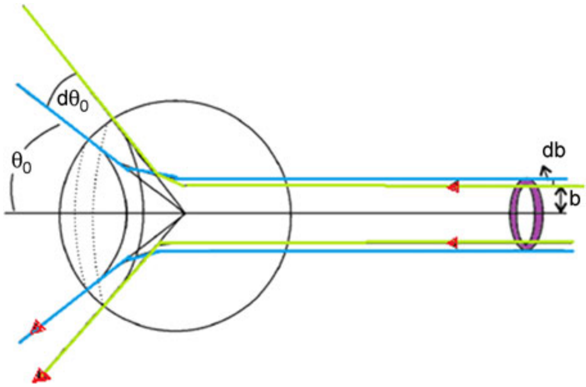


Fig. 1.22 Particles passing through the ring of radii b and $b + db$ are scattered in the angular interval $\theta_0 + d\theta_0$ and θ_0

direction per second. Near the centre of force a beam particle bends around and as it escapes from the field of force it once again describes a straight line. Because the force is central, one can expect an azimuthal symmetry in scattering about an axis along the beam direction. Assuming that the scattering is independent of β the element of solid angle becomes $d\Omega = 2\pi \sin\theta_0 d\theta_0$. Consider a uniform beam of particles of same energy directed towards a force centre, with impact parameters b and $b + db$. Such particles are seen to pass through a ring of radii b and $b + db$, the ring being perpendicular to the beam direction and symmetrical about an axis passing through the nucleus, and has an area of $2\pi b db$. Now, particles of given energy and impact parameter have a unique angle of deflection determined by formula (1.53). Therefore, particles passing through this ring must be scattered into the solid angle lying between θ_0 and $\theta_0 + d\theta_0$ (Fig. 1.22). Since the number of particles must be conserved

$$2\pi b db I = -2\pi \sin\theta_0 d\theta_0 I \sigma(\theta_0) \quad \text{or}$$

$$\sigma(\theta_0) = -\frac{b db}{\sin\theta_0 d\theta_0} \tag{1.56}$$

The negative sign is introduced in (1.56) due to the fact that an increase in b implies a decrease in θ_0. Rewriting (1.53)

$$b = \frac{R_0}{2} \cot\frac{\theta_0}{2} \tag{1.57}$$

Hence

$$db = -\frac{R_0}{4} \operatorname{cosec}^2\frac{\theta_0}{2} d\theta_0 \tag{1.58}$$

Using (1.57) and (1.58) in (1.56) and noting that $\sin\theta_0 = 2\sin\frac{\theta_0}{2} \cdot \cos\frac{\theta_0}{2}$, we get after simplification:

$$\sigma(\theta_0) = \frac{R_0^2}{16\sin^4\frac{\theta_0}{2}} = \frac{1}{16}\left[\frac{zZe^2}{T_0}\right]^2 \frac{1}{\sin^4\frac{\theta_0}{2}}$$

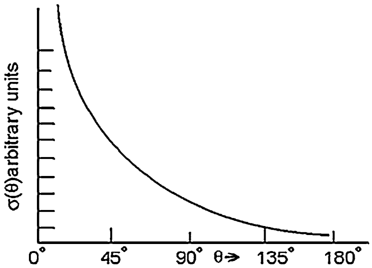


Fig. 1.23 Differential cross-section (in arbitrary units) as a function of scattering angle θ in the LS

$$\sigma(\theta) = \frac{1}{4}\left[\frac{zZe^2}{mv_0^2}\right]^2 \frac{1}{\sin^4\frac{\theta_0}{2}} \quad \text{(Rutherford's scattering formula)} \tag{1.59}$$

This is the famous Rutherford's scattering formula. Henceforth, the suffix 0 is dropped off in θ_0. The expected differential cross-section as a function of scattering angle given by (1.59) is shown in Fig. 1.23. Observe that the differential cross-section falls off rapidly with increasing angle, the scattering thus being predominantly in the forward direction.

Formula (1.59) also shows that $\sigma(\theta)$ will be greater for targets and incident particles of higher atomic number and that it will be more important for low energy particles.

For the purpose of numerical calculations (1.59) can be written in the form:

$$\sigma(\theta) = 1.295\left(\frac{zZ}{T}\right)^2 \frac{1}{\sin^4\theta/2}\ \text{Mb/sr} \tag{1.60}$$

where T is in MeV.

1.3.2 Darwin's Formula

Rutherford's formula which takes into account the recoil of the nucleus is due to Darwin (see Example 1.18)

$$\sigma(\theta) = \left(\frac{zZe^2}{mv^2}\right)^2 \frac{1}{\sin^4\theta} \frac{[\cos\theta \pm (1-\gamma^2\sin^2\theta)^{1/2}]^2}{(1-\gamma^2\sin^2\theta)^{1/2}} \tag{1.61}$$

where M is the mass of the target nucleus and m is the mass of the incident particle, and $\gamma = m/M$. If $\gamma < 1$, the positive sign should be used only before the square root. If $\gamma > 1$ the expression should be calculated for positive and negative signs and the results are added to obtain $\gamma(\theta)$. For $\gamma = 1$

$$\sigma(\theta) = \left[\frac{zZe^2}{T}\right]^2 \frac{\cos\theta}{\sin^4\theta} \tag{1.62}$$

1.3.3 Mott's Formula

If the scattered and the scattering particles are identical (Indistinguishable particles), the quantum mechanical exchange effects must be taken into account. The scattering formula due to Mott is

$$\sigma(\theta) = \frac{z^2 Z^2 e^4 \cos\theta}{T^2}\left\{\frac{1}{\sin^4\theta} + \frac{1}{\cos^4\theta}\begin{bmatrix}+2\\-1\end{bmatrix}\left[\frac{1}{\sin^2\theta\cos^2\theta}\frac{\cos\gamma^2 Z^2 e^2}{\hbar v}\ln tg^2\theta\right]\right\} \tag{1.63}$$

where h is Planck's constant. $+2$ is put infront of the square brackets if the particles have zero spin, and -1, if their spin is $\frac{1}{2}$.

1.3.4 Cross-Section for Scattering in the Angular Interval θ' and θ''

The cross-section $\sigma(\theta', \theta')$ per nucleus for scattering between angle θ' and θ'' is given by:

$$\sigma\left(\theta', \theta''\right) = \int_{\theta'}^{\theta''} \sigma(\theta) d\Omega = 2\pi \int_{\theta'}^{\theta''} \sin\theta\sigma(\theta) d\theta = \frac{2\pi R_0^2}{16}\int_{\theta'}^{\theta''} \frac{\sin\theta d\theta}{\sin^4\theta/2}$$

where (1.59) has been used. As $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\begin{aligned}\sigma\left(\theta', \theta''\right) &= \frac{\pi}{4}R_0^2\int_{\theta'}^{\theta''} \operatorname{cosec}^2(\theta/2)\cot(\theta/2) d\theta\\ &= \frac{\pi}{2}R_0^2\int_{\theta''}^{\theta''} \cot(\theta/2) d\cot(\theta/2)\\ \sigma\left(\theta', \theta''\right) &= \frac{1}{4}\pi R_0^2\left(\cot^2\theta'/2 - \cot^2\theta''/2\right)\end{aligned} \tag{1.64}$$

In particular, $\sigma(90°, 180°)$ the cross-section for scattering for angles greater than 90° is given by setting $\theta' = 90°$ and $\theta'' = 180°$ in (1.64)

$$\sigma\left(90°, 180°\right) = \frac{\pi R_0^2}{4} = \frac{\pi}{4}\left(\frac{zZe^2}{T_0}\right)^2 \tag{1.65}$$

1.3.5 Probability of Scattering

Consider a box of face area 1 cm^2 and length λ cm, so that its volume becomes λ cm^3. Let a beam of particles be incident on its face. By definition λ is such a

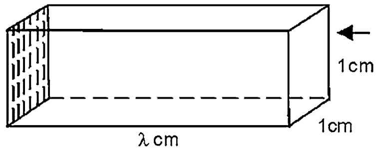


Fig. 1.24 A box of face area 1 cm^2 and length λ cm containing n atoms is exposed to a beam of particles

length that on an average the particle suffers the given type of scattering, i.e. λ is the mean-free-path. If there are n number of atoms per cm^3, the number of atoms inside the box of volume λ cm^3 will then the equal to λn. The cross-section arising from all these atoms will then be equal to $\lambda n \sigma(\theta', \theta'')$. Imagine all the atoms inside the box to be pushed on the rear surface of the box (Fig. 1.24). The total area corresponding to the cross-section of all the atoms must be such as to completely fill up area of 1 cm^2 since our assumption demands that on an average one scattering of the given type will occur when the incident particle passes through λ cm

$$\therefore \quad \lambda n \sigma\left(\theta' \theta''\right) = 1 \quad \text{or}$$

$$n\sigma\left(\theta', \sigma''\right) = \frac{1}{\lambda}$$

If the foil is only t cm thick, then the probability of scattering between θ' and θ'' will be:

$$P = t/\lambda = nt\sigma\left(\theta, \theta''\right) \tag{1.66}$$

1.3.6 Rutherford Scattering in the LS and CM System

So far, we have considered the scattering of particles from massive target nuclei so that the recoil of the latter can be neglected altogether. However, if a light target be considered then the target nucleus would necessarily recoil due to the collision and the analysis of the collision is rendered fairly complicated when done in the lab system. Figure 1.25 shows for definiteness the elastic scattering of an α-particle ($m_1 = 4$) with a carbon nucleus ($m_2 = 12$) originally stationary seen in the lab system. The α particle moves with velocity u_1, and makes an impact parameter b. Since m_2 is assumed to be stationary, the relative velocity of approach is also u_1.

The centre of mass (indicated by CM in the diagram) moves with constant velocity $v_c = m_1 u_1/(m_1 + m_2)$, before, during and after the collision, which is always directed parallel to the incident direction of m_1. In the chosen example, v_c is one-fourth of the initial velocity of the α particle. We have seen that the analytical relationships which connect the scattering angle θ and ϕ with the impact parameter b and with the charges, masses and velocities of m_1 and m_2 are too complicated to be of any general use. Observe that after the collision the initial direction of m_2 is away from that of m_1. This is a simple consequence of Coulomb's repulsion between the two nuclei. It must be pointed out that the trajectories are no longer simple

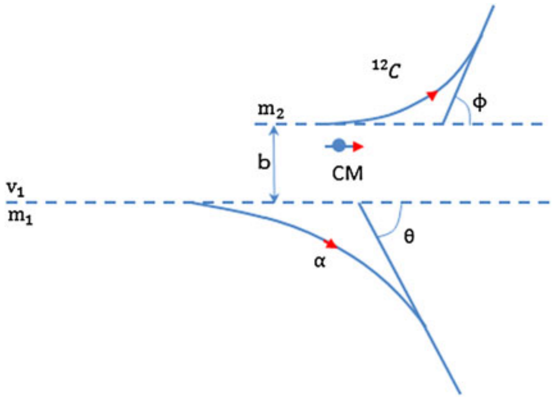


Fig. 1.25 Scattering of α particles with a carbon nucleus in the LS

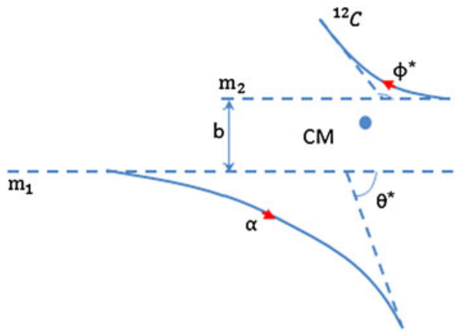


Fig. 1.26 Scattering of α particles with a carbon nucleus in the CMS

hyperbolas in the lab system. In the CM system, no distinction is made between the projectile and the target particles, see Fig. 1.26. The relative velocity of the particles is, $v(\text{rel}) = u_1^* + u_2^* = (u_1 - u_c) + u_c = u_1$, which is identical with that in the lab system.

There is complete symmetry in the scattering of the particles in the CM system. Both the particles approach each other with equal and opposite momentum before the collision and recede with equal and opposite momentum after the collision. In the event of elastic scattering, the respective speeds of the particles remain unaltered before and after the collision. Both are deflected through the same angle measured with their respective original direction. Their centre of mass remains at rest throughout the collision. Each of the particles describes a hyperbola. The collision diameter, impact parameter and eccentricity of the orbit are the same for both the particles, Fig. 1.27. In our example, α particle traverses its hyperbolic path r_1 about the centre of mass, while the carbon nucleus also traverses a similar path, $r_2 = r_1 m_1/m_2 = r_1/3$, on the other side of the centre of mass. The line joining the positions of the two particles passes through the centre of mass at all times. The angular momentum about the centre of mass evaluated in the CM system (Fig. 1.27)

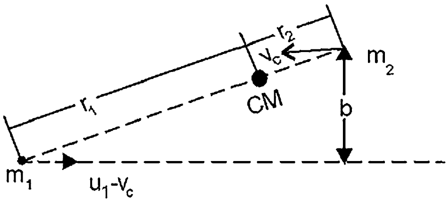


Fig. 1.27 Angular momentum about the centre of mass in the CMS

is

$$J = m_1(u_1 - v_c)\frac{r_1 b}{r_1 + r_2} + \frac{m_2 v_c r_2 b}{r_1 + r_2} = \frac{m_1 u_1 r_1 b}{r_1 + r_2} + \frac{v_c b}{r_1 + r_2}(m_2 r_2 - m_1 r_1)$$
$$= \frac{m_1 u_1 r_1 b}{r_1 + r_2}$$

Since $m_1 r_1 = m_2 r_2$

$$\frac{r_1}{r_1 + r_2} = \frac{m_2}{m_1 + m_2}$$

$$J = m_1 u_1 \frac{b m_2}{m_1 + m_2} = \mu u_1 b$$

where μ is the reduced mass. The angular momentum J of this system of two particles is a constant of their motion since no external torques act on the system. The angular momentum taken about the centre of their mass has the same value both in the lab system and CM system since these two systems differ only in regard to the translation velocity of the centre of mass (v_c in the lab system and zero in the CM system).

1.3.7 Validity of Classical Description of Scattering

We must be able to form a wave packet which is narrower than the distance of the closest approach, otherwise there is no way to make sure that the particle experiences a definitely predictable force from which the deflection can be calculated classically. To obtain a rough estimate of the validity of the classical description, we can safely assume that the distance of closest approach is of the same order of magnitude as the impact parameter b. In order to form a wave packet that is smaller than b, it is of course necessary that one uses a range of wavelengths of the order of b or smaller. Thus the first requirement is that the momentum of the incident particles be considerably larger than $p = \hbar/b$. Moreover, in defining the position of this packet will make the momentum of the particle uncertain by a quantity much greater than $\delta p = \hbar/b$. This uncertainty will cause the angle of deflection to be made uncertain by a quantity much greater than $\delta\theta = \delta p/p$. In order that the classical description be applicable, the above uncertainty ought to be a great deal smaller than the deflection itself; otherwise the entire calculation of the deflection by classical method will be meaningless. This requirement is, however, equivalent to the requirement that the uncertainty in the momentum be much smaller than the net momentum, Δp, transferred during the collision, or that

$$\delta p/\Delta p = \hbar/b\Delta p \ll 1 \tag{1.67}$$

Now, for elastic scattering, for small angles

$$\Delta p = 2p\sin(\theta/2) \simeq p\theta \tag{1.68}$$

Also from (1.53)

$$\theta = zZe^2/Tb \tag{1.69}$$

Combining (1.68) and (1.69) and noting that $p/T = 2/v$, the condition is,

$$2zZe^2/\hbar v \gg 1 \quad \text{or} \quad 2zZ/137\beta \gg 1 \tag{1.70}$$

For 5 MeV α and gold nucleus ($Z = 79$) as the target, $\beta = 0.05$, and the left hand side of (1.67) becomes 46, a value which is much greater than unity, so that classical description of scattering is fully valid.

1.3.8 Coulomb Scattering with a Shielded Potential Under Born's Approximation

It is always an abstraction to assume that the Coulomb force continues to be unmodified out to arbitrarily large distance. Thus the Coulomb force resulting from distances of the order of a few atomic radii is screened or shielded by the atomic electrons. The resulting shape of the potential may be approximated by the shielded Coulomb potential of the form

$$V = \frac{zZe^2}{r}\exp(-r/r_0) \tag{1.71}$$

The exponential factor causes the force to become negligible when the factor r/r_0 is much greater than unity. According to the Born approximation, the expression for the differential cross-section is given by

$$\sigma(\theta) = \frac{4\pi^2 m^2}{h^4}\left[V(p - p_0)\right]^2 \tag{1.72}$$

where the momentum transfer is

$$|p - p_0| = 2p\sin(\theta/2) \tag{1.73}$$

and

$$V(p - p_0) = zZe^2\int \exp\left[i(p - p_0)\right]\cdot r\hbar\frac{\exp(-r/r_0)dr}{r} = \frac{4\pi zZe^2}{|\frac{p-p_0}{\hbar^2}|^2 + \frac{1}{r_0^2}} \tag{1.74}$$

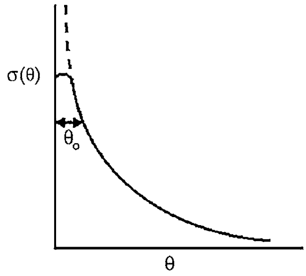


Fig. 1.28 Rutherford scattering with a shielded potential

Letting $r_0 \rightarrow \infty$, and combining (1.72), (1.73) and (1.74), we get exactly the same expression as (1.59), i.e. Rutherford scattering law. Thus, classical mechanics and quantum mechanics give the same result for the Rutherford scattering.

The general appearance of the cross-section for a shielded Coulomb force as a function of angle is shown in Fig. 1.28. The curve rises steeply with decreasing θ, as is characteristic of the Rutherford cross-section, until

$$\sin\frac{\theta_0}{2} \cong \frac{\hbar}{2pr_0}$$

For angles smaller than θ_0, the rise of $\sigma(\theta)$ is comparatively small. Thus, θ_0, may be regarded as a sort of minimum angle, below which Rutherford scattering ceases, as a result of the shielding effects. With a shielded Coulomb potential, θ_0 will approach zero with increasing b much more rapidly, as soon as b goes beyond the shielding radius. In fact, shortly beyond the shielding radius, the entire scattering effect can be neglected. The minimum angle below which the cross-section ceases to increase is given by setting $b = r_0$ in Eq. (1.69) adapted for small angle approximation, $\theta(\min) = zZe^2/Tr_0$.

1.3.9 Discussion of Rutherford's Formula

The formula fails for indistinguishable particles and also for relativistic particles. The derivation ignores spin interaction. Formula (1.59) predicts pronounced scattering in the forward direction, i.e. small angle scattering is favoured. Spin interaction, however, affects only the large angle scattering. Screening effect of electrons has been ignored which tends to reduce the effective charge of the nucleus. This effect is small for small impact parameters (large θ) and will clearly manifest itself in heavy atoms in which K electrons are very close to the nucleus.

Rutherford's formula is valid only for single scattering. It is, therefore, necessary to use thin foils; otherwise, multiple scattering will result from the superposition of successive single scatterings. Scattering due to orbital electrons may be ignored since maximum angle of scattering will be typically $<10^{-4}$ radians.

If the mass of the incident particle cannot be neglected in comparison with the mass of the target nucleus, then (1.59) is still valid provided the energy T now refers to the centre of mass system, and similarly the angle.

The total cross-section σ is given by

$$\sigma = 2\pi \int_0^{\pi} \sigma(\theta) \sin\theta d\theta \tag{1.75}$$

By inserting (1.59) in (1.75), it is seen that $\sigma \to \infty$. This is the direct consequence of the long range character of Coulomb forces. But physically, at ranges (impact parameters) comparable with the atomic radii, the force field is effectively cut-off, leading to a finite cross-section. In other words, divergence in σ which results at $\theta = 0$ does not occur in practice, since infinite impact parameters are not effective owing to the shielding of the nucleus by the orbital electrons.

1.3.10 The Scattering of α Particles and the Nuclear Theory of Atom

The experiments of Geiger and Marsden suggested that if alphas passed through thin foils of platinum then about 1 in 8000 suffered a deflection larger than 90°. From the then existing Thompson's theory of atom (protons and electrons uniformly distributed in the atom), it appeared that the electric fields are weak, much too weak to give rise to such deflections. It was conjectured that a cumulative effect of a large number of small angle scatters might result in a large deflection. But it was not possible to explain deflections of this magnitude. To explain this result, Rutherford postulated that positive charge is concentrated in a very small volume called the nucleus. It was then shown that all the wide angle scatters could be explained due to the existence of the nucleus alone, and that the screening effect due to the orbital electrons could be ignored. Rutherford used the scintillation method to count the scattered alphas, and an excellent agreement was made with the theory. The results were so satisfactory that Rutherford used his formula to verify the atomic numbers of some of the elements that were already known from Mosley's work on X-ray spectra of elements, and an extraordinary agreement was obtained. The dependence of scattering on energy and angle left no doubt as to the validity of Coulomb law of force, the distance of closest approach being 70 to 140 fm (1 fermi $= 10^{-13}$ cm). For alpha scattering against gold foils, the scattering was well represented by the Coulomb law of force when the closest distance of approach was 32 fm. It was, therefore, concluded that the radius of gold nucleus is not greater than this value (actual value being about 8 fm). Thus, any departure from Rutherford's law of scattering is an indication that the incident particle is coming into contact with the target nucleus. Careful experiments can, therefore, give reasonably good estimates of nuclear sizes. For the range of energies considered (alphas from radioactive substances), no departure from Coulomb inverse square law of forces was observed, for very close distance of collision. The alpha scattering was found to be strictly governed by the Rutherford scattering law for targets ranging from copper to uranium. In the mean time, Blacket obtained an independent verification from Wilson chamber photographs by determining the angular distribution at various energies. But experiments with light elements revealed serious departure from the Rutherford scattering law at large angles. Observe that R_0 (minimum distance of approach) is proportional to Z (charge of the target nucleus) while the nuclear radii are roughly proportional to $Z^{\frac{1}{3}}$ for light and medium elements. The latter, therefore, decreases much slower than the former. Consequently, alphas could penetrate the nuclei of light elements. It was also observed that when alphas of larger energies were employed, a more rapid departure from the Rutherford law of scattering was approached, the observed number being greater than the predicted number. These results confirmed the general result that when the particles approach close to the nucleus the simple Coulomb law is no longer obeyed and that no known law concerning the electrical forces could explain the observed scattering. Explanation in terms of magnetic interaction between the alpha particle and the nucleus was ruled out since it was already known that alpha particle has zero spin, and therefore does not have any magnetic moment. Further, the validity of the classical formula of Rutherford could not be questioned since quantum mechanics also gives precisely the same result. The only answer to the anamolous scattering was to postulate the existence of strong short range attractive nuclear forces. For large b the angle θ will be given by pure Coulomb forces. As b decreases θ would increase, but not so rapidly as in the pure Coulomb field. This has the consequence of more particles to be scattered at small angles and fewer at large angles. As b decreases further, the attractive nuclear forces may more than compensate for the repulsive Coulomb forces. The angle of scattering may therefore decrease rather than increase, and in this case there will be a maximum angle of scattering. For very small values of b, the particles may get scattered at random and in some cases may get captured.

Example 1.7 If σ_g is the geometrical cross-section for uncharged particles for hitting a nucleus, show that for positively charged particles, the cross-section will decrease by the factor $(1 - \frac{R_0}{R})$, where $R_0 = zZe^2/4\pi\varepsilon_0 K_0 R =$ radius of the nucleus, $\sigma_g = \pi R^2$ and K_0 is the kinetic energy.

Solution When the charged particle just grazes the nucleus

$$r(\min) = R = \frac{1}{2}R_0\left[1 + \sqrt{1 + \frac{4b^2}{R_0^2}}\right] \tag{1.55}$$

whence we obtain $b^2 = R^2 - RR_0$.

Denoting the cross-section by $\sigma = \pi b^2$, it follows that $\sigma/\sigma_g = \pi b^2/\pi R^2 = 1 - R_0/R$.

Example 1.8 Alphas of 8.3 MeV bombard an aluminum foil. The scattered alphas are observed at an angle of 60°. Calculate the minimum distance of approach in this case.

Solution $T_0 = 8.3$ MeV; $z = 2$, $Z = 13$

$$\frac{R_0}{2b} = \tan\frac{\theta}{2} = \tan\frac{60}{2} = \tan 30^\circ = \frac{1}{\sqrt{3}} \tag{1}$$

$$r(\min) = \frac{R_0}{2}\left[1 + \sqrt{1 + \frac{4b^2}{R_0^2}}\right] \tag{2}$$

Using (1) in (2)

$$r(\min) = 3R_0/2$$

$$R_0 = \frac{1.44zZ}{T_0} = \frac{1.44 \times 2 \times 13}{8.3} = 4.51 \text{ fm}$$

$$r(\min) = 1.5 \times 4.51 = 6.76 \text{ fm}$$

Example 1.9 Find the probability of scattering of alpha particles of energy 5 MeV through an angle greater than 90° in their passage through a foil of gold of thickness 4×10^{-5} cm. Given, Avogadro's number $N_0 = 6 \times 10^{23}$; $A = 196$; $Z = 79$; electronic charge $e = 1.6 \times 10^{19}$ C; $1/4\pi\varepsilon_0 = 9 \times 10^9$ N m^2/C^2, density of gold = 19.3 g/cm^3.

Solution

$$N = \text{Number of atoms/cm}^3 = (\text{number of atoms/g})\ \left(\text{g/cm}^3\right)$$

$$= \frac{6 \times 10^{23} \times 19.3}{196} = 5.91 \times 10^{22}$$

$$\sigma\left(90^\circ, 180^\circ\right) = \frac{\pi}{4}\left[\frac{1.44zZ}{T_0}\right]^2 = \frac{\pi}{4}\left[\frac{1.44 \times 2 \times 79}{5}\right]^2$$

$$= 1625 \text{ fm}^2 = 16.25 \times 10^{-24} \text{ cm}^2$$

If λ is the mean-free-path, $\frac{1}{\lambda} = N\sigma = 5.91 \times 10^{22} \times 16.25 \times 10^{-24} = 0.961 \text{ cm}^{-1}$.

Required probability $p = t/\lambda = 4 \times 10^{-5} \times 0.961 = 3.84 \times 10^{-5}$. In other words, one alpha in $1/(3.84 \times 10^{-5})$ or 26000, gets scattered at an angle greater than 90°.

Example 1.10 If the radius of gold nucleus ($Z = 79$), is 8×10^{-15} m, what is the minimum energy that the a particle should have to just reach it? Give your answer in MeV.

Solution Set

$$R = R_0 = 8 \text{ fm}$$

$$T = \frac{1.44zZ}{R_0} = \frac{1.44 \times 2 \times 79}{8} = 28.44 \text{ MeV}$$

Example 1.11 The following counting rates (in arbitrary units) were obtained when a particles were scattered through 180° from a thin gold ($Z = 79$) target. Deduce a value for the radius of a gold nucleus from these results.

Energy of α particle (MeV)	8	12	18	22	26	27	30	34
Counting rate	30300	13400	6000	4000	2800	33	4	0.4

Solution Since at a given angle, the counting rate is inversely proportional to the square of energy, we can calculate the expected counting rate for various energies, assuming that at 8 MeV (lowest energy) the counting rate N_8 is in agreement with the expected value. In the table below are displayed the calculated counting rates with the aid of the formula, $N = N_8(8/T)^2$, where T is in MeV.

T in MeV	8	12	18	22	26	27	30	34
N (cal)	30300	13100	5990	4010	2867	2667	2157	1679
N (obs)	30300	13400	6000	4000	2800	33	4	0.4

Comparison between the calculated and observed counting rates indicates that departure from Rutherford scattering begins at 26 MeV. Since scattering angle is $\theta = 180°$, we are concerned with head-on collisions. Hence

$$R_0 = \frac{1.44zZ}{T_0} = \frac{1.44 \times 2 \times 79}{26} = 8.75 \text{ fm}$$

Hence, the radius of the gold nucleus is 8.75 fm.

Example 1.12 (a) If a gold foil is bombarded by 5.4 MeV a particles, determine the distance of closest approach (b) what is the deflection of the alpha particle when the impact parameter is equal to this distance?

Solution

(a) Distance of closest approach

$$R_0 = \frac{1.44zZ}{T_0} = \frac{1.44 \times 2 \times 79}{5.4} = 42.1 \text{ fm}$$

(b)

$$\tan\frac{\theta}{2} = \frac{R_0}{2b} = \frac{R_0}{2R_0} = \frac{1}{2}$$
$$\theta = 53°8'$$

Example 1.13 To what minimum distance will an alpha particle of energy 0.4 MeV approach a stationary Li^7 nucleus in the case of a head-on collision? Take the nuclear recoil into account.

Solution We work out in the CM system. Equating the potential energy at the closest distance of approach R_0 to the initial K.E.

$$\frac{1.44zZ}{R_0} = \frac{1}{2}\mu v^2 = \frac{1}{2}\frac{m_1 m_2 v^2}{(m_1+m_2)} = \frac{T_0 m_2}{m_1+m_2} \quad \text{or}$$

$$R_0 = \frac{1.44zZ}{T_0}\left(1+\frac{m_1}{m_2}\right) = \frac{1.44\times 2\times 3}{0.4}\left(1+\frac{4}{7}\right) = 33.9\ \text{fm}$$

Example 1.14 A narrow beam of alpha particles with kinetic energy $T = 600$ keV falls normally on a golden foil incorporating $n = 1.1 \times 10^{19}$ nuclei/cm^2. Find the fraction of alpha particles scattered through the angles $\theta < \theta_0 = 20°$.

Solution Given $nt = 1.1 \times 10^{19}$ nuclei/cm^2

$$\Delta N/N = 1 - \frac{\pi}{4}R_0^2 \cot^2\frac{\theta}{2}\cdot nt = 1 - \frac{\pi}{4}\left(\frac{1.44zZ}{T_0}\right)^2 \cot^2\frac{\theta}{2}\cdot nt$$

$$= 1 - \frac{\pi}{4}\left(\frac{1.44\times 2\times 79}{0.6}\right)^2 \cot^2\frac{20}{2} \times 1.1\times 10^{19}\times 10^{-26} = 0.6$$

The factor 10^{-26} has been introduced to convert fm^2 into cm^2.

Example 1.15 A narrow beam of protons with kinetic energy $T = 1.4$ MeV falls normally on a brass foil whose mass thickness $\rho t = 1.5$ mg/cm^2. The weight ratio of copper and zinc in the foil is equal to 7 : 3, respectively. Find the fraction of the protons scattered through the angles exceeding $\theta = 30°$.

Solution

$$\frac{\Delta N}{N} = \frac{\pi}{4}\frac{10^{-26}}{T^2}\times 1.44^2\left(\frac{0.7Z_1^2}{M_1}+\frac{0.3Z_2^2}{M_2}\right)\rho t N_0 \cot^2\frac{\theta}{2}$$

where Z_1 and Z_2 are the atomic numbers of copper and zinc, M_1 and M_2 are their molar masses, N_0 is the Avagadro's number. The factor 10^{-26} is introduced to express fm^2 as cm^2

$$\frac{\Delta N}{N} = \frac{(1.44)^2}{(1.4)^2}\left[\frac{0.7\times 29^2}{63.55}+\frac{0.3\times 30^2}{65.38}\right]\times 1.5\times 10^{-3}\times 6\times 10^{23}$$

$$\times (3.732)^2 \times 10^{-26}$$

$$= 1.4\times 10^{-3}$$

Example 1.16 Find the effective cross-section of a uranium nucleus corresponding to the scattering of alpha particles with kinetic energy $T = 1.5$ MeV through the angles exceeding $\theta = 60°$.

Solution

$$\sigma(\theta, \pi) = \frac{\pi}{4}\left(\frac{1.44zZ}{T}\right)^2 \cot^2\frac{\theta}{2} = \frac{\pi}{4}\left(\frac{1.44 \times 2 \times 92}{1.5}\right)^2 \cot^2 30°$$
$$= 73480 \text{ fm}^2 = 735 \text{ b}$$

Example 1.17 The effective cross-section of a gold nucleus corresponding to the scattering of monoergic alpha particles within angular interval from 90° to 180° is equal to $\Delta\sigma = 0.5$ kb. Find (a) the energy of alpha particles (b) the differential cross-section of scattering $\sigma(\theta)$ (kb/sr) corresponding to the angle $\theta = 60°$.

Solution

(a)

$$\sigma\left(90°, 180°\right) = \frac{\pi}{4}\left(\frac{1.44zZ}{T}\right)^2$$

$$T = \frac{\sqrt{\pi} \times 1.44zZ}{2 \times \sqrt{\sigma(90°, 180°)}} = \frac{\sqrt{\pi} \times 1.44 \times 2 \times 79}{2 \times \sqrt{5 \times 10^4}} = 0.9 \text{ MeV}$$

(b)

$$\frac{d\sigma}{d\Omega} = 1.295\left(\frac{zZ}{T}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} = 1.295\left(\frac{2 \times 79}{0.9}\right)^2 \frac{1}{\sin^4\frac{6}{2}}$$
$$= 0.638 \times 10^6 \text{ mb/sr} = 0.64 \text{ kb/sr}$$

Example 1.18 Derive Darwin's formula for scattering (modified Rutherford's formula which takes into account the recoil of the nucleus).

Solution

$$\sigma(\theta) = \frac{(1 + \gamma^2 + 2\gamma \cos\theta^*)^{\frac{3}{2}}}{|1 + \gamma \cos\theta^*|}\sigma\left(\theta^*\right) \tag{1}$$

Now, Rutherford's formula for CMS is

$$\sigma\left(\theta^*\right) = \frac{1}{4}\left(\frac{zZe^2}{\mu v^2}\right)^2 \frac{1}{\sin^4\frac{\theta^*}{2}} \tag{2}$$

Also

$$\sin^4 \frac{\theta^*}{2} = \frac{1}{4} \frac{\sin^4 \theta^*}{(1+\cos\theta^*)^2} \quad \text{and} \tag{3}$$

$$\mu = \frac{mM}{m+M} = \frac{m}{1+\gamma} \tag{4}$$

where M and m are the target and incident particle masses respectively and $\gamma = m/M$

$$\tan\theta = \frac{\sin\theta^*}{\gamma + \cos\theta^*} \tag{5}$$

Squaring (5) and expressing it as a quadratic equation and solving it we get

$$\cos\theta^* = \gamma \sin^2\theta \pm \cos\theta\sqrt{1-\gamma^2\sin^2\theta} \tag{6}$$

combining (1), (2), (3), (4) and (6), and after some algebraic manipulations we get

$$\sigma(\theta) = \left(\frac{zZe^2}{mv^2}\right)^2 \frac{1}{\sin^4\theta} \frac{[\cos\theta \pm \sqrt{1-\gamma^2\sin^2\theta}]^2}{\sqrt{1-\gamma^2\sin^2\theta}} \tag{7}$$

This is Darwin's formula. For $m \ll M$, $\gamma \to 0$ and (7) reduces to the usual Rutherford formula.

1.4 Multiple Scattering

1.4.1 Mean Scattering Angle

A charged particle in passing through a thick medium is scattered through an angle θ. The observed scattering may be the result of cumulative effect of a number of small deflections produced by different atomic nuclei in the matter traversed, or it may be a single deflection through an angle θ produced by a single nucleus. The first type of scattering is called multiple or plural, according to the number of contributing collisions is large or small. The second type is referred to as single scattering. Which process is mainly operative depends on the nature and velocity of the scattered particle, the matter traversed and the scattering angle. In the simple treatment, it is assumed that Θ is distributed about $\Theta = 0$, according to the Gaussian law, i.e. the probability for scattering in the angular interval Θ and $\Theta + d\Theta$ is

$$P(\Theta)d\Theta = \text{const} \cdot \exp\left(-K\Theta^2\right)d\Theta \tag{1.76}$$

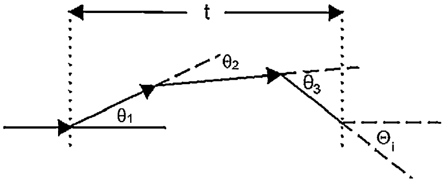


Fig. 1.29 Multiple scattering Θ_i resulting from the superposition of single scatterings, $\theta_1, \theta_2, \ldots, \theta_i$

where K is a constant. The single scattering is governed by the Rutherford law of scattering, which for small angles has the form

$$P(\theta)d\theta = \text{const} \cdot \frac{d\theta}{\theta^3} \tag{1.77}$$

Let θ_i be the deflection in the ith collision. Let there be q collisions in a traversal of t cm. Since small angles are vectors (Fig. 1.29)

$$\Theta_q = \sum_{i=1}^{q} \theta_i$$

Take the dot product

$$\Theta_q^2 = \sum_{i=1}^{q} \theta_i^2 + \sum_{i \neq j}^{q} \theta_i \theta_j$$

In averaging over many traversals, θ_j is positive as many times as it is negative; and the second summation drops off. Thus

$$\langle \Theta_q^2 \rangle = \sum_{i=1}^{q} \theta_1^2$$

Since statistically, the individual events do not differ, $\theta_i^2 = \theta^2$ we can therefore write

$$\langle \Theta^2 \rangle = q \langle \theta^2 \rangle \tag{1.78}$$

Now, Rutherford scattering for small angles can be written as

$$\sigma(\theta) = 4\left(\frac{zZe^2}{pv}\right)^2 \frac{1}{\theta^4} \tag{1.79}$$

The probability for the particle to be scattered into solid angle $d\Omega = 2\pi \sin\theta d\theta$, is given by $\sigma(\theta)2\pi\theta d\theta$ for small θ. Let there be N atoms/cm^3

$$\langle \theta^2 \rangle = \frac{\int_{\theta_{\min}}^{\theta_{\max}} \theta^2 f(\theta) d\theta}{\int_{\theta_{\min}}^{\theta_{\max}} f(\theta) d\theta} = \frac{\int \theta^2 f(b) db}{\int f(b) db} = \frac{\int_{b_{\min}}^{b_{\max}} \frac{4z^2Z^2e^4}{p^2 v^2 b^2} 2\pi N t db}{\int 2\pi b db N t}$$

since $\tan\frac{\theta}{2} = \frac{zZe^2}{Mbv^2}$ and for small θ

$$\theta^2 = \frac{4z^2Z^2e^4}{p^2v^2b^2} \tag{1.80}$$

The denominator is nothing but q. We, therefore, find after substituting (1.79) in (1.80)

$$\langle\Theta^2\rangle = q\langle\theta^2\rangle = 8\pi Nt\left(\frac{zZe^2}{pv}\right)^2 \ln\frac{\theta(\max)}{\theta(\min)} \tag{1.81}$$

We have assumed that the particle is sufficiently energetic so that the velocity does not change over the considered traversal. We also ignore the scattering off the electrons, since it is unimportant. Observe that scattering off the nuclei is proportional to Z^2, whilst for electrons, it is proportional to Z. In hydrogen, the scattering off electrons, however, will be important. We shall now consider the limits $\theta(\max)$ and $\theta(\min)$. Nuclear scattering at large distance is reduced by the electrostatic shielding of the nucleus by its electrons. The electrostatic shielding reduces the scattering of distant particles but it does not reduce the energy loss. Thus, a primary particle travelling at a distance such that the fields of electrons and nuclei compensate each other almost completely is still capable of transferring energy to the atom. The physical reason why screening reduces scattering much more than the energy transfer is as follows. A fast particle passing near an atom transfers a certain amount of momentum to each electron and also to the nucleus. The angle of scattering is, however, determined by the transverse component of the total recoil. Due to the opposite sign of charge, the electrons recoil in the opposite direction to the nucleus and if the fast particle passes at a sufficient distance, the transverse components of the recoiling electrons cancel the transverse components of the nuclear recoil and thus no scattering results. The total energy transfer is equal to

$$\sum_{i=1}^{Z}\frac{p_i^2}{2m_e} + \frac{P_{nuc}^2}{2M_{nuc}} \tag{1.82}$$

The energy transfer to any of the electrons or to the nucleus is positive and is not affected by the presence of other particles except for the small effects of the binding forces. In other words, a particle passing near an atom suffers $Z+1$ collisions with the constituents of the atom and loses energy to everyone of them. The $Z+1$ angles of scattering, however, tend to compensate each other and therefore the scattering is reduced strongly by shielding. Now by (1.81), the root mean square multiple scattering angle is given by

$$\sqrt{\langle\Theta^2\rangle} = \sqrt{\frac{8\pi Z^2z^2e^4Nt}{p^2v^2}\ln\frac{b(\max)}{b(\min)}} = K\frac{\sqrt{t}ze}{pv} \tag{1.83}$$

where

$$K = \sqrt{8\pi N Z^2 e^2 \ln \frac{b(\text{max})}{b(\text{min})}} \tag{1.84}$$

is called the scattering constant.

1.4.2 *Choice of* b(max) *and* b(min)

We may choose

$$b(\text{max}) = \frac{a_o}{Z^{1/3}} \tag{1.85}$$

where a_0 is the Bohr radius. This is justified by Fermi-Thomas model of the atom, since the right hand side of (1.85) represents the radius of the atom.

The limit on b(min) is dictated by the finite size of the nucleus. Thus, b(min) is greater than $1.3 \times 10^{-13} A^{1/3}$ cm. An alternative criterion would be to avoid counting deflections with $\Theta > 1$ radian. A rough criterion is provided by restricting the individual single scatterings to $\theta < 1$. Now for small scattering angles, formula (1.53) reduces to

$$\theta = \frac{2zZe^2}{mv^2 b} \tag{1.86}$$

Putting $\theta = 1$ radian and $p = mv$, we obtain the rough criterion

$$b(\text{min}) = \frac{2zZe^2}{pv} \tag{1.87}$$

Fortunately, the results are insensitive to the choice of b(max) and b(min)

$$\frac{b(\text{max})}{b(\text{min})} \simeq \frac{\text{Atomic dimension}}{\text{Nuclear dimension}} = \frac{10^{-8}\ \text{cm}}{10^{-12}\ \text{cm}} = 10^4, \quad \text{and so } \ln 10^4 \simeq 10$$

Thus, $\sqrt{\langle\Theta\rangle^2}$ is very insensitive to the logarithmic term in (1.83) which is of the order of 10. The main dependence comes from the factors outside the logarithmic term. Further, in view of (1.87) the scattering constant K is a slow varying function of the particle velocity. Observe that $\sqrt{\langle\Theta\rangle^2}$ is directly proportional to the charge of the scattering nuclei, and the charge of the incident particles, inversely proportional to the energy of the incident particles and directly proportional to the square root of the thickness of the absorber.

1.4.3 Mean Square Projected Angle and the Mean Square Displacement

Consider a charged particle traversing an absorber of thickness t. Assume that the collisions take place at depths $X_1, X_2, \ldots$, resulting in deflections $\theta_1, \theta_2, \ldots$. The azimuth of the deflection will change after each collision, the subsequent azimuths being $\phi_1, \phi_2, \ldots$, the projected angle of deflection is given by

$$\Theta_P = \sum_{i=1}^{q} \theta_i \cos\phi_i$$

We have to average over the parameters of the single collisions. Since the azimuths can be taken as independent, we have

$$\langle \cos\phi_i \cos\phi_j \rangle = \frac{1}{2}\delta_{ij} \tag{1.88}$$

where δ_{ij} is the Kronecker delta. Hence

$$\langle \Theta_P^2 \rangle = \frac{1}{2} q \langle \theta^2 \rangle = \frac{1}{2}\langle \Theta^2 \rangle \tag{1.89}$$

Observe that the most probable value of Θ or Θ_P is zero. However, $\langle \Theta \rangle$ and $\langle \Theta^2 \rangle$ are necessarily positive, whereas $\langle \Theta_P \rangle$ is zero.

Similarly the mean square projected displacement is equal to half the mean square unprojected displacement

$$\langle y^2 \rangle = \frac{1}{2}\langle r^2 \rangle$$

Now, $y = \sum_{i=1}^{q} X_i \theta_i \cos\phi_i$

$$\therefore \quad y^2 = \sum_{i \neq j} x_i x_j \theta_i \theta_j \cos\phi_i \cos\phi_j$$

Since x and ϕ are independent, using relation (1.88)

$$\langle y^2 \rangle = \frac{1}{2} \sum_i \langle x_i^2 \rangle \langle \theta^2 \rangle$$

Now

$$\langle x_i^2 \rangle = \langle X^2 \rangle = \frac{1}{t}\int_0^t x^2 dx = \frac{t^2}{3}$$

$$\langle y^2 \rangle = \frac{t^2}{6} q; \qquad \langle \theta^2 \rangle = \frac{t^2}{6}\langle \Theta^2 \rangle \tag{1.90}$$

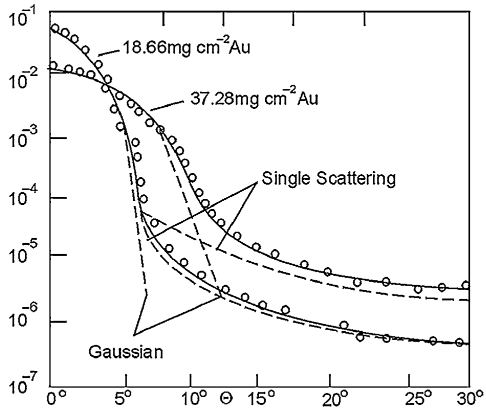


Fig. 1.30 Angular distribution of electrons scattered from AU at 15.7 MeV. *Solid lines* indicate the distribution expected from the Moliere theory for small-and large-angle multiple scattering, with an extrapolation in the transition region: *dashed lines*, the distributions according to the Gaussian and single scattering theories. The ordinate scale gives the logarithm of the fraction of the beam scattered within 9.696×10^{-3} sr (Birkhoff)

Also

$$\langle r^2 \rangle = \frac{t^2}{3} \langle \Theta^2 \rangle \tag{1.91}$$

Expressions (1.90) and (1.91) are of great interest in the cosmic ray shower theory and experiments. Figure 1.30 shows the contribution from multiple scattering (Gaussian) at small angles and single scattering (Rutherford) at large angles. Kamal, Rao and Rao (Fig. 1.31) have compared the experimental distributions of $\overline{D}$ the average of 'Second differences' of 17.2 GeV/c beam tracks in photographic emulsions for cell lengths $t = 4$, 6, 8, 10, 20, and 30 mm and compared with Moliere's theory. The quantity D is related to the projected angular deflection and is obtained from the y-coordinates of the track; $D_i = y_{i+1} - 2y_i + y_{i-1}$, where y_i is the ith coordinate of the track at constant x-intervals called cell length t. In order to avoid very large scattering angles, a $4\overline{D}$ cut-off is usually employed, a procedure in which all deflections larger than four times the mean second difference are eliminated. This is also indicated in the figure. Moliere's probability function (Gaussian function plus the single scattering tail) for the second differences was computed from the work of Scott. A good agreement was found between theory and observations.

In conclusion, we may point out that Rutherford used extremely thin foils for his classical experiments on alpha scattering in order to avoid the contribution of multiple scattering.

From (1.83) it is clear that the determination of root-mean-square angle permits one to estimate the energy of the particle.

Protons and electrons of the same energy will have the same root-mean-square angle of scattering, but their ionization would be different since their velocities would be different. Thus joint measurements of multiple scattering and ionization

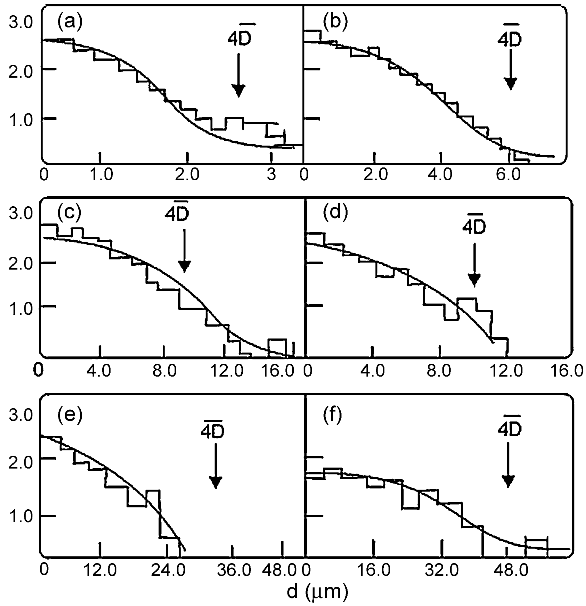


Fig. 1.31 Multiple scattering distribution for various cell lengths, (**a**) 4 mm; (**b**) 6 mm; (**c**) 8 mm; (**d**) 10 mm; (**e**) 20 mm; (**f**) 30 mm Moliere's Gaussian function plus single scattering tail [2]

permit us to estimate the mass of the particle and identify it. This method is particularly suitable for particles which are not too energetic and at the same time are not brought to rest within the stack of emulsions.

The existence of multiple scattering can create problems in the curvature measurements of tracks in a cloud chamber. In certain cases the multiple scattering may be so severe that spurious curvatures are observed even in the absence of magnetic fields. In the case of bubble chambers, curvature measurements under magentic fields are rendered difficult when a heavy liquid like xenon is used. It is also implied that curvature measurement in photographic emulsions under pulsed magnetic fields are limited owing to severe multiple scattering by the heavy nuclei of silver and bromine.

The phenomenon of multiple scattering leads to an interesting observation in cosmic ray showers. Owing to multiple scattering in air, the electrons in the shower undergo lateral displacement from the original path through several meters as they traverse down the atmospheric depth (see expressions (1.90) and (1.91)).

1.5 Theory of Ionization

1.5.1 Bohr's Formula

Charged particles in their passage through a medium lose their energy mostly through excitation of atoms and ionization (collision) processes. The collision process is only one of several mechanisms by which charged particles may lose energy.

In the case of electrons it constitutes the most important source of energy loss only for relatively small energies. At energies of the order of 10 to 100 MeV, radiation losses overtake the collision losses, depending on the Z of the absorber. For muons, collision losses remain dominant up to energies of the order of 100 or 1000 MeV. For protons, radiation losses are never significant, but the occurrence of nuclear collisions overshadows the collision losses at energies of the order of 1000 MeV or greater. Energy loss by the emission of Cerenkov radiation is negligible except at very high energies. Thus, in general, collision losses represent the most important source of energy loss only for energies smaller than a certain value that depends on the nature of the particles.

Most of the electrons ejected in ionization processes have energies very small compared with the energy of the primary particle. Nevertheless, they are able to produce several ion pairs before coming to rest. The total specific ionization consists of two parts (i) primary specific ionization which is defined as the average number of ion pairs produced per $\mathrm{g\,cm^{-2}}$ (ii) secondary specific ionization which refers to the average number of ion pairs per $\mathrm{g\,cm^{-2}}$ by all the secondary electrons and tertiary electrons and radiation. The total ionization implies the sum of (i) and (ii). The total ionization is roughly three times the primary ionization. When the primary particle is absorbed its energy is dissipated in exciting the atoms and in producing secondary electrons partly by collisions and partly by radiation. The secondary electrons radiation will excite more atoms and produce tertiary electrons and photons and so on. It is clear that an electron will continue to lose its energy in elastic collisions as long as its energy is in excess of the lowest excitation potential of the atoms and the photons will be absorbed as long as their energy is greater than the threshold energy for minimum ionization potential. In the event an atom gets into an excited state by an inelastic collision with an electron or by the absorption of a photon it immediately loses the excitation energy by the emission of photons or Auger electrons. The fraction of the initial energy that is used in producing ionization is not appreciably affected by the nature of the primary particle nor by its energy as most of the ionization and excitation processes are caused by electrons of low energy. The classical ionization formula is originally due to N. Bohr. The basic assumptions made in the derivation are (i) electrons are free (ii) the incident particle remains undeviated throughout its motion.

The velocity acquired by the electron in an elastic collision is given by (1.27),

$$v_2 = 2v_c \cos\phi = \frac{2vm_1 \cos\frac{1}{2}\phi^*}{m_1 + m_2}$$

where $v = u_1$. Since $m_2 \ll m_1$, $v_2 \simeq 2v\cos\frac{1}{2}\phi^*$.

The energy imparted to the electron is

$$T = \frac{1}{2}mv_2^2 = 2mv^2\cos^2\frac{1}{2}\phi^* \tag{1.92}$$

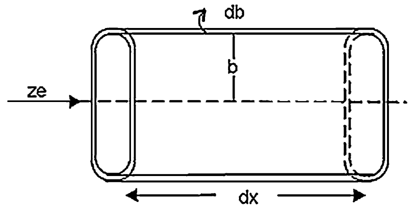


Fig. 1.32 Atoms lying within impact parameters b and $b + db$

Since

$$\phi^* = \pi - \theta^*, \qquad \cos^2 \frac{1}{2}\phi^* = \frac{1}{1 + \cot^2 \frac{1}{2}\theta^*}$$

But scattering angle in the CMS is related to the impact parameter by $\cot \frac{1}{2}\theta^* = 2b/R_0$. Hence,

$$T = \frac{2z^2 e^4}{mv^2[R_0^2/4 + b^2]} \tag{1.93}$$

where we have used the fact that $R_0 = 2zZe^2/\mu v^2$ and the $Z = -1$ for target electron, and $\mu = m$, since the incident particle is considered much more massive than the electron. Let there be n electron/cm^3 of the medium consider a differential element of length dx along the path of the incident particle. The number of electrons situated between the impact parameters b and $b + bd$ over a length dx is given by $2\pi bdbndx$ (Fig. 1.32). The energy imparted to electrons for this range of b's is given by multiplying this number of electrons by T give by (1.93); but this is equal to the energy lost by the primary particle by traversing the element of length dx. We can, therefore, write:

$$-dE/dx = \int_0^{b(\max)} \frac{4\pi n z^2 e^4 b db}{mv^2[b^2 + R_0^2/4]} = \frac{4\pi n z^2 e^4}{mv^2} \ln\left[\frac{2b(\max)}{R_0}\right] \tag{1.94}$$

The underlying assumption that electrons are free is only approximately correct. Actually they are bound to the atoms and can be considered free only if the collision time is short compared with the period of revolution. On the other hand, if the collision time is long compared with the period of revolution, the electrons do not absorb any energy at all. Let $b(\max)$ represent the impact parameter for which the collision time $\tau = 1/\nu$, where ν is the orbital frequency of the electron

$$\text{Impulse} = \int F dt = \text{momentum acquired by the electron} \tag{1.95}$$

or

$$\frac{ze^2\tau}{b^2(\max)} = \sqrt{2mT(\max)} = \sqrt{4z^2e^4/v^2b^2(\max)} \tag{1.96}$$

Hence

$$b(\max) = v/2\nu \tag{1.97}$$

$$-dE/dx = \frac{4\pi n z^2 e^4}{mv^2} \ln\left[\frac{mv^3}{2\nu/ze^2}\right] \quad \text{(Bohr's formula)} \tag{1.98}$$

The negative sign implies that as x increases, E decreases. The quantity $-dE/dx$ is called the linear stopping power and is defined as amount of energy lost per unit length in the absorber. When x is measured in g/cm^2, then this quantity is called the mass stopping power.

Bohr's classical formula is valid provided the particle velocity is larger than the orbital electron velocity. The value of b(max) given in Bohr's classical formula corresponds to such low energy transfers that they are far less than the ionization potential and are therefore incompatible with the acceptable theory of atomic structure. For this reason, the classical theory predicts too great energy loss by high velocity particles.

A quantum mechanical formula which is more exact is due to H.A. Bethe:

$$-dE/dx = \frac{4\pi z^2 e^4 n}{mv^2}\left[\ln\frac{2mv^2}{I} - \ln\left(1-\beta^2\right) - \beta^2\right] \tag{1.99}$$

where $B = \frac{v}{c}$, $I =$ mean ionization potential of the atoms of the medium; $I = KZ$, $K = 13.5$ volts. The derivation assumes that the particle is a point charge, and that the spin and magnetic moment are disregarded. Observe that the quantity $-dE/dx$ which represents ionization loss per unit length, is a function of velocity of the particle and its charge but is independent of its mass. Bohr's formula (1.98) is not applicable for incident electrons for two reasons: (a) the derivation assumes that the incident particle is undeflected during the collision which is not correct for an electron; (b) for identical particles exchange phenomenon must be considered.

The last two terms in the bracket of (1.99) almost cancel out at low velocities (small β). Since the logarithmic term is quite a slow varying function of velocity, the main variation of $-dE/dx$ comes from the factor $1/v^2$. At very low velocities, the energy loss must go down because of the capture of the electrons by the incident particle. This is not considered in the quantum mechanical formula which can be relied on up to 5 MeV α's or 1.3 MeV protons. As the velocity of the incident particle decreases to very low values, various complicated effects enter the energy loss mechanism. When the incident velocity becomes comparable with the K-shell electron velocity, energy transfer to the K-shell electrons becomes difficult. The electrons effective for energy loss are those with velocities smaller than $v = \sqrt{I/2m}$. At low energies, the charge transfer process becomes more important than ionization process. The atom or the ion formed by capturing an electron may lose the electron. When the particle velocity is significantly greater than Bohr's orbital velocity for the K-shell electron, the electron loss dominates over electron capture. This corresponds to 25 keV proton energy or 400 keV α energy.

At higher velocities, the terms $\ln v^2$ and $\ln(1-\beta^2)$ in the square brackets of (1.99) become important. The ionization vs velocity curve (Fig. 1.33) passes through a broad minimum as $\beta \to 1$. The origin of the rise of ionization is due to the Lorentz

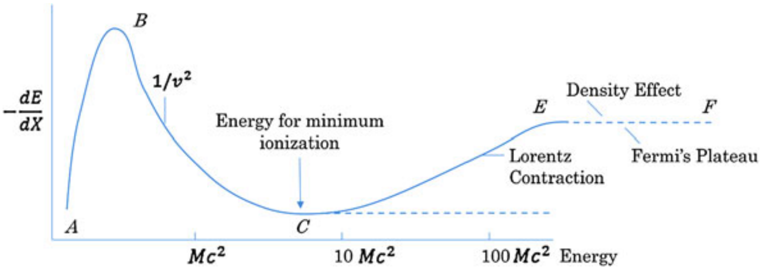


Fig. 1.33 The *curve* BCD gives the $1/v^2$ dependence. At relativistic energies v changes little and CD is asymptotic to $v = c$. At relativistic energies, the log term in $v^2/(1-\beta^2)$ changes, and increases as $v \to c$, giving the rise at the *curve from* C to E. At very low energies (*region* AB) Eq. (1.99) breaks down because the particle has velocity comparable to that of the orbital electrons in the absorber, and the efficiency of energy exchange is much lower. The particle itself captures electrons and spends part of its time reduced change

contraction of Coulomb field of the incident particle which makes possible the energy transfer to the electrons at greater distance from the particle path. At exceedingly high velocities, however, the 'density effect' limits the energy transfer. This is also called the polarizability effect. In the derivation of (1.99), the atoms have been considered isolated. This is justifiable so long the medium is a gas. In a condensed medium, the atoms may still be considered as isolated in close collisions, but when the impact parameter is larger than the atomic distances, the screening of the electric field due to the simultaneous movement of the electrons of the neighbouring atoms becomes important and this has the consequence of lessening the magnitude of $-dE/dx$. At ultra-relativistic velocities, the curve, therefore, gets saturated to a plateau value called the Fermi plateau (in Fig. 1.33). It may be pointed out that in a cloud chamber the plateau may be higher by 50 percent compared to the trough whilst in photographic emulsions it may be higher only by about 10 percent. This is because the density effect in the former is much less than in the latter.

1.5.2 Range-Energy-Relation

When other types of energy losses are negligible compared with the collision loss, fluctuation in the energy loss are small and in a given material all particles of a given energy have almost the same fixed range. The range is defined as the total distance traversed by the particle along its track till its velocity becomes zero. In principle it should be possible to integrate (1.99) and obtain a relation between the range and the energy of the particle. There are two difficulties with this procedure, first the integration is cumbersome, second at very low velocities the phenomenon of electron capture and the uncertainties in the ionization potential render the calculations exceedingly doubtful. In practice, one uses an empirical relation of the form:

$$E = Kz^{2n} M^{l-n} R^n \tag{1.100}$$

where E is the kinetic energy of the particle corresponding to the range R. M is the mass of the particle in terms of proton mass. K and n are empirical constants that depend on the nature of the absorber. The form of (1.100) ensures that the quantity $dE/dR = z^2 f(v)$ and $\neq f(M)$, as desired. Since $-dE/dx = z^2 f(v)$, $dx = -dE/z^2 f(v) = Mf'(v)dv/z^2$, where $f(v)$ and $f'(v)$ are some functions of velocity of the particle. Therefore

$$R = \int_0^R dx = \frac{M}{z^2}\int_0^v f'(v)dv = \frac{M}{z^2} f''(v) \tag{1.101}$$

where $f'(v)$ is still another function of velocity.

Consider two particles of masses M_1 and M_2 having charges z_1 and z_2. Let their initial velocities be the same, and their ranges R_1, and R_2, respectively. It follows from (1.101) that the expected ratio of their ranges would be

$$\frac{R_1}{R_2} = \frac{M_1 z_2^2}{M_2 z_1^2} \tag{1.102}$$

In particular, if the ranges of two tracks of singly charged particles from the point of equal ionization (equal velocity) are known then

$$\frac{R_1}{R_2} = \frac{M_1}{M_2} \tag{1.103}$$

This technique was employed for the mass determination of π meson in the historical experiment of Powell, Occhialini and Lattes, using photographic emulsions. Comparison was made with proton tracks having the same initial ionization.

Example 1.19 The range of a low energy proton is 1500 μm in nuclear emulsions. A second particle whose initial ionization is same as the initial ionization of proton has a range of 228 μm. What is the mass of this particle? (The rate at which a singly ionized particle loses energy E by ionization along its range is given by $dE/dR = K/(\beta c)^2$ MeV μm^{-1} where βc is the velocity of the particle, and K is a constant depending only on emulsions; the mass of proton is 1837 mass of electron.)

Solution Using (1.103)

$$M_2 = \frac{R_2}{R_1} M_1 = \frac{228 \times 1837}{1500} = 279 m_e$$

The particle is identified as π meson (pion).

Example 1.20 α particles and deuterons are accelerated in a cyclotron under identical conditions. The extracted beam of particles is passed through an absorber. Show that the expected range of deuterons is twice that of α particles.

Solution The condition for a circular orbit in a magnetic field (induction B) is

$$Bzev = mv^2/r$$

Since B and r are same for both d and α

$$v_d = \frac{Bzer}{m_d} \quad \text{and} \quad v_\alpha = \frac{B(2e)r}{m_\alpha}$$

Since $m_\alpha = 2m_d$, it follows that $v_d = v_\alpha$.

From (1.102)

$$\frac{R_d}{R_\alpha} = \frac{m_d}{m_\alpha}\frac{2^2}{1^2} = 2$$

Example 1.21 α-particles have an initial energy of 8.5 MeV and a range in standard air of 8.3 cm. Find their energy loss per cm in standard air at a point 4 cm distant from a thin source.

Solution The range-energy-relation is

$$E = Kz^{2n}M^{1-n}R^n \tag{1}$$

$$\frac{dE}{dR} = nKz^{2n}M^{1-n}R^{n-1} = \frac{nE}{R} \tag{2}$$

Let $E_1 = 8.5$ MeV and $R_1 = 8.3$ cm. On moving away 4 cm from the source $R_2 = 8.3–4.0 = 4.3$ cm. Let the corresponding energy be E_2

$$dE_2/dR = nE_2/R_2 \tag{3}$$

$$dE_1/dR = nE_1/R_1 \tag{4}$$

Therefore

$$\frac{dE_2/dR}{dE_1/dR} = \frac{E_2R_1}{E_1R_2} \tag{5}$$

Also

$$\frac{dE_2/dR}{dE_1/dR} = \frac{1/v_2^2}{1/v_1^2} = \frac{v_1^2}{v_2^2} = \frac{E_1}{E_2} \tag{6}$$

From (5) and (6)

$$\frac{E_1}{E_2} = \sqrt{\frac{R_1}{R_2}} = \sqrt{\frac{8.3}{4.3}} \tag{7}$$

Using (1)

$$\frac{E_1}{E_2} = \left(\frac{R_1}{R_2}\right)^n \tag{8}$$

Comparing (7) and (8), $n = \frac{1}{2}$. From (8) or (7)

$$E_2 = E_1 (R_2/R_1)^{\frac{1}{2}} = 8.5\sqrt{\frac{4.3}{8.3}} = 6.12 \text{ MeV}$$

$$\frac{dE_2}{dR} = \frac{nE_2}{R_2} = \frac{0.5 \times 6.12}{4.3} = 0.71 \text{ MeV/cm}$$

1.5.2.1 Range in Air—Geiger's Rule

If we ignore the logarithmic term in the formula for $-dE/dx$, then $dE/dx \propto 1/v^2$ or $R \propto v^4$ for the low energy region. A better approximation is provided by the formula

$$R = \text{const} \cdot v^3 \quad \text{(Geiger's rule)}$$

This formula is valid for 4–10 MeV α particles. At higher energy the exponent changes. A Formula which gives the range of α's in air at 15 °C and atmospheric pressure is

$$R = 0.32 \text{ (MeV)}^{3/2} \text{ cm} \quad \text{(alphas in air)}$$

This formula is correct to about 10 per cent in the low energy region but breaks down for relativistic velocities. Figure 1.34 shows the range energy curves for protons and Fig. 1.35 for alpha particles in air at 15 °C and 760 mm pressure.

1.5.2.2 The Bragg-Kleeman Rule

This rule permits one to convert the range R_1, in medium 1 of known density ρ_1 and atomic weight A_1 to range R_2 in medium 2 of known density ρ_2 and atomic weight A_2

$$\frac{R_2}{R_1} = \frac{\rho_1}{\rho_2}\frac{\sqrt{A2}}{\sqrt{A1}} \quad \text{(Bragg-Kleeman rule)} \tag{1.104}$$

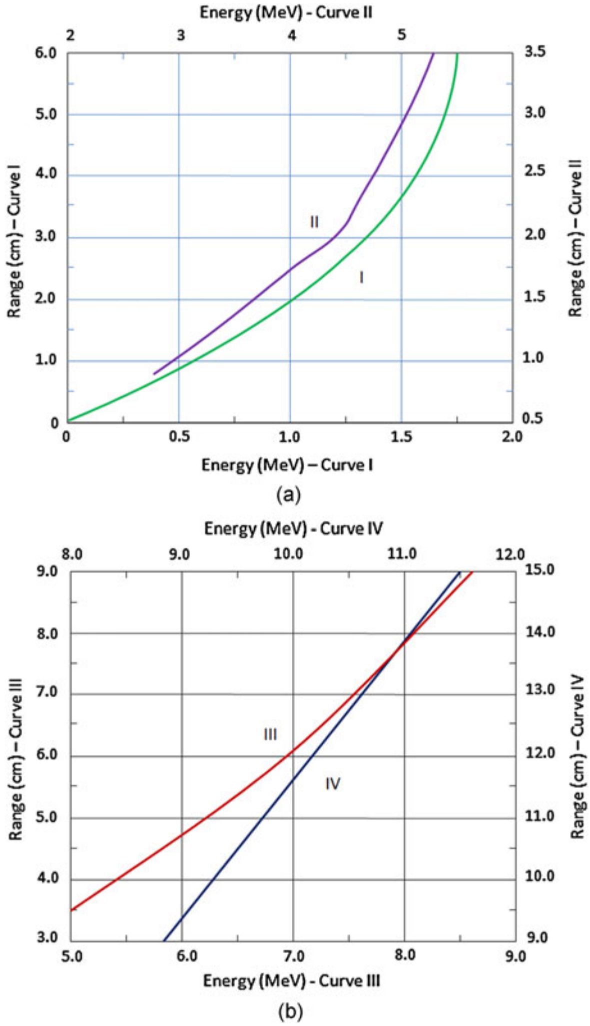
This rule is correct to within 15 per cent. As an example, for air $\sqrt{A_1} = 3.81$ and $\rho_1 = 1.226 \times 10^{-4}$ g/cm^3 at 15 °C, 76 cm of Hg. Then $R_2 = 3.2 \times 10^{-4} \times \sqrt{A_2} R(\text{air})/\rho_2$. For aluminum $A_2 = 27$ and $\rho_2 = 2.7$, so that in aluminum the range of α-particles and protons (1–10 MeV) is about $1/1600$ of the range in air.

Example 1.22 Compare the stopping power of a 3 MeV proton and a 6 MeV deuteron in the same medium.

Solution

$$v_p = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2 \times 3}{1}} = \sqrt{6}, \quad \text{and} \quad v_d = \sqrt{\frac{2 \times 6}{2}} = \sqrt{6}$$

Fig. 1.34 Range-energy relation for protons in air at 15 °C, 760 mm pressure up to 11.8 MeV



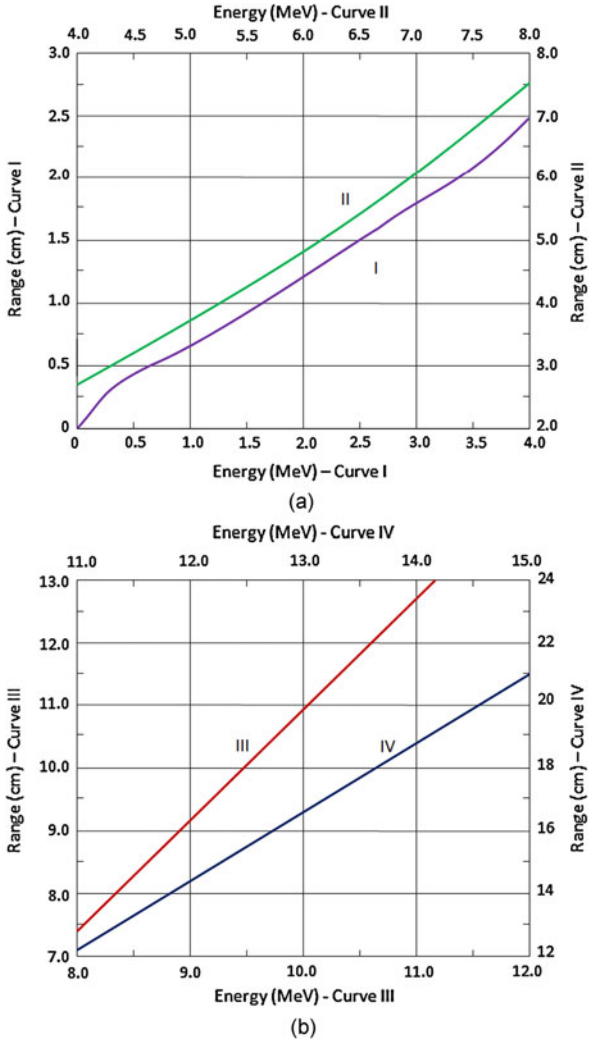
Since the velocities are same and also both proton and deuteron are singly charged particles, their stopping powers are the same.

Example 1.23 Show that the specific ionization of a 320 MeV α particle is approximately equal to that of a 20 MeV proton.

Solution

$$-\frac{dE}{dx} \propto \frac{z^2}{v^2} \quad \text{or} \quad \propto \frac{Mz^2}{E}$$

Fig. 1.35 Range-energy relation for alpha-particles in air at 15 °C, 760 mm pressure up to 15 MeV



$$\text{for } \propto\text{'s,} \qquad -\frac{dE}{dx} \propto \frac{4 \times 2^2}{320} = \frac{1}{20}$$

$$\text{for } p\text{'s,} \qquad -\frac{dE}{dx} \propto \frac{1 \times 1^2}{20} = \frac{1}{20}$$

Thus the specific ionization is same.

Example 1.24 If the range is multiplied by density, equivalent thickness in g/cm^2 is obtained. Calculate the thickness of aluminum that is equivalent in stopping power of 1 cm of air. Given the relative stopping power for aluminum $S = 1700$ and its density $= 2.7$ g/cm^3.

Solution

$$R(\text{Al}) = \frac{R(\text{air})}{S} = \frac{1}{1700}\ \text{cm}$$

$$R(\text{Al}) = \frac{2.7}{1700}\ \text{g/cm}^2 = 1.59 \times 10^{-3}\ \text{g/cm}^2$$

Example 1.25 Calculate the minimum energy an α particle can have and still be counted with a GM counter if the counter window is made of stainless steel ($A \approx 56$) with 2 mg/cm^2 thickness.

Solution For steel

$$R_s\ (\text{cm}) = R_s\ \left(\text{g/cm}^2\right)/\rho_s = 2 \times 10^{-3}/\rho_s$$

Equivalent range for air

$$R_a = \frac{R_s \rho_s \sqrt{A_a}}{\rho_a \sqrt{A_s}} = \frac{2 \times 10^{-3} \times \sqrt{14.5}}{1.226 \times 10^{-3} \times \sqrt{56}} = 0.83\ \text{cm}$$

$$E = \left(\frac{R}{0.32}\right)^{2/3} = \left(\frac{0.83}{0.32}\right)^{2/3} = 1.89\ \text{MeV}$$

α's of energy greater than 1.89 MeV will be counted.

Example 1.26 Calculate the range of 4 MeV α particles in air of 760 mm of Hg pressure and 15 °C temperature.

Solution Use the formula

$$R = 0.32(E)^{\frac{3}{2}}\ \text{cm} = 0.32(4)^{\frac{3}{2}} = 2.56\ \text{cm}$$

Example 1.27 Calculate the range in aluminum of a 5 MeV a particle if the relative stopping power of Aluminum is 1700.

Solution Relative stopping power $S = R(\text{air})/R(\text{Al})$. But,

$$R(\text{air}) = 0.32(5)^{\frac{3}{2}} = 3.578\ \text{cm}$$

$$R(\text{Al}) = \frac{3.578\ \text{cm}}{1700} = 21\ \mu\text{m}$$

Example 1.28 The range of 5 MeV a's in air at NTP is 3.8 cm. Estimate the range of 10 MeV a's using Geiger-Nuttal law.

Solution According to Geiger's rule, $R \propto v^3$, or $R \propto E^{\frac{3}{2}}$

$$\frac{R_2}{R_1} = \left(\frac{E_2}{E_1}\right)^{3/2} = \left(\frac{10}{5}\right)^{3/2} = 2\sqrt{2}$$

$$R_2 = 2\sqrt{2}R_1 = (2\sqrt{2})(3.8) = 10.75 \text{ cm}$$

Example 1.29 Mean ranges of a particles in air under standard conditions is defined by the formula R (cm) $= 0.98 \times 10^{-27} v_0^3$, where v_0 (cm/s) is the initial velocity of an alpha particle. Using this formula, find an α-particle with initial kinetic energy 7.0 MeV (a) its mean range (b) the average number of ion pairs formed by the given a-particle over the whole path as well as over its first half, assuming the ion pair formation energy to be equal to 34 eV.

Solution

(a)

$$v_0 = \sqrt{\frac{2T}{m}} = c\sqrt{\frac{2T}{mc^2}} = c\sqrt{\frac{2 \times 7}{3726}} = 0.061c$$

$$R = 0.98 \times 10^{-27} \times \left(3 \times 10^{10} \times 0.061\right)^3 = 6 \text{ cm}$$

(b) (i) Total number of ion pairs $= \frac{7 \times 10^6}{34} = 2.06 \times 10^5$

(ii) For $R = 3$ cm range, $3 = 0.98 \times 10^{-27} v_0^3$, or $v_0 = 1.45 \times 10^9$ cm/s. Corresponding energy at mid path is

$$E = \frac{1}{2}Mv^2 = \frac{1}{2}Mc^2(v/c)^2 = \frac{1}{2} \times 3726 \times (0.048)^2 = 4.39 \text{ MeV}$$

Energy lost in the first half of the path, $\Delta E = 7.0–4.39 = 2.61$ MeV.
Number of ion pairs over the first half of the path $= \frac{2.61}{34} \times 10^6 = 7.67 \times 10^4$.

Example 1.30 Assuming that ^{14}C and ^{14}N nuclei are both accelerated to an energy of 40 MeV and are then allowed to pass through a thin foil. If the ^{14}C nuclei lose 2 MeV, how much energy will the ^{14}N nuclei lose?

Solution

$$\frac{-dE}{dx} \propto \frac{z^2}{v^2} \quad \text{or} \quad \propto Z^2 \frac{M}{E}$$

As $\frac{M}{E}$ is the same for the nuclei

$$(-dE/dx)_{\mathrm{N}} = \frac{z_{\mathrm{N}}^2}{z_{\mathrm{C}}^2}(-dE/dx)_{\mathrm{C}} = \frac{7^2 \times 2}{6^2} = 2.72 \text{ MeV}$$

Example 1.31 Protons and deuterons have the same kinetic energy when they enter a thin sheet of material. How are their energy losses related?

Solution

$$-\frac{dE}{dx} \propto \frac{z^2}{v^2} \quad \text{or} \quad \propto \frac{M}{E}$$

as both P and d have the same z. Also both have same energy E. Therefore, $(-\frac{dE}{dx})_d = 2(\frac{dE}{dx})_p$.

Example 1.32 If protons and deuterons lose the same amount of energy when they enter a thin sheet of material, how are their energies related?

Solution

$$\left(-\frac{dE}{dx}\right) \propto \frac{M}{E}$$

$$\frac{M_p}{E_p} = \frac{M_d}{E_d}$$

$$E_d = \frac{M_d}{M_p} E_p = \frac{2}{1} E_p = 2E_p$$

1.5.3 Energy Loss to Electrons and Nuclei

For fast charged particles the energy loss results more from electron collisions than nuclear collisions. The latter affect stopping mainly for relatively low velocities and large charges of incident particles. For helium ions of energy larger than 0.5 MeV, even in heavy materials like silver and gold, the nuclear collisions do not account for more than 0.5 per cent of the total energy losses. For heavy ions with relatively low velocities, the contribution of nuclear collisions becomes increasingly important with charge. However, in this case too the collisions with electrons is the dominant process for the energy loss. Thus, for example, in the case of quadruply ionized carbon and oxygen ions in metals, nuclear collisions contribute only to the extent of a few per cent of the energy loss.

1.5.4 Energy Loss of Heavy Fragments

Heavy ions such as ^{12}C, ^{16}O, ^{40}A, ^{85}Kr are slowed down predominantly by ionization in much the same way as alpha particles. The only difference is that z is replaced by $z_{eff} = f(\beta)z$, where $f(\beta)$ is an increasing function of velocity reaching its limiting value of 1 for $\beta = 2z/137$. At very low incident particle velocities various complicated effects enter the energy loss mechanism. When the velocity approaches that of K-shell electron, energy transfer to K-electrons becomes difficult. The energy at which the energy loss attains maximum value is given by $E(\max) \cong \frac{1}{2}Mc^2(1/137)^2 Z^{2/3}$, where M is the mass of the incident particle, Z being the atomic number of the target. At velocities (v) less than Bohr's orbital velocity (u) for K-electron, the incident particle tends to capture an electron (s) from the atom, resulting in the decrease of the effective charge of the incident particle. This is called 'pick-up' process. It may also lose the captured electron. The pick-up process becomes a highly probable process for velocity $v \approx u$, where $u = zc/137 = 0.22 \times 10^9$ cm/s for protons ($E_p = 25$ keV) and $= 0.44 \times 10^9$ cm/s for alphas ($E_\alpha = 400$ keV). Towards the end of the range, as the velocity decreases, the stopping power increases reaching the maximum value for $\beta = 0.037$ for carbon and 0.059 for argon-40 ions, which correspond to 8 and 65 MeV energy respectively. At lower energy the stopping power decreases as the ions are further slowed down, since the decrease of nuclear charge overcompensates the opposing effect of diminishing velocity. This phenomenon is beautifully demonstrated by the thinning down of very heavy ion tracks just before they are arrested in photographic emulsions. The extreme case is furnished by the fission fragments. Their effective charge is large reaching about $20e$ at the beginning of the range, and nuclear collisions are an important source of energy loss. If a fragment of atomic number z crosses a medium of atomic number Z and nuclear mass m_2, the specific energy loss is

$$\frac{-dE}{dx} \propto \frac{z^2 Z^2}{m_2 v^2} \quad \text{(nuclear)} \tag{1.105}$$

whereas the loss to electrons is

$$-\frac{dE}{dx} \propto z_{eff}^2 \frac{Z}{mv^2} \quad \text{(electronic)} \tag{1.106}$$

Equation (1.105) applies to close nuclear collisions where the entire charges of the fragments and the target are effective. In the case of electronic collisions, only the net charge z_{eff} of the fragment is effective, since it carries with it certain number of electrons, and further the target electrons have unit charge. The factor Z in (1.106) arises from the presence of Z electrons/nucleus. The two energy losses may be comparable, but only a few nuclear collisions are responsible for the nuclear component of energy loss whilst in the electronic collisions, the loss is uniformly distributed along the path. The peculiar branches observed in the cloud chamber photographs of fission fragments have their origin in nuclear collisions. The concentration of nuclear energy loss in a limited number of events leads to the enormous spread of ranges of fission fragments of the same energy, a phenomenon called 'straggling'.

It is of interest to point out that heavy ions in passing through crystalline solids lose energy differently depending on the orientation of the trajectory with respect to the axes of a single crystal. For example, 40 keV ^{85}Kr ions are found to penetrate the face centred cubic lattice of aluminum crystals for about 4000 Å in the direction perpendicular to the (101) face but only 1500 Å in the direction perpendicular to the (111) face. This is because the number of atoms encountered in these two cases is not same.

1.5.5 Energy Loss of Electrons

It was pointed out that in the case of heavy ions, ionization is the dominant mode of energy loss. However, for electrons, the energy loss is complicated due to an additional mechanism of loss through radiation, a phenomenon called Bremsstrahlung. At low energies ($E < 2mc^2$) the ionization loss dominates over that due to radiation. The problem of energy loss of electrons by ionization follows similar to that of heavy ions, but the treatment differs in two important respects. They are the identity of particles which participate in the collisions, and secondly their reduced mass.

The formula for non-relativistic electrons is:

$$\frac{-dE}{dx} = \frac{4\pi e^4 n}{mv^2}\left[\ln\frac{mv^2}{2I} - \frac{1}{2}\ln 2 + \frac{1}{2}\right] \tag{1.107}$$

Except for small differences in the terms within the square brackets, formula (1.107) bears a striking resemblance to (1.99). We, therefore, conclude that the non-relativistic electrons lose their energy by ionization at the same rate as the protons.

The relativistic formula for electrons is

$$-\frac{dE}{dx} = \frac{4\pi e^4 n}{mc^2}\left[\ln\frac{2mc^2}{I} + \frac{3}{2}\ln\gamma + \frac{1}{16}\right] \tag{1.108}$$

and that for protons

$$-\frac{dE}{dx} = \frac{4\pi e^4 n}{mc^2}\left[\ln\frac{2mc^2}{I} + 2\ln\gamma - 1\right] \tag{1.109}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor. At equal velocities, formulae (1.108) and (1.109) agree within 10 per cent.

1.6 Delta Rays

1.6.1 Energy Spectrum

In the collision of a charged particle with an atom, one or more electrons are ejected. The more energetic ones of these are called Delta rays and are responsible for the secondary ionization, i.e. the production of further ions due to the collision of delta rays with the atoms of the medium. In what follows we shall be concerned with delta rays of energy larger than the ionization potential of the atoms of the medium. The binding energy of the electron is, therefore, ignored and the collision between the incident particle and the electrons is considered as approximately elastic. From (1.27) the kinetic energy of the ejected electron ($m_2 \ll m_1$)

$$W = 2mv^2 \cos^2 \phi \tag{1.110}$$

where $m = m_2$ is the electron mass, ϕ is the angle of emission of electron, and v is the velocity of the incident particle. The maximum energy, $W(\max) = 2mv^2$ (non-relativistically). Now, for the recoil particle (electron) $\phi = \frac{1}{2}\phi^*$ and $\phi^* = \pi - \theta^*$, and so

$$\cos^2 \phi = \sin^2 \frac{1}{2}\theta^* \tag{1.111}$$

$$W = 2mv^2 \sin^2 \frac{1}{2}\theta^* \tag{1.112}$$

$$dW = mv^2 \sin\theta^* d\theta^* \tag{1.113}$$

But Rutherford's formula for scattering in the CMS is

$$\sigma\left(\theta^*\right) = \frac{d\sigma}{d\Omega^*} = \frac{1}{4}\frac{z^2 e^4}{\mu^2 v^4 \sin^4 \frac{1}{2}\theta^*} \tag{1.114}$$

where we have put $Z = -1$. Since the electron mass is negligible compared to that of the incident particle, $\mu \cong m$. Also, the element of solid angle $d\Omega^* = 2\pi \sin\theta^* d\theta^*$. Therefore,

$$d\sigma = \frac{2\pi \sin\theta^* d\theta^* z^2 e^4}{4m^2 v^4 \sin^4 \frac{1}{2}\theta^*} \tag{1.115}$$

Using (1.112) and (1.113) in (1.115)

$$\frac{d\sigma}{dW} = \frac{2\pi z^2 e^4}{mv^2 W^2} \quad \text{(differential energy spectrum)} \tag{1.116}$$

This gives us the cross-section for finding the delta rays of energy W per unit of energy interval.

1.6.2 Angular Distribution

Using the relations (1.111) and (1.115) and the expression for the element of solid angle in the lab system $d\Omega = 2\pi \sin\phi d\phi$, we obtain the differential cross-section for the delta rays in the LS:

$$\sigma(\phi) = \frac{z^2 e^4}{m^2 v^4 \cos^3 \phi} \tag{1.117}$$

where we have used the relations $\phi = \frac{1}{2}\phi^* = \frac{1}{2}\pi - \frac{1}{2}\theta^*$. It follows that most of the delta rays are emitted at large angles with correspondingly small energy. Note that $\phi(\max) = 90°$ for which $W = 0$. The fact that the delta rays can be emitted only in the forward hemisphere implies that one can find the direction of the primary.

1.6.3 Delta Ray Density

For a 5 MeV proton $W(\max) = 2mv^2 = 4Tm/M = 4 \times 0.51 \times 5/940$ MeV $=$ 10.85 keV. From (1.116), it is evident that the number of delta rays per cm of path is inversely proportional to the primary energy; also it is greater for heavy primaries. The observation of delta ray per cm density is very useful in establishing the charge of heavy nuclei in cosmic radiation. The total number of δ-rays/cm with energy $> W_1$, is given by integrating (1.116) between the limits $2mv^2$ corresponding to the maximum energy of delta rays and some arbitrarily lower value W_1, and multiplying the result by N, the number of electrons per cm^3. This follows from the fact that $n(T, v)$, the number of δ-rays ejected in 1 cm $= 1/\lambda = \Sigma = N\sigma$. We thus have:

$$n(T, v) = \frac{2\pi N e^4 z^2}{mv^2}\left(\frac{1}{W_1} - \frac{1}{2mv^2}\right) \tag{1.118}$$

Below the lower limit W_1, the δ-rays are not recorded. Clearly, $n(T, v)$ is an arbitrary quantity as it depends on the choice of W_1. It follows that for particles of identical velocities but of different charges, $n(T, v)$ varies as z^2 and the distributions of the values of $n(T, v)$ along the tracks of the particles would, apart from statistical fluctuations, be similar in form. It is also seen that at a velocity less than $v_c = \sqrt{(w_1/2m)}$ the primary would not produce δ-rays with energy $> w_1$. Above the critical value, the density would increase at a rate which depends on the variation of the velocity of the particle along the track. The maximum value is attained for $v = \sqrt{(w_1/m)}$ which is simply obtained by maximizing n with respect to v. After this, it varies approximately as $1/v^2$, as the second term in the brackets becomes practically constant. The resulting distribution would thus increase to a maximum and then slowly decrease (Fig. 1.36). The maximum value $n(\max)$ for a given particle of z_2 may be compared with that obtained from similar observations on the tracks of particles of known charge z_1. Thus, the unknown charge z_2 may easily be obtained from the following condition:

$$n_2/n_1 = z_2^2/z_1^2 \tag{1.119}$$

It may be pointed out that this condition is also fulfilled for relativistic particles. It is also possible to determine the mass of the primary particle by measuring the emission angle and the energy of the delta ray caused by the particle whose momentum

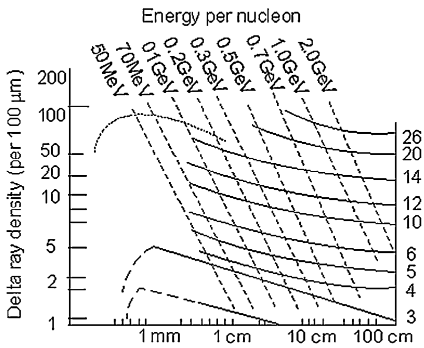


Fig. 1.36 Variation of δ-ray density with range for nuclei of charges 2 to 26

$p = Mv$ is known. We can rewrite (1.110)

$$W = 2mv^2 \cos^2 \phi = \frac{2mp^2}{M^2} \cos^2 \phi \tag{1.120}$$

From the measurement of W, ϕ and p, the mass M of the primary particle can be deduced. This method is specially suited when the conventional methods do not permit the particles to be identified. For example, in bubble chambers, this method is commonly employed for the estimation of contamination of pions or muons in kaon or antiproton beams.

1.7 Straggling

1.7.1 Theory

Identical charged particles, having the same initial velocity, do not have exactly the same ranges. In other words, for a given energy loss the path length fluctuates. This phenomenon is called *Range straggling*. Also, for a given path length the ionization loss and therefore the energy loss fluctuate. This is called *Energy straggling*. There is an intimate relation between the two. The observed ranges of individual particles from any mono-energetic source will show a substantially normal distribution about the mean range. The standard deviation of this distribution is of the order of 1 per cent for a few MeV alphas in any absorber. The distribution is due to the statistical fluctuations in the individual collisions between the charged particle and atomic electrons, which are finite in number. The nuclear collisions, fewer in number, which may cause substantial loss of energy specially towards the end of the ranges, contribute to the short range tail of the distribution. For small energies, however, this will be a small contribution and the distribution may be taken as approximately symmetrical. The harder collisions account for most of the straggling and because very hard collisions are few in number, the actual distribution is some what asymmetric, with a longer tail in the direction of short ranges and with a mean range slightly shorter than the modal range.

1.7.2 Energy Straggling

The energy straggling is produced when an initially mono-energetic beam of particles traverses a given thickness of the absorber. Let A_x be the number of collisions per unit path length in which an energy between W, and $W + dW$ is transfered. Then, from (1.116) we have

$$A_x = \frac{2\pi N z^2 e^4 W}{m v^2 W_x^2} \tag{1.121}$$

where N is the number of electrons/cm^3. The energy transfer in a distance Δr is given by

$$\Delta E = \sum_x A_x W_x \Delta r \tag{1.122}$$

$$\Delta E/\Delta r = \sum_x A_x W_x \tag{1.123}$$

When the number of collisions is large, we may use integration rather than summation

$$\frac{dE}{dr} = \frac{2\pi N z^2 e^4}{m v^2} \int_{W(\min)}^{W(\max)} \frac{dW}{W_x} \tag{1.124}$$

The statistical fluctuations in energy loss ΔE arise from fluctuations about the average number of collisions $A_x \Delta r$. We assume that the collisions are randomly distributed and that the S.D. is given by $\sqrt{A_x \Delta r}$. The S.D. of the energy loss is then $W_x \sqrt{A_x \Delta r}$. The variance for all types of collisions is then given by the summation of the individual variances

$$\sigma^2 = \Delta r \sum_x W_x^2 A_x = \frac{2\pi N z^2 e^4 \Delta r}{m v^2} \int_{W(\min)}^{W(\max)} dW$$
$$= \frac{2\pi N Z^2 e^4 \Delta r}{m v^2} \left[W(\max) - W(\min) \right]$$

where we have replaced the summation by integration. Since $W(\min) \ll W(\max) = 2mv^2$

$$\sigma^2 = 4\pi N z^2 e^4 \Delta r \tag{1.125}$$

If it is assumed that the actual energy loss has a Gaussian distribution around the average value E_0, the use of expression (1.125) for the S.D. in energy loss leads to

$$P(E)dE = \frac{dE}{\sqrt{8\pi^2 z^2 e^4 N t}} \exp - \left[\frac{(E - E_0)^2}{8\pi N z^2 e^4 t} \right] \tag{1.126}$$

where t is the absorber thickness.

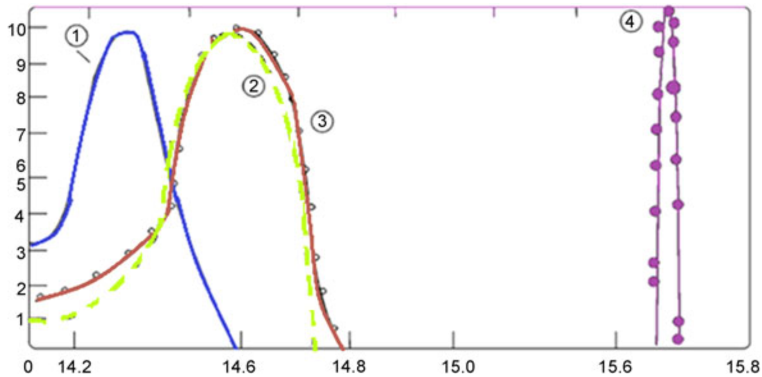


Fig. 1.37 Energy distribution of an 'unobstructed' electron beam and the calculated and experimental distributions of electrons that have passed through 0.86 $\mathrm{g\,cm^{-2}}$ of aluminum. (**1**) Landau theory without density correction; (**2**) Landau theory with Fermi density correction; (**3**) experiment; (**4**) incident beam [1]

In the case of fission fragments large energy losses in individual nuclear collisions give rise to a tail on the side of higher energy losses of the distribution. The straggling effects are much more important for electrons than for heavy particles, because an electron may lose even half its energy in a single elastic collision, where as a heavy particle may lose only a fraction of its energy. Radiation losses add further to the electron straggling. Thus electron straggling reaches values of the order of 0.2 of the total energy loss. Figure 1.37 shows the energy distribution of electrons before they have entered the absorber and after they have traversed 0.86 $\mathrm{g/cm^2}$ thickness of aluminium.

1.7.3 Range Straggling

The fluctuations in range and energy loss are related. Denoting the S.D. of energy and range by σ_E and σ_R respectively, we can use the formula for the propagation of errors and write:

$$\sigma_R^2 = (dR/dE)^2 \sigma_E^2 \tag{1.127}$$

Using (1.125), we get

$$\sigma_R^2 = (dR/dE)^2 4\pi N z^2 e^4 dR \tag{1.128}$$

Writing $dR = (dR/dE)dE$, we get the result:

$$\sigma_R^2 = 4\pi N z^2 e^4 \int_0^{E_0} \left(\frac{dE}{dR}\right)^{-3} dE \tag{1.129}$$

This relation is not applicable to heavy ions and fission fragments that undergo excessive straggling owing to the occurrence of single nuclear collisions. Assuming

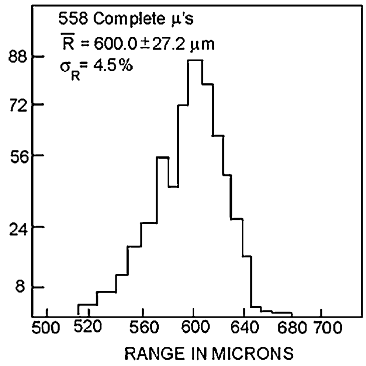


Fig. 1.38 Measured ranges of muons from π–μ decay in emulsions of standard composition

that the ranges of individual particles are distributed about the mean range in a Gaussian way, the probability that the individual range falls between R and $R + dR$ is

$$P(R)dR = \frac{dR}{\sigma_R\sqrt{2\pi}} \exp -\left[(R - \overline{R})^2/2\sigma_R^2\right] \tag{1.130}$$

For α particles from Polonium, $E_0 = 5.3$ MeV, $\overline{R} = 3.84$ cm in air, the corresponding $\sigma_R = 0.036$ cm and the ratio $\sigma_R/\overline{R} = 0.9$ %. Figure 1.38 shows the histogram of ranges of μ mesons produced in the decay of π^+ mesons at rest. Since the π mesons decay by a two-body process, μ^+ is produced with unique energy (4.27 MeV). The mean range in photographic emulsions is found to be 600 μm. The S.D. of the range distribution is found to be, $\sigma_R = 2.7$ μm; this gives $\sigma_R/\overline{R} = 0.045$, or 4.5 percent.

1.7.3.1 The Range Straggling Parameter

This is related to S.D. by

$$\alpha_0 = \sqrt{2}\sigma_R \tag{1.131}$$

Several common types of particle detectors measure the integrated number of particles. The particles that are still present in the collimated beam having ranges equal or greater than R is given by

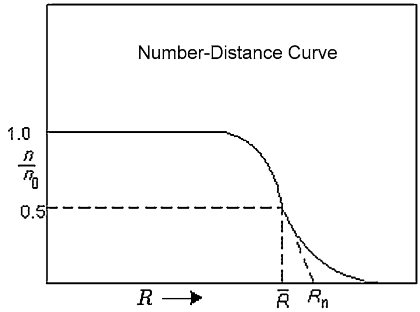
$$n = n_0 - \int_{-\infty}^{R} dn$$

where dn/n_0 is given by the normal distribution

$$\frac{dn}{n_0} = \frac{1}{\alpha\sqrt{\pi}} \exp\left[-(R - \overline{R})^2/\alpha^2\right] dR \tag{1.132}$$

dn is the actual range between R and $R + dR$, n_0 is the total number of particles initially present, and α is the half width of the range distribution at $1/e$ of the max-

Fig. 1.39 The extrapolated number-distance range R_n exceeds the mean range $\overline{R}$ by 0.886α, where α is the range-straggling parameter



imum. Although the normal distribution is non-integrable, its value can be found from standard tables.

The number-distance curve, n/n_0 against R, is indicated in Fig. 1.39. Its slope $(dn/n_0)/dR$ at the mean range $R = \overline{R}$ is $1/\alpha\sqrt{\pi}$. As the central portion of the number-distance curve is approximately linear, it can be extrapolated to cut the range axis at $R = R_n$. This is called extrapolated range. From Fig. 1.39, we find the relation between R_n and $\overline{R}$

$$\frac{\frac{1}{2}}{R_n - \overline{R}} = \frac{1}{\alpha\sqrt{\pi}} \tag{1.133}$$

whence the mean range, in term of the measured extrapolated range R_n and straggling parameter a, is

$$\overline{R} = R_n - \frac{1}{2}\sqrt{\pi}a = R_n - 0.886\alpha \tag{1.134}$$

1.7.3.2 Deduction of Ranges Parameter

For particles of charge ze and mass M but the same initial velocity v_0 as alpha-particles

$$\sigma_R = \left[4\pi N z^2 e^4 \int_0^{E_0} \left(\frac{dE}{dR}\right)^{-3} dE\right]^{1/2}$$

$$\frac{dE}{dR} = z^2 f(v_0)N, \quad \text{since } \frac{dE}{dR} = \frac{2\pi z^2 e^4 N}{mv^2} \int_{w(\min)}^{w(\max)} \frac{dw}{w}$$

$$dE = d\left(Mv^2\right) = Mvdv \quad \text{and} \quad \sigma_R = \frac{\sqrt{M}}{Nz^2} f(v_0, I)$$

$$R = \frac{Mf'(v_0, I)}{Nz^2} \tag{1.135}$$

$$\frac{\sigma_R}{R} = \frac{1}{\sqrt{M}} f''(v_0, I) \tag{1.136}$$

where f, f', f'' are complicated functions of the initial velocity v_0, mean excitation and ionization potential I. The function f'' and hence σ_R/R are independent of N and of z but is found to decrease slowly with increasing I.

For particles of mass M having the same initial velocity v_0 as the α particles,

$$\frac{(\alpha_0/R)_M}{(\alpha_0/R)_{\alpha\ particle}} = \sqrt{\frac{4}{M}} \tag{1.137}$$

It follows that protons will have about twice the range straggling parameter of α-particles which have the same initial velocity and hence about the same range.

1.8 Cerenkov Radiation

Electromagnetic radiation is emitted when a charged particle passes through a medium in which its velocity $v = \beta c$ exceeds the phase velocity c/μ, where μ is the refractive index of the medium. This observation was discovered by Cerenkov and was explained theoretically by Frank and Tamm. The effect was first observed in the experiments of Cerenkov who was investigating the glow in pure liquids caused by γ rays from radium. Vavilov and Cerenkov showed that the radiation is not due to luminescence (emission from excited atoms and molecules of the medium) but due to the passage of knock-on electrons produced in Compton scattering of γ rays. The radiation is instantaneous and possesses a sharply pronounced spatial symmetry.

When relativistic charged particles are incident on a transparent dielectric, the velocity of the particle is substantially unchanged except for the ionization and radiation losses. On the other hand, the electric field due to the charge of the particle and the magnetic field produced by the moving charge are propagated through the medium with velocity of only c/μ. The resulting electromagnetic radiation is cancelled in all directions if $\beta\mu < 1$; however, if $\beta\mu > 1$, constructive interference can take place in one direction defined by angle θ (Fig. 1.40). When $\beta c > c/\mu$, i.e. the particle velocity exceeds the velocity of light in the medium it is as if the particle runs away from its own slower electromagnetic field, resulting in the emission of all frequencies for which $\beta\mu > 1$. The resulting radiation called Cerenkov radiation is emitted on a conical surface BDA of half angle a_0. Figure 1.40 gives the Huyghens' construction for the electromagnetic waves emitted by the particle along its path. The particle is at A at $t = 0$; and at a later time it moves on to D such that $AD = \beta ct$. The front of the electromagnetic wave lies on the surface of the cone of half angle α. Consequently the corresponding rays of light make an angle θ with the path of the particle. The axis of the cone coincides with the direction of the incident particle and the half angle of the cone is determined by:

$$\sin\alpha = \cos\theta = \frac{(c/\mu)t}{\beta ct} = \frac{1}{\beta\mu} \tag{1.138}$$

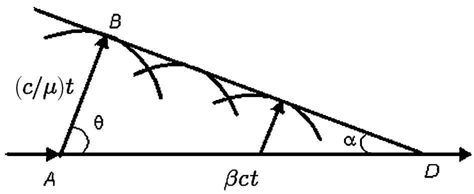


Fig. 1.40 Huyghens' construction for electromagnetic waves emitted by a moving charged particle

This follows from the condition that the optical difference in the path of the waves emitted by the moving particle at various points of its trajectory is equal to zero. The light is polarized with its electric vector in the plane of the conical surface and radially directed along DB. The conical distribution of the Cerenkov radiation has a natural half width of the order of a few degrees. This is attributed to the occurrence of successive changes in particle velocity when photons are emitted. The phenomenon is analogous to the V-shaped shock wave observed in acoustics when a projectile or an aeroplane travels with supersonic velocity. Apart from (1.138), there are two other conditions that must be fulfilled to achieve coherence. These are (i) pathlength of the particle in the medium must be large compared with the wavelength of the radiation, otherwise diffraction effects become dominant and (ii) velocity of the particle must remain constant during its passage through the medium.

For a medium of a given refractive index μ, there is a threshold velocity $\beta(\min) = 1/\mu$, below which no radiation is emitted. At this critical velocity, the direction of radiation coincides with that of the particle. For glass ($\mu = 1.5$), $\beta(\min) = 0.667$, corresponding to 200 keV electrons or 320 MeV protons. As the refractive index decreases, the threshold velocity increases. For an ultra-relativistic particle, for which $\beta \simeq 1$, there is a maximum angle of emission given by $\theta(\max) = \cos^{-1}(1/\mu)$.

Fermi showed that Cerenkov radiation results from small energy transfers to distant atoms due to the fast moving charged particles which is subsequently emitted as a coherent radiation. Thus the emission of Cerenkov radiation is a particular form of energy loss in extremely soft collisions. The classical theory of Cerenkov effect is originally due to Frank and Tamm and is justified by the quantum theory. Since the radiation in question is believed to be the result of the interaction with the medium as a whole and not due to the interaction of particles with individual atoms, the medium is considered as continuous and is characterized by the macroscopic parameter, the dielectric constant or by the refractive index. It is shown that the rate of energy loss per unit path length is given by:

$$-\frac{dE}{dx} = \frac{4\pi^2 z^2 e^2}{c^2} \int_{\beta\mu>1} \left[1 - \frac{1}{\beta^2\mu^2}\right] \nu d\nu \text{ ergs/cm} \tag{1.139}$$

where ze is the charge of the particle and ν is the frequency of the emitted radiation. The integration is to be carried over all frequencies for which $\beta\mu > 1$. For glass or Lucite, the energy loss by Cerenkov radiation is of the order of 1 keV/cm, a value which is much less than that incurred in ionization or radiation. Nonetheless the radiation is readily detected as a large number of photons are produced in the visible region. Formula (1.139) shows that the Cerenkov radiation is independent of the rest mass of the moving particle and depends only on the particle's charge and velocity, apart from the refractive index of the medium. The mean number of photons of frequency ν and $\nu + d\nu$ in the visible region per cm is calculated in Example 1.35, under the assumption that μ is independent of ν in the considered range of frequencies. Hence

$$N(\nu)d\nu = \frac{4\pi^2 z^2 e^2}{hc^2}\left(1 - \frac{1}{\mu^2\beta^2}\right)d\nu = \frac{2\pi z^2 d\nu \sin^2\theta}{137c} \tag{1.140}$$

The radiation has continuous spectrum, with components of all frequencies for which the refractive indices are higher than $1/\beta$. Equation (1.139) shows via the term $\nu d\nu$ which is proportional to $d\lambda/\lambda^3$ that the energy per wavelength interval $d\lambda$ is proportional to $1/\lambda^3$. Also, (1.140) shows through the term $d\upsilon$ that the number of quanta per cm per wavelength interval is proportional to $1/\lambda^2$. It follows that shorter wavelengths are preferred and the Cerenkov radiation appears visually as bluish white.

The density effect is closely connected with the phenomenon of Cerenkov effect. It was first pointed out by Bohr that the intricate relationship between the density effect and the Cerenkov effect is such that the entire contribution to the most probable energy loss from the minimum out to the beginning of the Fermi plateau in the ionization curve is due to Cerenkov effect.

Example 1.33 Pions and muons each of 160 MeV/c momentum pass through a transparent material. Find the range of the index of refraction of this material over which the muons alone give Cerenkov light. Assume $m_\pi c^2 = 140$ MeV, $m_\mu c^2 = 106$ MeV.

Solution Momentum, $p = m\beta\gamma c$. Therefore, $\frac{\beta}{\sqrt{1-\beta^2}} = \frac{cp}{mc^2}$

$$\text{Pions:} \quad \frac{cp}{m_\pi c^2} = \frac{160}{140} = \frac{8}{7} = \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\beta_\pi = 0.7525; \qquad \mu_\pi = \frac{1}{\beta_\pi} = \frac{1}{0.7525} = 1.33$$

$$\text{Muons:} \quad \frac{cp}{m_\mu c^2} = \frac{160}{106} = \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\beta_\mu = 0.8336; \qquad \mu_\mu = \frac{1}{\beta_\mu} = \frac{1}{0.8336} = 1.2$$

Therefore, the range of the index of refraction of the material over which the muons alone give Cerenkov light is 1.2–1.33.

Example 1.34 A beam of protons moves through a material whose refractive index is 1.6. Cerenkov light is emitted at an angle of 15° to the beam. Find the kinetic energy of the proton in MeV.

Solution

$$\beta = \frac{1}{\mu \cos\theta} = \frac{1}{1.6 \cos 15^\circ} = 0.647$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(0.647)^2}} = 1.31$$

$$K.E. = (\gamma - 1)mc^2 = (1.31 - 1) \times 938 = 292 \text{ MeV}$$

Example 1.35 The rate of loss of energy by production of Cerenkov radiation is given by the relation

$$-dW/dl = \frac{z^2 e^2}{c^2} \int \left(1 - \frac{1}{\beta^2 \mu^2}\right) \omega d\omega \text{ erg cm}^{-1}$$

where βc is the velocity of the of charge ze, μ is the refractive index of the medium and $\omega/2\pi$ is the frequency of radiation. Estimate the number of photons emitted in the visible region, per cm of track, by a particle having $\beta = 0.8$ passing through glass ($\mu = 1.5$). The fine structure constant $\alpha = e^2/\hbar c = 1/137$.

Solution For electron, $z = -1$ and since $\omega = 2\pi\nu$, the given expression becomes upon integration between the frequencies ν_1 and ν_2

$$-dW/dl = \frac{4\pi^2 e^2}{c^2} \left(1 - \frac{1}{\beta^2 \mu^2}\right) \frac{(\nu_2^2 - \nu_1^2)}{2}$$

where we have assumed that μ is independent of ν.

Calling the average photon energy as $h\overline{\nu} = \frac{1}{2}h(\nu_1 + \nu_2)$, the average number of quanta emitted per cm is

$$N = \frac{1}{h\overline{\nu}} \left(\frac{-d\nu c}{dl}\right) = \frac{4\pi^2 e^2}{hc^2} \left(1 - \frac{1}{\beta^2 \mu^2}\right) (\nu_2 - \nu_1)$$

$$= \frac{2\pi}{137} \left(1 - \frac{1}{\beta^2 \mu^2}\right) \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$$

where $\lambda = c/\nu$ is the vacuum wavelength and μ is the average refractive index over the wavelength interval from $\lambda_2 = 4000$ Å to $\lambda_1 = 8000$ Å and $\beta\mu = 0.8 \times 1.5 = 1.2$.

$$N = \frac{2\pi}{137} \left(1 - \frac{1}{1.2^2}\right) \left(\frac{1}{4000 \times 10^{-8}} - \frac{1}{8000 \times 10^{-8}}\right) = 175 \text{ photons}$$

1.9 Identification of Charged Particles

In numerous investigations in nuclear physics and particle physics it is necessary to determine the nature and energy of charged particles. In order to identify a particle, its charge and mass must be determined. The charge may be determined by δ-ray density or width measurements in photographic emulsions or pulse height in scintillation counters or proportional counters or solid state detectors.

Assuming that the charge is known, the mass of the particle can be determined from the simultaneous measurements of at least two dynamical quantities such as momentum and velocity, momentum times velocity and $-dE/dX$, $-dE/dX$ and E.

1.9.1 (a) Momentum and Velocity

Momentum can be determined from curvature measurement in a cloud chamber or in a bubble chamber with low Z liquid with known magnetic field or in photographic emulsions with pulsed magnetic field.

Velocity may be estimated by the estimation of ionization through drop density in a cloud chamber, bubble density in a bubble chamber, grain density, blob density or mean gap length in photographic emulsions or by Cerenkov counters or time-of-flight method.

1.9.2 (b) Momentum Times Velocity ($p\beta$) and Velocity

The product $p\beta$ is determined from the mean scattering angle in emulsions for energetic particles. The velocity is measured as in Sect. 1.9.1.

1.9.3 Energy and Velocity

Energy may be determined from range measurements for low energy particles and velocity as in Sect. 1.9.1.

1.9.4 Simultaneous Measurement of dE/dx and E

This method is widely used in the study of nuclear reactions using solid state detectors, since for non-relativistic particles the product EdE/dx is proportional to z^2M. A simultaneous measurement of E and dE/dx and their product permits the separation of the particles according to their masses in a wide range of energy variations.

1.9.5 Energy and Emission Angle

Energy and emission angle measurement of knock-on electron (δ-ray) together with the momentum measurement of the beam particles.

This method is very useful in case the conventional methods are not available. The method is explained in Sect. 1.6.3.

1.10 Bremsstrahlung

In their passage through matter, electrons lose energy in two ways (i) ionization (which was referred to in Sect. 1.5.1) and (ii) radiation or Bremsstrahlung. The electrons undergo radiative collisions mainly with the atomic nuclei of the medium. In the vicinity of the nucleus of charge Ze, the incident particle of charge ze and mass m undergoes acceleration which is proportional to zZ/m. According to electrodynamics, a charged particle undergoing acceleration emits radiation. This is called Bremsstrahlung or braking radiation whose spectrum has the form dE/E where E is the photon energy. The photon energy spectrum extends from low energy to the maximum value equal to the particle energy, with the low energy photons being preferably emitted (see Fig. 1.41 for typical energy spectrum). The radiation intensity is proportional to z^2Z^2/m^2. This then means that under identical conditions, radiation losses are 3×10^6 times as much for electron as for a proton. The total average energy loss per path length dx integrated over all frequencies is given by

$$-(\overline{dE})_{rad} = \frac{4Z(Z+1)}{137} NEr_e^2 \ln \frac{183}{Z^{1/3}} dx \tag{1.141}$$

where N is the number of nuclei per cm^3, E is the energy of electron and $r_e = e^2/mc^2$ is the classical electron radius.

Since an electron may lose appreciable energy in a single collision, the actual energy loss may vary significantly from the average value given by (1.141). This also implies that the range straggling of electrons would be so great that the definition of mean range would hardly be meaningful. If we define the radiation length X_0 by

$$\frac{1}{X_0} = \frac{4Z(Z+1)}{137} r_e^2 N \ln \frac{183}{Z^{1/3}} \tag{1.142}$$

we can write from (1.141)

$$\frac{dE}{dx} = -\frac{E}{X_0} \tag{1.143}$$

Integrating (1.143), we find the average energy of a beam of electrons of initial energy E_0 after traversing a thickness x of medium by the expression

$$\langle E \rangle = E_0 \exp(-x/X_0) \tag{1.144}$$

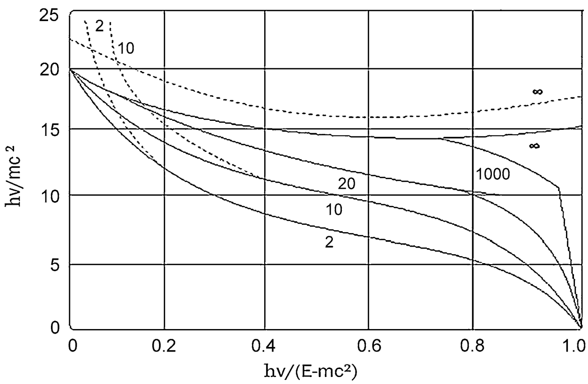


Fig. 1.41 Energy distribution of the radiation emitted by an electron. Ordinate intensity of radiation (quantum energy times number of quanta) per unit frequency interval. Abscissa, energy of emitted quantum as a fraction of the energy of the emitting electron. The *numbers on the curves* indicate the energy of the electron in units of mc^2. *Solid curves* for lead, including effect of screening. *Dotted curves* are without screening, valid for all Z [3]

For $x = X_0$, $\langle E \rangle = E_0/e$, where e is the exponential. This suggests that the radiation length X_0 may be simply defined as that thickness of the medium which reduces the beam energy by a factor of e. Since the thickness x can be measured in cm or g/cm^2 (which is obtained by multiplying the thickness in cm by the density of the medium) X_0, is expressed in corresponding units. At low electron energies ($E \ll mc^2$), the electrons lose their energy predominantly through excitation and ionization, and radiation loss is unimportant. The energy loss by ionization and excitation is proportional to Z and is practically constant at high energies as it increases only logarithmically with energy. On the other hand, radiation losses are proportional to Z^2 and increase linearly with energy. Thus, the radiation loss predominates at high energies. It is apparent that at some energy E_c, called the *critical energy*, $E_{rad} = E_{ion}$. It can be shown that roughly

$$\frac{(dE/dx)_{rad}}{(dE/dx)_{ion}} = \frac{EX}{600} \tag{1.145}$$

so that E_c (in MeV) $= 600/x$. The radiation lengths X_0 and the critical energy E_c, for some of the materials are shown in Table 1.1. Observe that X_0, decreases rapidly with increasing Z.

1.11 Questions

1.1 Why in Rutherford scattering the presence of orbital electrons in the target atom is ignored?

1.2 The total cross-section for Rutherford scattering is infinite. What is the physical reason?

Table 1.1 Radiation lengths and critical energy in different elements

Element	Z	X_0 (g cm^{-2})	E_c (MeV)
Hydrogen	1	58	340
Carbon	6	42.5	103
Air	7.2	36.5	83
Aluminium	11	23.9	47
Iron	26	13.8	24
Lead	82	5.8	6.9

1.3 Why in the famous a-scattering experiment thin foils were used for the target?

1.4 If the incident electron enters the nucleus, would the Coulomb's inverse square law between the charges be still valid? If not, how would it be modified for a nucleus in which the charge is uniformly distributed?

1.5 Why does the ionization fall off for very low particle velocity?

1.6 The inverse square velocity law for ionization would suggest that the rate of energy loss is greater at low speeds, since the time spent by the incident particle in the vicinity of the electron is longer. Is this reasonable? In the same manner would a slow moving heavenly object raise larger tides on approaching close to the earth compared to a fast moving one?

1.7 What is the physical origin of the rise in the $-dE/dx$ curve beyond the minimum?

1.8 At relativistic velocities, the $-dE/dx$ curve saturates to a plateau. What is the origin of the plateau?

1.9 In the cloud chamber studies of ionization, the plateau-to-trough ratio for the $-dE/dx$ curve might be as large as 1.5, but in photographic emulsions it is no more than 1.1. Explain.

1.10 How does the percentage straggling compare for ^{3}H and ^{3}He nuclei of the same initial velocity?

1.11 A cloud chamber photograph shows an alpha track which after certain distance gets thinned down and then disappears. It again re-appears before it stops. What is happening?

1.12 The range of a proton of few MeV is a measure of its initial energy. The energy thus estimated would be close to the actual value within few per cent. However, in the case of electrons of similar energy, the energy thus estimated can hardly be reliable. Explain.

1.13 The tracks of fission fragments often leave peculiar branches before coming to a rest. Explain.

1.14 A water cooled nuclear reactor appears bluish. What could be the origin of this colour?

1.15 A charged particle moves swiftly with uniform velocity in a vacuum. Would it radiate?

1.16 What is the dominant mechanism for energy loss for electrons of energy (a) <1 MeV, (b) 200–500 MeV?

1.12 Problems

1.1 Show that in an elastic collision, the ratio of the kinetic energy K'/K can be expressed through $\alpha = M/m$ and $y = \cos\theta$ as

$$\frac{K'}{K} = (1+\alpha)^{-2}\left[2y^2 + \alpha^2 - 1 + 2y\sqrt{a^2 + y^2 - 1}\right]$$

1.2 A body of mass M rests on a smooth table. Another of mass m moving with a velocity u collides with it. Both are perfectly elastic and smooth and no rotations are set up by the collision. The body M is driven in a direction at an angle ϕ to the previous line of motion of the body m. Show that its velocity is

$$\frac{2mu}{M+m}\cos\phi$$

1.3 A nucleus A of mass $2m$ moving with velocity u collides inelastically with the nucleus B of mass $10m$. After the collision, the nucleus A travels at 90° with the incident direction, while B proceeds at an angle 37° with the incident direction. (a) Find the speeds of A and B after the collision. (b) What fraction of the initial kinetic energy is gained or lost due to the collision?
[Ans. (a) $v_A = 3u/4$; $v_B = \frac{u}{4}$; (b) 1/8]

1.4 A beam of alphas gets scattered from a hydrogen target. What is the maximum angle of scattering?
[Ans. approximately 15°]

1.5 An alpha particle fired into a cloud chamber undergoes an elastic collision with a nucleus of the gas used to fill the chamber. The collision is recorded photographically as a forked track. Measurements from the photograph show that the collision deviated the alpha-particle at 60° and that the struck nucleus recoiled at an angle of 30° with the direction of motion of the incident alpha-particle. Assuming that the struck nucleus is initially at rest, calculate:

(a) The mass number of the gas used to fill the chamber.
(b) The ratio of the velocity of projection of the struck nucleus to the velocity of the incident alpha particle.

[Ans. (a) 4; (b) $\sqrt{3}$]

1.6 A particle of mass m makes an elastic collision with a proton, initially at rest. The proton is projected at an angle 22.1° whilst the incident particle is scattered through an angle 5.6° with the incident direction. Estimate m in atomic mass units.
[Ans. 7.8]

1.7 Consider an elastic collision between an incident particle having mass m and a target particle of mass M such that $m > M$. Show that the largest possible scattering angle $\theta(\max)$ in the lab system is given by: $\sin\theta(\max) = M/m$; and that this corresponds to C-system angle $\cos\theta^*(\max) = -M/m$. Also show that the maximum recoil angle $\phi(\max)$ is given by $\sin\phi(\max) = \sqrt{(m-M)/2m}$. Calculate the angle $\theta(\max) + \phi(\max)$ for elastic collisions between the incident deuterons and the target protons.
[Ans. 60°]

1.8 A billiard ball moving at a speed of 2.5 m/s makes a glancing collision with another identical ball initially at rest. After the collision, one ball is observed to move with a speed 2 m/s at an angle 37° with the original direction of motion. Find the speed of the other ball and the angle at which it moves. What is the nature of the collision?
[Ans. 1.5 m/s, 53°, elastic]

1.9 If a particle of mass m moving with kinetic energy K_0 makes elastic collision with a target particle of mass M initially at rest, such that the scattered particle is deflected at an angle θ in the lab system and has θ^* in the centre of mass system and has a kinetic energy K in the lab system, show that:

$$\frac{K}{K_0} = \frac{1}{(M+m)^2}\left[m\cos\theta + M\cos\left(\theta^* - \theta\right)\right]^2$$

1.10 A particle of mass m and initially of velocity u makes an elastic collision with a particle of mass M initially at rest. After the collision m is deflected through lab angle 90° with speed $u/\sqrt{3}$. The particle M recoils with speed v at a lab angle ϕ with the incident direction. Find (a) M/m, (b) v/u, (c) ϕ, (d) θ^*, (e) ϕ^*.
[Ans. (a) 2, (b) $1/\sqrt{3}$, (c) 30°, (d) 120°, (e) 60°]

1.11 A deuteron of velocity u strikes another deuteron (twice the mass of proton) initially at rest. As a result of the collision, a proton is produced which moves off at 45° with respect to the direction of incidence. The other product of this rearrangement collision is triton (three times the mass of proton). Assuming that this

collision may be approximated to an elastic collision, calculate the speed and direction of triton in the lab and CM system.
[Ans. 0.48 u, 34° in the lab system and $u/2\sqrt{3}$, 111° in the CM system]

1.12 An α-particle from a radioactive source collides with a stationary proton and continues with a deflection of 13.9°. Find the direction in which the proton moves.
[Ans. 30°]

1.13 When α-particles of kinetic energy 30 MeV pass through a gas, they are found to be elastically scattered at angles up to 30° but not beyond. Explain this, and identify the gas. In what way, if any, does the limiting angle vary with energy?
[Ans. Deuterium, does not vary]

1.14 A perfectly smooth sphere of mass m, moving with velocity v collides elastically with a similar but initially stationary sphere of mass m_2 ($m_1 > m_2$) and is deflected through an angle θ_L. Describe how this collision would appear in the centre of mass frame of reference and show that the relation between θ_L and the angle of deflection θ_M, in the centre of mass frame is

$$\tan\theta_L = \frac{\sin\theta_M}{[M_1/M_2 + \cos\theta_M]}$$

Also show that θ_L cannot be greater than about 19.5° if $M_1/M_2 = 3$.

1.15 Show that the maximum velocity that can be imparted to a proton at rest by a non-relativistic alpha particles is 1.6 times the velocity of the incident alpha particle.

1.16 Show that for low energy p–p scattering $\sigma(\theta) = 4\sigma(\theta^*)$ where the differential cross-sections $\sigma(\theta)$ and $\sigma(\theta^*)$ refer to the Lab and CMS, respectively.

1.17 (a) Compute the distance of closest approach in collisions between α-particles of energy 8.9 MeV and nuclei of $^{208}_{82}$Pb.
(b) How is this distance related to the radius of lead nucleus?
(c) What is the deflection of the α-particle when the impact parameter is equal to this distance?
[Ans. (a) 26.5 fm, (b) 7.7 fm, (c) 53°]

1.18 A beam of α-particles of kinetic energy 4.5 MeV passes through a thin foil of $^9_4$Be. The number of alphas scattered between 60° and 90° and between 90° and 120° is measured. What would be the ratio of these numbers?
[Ans. 3]

1.19 If the probability of α-particles of energy 8 MeV to be scattered through an angle greater than θ on passing through a thin foil is 10^{-3} what is it for 4 MeV protons passing through the same foil?
[Ans. 10^{-3}]

1.20 What α-particle energy would be necessary in order to explore the field of force within a radius of 10^{-12} cm of the centre of nucleus of atomic number 80, assuming classical mechanics to be adequate?
[Ans. 30 MeV]

1.21 In an elastic collision with a heavy nucleus, when the impact parameter b is just equal to the collision radius $\frac{1}{2}R_0$, what is the value of the scattering angle θ^* in the CMS?
[Ans. 90°]

1.22 In the elastic scattering of deuterons of 5.9 MeV from $^{208}_{82}$Pb, the differential cross-section is observed to deviate from Rutherford's classical prediction at 52°. Use the simplest classical model to calculate the closest distance of approach d to which this angle of scattering corresponds. You are given that for an angle of scattering θ, d is given by $\frac{1}{2}d_0(1 + \mathrm{cosec}\,\frac{1}{2}\theta)$, where d_0 is the value of d in a head-on collision.
[Ans. 32.8 fm]

1.23 20000,1 MeV α-particles are incident normally on a 0.004 mm thick copper plate. Using the small angle approximation, calculate the number of α-particles scattered in the angular range 5°–10°. Assume the copper nuclei to act as point charges and neglect nuclear forces. Density of $^{66.6}_{29}$Cu $= 8.9$ g cm^{-3}; Avagadro's number $= 6 \times 10^{23}$ (g molecule), $e = 1.6 \times 10^{-19}$ C; 1 eV $= 1.6 \times 10^{-19}$ J.
[Ans. 7894]

1.24 Given that the angle of scattering is $2\tan^{-1}(a/2b)$, where 'a' is the least possible distance of approach, and b is the impact parameter. Calculate what fraction of a beam of 1.0 MeV deuterons will be scattered through more than 90° by a foil of thickness 10^{-5} cm of a metal of density 5 g cm^{-3} atomic weight 100 and atomic number 50.
[Ans. 1.22×10^{-5}]

1.25 Show that the differential cross-section for the recoil nucleus in the lab system is given by

$$\sigma(\phi) = \left(zZe^2/2T\right)^2 \frac{1}{\cos^3\phi}$$

1.26 An electron of energy 10 keV approaches a bare nucleus ($Z = 20$) with an impact parameter corresponding to an orbital angular momentum $\hbar$. Sketch the form of the potential energy curve for the electron trajectory and calculate the distance from the nucleus at which this has a minimum (take $\hbar = 10^{-34}$ J s, $e = 1.6 \times 10^{-19}$ C and $m = 10^{-30}$ kg).
[Ans. 0.19 Å]

1.27 A beam of protons of 5 MeV kinetic energy traverses a gold foil. One particle in 5×10^6 is scattered so as to hit a surface 0.5 cm^2 in area at a distance 10 cm from the foil and in a direction making an angle of 60° with the initial direction of the beam. What is the thickness of the foil?
[Ans. 0.0066 μm]

1.28 A narrow beam of protons with velocity $v = 6 \times 10^6$ m/s falls normally on a silver foil of thickness $t = 1.0$ μm. Find the probability of the protons to be scattered into the backward hemisphere ($\theta > 90°$).
[Ans. 0.006]

1.29 A narrow beam of alpha particles with K.E. 0.5 MeV falls normally on a golden foil whose thickness is 1.5 mg/cm^2. The beam intensity is 5×10^5 particles per sec. Find the number of alpha particles scattered by the foil during the time interval of 30 minutes into angular interval 59–61°.
[Ans. 1.6×10^6]

1.30 A narrow beam of alpha particles falls normally on a silver foil behind which a counter is set to register the scattered particles. On substitution of platinum foil of the same mass thickness for the silver foil, the number of alpha particles registered per unit time increases 1.52 times. Find the atomic number of platinum, assuming the atomic number of silver and the atomic masses of both platinum and silver to be known.
[Ans. 78]

1.31 A narrow beam of alpha particles with kinetic energy 1.0 MeV falls normally on a platinum foil which is 1.0 μm thick. The scattered particles are observed at an angle of 60° to the incident beam direction by means of a counter with a circular sensitive area 1.0 cm^2 located at a distance 10 cm from the scattering section of the foil. What fraction of scattered alpha particles enters the counter? Assume the density of platinum as 21.5 g/cm^3.
[Ans. 3.33×10^{-5}]

1.32 Singly charged particles of masses m_1 and m_2 enter a medium with the same velocity. Show that the ratio of their ranges $R_1/R_2 = m_1/m_2$.

1.33 Show that a deuteron of energy E has twice the range of a protons of energy $E/2$.

1.34 If the mean range of 8 MeV proton in a medium in 0.30 mm, calculate the mean range of 16 MeV deuterons and 32 MeV α-particles.

1.35 An alpha particle moving with velocity 2×10^9 cm/sec, loses energy 0.066 MeV/mm by ionization in air and has range 7.86 cm in air. (a) Find the rate of

loss of energy per mm in air for proton and deuteron moving with the same initial velocity as alpha particle. (b) Find the range of proton and deuteron.
[Ans. (a) 0.0165 MeV/mm for both, (b) 7.86 cm, 15.72 cm]

1.36 Estimate by the Bragg-Kleeman rule the mean range of 12 MeV deuterons in cobalt, if their mean range in air at 15 °C, 760 mm Hg is 93 cm. Assume the density of cobalt to be 8.6 g/cm^3.
[Ans. 0.0266 cm]

1.37 Show that the range of α-particles and protons of energy 1 to 10 MeV in aluminium is $1/1600$ of the range in air at 15 °C, 760 mm of Hg.

1.38 Show that the straggling of a beam of ^{4}He is smaller than that of ^{3}He of equal range.

1.39 Compute the energy loss and the approximate number of quanta of visible light ($\lambda = 4000$ to 7000 Å) as Cerenkov radiation by a 20 MeV electron in traversing 1 cm of Lucite. Assume the chemical composition of Lucite to be ($C_5H_8O_2$), and the refractive index $\mu = 1.5$.
[Ans. 660 eV/cm, 270 quanta/cm]

1.40 Show that the order of magnitude of the ratio of the rate of loss of kinetic energy by radiation for a 10 MeV deuteron and a 10 MeV electron passing through lead is 10^{-5}.

1.41 Compute the energy loss and the approximate number of quanta of visible light ($\lambda = 4000$ to 7000 Å) as Cerenkov radiation by a 20 MeV electron in traversing 1 cm of Lucite. Assume the chemical composition of Lucite to be ($C_5H_8O_2$), and the refractive index $\mu = 1.5$.

1.42 Extensive air showers in cosmic rays consist of a 'soft' component of electrons and photons, and a 'hard' component of muons. Suppose at the sea level the central core of a shower consists of a narrow vertical beam of muons of energy 100 GeV which penetrate the interior of the earth. Assuming that the ionization loss in rock is constant at 2 MeV g^{-1} cm^2, and the rock density is 3.0 g cm^{-3}, find the depth of the rock through which the muons can penetrate.
[Ans. 160 m]

References

1. E.L. Goldwasser, F.E. Mills, A.O. Hanson, Phys. Rev. **88**, 1137 (1952)
2. A.A. Kamal, G.K. Rao, Y.V. Rao, Osmania University
3. Proc. R. Soc. A **146**, 96 (1934)

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