

Chapter 2

Black Holes in General Relativity

Abstract As a prelude to our study of quantum black holes, in this chapter we briefly review some of the key features of black holes in general relativity. We focus on specific examples of black hole metrics in four and higher space-time dimensions, which will be needed later in the book. The chapter closes with the laws of black hole mechanics, which raise a curious analogy with the laws of thermodynamics.

2.1 Introduction

Before we can discuss the quantum physics of black holes, we first need to describe their classical behaviour, that is, as objects in Einstein's theory of general relativity. General relativity describes the force of gravity as the curvature of the fabric of space-time. This curvature is caused by matter (or energy, the two being equivalent in relativity due to Einstein's famous equation $E = mc^2$) and is governed by Einstein's equations of general relativity:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (2.1)$$

where the Einstein tensor $G_{\mu\nu}$ describes the space-time curvature and the stress-energy tensor $T_{\mu\nu}$ the matter or energy. The curvature of space-time in turn affects the paths of particles, which is particularly relevant for black holes.

In this chapter we provide a brief overview of some of the features of black holes in general relativity, focusing on those aspects needed for later chapters. Further details of black holes in general relativity can be found in the many excellent books available, see for example the classics (Hawking and Ellis 1975; Misner et al. 1973; Wald 1984), modern textbooks (Carroll 2003; Hartle 2002; Hobson et al. 2006) and works specifically on black holes (Frolov and Novikov 1998; Frolov and Zelnikov 2011; Raine and Thomas 2005; Townsend 1997). We begin with descriptions of black holes in four space-time dimensions before discussing some examples of

higher-dimensional black holes. The chapter closes by considering four laws of black hole mechanics, which reveal an intriguing similarity to the laws of thermodynamics.

2.2 Four-dimensional Black Holes

We begin by briefly reviewing black hole solutions of four-dimensional general relativity. Standard theorems (see, for example, Hawking and Ellis (1975)) govern the properties of four-dimensional black hole solutions of general relativity in either a vacuum or coupled to an electromagnetic field. In particular, such black holes are either *static* and *spherically symmetric*, or *rotating* and *axisymmetric*.

2.2.1 Schwarzschild Black Hole

The very first example of a black hole metric (although the nature of the metric as describing a black hole was not understood for several decades) was found by Schwarzschild not long after the publication of Einstein's theory of general relativity (Schwarzschild 1916). The Schwarzschild metric is given by:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \quad (2.2)$$

where

$$d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \quad (2.3)$$

is the metric on the 2-sphere. Here (r, θ, φ) are the usual spherical polar co-ordinates. In (2.2), the constant $M > 0$ is the mass of the black hole. The metric (2.2) is a solution of Einstein's Equations (2.1) in the absence of matter, so that the stress-energy tensor $T_{\mu\nu} = 0$. The metric (2.2) is static and spherically symmetric. As $r \rightarrow \infty$, the metric (2.2) approaches the flat-space Minkowski metric, and, accordingly, the Schwarzschild black hole is *asymptotically flat*.

From (2.2), it can be seen that the Schwarzschild metric becomes singular when $r = r_H = 2M$ and when $r = 0$. The apparent singularity at $r = 2M$ is a co-ordinate singularity which can be removed by changing to an alternative co-ordinate system. However there is a *curvature singularity* at $r = 0$ which cannot be removed by a change of co-ordinates.

The curvature of space-time represented by the metric (2.2) modifies the paths of particles. Since, in relativity, particles cannot travel faster than light, we consider the directions of light rays. If a flash of light is emitted from a point in flat space, the light rays will travel outwards from that point in all directions equally. The light rays form what is known as the *light cone* at that point. Since a particle cannot travel

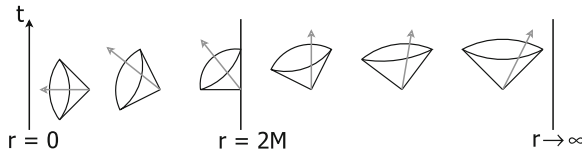


Fig. 2.1 Light cones in the Schwarzschild geometry. The co-ordinate r is the horizontal axis, and t is the vertical axis. The cones at each point are the directions of light rays emanating from that point. A possible particle path at each point is shown in grey. Particle paths must lie inside the light cone at each point because particles cannot travel faster than light

faster than light, its path must lie inside the light cone at each point. On the right of Fig. 2.1, far from the black hole, space-time is flat and the light-cones are at 45° degrees to the vertical. As we move from right to left in the diagram, it can be seen that the light cones tilt over due to the curvature of the space-time. At each point in Fig. 2.1, we have shown in grey a possible particle path, which must lie inside the light-cone. Inside $r = 2M$ we see that the light cones have tilted over sufficiently that particle paths must be directed towards the left, that is, towards the black hole, and it is not possible for a particle or light ray to move to the right away from the black hole. The surface $r = 2M$ is the *event horizon*, which forms the boundary of the black hole. The length $r_s = 2M$ is known as the *Schwarzschild radius* of the black hole.

In order to show the complete space-time structure, a Penrose diagram is useful. To draw the Penrose diagram, a conformal transformation is made which brings infinity ($r \rightarrow \infty$) into a finite region. The details of the construction can be found in standard books on the subject, such as Carroll (2003), Hawking and Ellis (1975), Raine and Thomas (2005). The Penrose diagram for Schwarzschild space-time is shown in Fig. 2.2. In a Penrose diagram, light rays travel at 45° to the horizontal, and time-like particle paths must have a slope of more than 45° to the horizontal. There are a number of important features of the Penrose diagram in Fig. 2.2. Firstly, the curvature singularity at $r = 0$ has become two space-like surfaces, depicted as the two horizontal lines at the top and bottom of the diagram. Secondly, the surfaces $r = 2M$ are lines at 45° which pass through the centre of the diagram. Thirdly, in the Penrose diagram, the complex nature of infinity can be clearly seen. Infinity ($r \rightarrow \infty$) consists of various points and surfaces: the points $i^\pm$ are future/past time-like infinity (through which particle paths ultimately pass); the points i^0 are space-like infinity; and the surfaces $I^\pm$ are future/past null infinity (through which light rays ultimately pass).

The nature of the surface at $r = 2M$ is evident from the Penrose diagram. Since the surface at $r = 2M$ is at 45° , any light ray or particle which enters into region 2 on the diagram must hit the singularity at $r = 0$ and cannot escape out to region 1. Region 1 is the exterior of the black hole and is the region of interest to external observers. Region 2 is the interior of the black hole. Region 3 is a time-reverse of a black hole, known as a *white hole*, which we do not consider further. Region 4 is a copy of region 1, with another asymptotic region. For black holes formed by matter

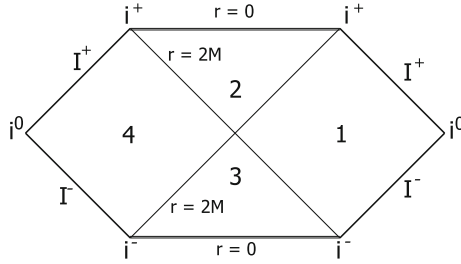


Fig. 2.2 Penrose diagram for Schwarzschild space-time. Time is on the vertical axis, increasing from the bottom to the top of the diagram. Each point on the diagram corresponds to a 2-sphere. The curvature singularity $r = 0$ takes the form of the two horizontal lines at the top and bottom of the diagram. The event horizon $r = 2M$ corresponds to the two lines at 45° passing through the centre of the diagram. The points and lines denoted $i^\pm$, i^0 and $I^\pm$ form infinity

collapsing under gravity, regions 3 and 4 are unphysical (see Fig. 3.1 in Chap. 3). For the remainder of this book, we are interested in region 1 of all black hole space-times, which is the region outside the event horizon.

2.2.2 Reissner-Nordström Black Hole

In the previous subsection we studied the metric of a static, spherically symmetric black hole in a vacuum. It is possible to couple a black hole to an electromagnetic field, which modifies the space-time geometry. The stress-energy tensor corresponding to an electromagnetic field with field strength tensor $F_{\mu\nu}$ is, in Gaussian units,

$$T_{\mu\nu}^{\text{EM}} = \frac{1}{4\pi} \left(F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \right). \quad (2.4)$$

Consider an electric field where the only non-zero component of the field strength is

$$F_{rt} = \frac{Q}{r^2}, \quad (2.5)$$

where Q is the electric charge. In this case the solution of Einstein's equations representing a static, spherically symmetric black hole takes the Reissner-Nordström form (Nordström 1918; Reissner 1916)

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_2^2. \quad (2.6)$$

Comparing (2.2) and (2.6), it can be seen that the metric components g_{tt} and g_{rr} are modified by the additional Q^2/r^2 term. The constant $M > 0$ is once again the mass of the black hole. There is a curvature singularity at $r = 0$.

For $M > |Q|$, there are now two values of r for which the metric component g_{tt} vanishes:

$$r_H = M + \sqrt{M^2 - Q^2}, \quad r_- = M - \sqrt{M^2 - Q^2}. \quad (2.7)$$

The larger of these two roots, r_H , is the location of the event horizon of the black hole. The smaller root, r_- , is a new type of horizon, known as either an *inner horizon* or a *Cauchy horizon*. If $M = |Q|$, the inner and event horizons merge and an *extremal* black hole results. For $M < |Q|$, there is no event horizon and we have a *naked singularity* at $r = 0$. Penrose diagrams for all these cases can be found in Hawking and Ellis (1975). We will only consider non-extremal black holes for the remainder of this book.

2.2.3 Kerr Black Hole

The two black hole metrics which we have considered thus far have both been static and spherically symmetric. The *Kerr(-Newman)* solution represents a black hole which is rotating. The metric takes the following form, in spheroidal polar co-ordinates (r, θ, φ) :

$$ds^2 = -\frac{\Delta}{\Sigma} \left[dt - a \sin^2 \theta d\varphi \right]^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} \left[(r^2 + a^2) d\varphi - a dt \right]^2, \quad (2.8)$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta. \quad (2.9)$$

The constant $M > 0$ is the mass of the black hole. The black hole is rotating, and has angular momentum $J = aM$, where $a > 0$ is a constant. The angle $\theta \in [0, \pi]$ corresponds to latitude: $\theta = 0$ is the rotation axis of the black hole and $\theta = \pi/2$ is the equatorial plane. The metric (2.8) does not depend on time t or the azimuthal (longitude) angle $\varphi \in [0, 2\pi]$. The metric is *stationary*; it is not static because $g_{t\varphi} \neq 0$.

The *Kerr* metric (Kerr 1963) is obtained by setting the charge $Q = 0$, and when $Q \neq 0$ we have the *Kerr-Newman* metric (Newman et al. 1965; Newman and Janis 1965). The Kerr metric is a solution of Einstein's equations (2.1) in a vacuum with $T_{\mu\nu} = 0$. The Kerr-Newman metric is a solution of Einstein's equations with an electromagnetic field, so the stress-energy tensor $T_{\mu\nu}$ has the form (2.4). The electromagnetic field is most compactly written in terms of an electromagnetic potential A_μ , which has non-zero components

$$A_t = -\frac{Qr}{\Sigma}, \quad A_\varphi = \frac{Qar}{\Sigma} \sin^2 \theta. \quad (2.10)$$

The components of the field strength $F_{\mu\nu}$ can be computed from the electromagnetic potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.11)$$

As well as an electric part A_t (as in the Reissner-Nordström solution), the electromagnetic potential also has a non-zero magnetic part A_φ , due to the rotation of the black hole.

The metric (2.8) has a curvature singularity when $\Sigma = 0$, that is, $r = 0$ and $\theta = \pi/2$. This is known as the *ring singularity* and its structure is explored in more detail in Hawking and Ellis (1975), O'Neill (1992). The metric component g_{rr} becomes singular when $\Delta = 0$. The values of r at which this happens are, for $M^2 > a^2 + Q^2$,

$$r_H = M + \sqrt{M^2 - a^2 - Q^2}, \quad r_- = M - \sqrt{M^2 - a^2 - Q^2}. \quad (2.12)$$

As with the Reissner-Nordström metric, the surface $r = r_H$ is the event horizon of the black hole and $r = r_-$ corresponds to the inner (or Cauchy) horizon. When $M^2 = a^2 + Q^2$, the event and inner horizons merge and the black hole is extremal. For $M^2 < a^2 + Q^2$, there is no event horizon and a naked singularity results.

The event horizon rotates with an angular speed

$$\Omega_H = \frac{a}{r_H^2 + a^2}. \quad (2.13)$$

The metric component g_{tt} vanishes on a surface $r = r_S$, where

$$r_S = M + \sqrt{M^2 - a^2 \cos^2 \theta - Q^2}, \quad (2.14)$$

which lies outside the event horizon $r = r_H$ (see Fig. 2.3). This surface is known as the *stationary limit surface*. The region between the event horizon and the stationary limit surface is known as the *ergosphere*. Within the ergosphere, it is not possible for

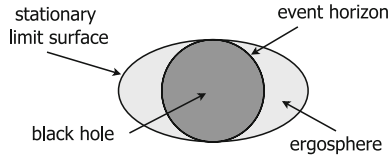


Fig. 2.3 Diagram showing the location of the stationary limit surface and ergosphere for a Kerr black hole. The stationary limit surface lies outside the event horizon of the black hole except on the axis of rotation, where the stationary limit surface touches the event horizon at its north and south poles

a particle to remain at fixed (r, θ, φ) . Instead, a massive particle must rotate in the same direction as the event horizon of the black hole. However, since the ergosphere lies outside the event horizon of the black hole, it is still possible for a particle inside the ergosphere to escape from the black hole out to infinity.

The rotation of the black hole described by the Kerr metric and the existence of an ergosphere have many important physical implications. An example is the *Penrose process* (Penrose and Floyd 1971), whereby the rotational energy of the black hole can be extracted. Suppose a particle inside the ergosphere splits into two particles. One of these particles, particle *A*, escapes to infinity, while the other, particle *B*, falls down the event horizon and enters the black hole. Imagine that particle *B*, which falls down the event horizon, has negative energy and angular momentum. Particle *B* having negative energy means that the energy required to move the particle from its location inside the ergosphere to infinity is greater than the rest-mass energy of the particle. Particle *B* having negative angular momentum simply means that it is rotating in the opposite direction to the black hole. If particle *B* has negative energy and angular momentum, particle *A*, which escapes to infinity, will have greater energy than the original particle. Since particle *B* enters the event horizon, the energy of the black hole and the angular momentum of the black hole will decrease. The rotational energy of the black hole has effectively been extracted and given to particle *A*.

Astrophysical black holes are expected to be rapidly rotating, and hence described by the Kerr metric (see, for example, Wiltshire et al. (2009) for more on Kerr black holes in astrophysics). We will also see in Chap. 5 that black holes formed by particle collisions will in general be rapidly rotating.

2.3 Higher-dimensional Black Holes

In Sect. 2.2 we have outlined some of the features of four-dimensional black holes in general relativity. Black holes in more than four space-time dimensions are complex objects in general, as many of the standard theorems governing the properties of four-dimensional black holes do not generalize to higher dimensions. In this section we will discuss the higher-dimensional generalizations of the four-dimensional Schwarzschild, Reissner-Nordström and Kerr black holes. For more information on the state-of-the-art in research on higher dimensional black holes, including such exotic objects as black rings and black Saturns, see the reviews by Emparan and Reall (2008) and Horowitz (2012). In this section we take the number of space-time dimensions to be $d = n + 4$, so that n represents the number of extra dimensions.

2.3.1 Spherically Symmetric Higher-dimensional Black Holes

The four-dimensional Schwarzschild metric (2.2) has a simple generalization to higher dimensions (Tangherlini 1963):

$$ds^2 = - \left[1 - \left(\frac{r_H}{r} \right)^{n+1} \right] dt^2 + \left[1 - \left(\frac{r_H}{r} \right)^{n+1} \right]^{-1} dr^2 + r^2 d\Omega_{n+2}^2, \quad (2.15)$$

where $d\Omega_{n+2}^2$ is the metric on the $(n+2)$ -sphere. The Schwarzschild-Tangherlini black hole shares many properties with the Schwarzschild black hole. There is an event horizon at $r = r_H$ and a curvature singularity at $r = 0$. The metric (2.15) is static, spherically symmetric and asymptotically flat. The mass M of the black hole is related to the event horizon radius r_H by (Myers and Perry 1986)

$$M = \frac{1}{16\pi} (n+2) r_H^{n+1} A_{n+2}, \quad (2.16)$$

where A_{n+2} is the area of a unit $(n+2)$ -sphere:

$$A_{n+2} = \frac{2\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \quad (2.17)$$

and $\Gamma(x)$ is the Γ -function. The Reissner-Nordström metric (2.6) also has a straightforward generalization to higher dimensions (Myers and Perry 1986) but we shall not consider that black hole further.

2.3.2 Higher-dimensional Rotating Black Holes

The generalizations of the neutral Kerr metric (2.8) to higher dimensions are known as *Myers-Perry* black holes (Myers and Perry 1986). The metrics are rather complicated, because a black hole in $(n+4)$ space-time dimensions has N_r independent axes of rotation, where $N_r = k+1$ if $n = 2k$ is even, and $N_r = k+2$ if $n = 2k+1$ is odd. Therefore a four-dimensional Kerr black hole ($n = 0$) has one possible axis of rotation (given by $\theta = 0$ in the Kerr metric (2.8)) but a seven-dimensional black hole has three perpendicular axes of rotation. The metric in the general case can be found in Myers and Perry (1986).

In this book we are interested in black holes formed by a collision of two particles which are moving in a four-dimensional subspace of the higher-dimensional space-time (see Chap. 5). In this situation, if the particles do not collide exactly head-on, the system consisting of the two particles will have non-zero angular momentum about an axis of rotation in the four-dimensional subspace. Therefore, by conservation of angular momentum, the resulting black hole will also have non-zero angular momentum about a single axis, which is in the four-dimensional subspace in which the particles collide. Such black holes are known as *singly rotating* black holes and in this case the general Myers-Perry metric simplifies.

The metric for a singly-rotating Myers-Perry black hole takes the form (Myers and Perry 1986)

$$\begin{aligned}
ds^2 = & - \left(1 - \frac{\mu}{\Sigma r^{n-1}}\right) dt^2 - \frac{2a\mu \sin^2 \theta}{\Sigma r^{n-1}} dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\
& + \left(r^2 + a^2 + \frac{a^2 \mu \sin^2 \theta}{\Sigma r^{n-1}}\right) \sin^2 \theta d\varphi^2 + r^2 \cos^2 \theta d\Omega_n^2,
\end{aligned} \tag{2.18}$$

where

$$\Delta = r^2 + a^2 - \frac{\mu}{r^{n-1}}, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \tag{2.19}$$

and $d\Omega_n^2$ is the metric on the n -sphere. The mass M and angular momentum J of the black hole are given by:

$$M = \frac{1}{16\pi} (n+2) A_{n+2} \mu, \quad J = \frac{2}{n+2} a M, \tag{2.20}$$

so that the parameter $\mu > 0$ governs the mass of the black hole and the parameter $a > 0$ its angular momentum.

The metric component g_{rr} diverges at the event horizon, $r = r_H$, where $\Delta = 0$. For $n > 0$, the function Δ has only one positive real root and therefore the black hole has only an event horizon and no inner horizon. For $n = 1$, the event horizon exists only if $a^2 < \mu$, otherwise there is a naked singularity. For $n > 1$, the event horizon exists for all values of a , so that there is no upper bound on the angular momentum of the black hole.

Higher-dimensional generalizations of the Kerr-Newman metric in closed form have to date eluded researchers. For general relativity with an electromagnetic field, solutions are known only numerically. Analytic solutions are known for higher-dimensional rotating black holes in low-energy models arising from string theory, but these have additional matter fields. See Allahverdizadeh et al. (2011) for a summary of what is known about higher-dimensional charged rotating black holes.

2.4 Brane Black Holes

We now turn to a particular class of higher-dimensional black holes, arising in brane worlds. Firstly we give a very brief summary of the key features of brane worlds before discussing a simple model of the geometry of brane black holes.

2.4.1 Brane Worlds

Brane worlds are a particular class of higher-dimensional theory. There are two main realizations of the brane world idea: the ADD model (Antoniadis et al. 1998; Arkani-Hamed et al. 1998, 1999) in which the extra dimensions are flat, and the

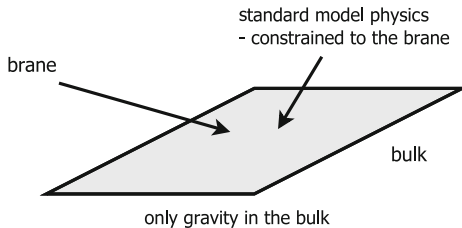


Fig. 2.4 Sketch of a brane world model. The particles and forces of standard model physics (that is, the electromagnetic and strong and weak nuclear forces) are constrained to the brane, only gravity may propagate in the bulk

Randall-Sundrum model (Mannheim 2005; Randall and Sundrum 1999, Randall and Sundrum 1999) in which the extra dimensions are warped (have curvature). These models were proposed as a solution to the hierarchy problem, which can be roughly stated as: why is the force of gravity so much weaker than the other fundamental forces? In brane world models the force of gravity is diluted because it probes the extra dimensions. Quantum gravity in brane world models is discussed in more detail in Chap. 6, here we focus on the nature of classical black holes in these models.

In this section we shall consider only black holes in ADD brane worlds where the extra dimensions have a flat geometry. Matter fields are constrained to be on a four-dimensional subspace (termed the *brane*) of a higher-dimensional space-time (called the *bulk*). Only gravitational degrees of freedom may propagate in the bulk (see Fig. 2.4 for a sketch).

2.4.2 Neutral Brane Black Holes

In the ADD brane world model, the brane can be thought of simply as a “slice” through the higher-dimensional bulk space-time. We shall model the brane itself as having no structure or tension. Now suppose that there is a higher-dimensional black hole in the bulk, as sketched in Fig. 2.5. Since gravity can propagate in the bulk, the higher-dimensional black hole is a solution of Einstein’s equations (2.1) in the bulk with $T_{\mu\nu} = 0$ on the right-hand-side (since there is no matter in the bulk, only gravity). The fact that the extra dimensions in the bulk are flat in the ADD model means that the black hole must be asymptotically flat. Assuming that the black hole is much smaller than the size of the extra dimensions, the metric of the higher-dimensional black hole is therefore described by the Myers-Perry metric (see Sect. 2.3.2). If we are interested in higher-dimensional black holes formed by collisions of particles on the brane, the black hole will have a single non-zero component of angular momentum, which will be parallel to the axis of rotation of the black hole in the brane. Therefore the appropriate metric to consider is the singly-rotating Myers-Perry black hole metric (2.18).

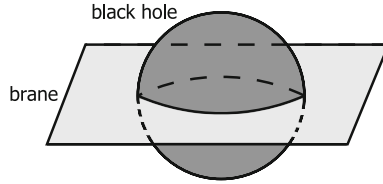


Fig. 2.5 Sketch of a higher-dimensional black hole in an ADD brane world with flat extra dimensions. On taking a slice through the higher-dimensional black hole, the geometry on the brane - a brane black hole - is revealed

To find the metric describing the black hole on the brane, we simply take a slice through the Myers-Perry metric (2.18), by fixing the values of the co-ordinates describing the extra dimensions. The $d\Omega_n^2$ term in the metric (2.18) corresponds to the extra dimensions and disappears when taking this slice, to give the metric of the brane black hole as:

$$ds^2 = - \left(1 - \frac{\mu}{\Sigma r^{n-1}} \right) dt^2 - \frac{2a\mu \sin^2 \theta}{\Sigma r^{n-1}} dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{a^2 \mu \sin^2 \theta}{\Sigma r^{n-1}} \right) \sin^2 \theta d\varphi^2. \quad (2.21)$$

If the higher-dimensional black hole is not rotating, then its metric will have the Schwarzschild-Tangherlini form (2.15). Taking a slice through the metric (2.15) gives:

$$ds^2 = - \left[1 - \left(\frac{r_H}{r} \right)^{n+1} \right] dt^2 + \left[1 - \left(\frac{r_H}{r} \right)^{n+1} \right]^{-1} dr^2 + r^2 d\Omega_2^2. \quad (2.22)$$

It is worth commenting that the metrics (2.21–2.22) are not solutions of the vacuum Einstein equations (2.1) in four space-time dimensions. Instead, if one calculates the Einstein tensor $G_{\mu\nu}$ for the metrics (2.21–2.22), there is a non-zero stress-energy tensor $T_{\mu\nu}^0$ on the right-hand-side of the Einstein equations (2.1) (Sampaio 2009). This represents an effective fluid seen by an observer on the brane due to the fact that the black hole is actually a higher-dimensional object, but the observer cannot directly probe the extra dimensions.

We use the metrics (2.21–2.22) as simple models for the geometry of brane black holes, particularly in Chap. 3.

2.5 Black Hole Mechanics and Thermodynamics

So far in this chapter we have studied stationary black hole solutions of Einstein's equations. We have not considered black holes as dynamical objects, changing in time. In Chap. 3 we will be interested in evolving black holes. In this section we

outline four laws of *black hole mechanics*, which govern aspects of how black holes evolve in time. We will then discover a surprising connection with thermodynamics.

2.5.1 Definitions

Before we can state the four laws of black hole mechanics, a few definitions are needed. Since the defining characteristic of a black hole is that it has an event horizon, the laws of black hole mechanics are essentially laws about the properties of event horizons. We now define two key quantities which can be ascribed to an event horizon.

The first key quantity is the *area* $\mathcal{A}_H$ of an event horizon. This is simply the area of the surface which forms the black hole event horizon. For a four-dimensional Kerr black hole with metric (2.8) the area of the event horizon is

$$\mathcal{A}_H = 4\pi \left(r_H^2 + a^2 \right), \quad (2.23)$$

which reduces to

$$\mathcal{A}_H = 4\pi r_H^2 \quad (2.24)$$

if the black hole is non-rotating (so that $a = 0$).

The next quantity we require is a measure of the acceleration due to gravity at the event horizon of the black hole, known as the *surface gravity* and denoted κ . We are used to the concept of the Newtonian acceleration due to gravity at the surface of a star or planet. In order to consistently define the corresponding quantity in relativity, we have to specify the observer (or frame) in which the acceleration is measured. It would be natural to define the surface gravity as the acceleration of a freely-falling object, instantaneously at rest, near the surface of the body, as measured by an observer at rest near the surface of the body. For a black hole this quantity is divergent at the event horizon, due to the fact that an observer at rest near the event horizon will have a divergent acceleration themselves. However, the acceleration at the event horizon as seen by an observer at infinity is finite, and this will be taken to be the definition of the surface gravity of a black hole. We note that this definition reduces to the usual Newtonian definition in the limit of small velocities and accelerations.

This definition can be used to find the surface gravity of the black holes we have considered earlier in this chapter: see, for example, the treatment in Raine and Thomas (2005). For a four-dimensional Kerr black hole with metric (2.8) the surface gravity is

$$\kappa = \frac{r_H - M}{2Mr_H}. \quad (2.25)$$

For a four-dimensional Schwarzschild black hole with $r_H = 2M$, this reduces to

$$\kappa = \frac{1}{4M}. \quad (2.26)$$

The surface gravity of the higher-dimensional Myers-Perry black holes discussed in Sect. 2.3.2 will also be needed in Chap. 3, and is given by (Myers and Perry 1986):

$$\kappa = \frac{(n+1)r_H^2 + (n-1)a^2}{2r_H(r_H^2 + a^2)}, \quad (2.27)$$

which reduces to (2.25) when $n = 0$.

Finally in this section, we define the *electrostatic potential* Φ_H at the event horizon. For a non-rotating black hole, Φ_H is simply equal to $-A_t$, the time component of the electromagnetic potential. For a rotating black hole,

$$\Phi_H = -A_t - \Omega_H A_\varphi, \quad (2.28)$$

where Ω_H is the angular speed of the event horizon. For a Kerr black hole or charged rotating brane black hole, both of which have the electromagnetic potential (2.10), the electrostatic potential on the horizon is

$$\Phi_H = \frac{Qr_H}{r_H^2 + a^2}, \quad (2.29)$$

which reduces to $\Phi_H = \frac{Q}{r_H}$ when the black hole is non-rotating.

2.5.2 Black Hole Mechanics

With the quantities defined in the previous subsection, we are now in a position to state the laws of black hole mechanics. For simplicity, we restrict attention to four-dimensional black hole solutions of Einstein's equations either in a vacuum or with an electromagnetic field. The four laws of black hole mechanics (Bardeen et al. 1973) are

Zeroth law The surface gravity κ of the event horizon of a stationary black hole is constant over the event horizon.

First law Any two neighbouring stationary, axisymmetric, black hole solutions are related by

$$\delta M = \frac{\kappa}{8\pi} \delta \mathcal{A}_H + \Omega_H \delta J + \Phi_H \delta Q. \quad (2.30)$$

Second law The area $\mathcal{A}_H$ of the event horizon of a black hole is a non-decreasing function of time. If two or more black holes coalesce, the area of the final event horizon is greater than the sum of the areas of the initial event horizons.

Third law It is not possible to form a black hole with a vanishing surface gravity κ in a finite number of operations.

It is clear from (2.25–2.26) that the zeroth law holds for Schwarzschild and Kerr black holes. The zeroth law has been proved in Bardeen et al. (1973) for a class of matter fields which includes electromagnetism (alternative proofs can be found in

DeWitt and DeWitt (1973)). The first law is proved in Bardeen et al. (1973), again for a range of matter fields which includes electromagnetism. The second law is known as the *area theorem* and was proved by Hawking (1972), with some assumptions on the matter fields which are satisfied by electromagnetism. The status of the third law is rather different; it is a postulate and there is currently no mathematical proof, and, equally, no contradiction is known.

Classical processes involving black holes must satisfy the first and second laws of black hole mechanics. For example, in the Penrose process (see Sect. 2.2.3) the event horizon area $\mathcal{A}_H$ does not decrease—see for example the discussion in Raine and Thomas (2005).

2.5.3 Black Hole Entropy and Thermodynamics

The naming and formulation of the laws of black hole mechanics in the previous subsection is rather suggestive of the laws of thermodynamics. In this subsection we explore this analogy further, although it is only in Chap. 3 that the comparison will come to fruition.

Firstly, we observe that classical black holes in general relativity must have an *entropy*. To see why, suppose that black holes did not have an entropy. In this case it would be possible to violate the second law of thermodynamics (that the entropy of the universe cannot decrease) by throwing some entropic matter into a black hole. Therefore black holes must have an entropy and the *generalized second law of thermodynamics* holds, which states that the total entropy of the universe does not decrease, where the total entropy of the universe is the entropy outside black hole event horizons plus the entropy of all the black holes in the universe.

The second law of black hole mechanics suggests that a quantity proportional to the area of the event horizon of a black hole plays the role of black hole entropy (Bardeen et al. 1973; Bekenstein 1973). Let us therefore, in general relativity, set the black hole entropy S_{BH} to be

$$S_{BH} = \mathcal{K} \mathcal{A}_H, \quad (2.31)$$

where $\mathcal{K}$ is a constant, independent of the nature of the black hole under consideration.

With this in mind, we now turn to the first law of black hole mechanics (2.30). In relativity mass M and energy E are equivalent (we are using units in which $c = 1$). The second and third terms on the right-hand-side of (2.30) represent the work done in either increasing the angular momentum of the black hole by an amount δJ or increasing the charge of the black hole by an amount δQ . Therefore, the first law of black hole mechanics (2.30) takes the form of the first law of thermodynamics:

$$\delta E = T \delta S + \delta W \quad (2.32)$$

where T is temperature, S is entropy and W is work. If the black hole has an entropy S_{BH} given by (2.31), then it appears that the quantity

$$T_{BH} = \frac{\kappa}{8\pi \mathcal{K}} \quad (2.33)$$

plays the role of *temperature* in the first law.

For the moment, let us take the approach that T_{BH} is a quantity which is analogous to temperature as far as the laws of black hole mechanics are concerned. The zeroth and third laws of black hole mechanics then immediately have the form of the first and third laws of thermodynamics respectively. The zeroth law implies that a stationary black hole is an equilibrium configuration with constant T_{BH} (corresponding to thermal equilibrium). The third law implies that a “zero temperature” configuration with $T_{BH} = 0$ cannot be reached in a finite number of steps.

2.6 Conclusions

In this chapter we have briefly reviewed the metrics describing four and higher-dimensional black holes in general relativity. We have considered static, spherically symmetric black holes and axisymmetric, rotating black holes. In four-dimensional classical general relativity, black holes are very simple objects, with Kerr-Newman black holes described by their mass, angular momentum and charge. This means that nearly all the information about what has disappeared down the event horizon is lost.

The chapter closed with a brief discussion of the laws of black hole mechanics, which have some similarities with the laws of thermodynamics. This analogy is rather intriguing, but, in general relativity, it is only a metaphor. The temperature of a black hole in general relativity is zero because the black hole, by definition, absorbs matter but emits nothing. We leave the tantalizing question of whether this analogy can be realized as something deeper to Chap. 3.

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