

Chapter 2

Generalized Weibull Distributions

2.1 Introduction

Since 1970s, many extensions of the Weibull distribution have been proposed to enhance its capability to fit diverse lifetime data and Murthy et al. (2004) proposes a scheme to classify these distributions. Over the last 40 years papers dealing with various extensions of the Weibull distribution and their applications number well over 4000.

A common factor among the generalized models considered in this chapter is that the Weibull distribution is a special case of theirs. Either $F(t)$ or $h(t)$ of these modified models are related to the corresponding function of the Weibull distribution in some way. The means of these distributions usually do not have a simple expression although their hazard rate functions $h(t)$ are more capable to model diverse problems than the Weibull model does.

2.2 Methods of Constructions

Generalized Weibull distributions can be constructed in many ways. Members of this family usually contain the standard Weibull model as a special case. The following is a brief sketch of some of the common methods. A new generalized distribution may be obtained by:

- adding a constant λ to the existing hazard rate $h(t)$ of the Weibull distribution or a generalized Weibull model,
- transformations (linear, inverse or log) of the Weibull random variable,
- transformations of the distribution $F(t)$ or the survival function $\bar{F}(t)$ of the Weibull distribution in such a way that the new function remains a distribution or survival function,
- competing risk approach (minimum of two or more Weibull variables),

- mixtures of two or more Weibull variables, mixtures of two or more generalized Weibull variables, mixing a Weibull distribution with a generalized Weibull distribution, etc.
- method of compounding such as letting $T = \min(X_1, X_2, \dots, X_N)$ where X 's are independent and identically distributed Weibull variables and N is the compounded variable,
- probability integral transform, e.g., let a new density be defined as $f(t) = \psi(G(t))g$ where G is cdf of the Weibull variable with its density function g , ψ is a probability density function having support on the unit interval, or
- by adding a frailty parameter or a tilt parameter, see Chap. 7 of Marshall and Marshall and Olkin (2007).

For a more detailed discussion on constructing life-time distributions, see Lai et al. (2011).

2.3 Four-Parameter Weibull Distribution

This model was proposed by Kies (1958) and its survival function is given by

$$\bar{F}(t) = \exp \left\{ -\lambda \left[\frac{(t-a)}{(b-t)} \right]^\alpha \right\}; \quad 0 < a \leq t \leq b < \infty, \lambda, \alpha > 0. \quad (2.1)$$

We note that the support of this distribution is a finite interval. Smith and Hoepfner (1990) also presented a similar four-parameter Weibull model. They also applied the model to fit fatigue and compliance calibration data.

2.4 Five-Parameter Weibull Distribution

Phani (1987) extended the model due to Kies (1958) with survival function given by

$$\bar{F}(t) = \exp \left\{ \frac{-\lambda(t-a)^{\alpha_1}}{(b-t)^{\alpha_2}} \right\}; \quad 0 \leq a < t < b < \infty, \alpha_1, \alpha_2, \lambda > 0. \quad (2.2)$$

The tensile strength of two groups of fused silica optical fibers has been analyzed by Phani (1987) using the above Weibull distribution function.

2.5 Truncated Weibull Distribution

A simple way to construct a new distribution is by truncating the density function, either from above or from below or both. Let G denote the two-parameter Weibull distribution function: $G(t) = 1 - \exp\{-(t/\beta)^\alpha\}$ with its corresponding density function

$g = (\alpha/\beta)(t/\beta)^{\alpha-1} \exp[-(t/\beta)^\alpha]$. Then a doubly truncated Weibull distribution can be specified by letting

$$f(t) = \frac{g(t)}{G(b) - G(a)} \quad (2.3)$$

and

$$F(t) = \frac{G(t) - G(a)}{G(b) - G(a)}, \quad (2.4)$$

respectively, for $0 \leq a < t < b < \infty$. The density function in (2.3) can be written explicitly as

$$f(t) = \frac{(\alpha/\beta)(t/\beta)^{\alpha-1} \exp[-(t/\beta)^\alpha]}{\exp\{-(a/\beta)^\alpha\} - \exp\{-(b/\beta)^\alpha\}}. \quad (2.5)$$

Two special cases are as follows:

1. $a = 0$ and $b < \infty$, a right truncated Weibull model emerges,
2. $a > 0$ and $b \rightarrow \infty$, a left truncated Weibull distribution results.

Note: The right truncated and left truncated Weibull distributions are also known as the upper truncated and lower truncated Weibull distributions, respectively.

We now restrict ourselves to discuss the upper truncated Weibull as the related properties of the lower truncated case can be obtained similarly.

2.5.1 Mean and Variance

General moment expressions were derived in McEwen and Parresol (1991). The mean is

$$\mu = \frac{\beta\Gamma(1 + 1/\alpha)}{1 - \exp[-(b/\beta)^\alpha]}$$

and the variance can be shown to be

$$\sigma^2 = \beta^2 \left\{ \Gamma(1 + 2/\alpha)(1 - \exp[-(b/\beta)^\alpha]) - \Gamma^2(1 + 1/\alpha) \right\} / (1 - \exp[-(b/\beta)^\alpha])^2.$$

2.5.2 Hazard Rate Function

The hazard rate function is given by

$$h(t) = \frac{(\alpha/\beta)(t/\beta)^{\alpha-1} \exp[-(t/\beta)^\alpha]}{G(b) - G(t)}, \quad 0 \leq t \leq b. \quad (2.6)$$

Because of the right truncation at $t = b$, it is clear from the above equation that $h(t) \rightarrow \infty$ as $t \rightarrow b$.

Like the standard Weibull case, the shape of $h(t)$ depends only on the shape parameter α . Zhang and Xie (2011) showed the following results:

Case 1: $\alpha \geq 1$.

- (i) In this case, $h'(t) > 0$ for $0 \leq t \leq b$, and hence $h(t)$ is an increasing function;
- (ii) $h(0) = 0$ if $\alpha > 0$ and $h(0) = 1/[\beta - \beta \exp[-(b/\beta)]]$, if $\beta = 1$.

Case 2: $\alpha < 1$.

- (i) In this case, $h(t) \rightarrow \infty$ for $t \rightarrow 0$ and $t \rightarrow b$;
- (ii) the hazard rate function has a bathtub shape.

In fact, McEwen and Parresol (1991) have already shown by heuristical arguments that the corresponding hazard rate function can be bathtub-shaped.

Zhang and Xie (2011) computed and presented the minimum values of $h(t)$ for different values of α , β and b .

2.5.3 Estimation of Model Parameters

Zhang and Xie (2011) presented two methods: (i) The graphical approach based on the Weibull probability plot for the truncated Weibull, and (ii) the maximum likelihood method.

2.5.4 Applications

Several applications were reviewed in Zhang and Xie (2011). The applications of the truncated Weibull distribution include:

- engineering fields,
- tree diameter and height distributions in forestry,
- fire size,
- high-cycle fatigue strength prediction, and
- seismological data analysis for earthquakes.

2.6 Inverse Weibull Distribution

The inverse Weibull model is also known as the reverse Weibull model in the literature.

Let X denote the 2-parameter Weibull model with distribution function $1 - e^{-(t/\beta)^\alpha}$. Define T by the inverse transformation:

$$T = \frac{\beta^2}{X}. \quad (2.7)$$

Then T has a distribution function given by

$$F(t) = \exp(-(\beta/t)^\alpha), \quad \alpha, \beta > 0, t \geq 0. \quad (2.8)$$

Alternatively, we may express (2.8) as

$$F(t) = \exp(-(t/\beta)^{-\alpha}), \quad \alpha, \beta > 0, t \geq 0. \quad (2.9)$$

The inverse Weibull is also known as type 2 extreme value or the Fréchet distribution. (Johnson et al. 1995, Chap. 22) whereas the distribution of the negative of the Weibull random variable is a type 3 extreme value distribution. Weibull and Fréchet (inverse Weibull) distributions are both special cases of the generalized extreme value distribution.

Erto (1989) has discussed the properties of this distribution and its potential use as a lifetime model. The maximum likelihood estimation and the least squares estimation of the parameters of the inverse Weibull distribution have been discussed by Calabria and Pulcini (1990).

2.6.1 Density Function

The density function of the inverse Weibull model is

$$f(t) = \alpha\beta^\alpha t^{-\alpha-1} e^{-(\beta/t)^\alpha}. \quad (2.10)$$

The density plots are given in Fig. 2.1.

2.6.2 Moments

The k th moment about 0 is

$$\mu'_k = \beta^k \Gamma(1 - k\alpha^{-1})$$

which does not have a finite value for $k \geq \alpha$. The reason for this is that the inverse Weibull distribution has a heavy right tail.

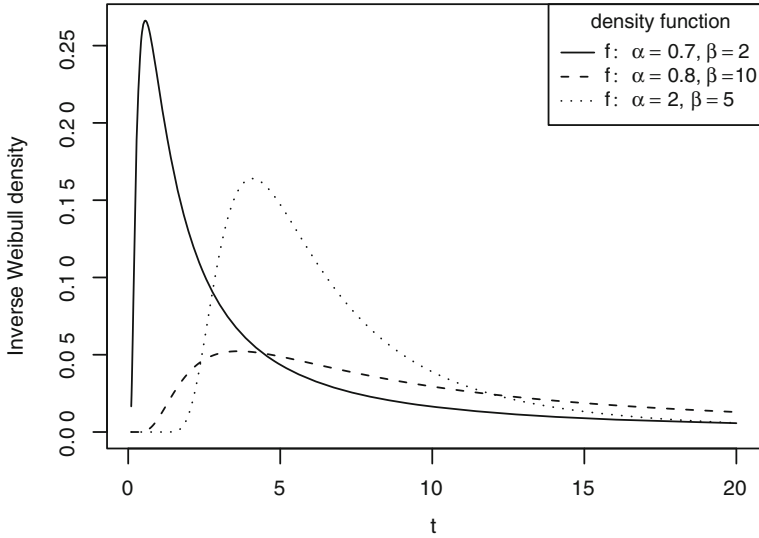


Fig. 2.1 Inverse Weibull density functions

2.6.3 Hazard Rate Function

The hazard rate function is given by

$$h(t) = \frac{\alpha \beta^\alpha t^{-\alpha-1} e^{-(\beta/t)^\alpha}}{1 - e^{-(\beta/t)^\alpha}}. \quad (2.11)$$

It can be shown that the inverse Weibull distribution generally exhibits a long right tail and its hazard rate function is similar to that of the log-normal and inverse Gaussian distributions.

Jiang et al. (2001) showed that the hazard rate function is unimodal (upside-down bathtub shaped) with the mode at $t = t_M$ given by the solution of the following equation

$$\frac{z(t_M)}{1 - e^{-z(t_M)}} = 1 + 1/\alpha \quad (2.12)$$

where $z(t) = (\beta/t)^\alpha$ and

$$\lim_{t \rightarrow 0} h(t) = \lim_{t \rightarrow \infty} h(t) = 0. \quad (2.13)$$

This is in contrast to the standard Weibull model for which the hazard rate is either decreasing (if $0 < \alpha < 1$), constant (if $\alpha = 1$) or increasing (if $\alpha > 1$).

Figure 2.2 gives the hazard rate plots of the inverse Weibull distribution.

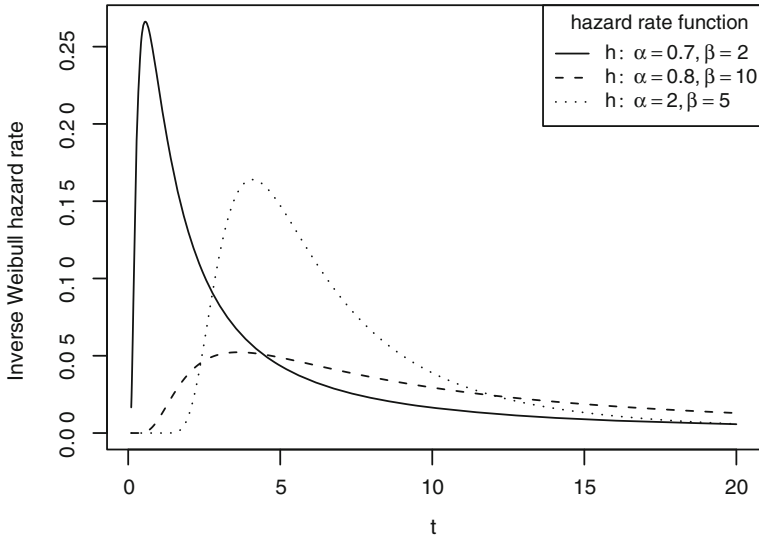


Fig. 2.2 Inverse Weibull hazard rate functions

We note with interest that the density functions and the hazard rate functions of the inverse Weibull model have similar shapes for a given pair of α and β .

2.6.4 Inverse Weibull Probability Plot

The inverse Weibull transform is given by

$$x = \log t, \quad y = -\log(-\log(F(t))). \quad (2.14)$$

The above transformation was first proposed by Drapella (1993). The plot y versus x is called the inverse Weibull probability plot (IWPP) plot. Under this transform, the inverse Weibull model as given in (2.9) also yields a straight line relationship

$$y = \alpha(x - \log \beta). \quad (2.15)$$

2.6.5 Applications

Keller et al. (1985) used the inverse Weibull distribution for reliability analysis of commercial vehicle engines.

2.6.6 Models Involving Two Inverse Weibull Distributions

This was investigated by Jiang et al. (2001), see also Chap. 3 of this monograph for a more detailed discussion.

2.6.7 The Generalized Inverse Weibull Distribution

de Gusmão et al. (2009) proposed a generalized inverse Weibull distribution by adding another parameter γ to the standard inverse Weibull:

$$\bar{F}(t) = 1 - \exp[-\gamma(\beta/t)^\alpha]. \quad (2.16)$$

When $\gamma = 1$, it clearly reduces to the inverse Weibull distribution.

Density Function

$$f(t) = \gamma\alpha\beta^\alpha t^{-(\alpha+1)} \exp[-\gamma(\beta/t)^\alpha]. \quad (2.17)$$

Moments

The k th moment about 0 is

$$\mu'_k = \gamma^\frac{k}{\alpha} \beta^k \Gamma(1 - k\alpha^{-1})$$

which does not have a finite value for $k \geq \alpha$ just as the inverse Weibull distribution.

Hazard Rate Function

The hazard rate function is given by

$$h(t) = \gamma\alpha\beta^\alpha t^{-(\alpha+1)} \exp[-\gamma(\beta/t)^\alpha] \{1 - \exp[-\gamma(\beta/t)^\alpha]\}^{-1}. \quad (2.18)$$

By differentiating the above $h(t)$ with respect to t , we can easily show that $h(t)$ is unimodal (upside-down bathtub shaped) with a maximum value at t^* , where t^* satisfies the nonlinear equation:

$$\gamma(\beta/t^*) \{1 - \exp[-\gamma(\beta/t^*)^\alpha]\} = 1 + \alpha^{-1}.$$

Estimate of Parameter

The maximum likelihood estimates of the parameters with censored data were obtained and studied in de Gusmão et al. (2009).

Further Extensions

de Gusmão et al. (2009) also considered the mixture of two generalized inverse Weibull distributions. Further, they also proposed the so called ‘The log-generalized inverse Weibull distribution’.

2.7 Reflected Weibull Distribution

Suppose X has a three-parameter Weibull distribution, then $T = -X$ has a reflected Weibull whose distribution function is

$$F(t) = \exp \left\{ - \left(\frac{\tau - t}{\beta} \right)^\alpha \right\}, \quad \alpha, \beta > 0, -\infty < t < \tau. \quad (2.19)$$

This is also known as type 3 extreme value distribution (Johnson et al. 1995, Chap. 22). The density function is given by

$$f(t) = \left(\frac{\alpha}{\beta} \right) \left(\frac{\tau - t}{\beta} \right)^{\alpha-1} \exp \left\{ - \left(\frac{\tau - t}{\beta} \right)^\alpha \right\}, \quad \alpha, \beta > 0, -\infty < t < \infty. \quad (2.20)$$

The hazard rate function is given by

$$h(t) = \left(\frac{\alpha}{\beta} \right) \left(\frac{\tau - t}{\beta} \right)^{\alpha-1} \frac{\exp \left\{ - \left(\frac{\tau - t}{\beta} \right)^\alpha \right\}}{1 - \exp \left\{ - \left(\frac{\tau - t}{\beta} \right)^\alpha \right\}}. \quad (2.21)$$

Strictly speaking, the reflected Weibull is not suitable for reliability modelling unless $\tau > 0$ and $(\tau/\beta)^\alpha \geq 9$ so that $\Pr(0 < T < \tau) \approx 1$.

The model is fitted to an observed age distribution of holders of a certain type of life insurance policy by Cohen (1973).

2.8 Log Weibull Distribution

This is an extreme value distribution derived from the logarithmic transformation of the two-parameter Weibull having distribution function as given in (1.4). The transformed variable has distribution function given by

$$F(t) = 1 - \exp \left\{ -\exp \left(\frac{t-a}{b} \right) \right\}, \quad -\infty < t < \infty, \quad (2.22)$$

where we have let $a = \log \beta$, $b = 1/\alpha$. This is also known as type 1 extreme value distribution or the Gumbel distribution. In fact, it is the most commonly referred to in discussions of extreme value distributions (Johnson et al. 1995, Chap. 22). The density function is already given in (1.7)—though in a different parametrization, i.e.,

$$f(t) = \frac{1}{b} \exp \left(\frac{t-a}{b} \right) \exp \left\{ -\exp \left(\frac{t-a}{b} \right) \right\}, \quad -\infty < t < \infty. \quad (2.23)$$

The hazard rate functions is given by

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{1}{b} \exp \left(\frac{t-a}{b} \right). \quad (2.24)$$

2.8.1 Applications

The log Weibull (Gumbel) distribution is an extreme value distribution. Thus it has been applied to many extreme value data such as flood flows, wind speeds, radioactive emissions, brittle strength of crystals, and etc. Chapter 22 Sect. 14 of Johnson et al. (1995) lists many applications of this distribution.

2.9 Stacy's Weibull Distribution

Stacy (1962) proposed a distribution which he called the ‘generalized gamma model’ with the pdf and cdf, given by

$$f(t) = \frac{\alpha \beta^{-\alpha c}}{\Gamma(c)} t^{\alpha c - 1} \exp \{ -(t/\beta)^\alpha \} \quad (2.25)$$

and

$$F(t) = \Gamma(c)^{-1} \gamma(c, (t/\beta)^\alpha), \quad (2.26)$$

respectively, for $\alpha, \beta, c > 0$, where $\gamma(\cdot, \cdot)$ denotes the incomplete gamma function defined by

$$\gamma(a, t) = \int_0^t x^{a-1} e^{-x} dx.$$

2.9.1 Special Cases

For $c = 1$, it reduces to the standard Weibull distribution. For $\alpha = 1$, it becomes the two-parameter gamma distribution. If $c \rightarrow \infty$, it becomes the lognormal distribution. For $c = 1/2$ and $\alpha = 2$, it reduces to the half-normal distribution. Further, the chi-square distribution and the Levy distribution are also included as special cases.

2.9.2 Hazard Rate Function

The hazard rate function $h(t)$ can be expressed as

$$h(t) = \frac{\alpha \beta^{-\alpha c} t^{\alpha c - 1} \exp \{-(t/\beta)^\alpha\}}{\Gamma(c) - \gamma(c, (t/\beta)^\alpha)}. \quad (2.27)$$

Glaser (1980) showed that the hazard rate function can exhibit various shapes as given below:

Case 1: $\alpha c < 1$.

- (a) If $\alpha < 1$, then $h(t)$ is decreasing.
- (b) If $\alpha > 1$, then $h(t)$ has a bathtub shape.

Case 2: $\alpha c > 1$.

- (a) If $\alpha > 1$, then $h(t)$ is increasing.
- (b) If $\alpha < 1$, then $h(t)$ has a upside-down bathtub shape.

Case 3: $\alpha c = 1$.

- (a) If $\alpha = 1$, $c = 1$, then $h(t)$ is a constant.
- (b) If $\alpha < 1$, then $h(t)$ is decreasing.
- (c) If $\alpha > 1$, then $h(t)$ is increasing.

McDonald and Richards (1987) and Pham and Almhana (1995) also considered the shape of the hazard rate function $h(t)$.

2.9.3 Estimation of Parameters

Parameter estimation for Stacy's generalized Weibull (gamma) distribution has been widely treated in the literature. The maximum likelihood method and the method of moment are the two common approaches but it has been known that there are various difficulties in implementing these methods to this distribution. Other methods such

as the heuristics and graphical methods were also proposed. For a literature review of these methods see Gomes et al. (2008).

Gomes et al. (2008) proposed a new method of estimation through the power transformation $T = X^c$ where X has Stacy's Weibull distribution as given in (2.25) above. Then T has a gamma distribution with shape parameter c and scale parameter β^α . Based on this they constructed a heuristic method called algorithm I.E.R.V. which involves looping around the shape parameter α .

2.9.4 Applications

Stacy's Weibull distribution has applications in many fields such as health costs, civil engineering (flood frequency, e.g. Pham and Almhana 1995), economics (income distribution, e.g., Klieber and Kotz 2003) and others.

2.10 Exponentiated Weibull Distribution

Mudholkar and Srivastava (1993) proposed a modification to the standard Weibull model through the introduction of an additional parameter ν ($0 < \nu < \infty$). The distribution function is

$$F(t) = [G(t)]^\nu = [1 - \exp\{-(t/\beta)^\alpha\}]^\nu, \quad \alpha, \beta > 0, t \geq 0, \quad (2.28)$$

where $G(t)$ is the standard two-parameter Weibull distribution. The support for F is $[0, \infty)$.

When $\nu = 1$, the model reduces to the standard two-parameter Weibull model. When ν is an integer, the model is a special case of the multiplicative model to be discussed in Sect. 3.3. The distribution has been studied extensively by Mudholkar and Hutson (1996), Jiang and Murthy (1999) and more recently Nassar and Eissa (2003).

2.10.1 Density Function

The density function is given by

$$f(t) = \nu\{G(t)\}^{\nu-1}g(t), \quad (2.29)$$

where $g(t)$ is the density function of the standard two-parameter Weibull distribution. So

$$f(t) = \frac{\alpha\nu}{\beta^\alpha} t^{\alpha-1} e^{-(t/\beta)^\alpha} \left(1 - e^{-(t/\beta)^\alpha}\right)^{\nu-1}. \quad (2.30)$$

Two special cases worth noting:

- (i) For $\alpha = 1$, the probability density function is

$$f(t) = \frac{\nu}{\beta} e^{-t/\beta} \left(1 - e^{-t/\beta}\right)^{\nu-1} \quad (2.31)$$

which is the exponentiated exponential distribution studied by Gupta et al. (1998).

- (ii) For $\alpha = 2$, we obtain the probability density function of the two-parameter Burr type X distribution:

$$f(t) = \frac{2\nu}{\beta^2} e^{-(t/\beta)^2} \left(1 - e^{-(t/\beta)^2}\right)^{\nu-1}. \quad (2.32)$$

We also note that from (2.30),

$$f(0) = \begin{cases} 0 & \text{if } \alpha\nu > 1, \\ \beta^{-\alpha} & \text{if } \alpha\nu = 1, \\ \infty & \text{if } \alpha\nu < 1. \end{cases} \quad (2.33)$$

The value of $f(0)$ will determine the value of the hazard rate function at $t = 0$ which in turn has an impact on the shape of the mean residual life MRL $\mu(t)$ of a lifetime distribution, see for example Theorem 4.2 of Lai and Xie (2006).

The density function is plotted for different parameter values in Fig. 2.3.

2.10.2 Hazard Rate Function

The hazard rate function was given in Mudholkar et al. (1995) and Mudholkar and Hutson (1996):

$$h(t) = \frac{\nu(\alpha/\beta)(t/\beta)^{\alpha-1} [1 - \exp(-(t/\beta)^\alpha)]^{\nu-1} \exp(-(t/\beta)^\alpha)}{1 - [1 - \exp(-(t/\beta)^\alpha)]^\nu}. \quad (2.34)$$

For small t , Jiang and Murthy (1999) have shown that

$$h(t) \approx \left(\frac{\alpha\nu}{\beta}\right) \left(\frac{t}{\beta}\right)^{\alpha\nu-1}. \quad (2.35)$$

In other words, for small t , $h(t)$ can be approximated by the hazard rate of a two-parameter Weibull distribution with shape parameter $(\alpha\nu)$ and scale parameter β .

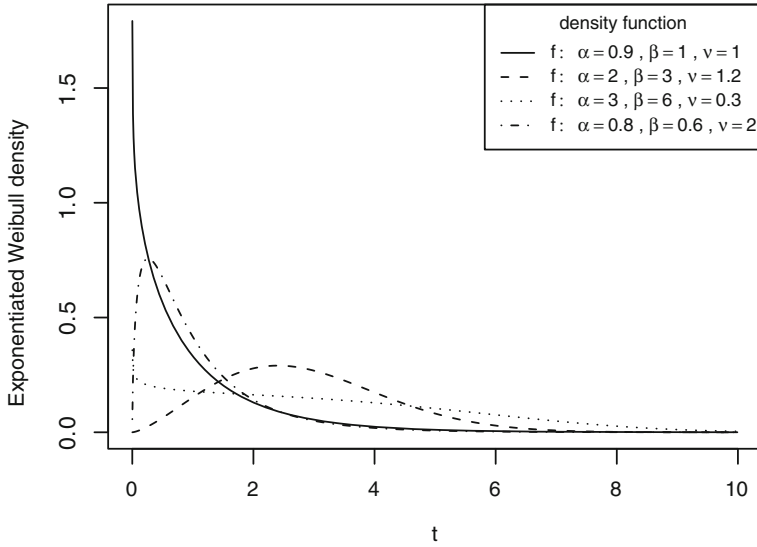


Fig. 2.3 Exponentiated Weibull density functions

For large t , i.e., $t \rightarrow \infty$, the term $\frac{\exp(-(t/\beta)^\alpha)}{1 - [1 - \exp(-(t/\beta)^\alpha)]^\nu}$ in (2.34) converges to $1/\nu$ by applying the L'Hospital's rule. It is now clear that (2.34) converges to

$$h(t) \approx \left(\frac{\alpha}{\beta}\right) \left(\frac{t}{\beta}\right)^{\alpha-1} \quad (2.36)$$

for large t .

In other words, for large t , $h(t)$ can be approximated by the hazard rate of a two-parameter Weibull distribution with shape parameter α and scale parameter β .

Mudholkar et al. (1995), Mudholkar and Hutson (1996) and Jiang and Murthy (1999) have all considered the shapes of $h(t)$ and its characterization in the parameter space. The shape of $h(t)$ does not depend on β and varies with α and ν . The characterization on the (α, ν) - plane is as follows:

- $\alpha \leq 1$ and $\alpha\nu \leq 1$: $h(t)$ monotonically decreasing.
- $\alpha \geq 1$ and $\alpha\nu \geq 1$: $h(t)$ monotonically increasing.
- $\alpha < 1$ and $\alpha\nu > 1$: $h(t)$ has an upside-down bathtub shape.
- $\alpha > 1$ and $\alpha\nu < 1$: $h(t)$ has a bathtub shape.

Figure 2.4 contains four plots of the the hazard rate of the exponentiated Weibull distribution.

The exponentiated Weibull was first introduced by Mudholkar and Srivastava (1993) for modeling bathtub shaped hazard rate data.

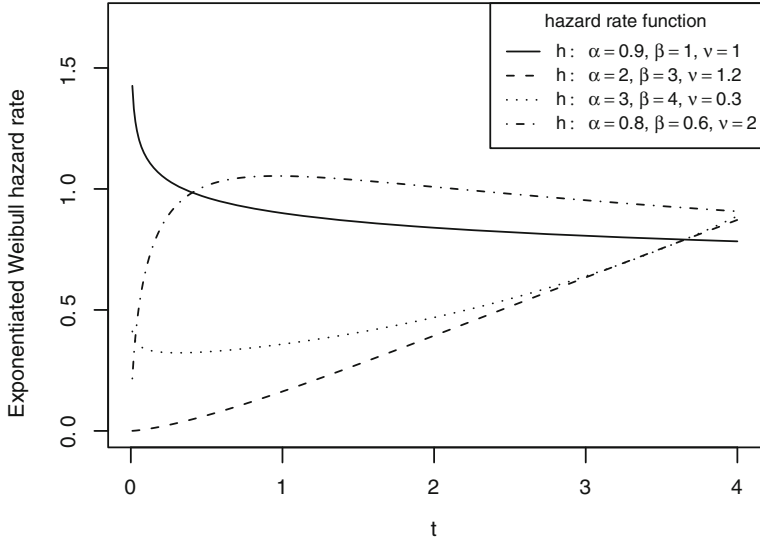


Fig. 2.4 Exponentiated Weibull hazard rate functions

2.10.3 Mean Residual Life Function

The mean residual life of the exponentiated Weibull has been given by Nassar and Eissa (2003) for positive integer ν as:

$$\begin{aligned} \mu(t) = \nu \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} / [(j+1)\bar{F}(t)] \left\{ \beta \Gamma\left(\frac{1}{\beta} + 1\right) [(j+1)^{-1/\alpha} \right. \\ \left. - (j+1)^{-1} \Gamma_{(j+1)\tau}\left(\frac{1}{\beta}\right) / \Gamma\left(\frac{1}{\beta}\right)] + t e^{-(j+1)\tau} \right\} - t, \end{aligned} \quad (2.37)$$

where $\tau = (t/\beta)^\alpha$. Thus, the mean of the distribution for $\nu \in \mathcal{N}^+$ is

$$\mu = \mu(0) = \nu \Gamma\left(\frac{1}{\alpha} + 1\right) \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} / [(j+1)^{-1/\alpha}]. \quad (2.38)$$

From the general theory that connecting bathtub (inverted bathtub) shaped hazard rate distributions with increasing (decreasing) then decreasing (increasing) mean residual life distributions (see Sect. 4.5 of Lai and Xie (2006) for details), Nassar and Eissa (2003) have established the shape of $\mu(t)$ as follows:

Let $\mu(t)$ be given as (2.38), then

- (i) $\mu = \alpha$ if and only if $\alpha = \nu = 1$.

- (ii) $\mu(t)$ is decreasing (increasing) if $\alpha \geq 1$ and $\alpha\nu \geq 1$ ($\alpha \leq 1$ and $\alpha\nu \leq 1$).
- (iii) $\mu(t)$ is decreasing and then increasing with a change point t_m if $\alpha < 1$ and $\alpha\nu > 1$.
- (iv) $\mu(t)$ is increasing and then decreasing with a change point t_m if $\alpha > 1$ and $\alpha\nu < 1$.

Nadarajah and Gupta (2005) obtained the k th moment about the origin as

$$\mu'_k = \nu\beta^k \Gamma\left(\frac{k}{\alpha}\right) \sum_{i=0}^{\infty} \frac{(1-\nu)_i}{i!(i+1)^{(k+\alpha)/\alpha}}, \quad k > -\alpha, \quad (2.39)$$

where $(1-\nu)_i = (a)_i = a(a+1) \dots a(a+i-1)$.

If ν is an integer, then

$$\mu'_k = \nu\beta^k \Gamma\left(\frac{k}{\alpha}\right) \sum_{i=0}^{\nu-1} \frac{(1-\nu)_i}{i!(i+1)^{(k+\alpha)/\alpha}}, \quad k > -\alpha, \quad (2.40)$$

which was established by Nassar and Eissa (2003). Equation (2.40) follows from (2.39) because $(1-\nu)_i = 0$ for all $i \geq \nu$.

Xie et al. (2004) have studied the change points of $h(t)$ and $\mu(t)$ in terms of individual model parameters. The difference D of two change points is tabulated for various combinations of the parameters.

2.10.4 Graphical Study

Jiang and Murthy (1999) discussed the shape of the WPP for the exponentiated Weibull family and gave a parametric characterization of its probability density and hazard rate functions. Their paper also deals with the issues relating to modeling a given data set and the problem of estimating the model parameters.

2.10.5 Applications

The model has been applied

- to model the bathtub failure rate behavior of the data in Aarset (1987) (Mudholkar and Srivastava 1993)
- to reanalyse the bus motor failure data (Mudholkar et al. 1995), and
- to analyse a flood data (Mudholkar and Hutson 1996).

2.11 Beta-Weibull Distribution

The distribution is a generalization of the exponentiated Weibull distribution discussed in the preceding section. Let $G(t)$ denote the standard two-parameter Weibull distribution function given as $G(t) = 1 - \exp\{-(t/\beta)^\alpha\}$.

The beta-Weibull was first proposed by Famoye et al. (2005) by coupling the beta density and the Weibull distribution function such that the distribution function of the new distribution is

$$F(t) = \frac{1}{B(a, b)} \int_0^{G(t)} u^{a-1} (1-u)^{b-1} du; \quad 0 < a, b < \infty, \quad (2.41)$$

where $B(a, b)$ denotes the beta function defined by $\Gamma(a)\Gamma(b)/\Gamma(a+b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$. The same distribution was also proposed by Wahed et al. (2009). Clearly, if $b = 1$, (2.41) reduces to the exponentiated Weibull distribution. In general, the distribution function cannot be expressed explicitly.

The basic properties are given in Famoye et al. (2005).

2.11.1 Density and Hazard Rate Function

The density function is rather simple:

$$f(t) = \frac{1}{B(a, b)} \frac{\alpha}{\beta} \left(\frac{t}{\beta}\right)^{\alpha-1} \left[1 - e^{-(t/\beta)^\alpha}\right]^{a-1} e^{-b(t/\beta)^\alpha}. \quad (2.42)$$

The hazard rate function can be obtained from the preceding two equations as

$$h(t) = \frac{f(t)}{\bar{F}(t)}.$$

It cannot be expressed explicitly.

Lee et al. (2007) examined the shapes of the hazard rate function and we now summarize them below:

- (a) $h(t)$ is a constant b/β if $a = \alpha = 1$,
- (b) $h(t)$ is a decreasing function if $a\alpha \leq 1$ and $\alpha \leq 1$,
- (c) $h(t)$ is an increasing function if $a\alpha \geq 1$ and $\alpha \geq 1$,
- (d) $h(t)$ has a bathtub shape if $a\alpha < 1$ and $\alpha > 1$, and
- (e) $h(t)$ has an upside-down bathtub shape if $a\alpha > 1$ and $\alpha < 1$.

2.11.2 Comparison with Exponentiated Weibull Distribution

Lee et al. (2007) carried out a simulation study to compare the beta-Weibull model with the exponentiated Weibull model. They concluded that the bias of the MLE from the beta-Weibull distribution is smaller than the bias of the exponentiated Weibull model with comparable standard errors when $b < 1$. The bias and the standard errors are, in general, smaller for the exponentiated Weibull distribution when $b \geq 1$. (Recall, the beta-Weibull becomes the exponentiated Weibull when $b = 1$.) For the three data sets analyzed in Lee et al. (2007), the estimates for the parameter b are less than 1 which suggests the usefulness of the beta-Weibull for describing survival data sets.

2.11.3 Applications of Beta-Weibull Model

The model is applied by Lee et al. (2007) to two censored data sets of bus-motor failures and a censored data set of Arm A (Efron 1988) of the head-and-neck cancer clinical trial.

Wahed et al. (2009) also fitted the model to a breast cancer data set and they concluded that the beta-Weibull family is a reasonable candidate for modeling survival data.

Log Beta-Weibull Model

Let $Y = \log T$ where T has the beta-Weibull distribution given above. Then Y has a log beta-Weibull distribution studied by Ortega et al. (2011). Based on Y , they also proposed and studied the log-beta Weibull regression model which they considered to be very suitable for modeling censored and uncensored data.

2.12 Extended Weibull Model of Marshall and Olkin

Marshall and Olkin (1997) proposed a modification to the standard Weibull model through the introduction of an additional parameter ν ($0 < \nu < \infty$). The model is given through its survival function function

$$\bar{F}(t) = \frac{\nu \bar{G}(t)}{1 - (1 - \nu) \bar{G}(t)} = \frac{\nu \bar{G}(t)}{G(t) + \nu \bar{G}(t)} \quad (2.43)$$

where $G(t)$ is the distribution function of the two-parameter Weibull and $\bar{F}(t) = 1 - F(t)$.

The parameter ν is called a tilt parameter in Marshall and Marshall and Olkin (2007).

The case when G is an exponential distribution function has been considered as the exponential-geometric distribution studied by Adamidis and Loukas (1998) and Marshall and Olkin (1997).

When $\nu = 1$, $\bar{F}(t) = \bar{G}(t)$ so the model reduces to the standard Weibull model.

2.12.1 Extended Weibull

Using (1.4) as G in (2.43), we then have the distribution function given by

$$F(t) = 1 - \frac{\nu \exp[-(t/\beta)^\alpha]}{1 - (1 - \nu) \exp[-(t/\beta)^\alpha]}. \quad (2.44)$$

Marshall and Olkin (1997) called this the extended Weibull distribution. The mean and variance of the distribution cannot be given in a closed form, but they can be obtained numerically. The model may be considered as a competitor to the three-parameter Weibull distribution defined in (1.19).

Ghitany et al. (2005) showed that this distribution can be obtained by compounding the Weibull extension model of Xie et al. (2002) with the exponential distribution. That is, the extended Weibull is the result of mixing the Weibull extension model of Xie et al. with the exponential density. Zhang et al. (2007) also investigated some reliability properties of this model.

The resulting density function associated with (2.44) is given by

$$f(t) = \frac{(\alpha\nu/\beta)(t/\beta)^{\alpha-1} \exp[-(t/\beta)^\alpha]}{[1 - (1 - \nu) \exp[-(t/\beta)^\alpha]]^2} \quad (2.45)$$

The plots of the density function for different parameter values are given in Fig. 2.5.

2.12.2 Shapes of the Hazard Rate Function

The hazard rate function that corresponds to (2.45) is

$$h(t) = \frac{(\alpha/\beta)(t/\beta)^{\alpha-1}}{1 - (1 - \nu) \exp[-(t/\beta)^\alpha]}. \quad (2.46)$$

Marshall and Olkin (1997) carried out a partial study on the shape of the hazard rate. They found that $h(t)$ is increasing when $\nu \geq 1$, $\alpha \geq 1$ and decreasing when $\nu \leq 1$, $\alpha \leq 1$. If $\alpha > 1$, then the hazard rate function is initially increasing and

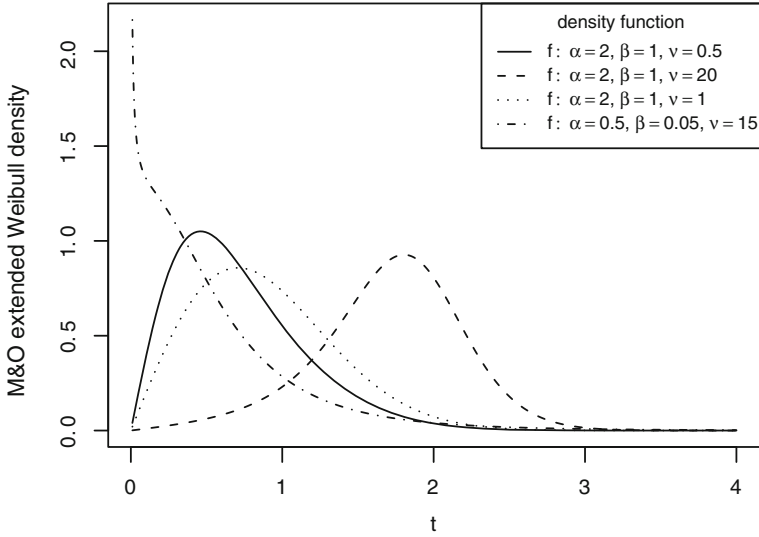


Fig. 2.5 M&O extended Weibull density functions

eventually increasing, but there may be an interval where it is decreasing. Similarly, when $\alpha < 1$, the hazard rate function is initially decreasing and eventually decreasing, but there may be an interval where it is increasing.

It is shown in Marshall and Olkin (1997) that $h(t)$ has modified bathtub shape (MBT, or N shape) for $\beta = 1$, $\alpha = 2$, $\nu = 0.05$ or $\nu = 0.1$.

Gupta et al. (2010) gave a complete study concerning the shapes of the hazard rate and their results were given in their Theorems 4–6 which we will summarize them below. In order to present these results, the shape parameter α is re-parameterized through the transformation $\psi = \frac{\alpha-1}{\alpha} e^{1/\alpha}$.

Case (i): $\nu > 0$, $\alpha > 1$.

1. $h(t)$ is initially increasing.
2. If $\nu \geq 1 - \psi$, then $h(t)$ is increasing.
3. If $\nu > 1 - \psi$, then $h(t)$ is (strictly) increasing and followed by a bathtub shape (i.e., it has an N shape). Such a shape is also called a modified bathtub shape.

Case (ii): $\nu > 0$, $0 < \alpha < 1$.

1. $h(t)$ is initially decreasing.
2. If $\nu \leq 1 - \psi$, then $h(t)$ is decreasing.
3. If $\nu < 1 - \psi$, then $h(t)$ is initially (strictly) decreasing and followed by an upside-down bathtub shape (i.e., it has a reflected N shape).

Case (iii): $\alpha = 1$.

1. If $0 < \nu < 1$, then $h(t)$ is decreasing.
2. If $\alpha > 1$, then $h(t)$ is increasing.

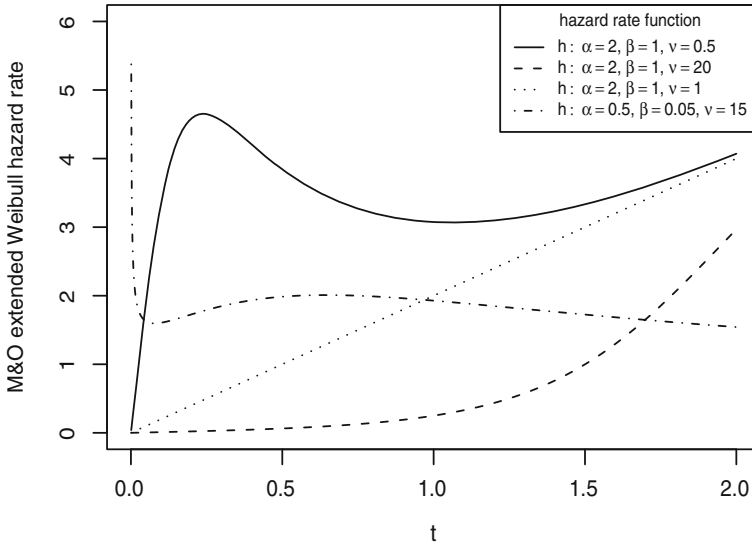


Fig. 2.6 M&O extended Weibull hazard rate functions

Of course, Case (iii) has already been established by Marshall and Olkin (1997). We also note that for $\nu = \alpha = 1$, the model reduces to the exponential distribution and thus $h(t)$ is a constant.

Figure 2.6 gives the plots of $h(t)$ from the extended Weibull model for various combination of parameter values.

2.12.3 Turning Points of $h(t)$

For non-monotonic types, the estimates of the turning points of $h(t)$ and their confidence intervals were provided by Gupta et al. (2010).

2.12.4 Mean Residual Life

In view of the complexity of the mean residual life function $\mu(t)$ of the Weibull distribution, we anticipate that the mean residual life function of the extended Weibull is worse in terms of complexity. Lai et al. (2004) have provided several plots of MRL for different combinations of parameter values from this distribution.

2.12.5 Application

Zhang et al. (2007) fitted the model to the failure times of a sample of devices from a field-tracking study of a large system.

2.13 The Weibull-Geometric Distribution

The Weibull-geometric distribution, abbreviated as WG, was proposed and studied by Barreto-Souza et al. (2010).

Let $\{X_i\}$ be independent and identically distributed Weibull random variables having shape and scale parameters α and β , respectively.

Consider $T = \min\{X_1, X_2, \dots, X_Z\}$ where Z is discrete random variable having a geometric distribution with probability function:

$$p(z) = (1 - p)p^{z-1}; \quad 0 < p < 1, z = 1, 2, \dots \quad (2.47)$$

Then T has the Weibull-geometric distribution with density function

$$f(t) = \alpha\beta^{-\alpha}(1 - p)t^{\alpha-1}e^{-(t/\beta)^\alpha} \left\{1 - pe^{-(t/\beta)^\alpha}\right\}^{-2}; \quad \alpha, \beta > 0. \quad (2.48)$$

The above model may arise from a situation where failure (of a device for example) occurs due to the presence of an unknown number, say Z , of initial defects of the same kind. The random variables X_i 's represent their lifetimes (with constant, increasing or decreasing hazard rate) and each defect can be detected only after causing failure, in which case it is repaired perfectly. Thus, the distributional assumptions given earlier lead to the WG distribution for modeling the time of the first failure.

In the derivation of the WG distribution, it is required that $0 < p < 1$. However, for negative value of p , Eq. (2.48) is still a proper density function. Thus we can extend the definition of the WG distribution in (2.48) for any $p < 1$.

Figure 2.7 gives the density plots with $\alpha = 5$ and $\beta = 1.111$ for different values of p .

2.13.1 Distribution Function

The cdf of the Weibull-geometric model has a simple form:

$$F(t) = \frac{1 - e^{-(t/\beta)^\alpha}}{1 - pe^{-(t/\beta)^\alpha}}, \quad (2.49)$$

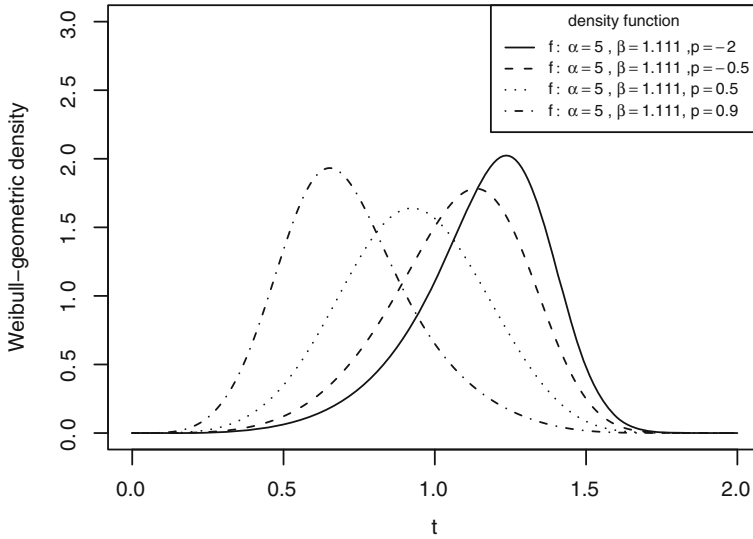


Fig. 2.7 Weibull-geometric density functions

and the corresponding survival function is

$$\bar{F}(t) = \frac{(1-p)e^{-(t/\beta)^\alpha}}{1-pe^{-(t/\beta)^\alpha}}. \quad (2.50)$$

2.13.2 Hazard Rate Function

The hazard rate function of the Weibull-geometric distribution is

$$h(t) = \alpha\beta^{-\alpha}t^{\alpha-1} \left\{1 - pe^{-(t/\beta)^\alpha}\right\}^{-1}. \quad (2.51)$$

The above function is decreasing for $0 < \alpha < 1$ and $p < 1$. However, for other parameter values, it can take different shapes. We use two figures to illustrate some of the possible shapes of the hazard function for selected values of α and β and $p = 0.2, 0.5, 0, 0.5, 0.9$.

Figure 2.8 gives the plots with $\alpha = 1.5$ and $\beta = 2$.

Figure 2.9 gives the plot for $\alpha = 5$ and $\beta = 1.111$.

These preceding two plots show that the hazard rate function of the WG distribution is much more flexible than some other generalized Weibull distributions.

We note that for $\alpha = 1.5, \beta = 2$ and $p = 0.9$, the hazard rate function is first increasing and then followed by a bathtub curve so it has a modified bathtub shape.

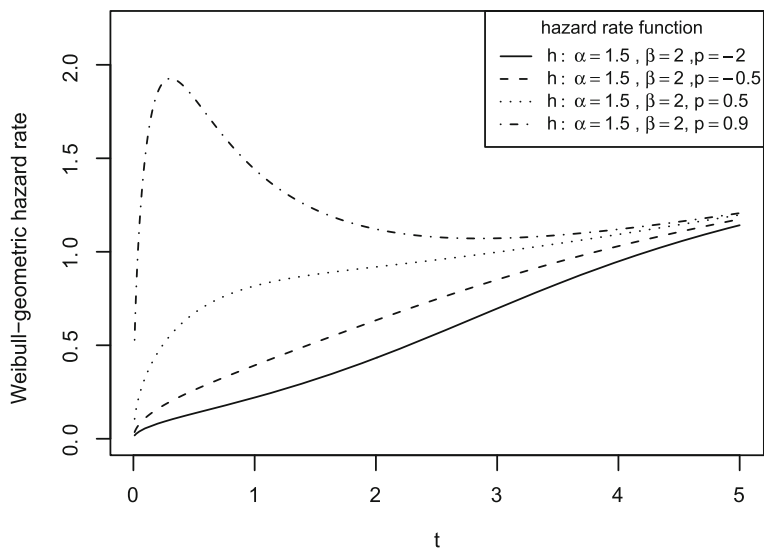


Fig. 2.8 Weibull-geometric hazard rate functions—I

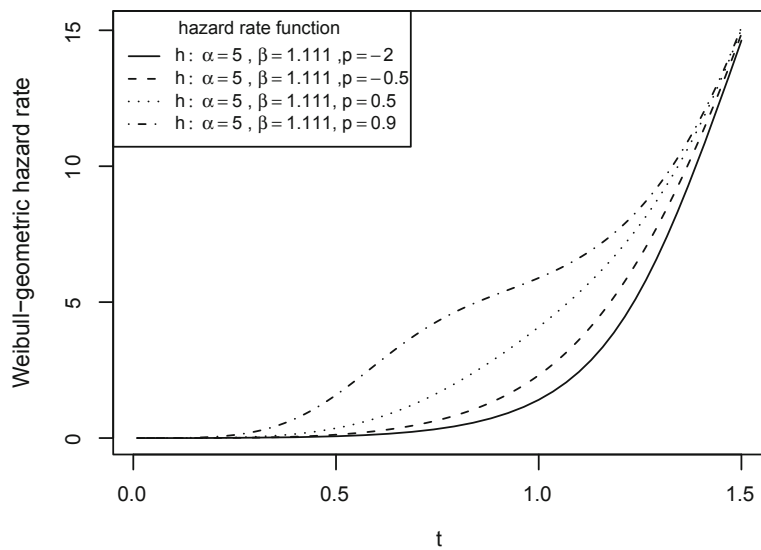


Fig. 2.9 Weibull-geometric hazard rate functions—II

2.13.3 Quantile Function and Moments

The quantile function is

$$Q(u) = \beta \left\{ \log \left(\frac{1 - pu}{1 - u} \right) \right\}^{1/\alpha} \quad (2.52)$$

and the k th moment about the origin 0 is

$$\mu'_k = (1 - p)\beta^k \Gamma(k/\alpha + 1) \Psi(p, k/\alpha, 1) \quad (2.53)$$

where $\Psi(z, s, a) = \{\Gamma(s)\}^{-1} \int_0^\infty t^{s-1} e^{-at} (1 - ze^t)^{-1} dt$; $z < 1, a, s > 0$ is known as the Lerch's transcendent function which can be computed via Mathematica or Maple.

2.13.4 Estimation of Parameters

Maximum likelihood estimates and other inferential properties were derived by Barreto-Souza et al. (2010).

2.13.5 Applications

Barreto-Souza et al. (2010) fitted the Weibull-geometric, extended exponential-geometric (EEG), and Weibull models to a real data set given in Meeker and Escobar (1998, p. 149). The data present the fatigue life (rounded to the nearest thousand cycles) for 67 specimens of Alloy T7987 that failed before having accumulated 300 thousand cycles of testing:

94, 118, 139, 159, 171, 189, 227, 96, 121, 140, 159, 172, 190, 256, 99, 121, 141, 159, 173, 196, 257, 99, 123, 141, 159, 176, 197, 269, 104, 129, 143, 162, 177, 203, 271, 108, 131, 144, 168, 180, 205, 274, 112, 133, 149, 168, 180, 211, 291, 114, 135, 149, 169, 184, 213, 117, 136, 152, 170, 187, 224, 117, 139, 153, 170, 188, 226.

It was found that the WG model gives a better fit than its competitors.

2.14 Weibull-Poisson Distribution

Hemmati and Khorram (2011) proposed a three-parameter ageing distribution using the same construction technique as in the preceding section.

Again, let X_i denote the standard Weibull variable and define $T = \min\{X_1, X_2, \dots, X_Z\}$ where Z denotes the zero truncated Poisson random variable having probability function given by

$$p(z) = e^{-\lambda} \lambda^z (1 - e^{-\lambda})^{-1} / z!; \quad \lambda > 0, z = 1, 2, \dots \quad (2.54)$$

It is easy to show that the resulting density function is given by

$$f(t) = \frac{\lambda \alpha}{\beta(1 - e^{-\lambda})} (t/\beta)^{\alpha-1} \exp\{-\lambda - (t/\beta)^\alpha + \lambda e^{-(t/\beta)^\alpha}\}; \quad \lambda, \alpha, \beta > 0. \quad (2.55)$$

The survival function is reasonably straightforward and it is given by

$$\bar{F}(t) = \left(1 - \exp\left\{\lambda e^{-(t/\beta)^\alpha}\right\}\right) / (1 - e^{-\lambda}). \quad (2.56)$$

It is clear from either (2.55) or (2.56) that the Weibull-Poisson distribution reduces to the Weibull distribution as $\lambda \rightarrow 0$.

2.14.1 Hazard Rate Function

The hazard rate function of the Weibull-Poisson model is given by

$$h(t) = \frac{\alpha \lambda (1 - e^{-\lambda}) (t/\beta)^{\alpha-1} \exp\{-\lambda - (t/\beta)^\alpha + \lambda e^{-(t/\beta)^\alpha}\}}{\beta (1 - e^{-\lambda}) (1 - \exp\{\lambda e^{-(t/\beta)^\alpha}\})}. \quad (2.57)$$

Hemmati and Khorram (2011) showed that $h(t)$ is increasing, decreasing or has a modified bathtub shape (N -shape, i.e., $h(t)$ strictly increasing then followed by a bathtub shape). Not that $h(t)$ is unable to achieve a bathtub shape.

2.15 Modified Weibull Distribution

Lai et al. (2003) proposed a three-parameter generalized Weibull model which they called the modified Weibull distribution. The distribution function is given by

$$F(t) = 1 - \exp(-at^\alpha e^{\lambda t}), \quad t \geq 0, \quad (2.58)$$

where the parameters $\lambda > 0$, $\alpha > 0$ and $a > 0$. For $\lambda = 0$, (2.58) reduces to a Weibull distribution. This simple generalization of the Weibull model is able to exhibit a bathtub shaped hazard rate function. We now study this model in some detail.

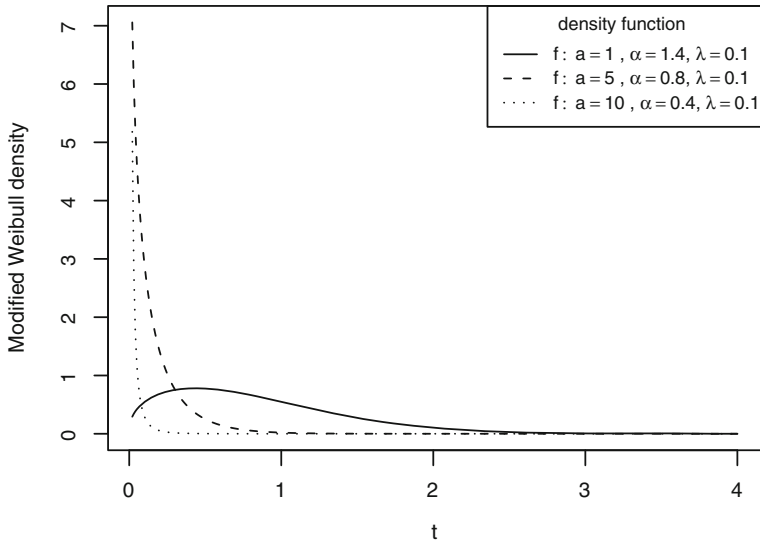


Fig. 2.10 Modified Weibull density functions

2.15.1 Density Function

The density function of the modified Weibull is given by

$$f(t) = a(\alpha + \lambda t)t^{\alpha-1}e^{\lambda t}\exp(-at^{\alpha}e^{\lambda t}). \quad (2.59)$$

Some density plots from this model are given in Fig. 2.10.

2.15.2 Moments

The moments of the distribution can be found in Nadarajah (2005).

2.15.3 Hazard Rate Function

The hazard rate function of this generalized Weibull model also has a simple form:

$$h(t) = a(\alpha + \lambda t)t^{\alpha-1}e^{\lambda t}. \quad (2.60)$$

Clearly the shape of $h(t)$ depends only on α because of through the term $t^{\alpha-1}$ and the remaining two parameters have no direct effect on the shapes.

Case (i): $\alpha \geq 1$.

1. $h(t)$ is increasing in t , implying an increasing hazard rate function, thus F is IFR.
2. $h(0) = 0$ if $\alpha > 1$ and $h(0) = a$ if $\alpha = 1$.
3. $h(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Case (ii): For $0 < \alpha < 1$.

1. $h(t)$ initially decreases and then increases, implying it has a bathtub shape.
2. $h(t) \rightarrow \infty$ as $t \rightarrow 0$ and $h(t) \rightarrow \infty$ as $t \rightarrow \infty$.
3. The change point t^* , the turning point of the hazard rate function, is

$$t^* = \frac{\sqrt{\alpha} - \alpha}{\lambda}. \quad (2.61)$$

The interesting feature of this hazard curve is that t^* increases as λ decreases. The limiting case when $\lambda = 0$ reduces to the standard Weibull distribution.

Figure 2.11 gives the hazard plots for the modified Weibull distribution.

Useful Periods

The ‘useful period’ of this lifetime distribution when its $h(t)$ has a bathtub shape is studied in detail by Bebbington et al. (2006).

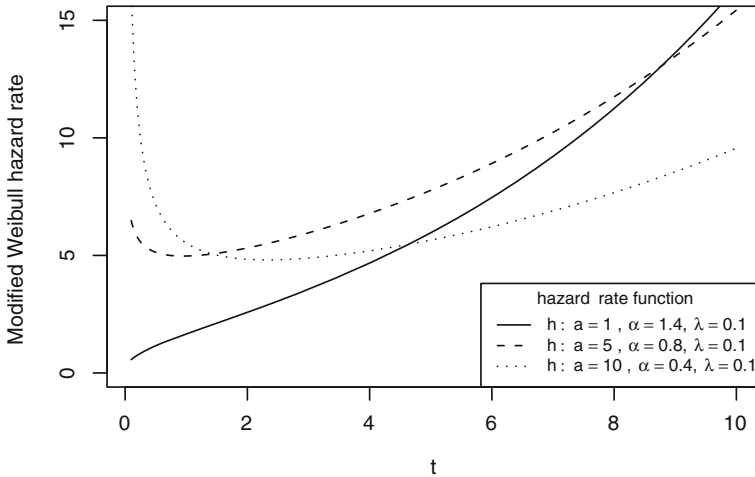


Fig. 2.11 Modified Weibull hazard rate functions

2.15.4 Mean Residual Life

In common with many other generalized Weibull distributions, the mean of this distribution does not have a closed form. Plots of the mean residual life function $\mu(t)$ can be found in Lai et al. (2004) or Xie et al. (2004) for some selected parameter values. The latter paper also computed the distance between the change points of $h(t)$ and $\mu(t)$.

2.15.5 Estimation of Parameters

A simple method for estimating the parameters of the modified Weibull model through a WPP was given in Lai et al. (2003). Bebbington et al. (2008) have suggested an empirical estimator for the turning point t^* and the theory was illustrated by means of real data set. A simulation study was conducted to assess the performance of the estimators in practice.

2.15.6 Estimates of Parameters for Progressively Type-II Censored Samples

Progressively type-II censoring is often used in life testing. Consider an experiment in which n units are placed on a life test. At the time of the first failure, k_1 units are randomly removed from the remaining $n - 1$ units. At the second failure, k_2 units from the remaining $n - 2 - k_1$ units are randomly removed. The life test continues until the m th failure at which time, all the remaining $k_m = n - m - k_1 - k_2 - \dots - k_{m-1}$ units are removed.

Ng (2005) studied the estimation of parameters based on a progressively type-II censored sample from the modified Weibull model specified by (2.58). He first obtained the maximum likelihood estimates of the model parameters. The estimators based on a least-squares fit of a multiple linear regression on a Weibull probability paper plot were also obtained and compared with the MLE via Monte Carlo simulations.

Recently, Jiang et al. (2010) also studied maximum likelihood estimation of the model parameters of the modified Weibull distribution with progressively type-2 censored samples. The property of the log-likelihood function was investigated by introducing a simple transformation to decrease the dimension of the parameter vector. Existence and uniqueness of the MLEs of the model parameters were proved. Several examples were presented to illustrate the uniqueness and existence property of the MLEs.

2.15.7 Application

Failure times data from Aarset (1987) was fitted by this modified Weibull model.

2.15.8 Competing Risk Models Involving Two Modified Weibull Distributions

Alwasel (2009) proposed a competing risk model involving two modified Weibull distributions. The maximum likelihood estimates of the six parameters (since each of modified Weibull models has three parameters) were also derived although not in a closed form. So a numerical method is required for computing the MLEs of these parameters.

2.15.9 Beta Modified Weibull Distribution

Silva et al. (2010) proposed a new distribution called the beta modified Weibull based on the following construction scheme:

$$F(t) = \frac{1}{B(a, b)} \int_0^{G(t)} u^{a-1} (1-u)^{b-1} du \quad (2.62)$$

where $G(t)$ denotes the cdf of the modified Weibull distribution. The density function is given by

$$f(t) = \frac{1}{B(a, b)} G(t)^{a-1} (1-G)^{b-1} g(u) du \quad (2.63)$$

where $g(t) = G'(t)$.

The model contains several well known sub-models including the beta-Weibull studied by Lee et al. (2007).

The hazard function $h(t)$ of the beta modified Weibull can be bathtub shaped, increasing, decreasing or inverted bathtub shaped (UBT) depending on the values of the parameters.

2.15.10 Generalized Modified Weibull Family

The survival function of the distribution proposed and studied by Carrasco et al. (2008) is

$$\bar{F}(t) = 1 - \left(1 - \exp \left\{ -at^\alpha e^{\lambda t} \right\} \right)^\beta, \quad \lambda \geq 0, \alpha, \beta, a > 0; t \geq 0. \quad (2.64)$$

Clearly, the distribution is a simple extension of the modified Weibull distribution of Lai et al. (2003) since (2.64) reduces to (2.58) when $\beta = 1$. In fact, it includes several other distributions such as type 1 extreme value, the exponentiated Weibull of Mudholkar and Srivastava (1993) as specified in (2.28) above, and others. An important feature of this lifetime (ageing) distribution is its considerable flexibility in providing hazard rates of various shapes.

2.16 Generalized Weibull Family

The so called generalized Weibull model was derived by Mudholkar and Kollia (1994) and Mudholkar and Hutson (1996) from the basic two-parameter Weibull distribution by appending an additional parameter. The quantile function for the new model is given by

$$\begin{aligned} Q(u) &= \beta \left[\theta 1 - (1 - u)^{1/\theta} \right]^{1/\alpha}, \quad \theta < \infty \\ &= \beta [-\log(1 - u)]^{1/\alpha}, \quad \theta \rightarrow \infty \end{aligned} \quad (2.65)$$

where the new parameter θ is unconstrained so that $-\infty < \theta < \infty$. It follows immediately that the survival function is

$$\bar{F}(t) = \left[1 - \left(\frac{t}{\beta} \right)^\alpha / \theta \right]^\theta; \quad \alpha, \beta > 0, \quad (2.66)$$

where the support is for $F(t)$ is $(0, \infty)$ for $\theta \leq 0$ and $(0, \beta\theta^{1/\alpha})$ for $\theta > 0$. So the support of the distribution can be a finite interval or unbounded.

Indeed the model reduces to the basic two-parameter Weibull when $\theta \rightarrow \infty$. Nikulin and Haghighi (2006) observed that the generalized Weibull distribution turns into the exponential if $\alpha = 1$ and $\theta \rightarrow \infty$, and the log-logistic distribution if $\theta = -1$, which is often used as a model in survival studies. Further, common distributions such as the lognormal and gamma distributions may be well approximated by this family. They also noted that if $\alpha \geq 0$ and $\theta < 0$, then the family coincides with Burr XII distributions.

2.16.1 Characterization of Hazard Rate

This generalized family not only contains distributions with bathtub and inverted (upside-down bathtub) hazard rate shapes, but also allows for a broader class of monotonic hazard rates. The model hazard rate is given by

$$h(t) = \frac{\alpha(t/\beta)^{\alpha-1}}{\beta(1 - (t/\beta)^\alpha/\theta)}. \quad (2.67)$$

The following classification was obtained by Mudholkar and Hutson (1996):

1. $\alpha < 1$ and $0 < \theta < \infty$: $h(t)$ has a bathtub shape.
2. $\alpha \leq 1$ and $\theta \leq 0$: $h(t)$ is decreasing in t .
3. $\alpha > 1$ and $-\infty < \theta < 0$: $h(t)$ has an inverted bathtub shape.
4. $\alpha \geq 1$ and $\theta \geq 0$: $h(t)$ is increasing in t .
5. $\alpha = 1$ and $\theta \rightarrow \infty$: F is exponential (i.e., $h(t)$ is a constant).

It was also shown that the generalized Weibull family (2.68) is closed under proportional hazards relationships, that is, for any $\nu > 0$, $\bar{F}(t)^\nu$ is also a member of the family (2.68).

Furthermore, for $\theta \leq 0$, (2.68) reduces to the hazard rate of the Burr XII distribution, see for example, Sect. 2.3.13 of Lai and Xie (2006).

2.16.2 Estimation of Parameters

The maximum likelihood estimates of the model parameters were obtained by Mudholkar and Hutson (1996).

2.16.3 Applications

Mudholkar and Hutson (1996) have successfully fitted the model to the two-arm clinical trials data (Head-and-Neck cancer trials) considered by Efron (1988).

2.17 Jeong's Extension of Generalized Weibull

Jeong (2006) extended the above generalized Weibull model by incorporating another parameter with the survival function given as

$$\bar{F}(t) = \exp \left[-\frac{\theta^{1-\tau} \{(t/\beta)^\alpha + \theta\}^\tau - \theta}{\tau} \right]; \quad 0 < \alpha, \beta, \theta < \infty, \quad -\infty < \tau < \infty. \quad (2.68)$$

The distribution becomes the standard Weibull distribution when $\tau = 1$. It reduces to the Mudholkar's generalized Weibull of (2.66) as $\tau \rightarrow 0$.

This new parameter family was motivated to parameterize the cumulative incidence function (in the context of survival analysis) completely.

The model has an application to breast cancer data.

2.17.1 Hazard Rate Function

The hazard rate function of Jeong's extension is

$$h(t) = \frac{\alpha(t/\beta)^\alpha \theta^{1-\tau}}{t [(t/\beta)^\alpha + \theta]^{1-\tau}}. \quad (2.69)$$

By differentiating $\log h(t)$ with respect to t and then setting the derivative to zero, we find the turning point $\beta \left\{ \frac{\theta(1-\alpha)}{\alpha\tau-1} \right\}^{1/\alpha}$ exists on the real axis only if (a) $\alpha < 1$ and $\alpha\tau > 1$ or (b) $\alpha > 1$ and $\alpha\tau < 1$.

When (a) holds, $h(t)$ has a bathtub shape and if (b) holds then it has an upside-down bathtub shape (unimodal). Otherwise, $h(t)$ can be a constant, decreasing or increasing.

2.17.2 Application

The model was applied to two breast cancer data sets from the National Surgical Adjuvant Breast and Bowel Project.

2.18 Generalized Power Weibull Family

Nikulin and Haghighi (2006) introduced a family of distributions with survival function

$$\bar{F}(t) = \exp \left\{ 1 - (1 + (t/\beta)^\alpha)^\theta \right\}; \quad \alpha, \beta, \theta > 0. \quad (2.70)$$

Obviously, the case $\theta = 1$ reduces it to the standard Weibull distribution.

The quantile function is given as below:

$$Q(p) = \beta \{ (1 - \log(1 - p))^{1/\theta} - 1 \}^{1/\alpha}; \quad 0 < p < 1. \quad (2.71)$$

2.18.1 Probability Density Function

The probability density function is given by

$$f(t) = \frac{\alpha\theta}{\beta^{\alpha-1}} t^\alpha \{1 + (t/\beta)^\alpha\}^{\theta-1} \exp \left\{ 1 - (1 + (t/\beta)^\alpha)^\theta \right\}. \quad (2.72)$$

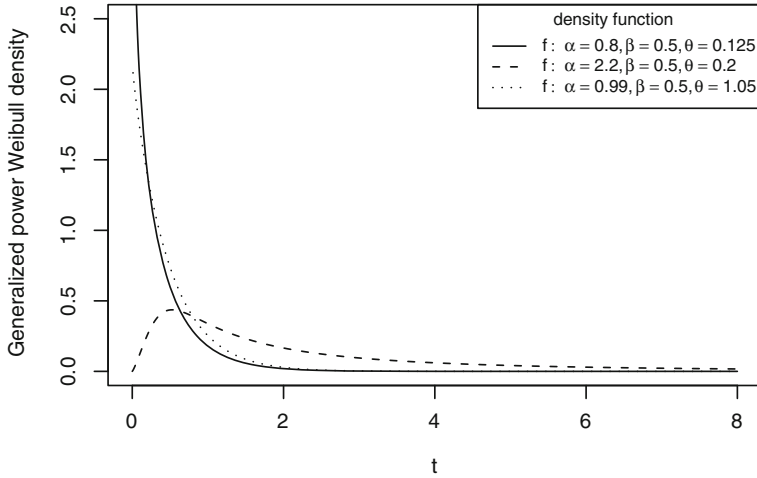


Fig. 2.12 Generalized power Weibull density functions

The density plots of the generalized power Weibull for three combinations of the parameters α , β and θ are given in Fig. 2.12.

2.18.2 Hazard Rate Function

The hazard rate of the generalized power Weibull family is

$$h(t) = \frac{\alpha\theta}{\beta^{\alpha-1}} t^{\alpha} \{1 + (t/\beta)^{\alpha}\}^{\theta-1}. \quad (2.73)$$

It has been shown in Nikulin and Haghighi (2006) that the hazard rate has nice and flexible properties. Depending on the values of the parameters, $h(t)$ can be constant, monotone (increasing, decreasing), unimodal or bathtub shaped.

More specifically, it has been shown that the hazard rate curve is

- (a) monotone increasing if either $\alpha > 1$ and $\alpha\theta > 1$ or $\alpha = 1$ and $\theta > 1$;
- (b) monotone decreasing if either $0 < \alpha < 1$ and $\alpha\theta < 1$ or $0 < \alpha < 1$ and $\alpha\theta = 1$;
- (c) unimodal (inverted bathtub shaped) if $\alpha > 1$ and $0 < \alpha\theta < 1$;
- (d) bathtub shaped $0 < \alpha < 1$ and $\alpha\theta > 1$.

The plots of hazard rates are given in given in Fig. 2.13.

Using a chi-squared goodness-of-fit test, Nikulin and Haghighi (2006) showed that the model provides a good fit to the well known randomly censored survival times data for patients at ARM A of the Head-and-Neck Cancer Trail.

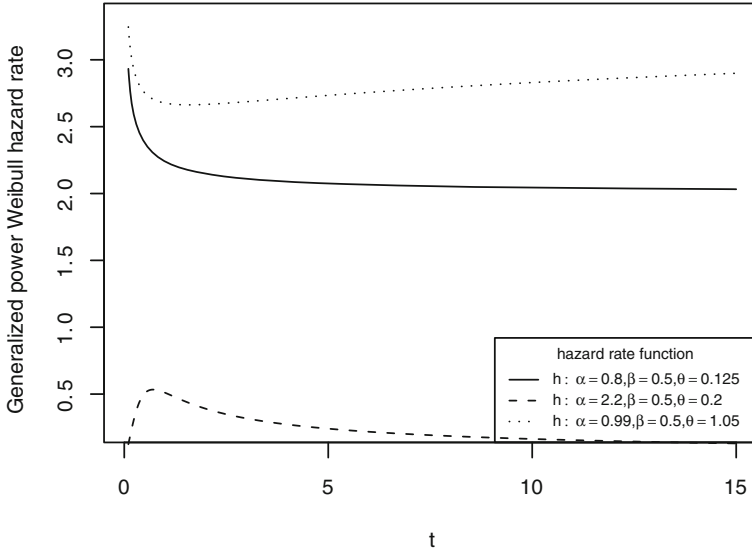


Fig. 2.13 Generalized power Weibull hazard rate functions

2.19 Slymen-Lachenbruch Modified Weibull Distribution

Slymen and Lachenbruch (1984) proposed a modified Weibull model with survival function given by

$$\bar{F}(t) = \exp \left\{ -\exp \left[c + \frac{\beta(t^\alpha - t^{-\alpha})}{2\alpha} \right] \right\}; \quad \alpha, \beta, c > 0. \quad (2.74)$$

Thus

$$\log[-\log \bar{F}(t)] = c + \frac{\beta(t^\alpha - t^{-\alpha})}{2\alpha}. \quad (2.75)$$

We can see that a Weibull type probability plot can be constructed from the last equation.

Hazard Rate Function

The hazard rate function can be expressed as:

$$h(t) = \frac{\beta}{2} (t^{\alpha-1} - t^{-\alpha-1}) \exp \left[c + \frac{\beta(t^\alpha - t^{-\alpha})}{2\alpha} \right]. \quad (2.76)$$

Slymen and Lachenbruch (1984) showed that if $2\{(\alpha+1)t^{-\alpha-2} - (\alpha-1)t^{\alpha-2}\} \times (t^{\alpha-1} - t^{-\alpha-1})^{-2}$ is a bounded function then $h(t)$ can have a bathtub shape.

2.20 Flexible Weibull

Bebbington et al. (2007a) introduced a distribution which is quite simple and yet very flexible to model reliability data. The proposed distribution has survival function given as

$$\bar{F}(t) = \exp\left(-e^{\alpha t - \beta/t}\right). \quad (2.77)$$

For $\beta = 0$, it reduces to type 1 extreme value distribution (Johnson et al. 1995, Chap. 22) so that it can be transformed to the standard Weibull and hence the name flexible Weibull was given.

When $\beta = \alpha$, it reduces to a special case of the Slymen-Lachenbruch modified Weibull model considered in the preceding section.

It follows easily from (2.77) that

$$\log[-\log(\bar{F}(t))] = \alpha t - \beta/t \quad (2.78)$$

showing that the two parameters α and β can be initially estimated through a probability plot.

2.20.1 Density Function

The density function corresponding to (2.77) is

$$f(t) = (\alpha + \beta/t^2) \exp(\alpha t - \beta/t) \exp\left(-e^{\alpha t - \beta/t}\right). \quad (2.79)$$

The plots of three density curves are given in Fig. 2.14.

2.20.2 Hazard Rate Function

Unlike many other generalized Weibull distribution, the hazard rate of this distribution is rather simple:

$$h(t) = (\alpha + \beta/t^2) \exp(\alpha t - \beta/t). \quad (2.80)$$

A distribution F is said to be IFRA (increasing failure rate average) if and only if $\int_0^t h(s)ds/t = -\log \bar{F}(t)/t$ is increasing in t . IFR (increasing average failure rate) is a subclass of IFRA. Note that the IFRA ageing class is featured prominently in Barlow and Proschan (1981).

It was shown by Bebbington et al. (2007a) that the distribution is IFR if $\alpha\beta \geq 27/64$, IFRA if and only if $\alpha\beta \geq 1/4$, which includes the IFR case, as required. We

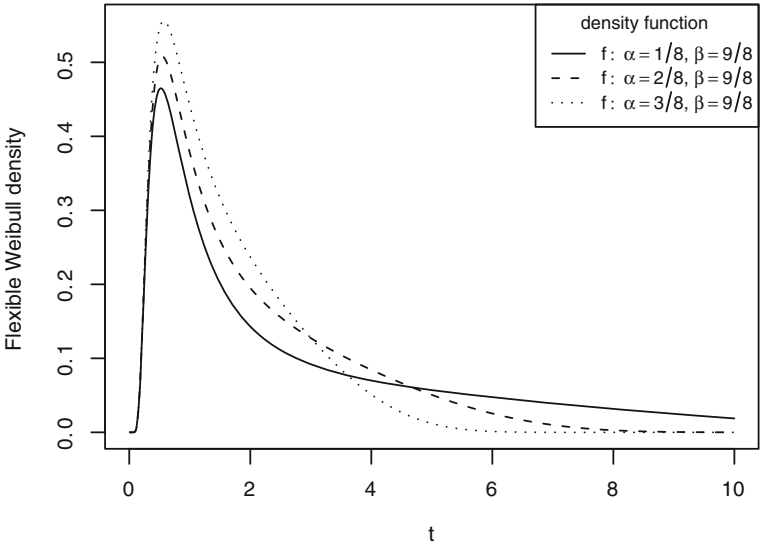


Fig. 2.14 Flexible Weibull density functions

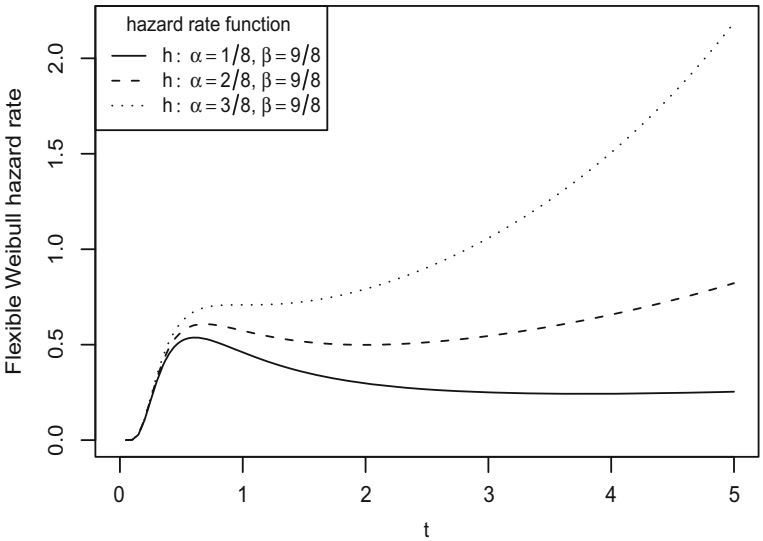


Fig. 2.15 Flexible Weibull hazard rate functions

see that if $1/4 < \alpha\beta \leq 27/64$, then the distribution is IFRA, but not IFR. Also, if $\alpha\beta < 27/64$, the hazard rate has a modified bathtub shape (i.e., h is first increasing followed by a bathtub shape).

The hazard rate plots are given in Fig. 2.15.

2.20.3 Parameter Estimation

Bebbington et al. (2007a) showed that α and β can be estimated through the Weibull-type probability plot using (2.78) above.

The maximum likelihood estimates of two parameters can also be obtained numerically through maximizing the likelihood function.

2.20.4 Applications

The flexible Weibull distribution has applications in:

- survival analysis for secondary reactor pumps (Bebbington et al. 2007a),
- human mortality study (Bebbington et al. 2007b),
- stop over duration of animals in the presence of trap-effects (Choqueta et al. 2013).

2.21 Weibull Extension Model

Another generalization of Weibull was introduced by Xie et al. (2002) and a detailed statistical analysis was given in Tang et al. (2003). In the latter, the authors simply referred their distribution as ‘the Weibull extension model’.

The distribution is in fact a generalization of the model studied by Chen (2000). The cumulative distribution function is given by

$$F(t) = 1 - \exp \left\{ -\lambda\beta \left[e^{(t/\beta)^\alpha} - 1 \right] \right\}, \quad t \geq 0, \alpha, \beta, \lambda > 0. \quad (2.81)$$

We see that the distribution approaches to a two-parameter Weibull distribution when $\lambda \rightarrow \infty$ with β in such a manner that $\beta^{\alpha-1}/\lambda$ is held constant.

For $\lambda = 1$, the above distribution is the exponential power distribution considered and studied by Smith and Bain (1975, 1976). The case with scale parameter $\beta = 1$ was considered by Chen (2000) who also considered the estimation of parameters.

2.21.1 Density Function

The probability density function is

$$f(t) = \lambda\alpha(t/\beta)^{\alpha-1} \exp \left\{ (t/\beta)^\alpha + \lambda\beta(1 - e^{(t/\beta)^\alpha}) \right\}. \quad (2.82)$$

Figure 2.16 gives the density plots.

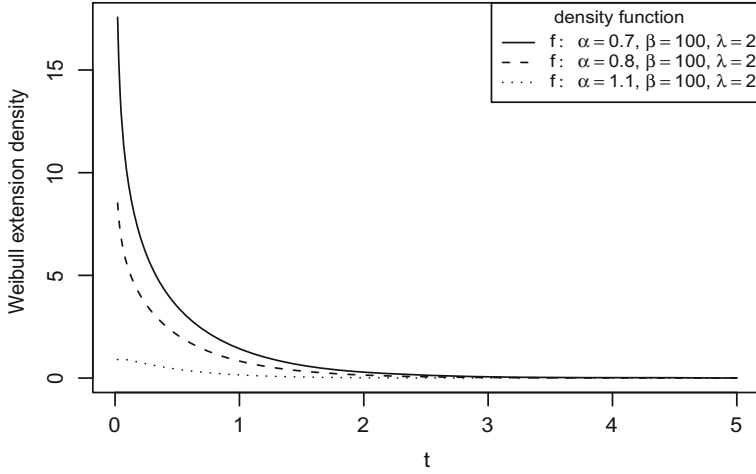


Fig. 2.16 Weibull extension density functions

2.21.2 Hazard Rate Function

The hazard rate function that corresponds to (2.81) is

$$h(t) = \lambda \alpha (t/\beta)^{\alpha-1} \exp \left[-(t/\beta)^\alpha \right]. \quad (2.83)$$

The shape of $h(t)$ depends only on the shape parameter α .

For $\alpha \geq 1$:

- (i) $h(t)$ is an increasing function;
- (ii) $h(0) = 0$ if $\alpha > 1$ and $r(0) = \lambda$, if $\alpha = 1$;
- (iii) $h(t) \rightarrow \infty$ as $t \rightarrow \infty$.

For $0 < \alpha < 1$:

- (i) $h(t)$ is decreasing for $t < t^*$ and increasing for $t > t^*$ with

$$t^* = \beta(1/\alpha - 1)^{1/\alpha}. \quad (2.84)$$

This implies that the hazard rate function has a bathtub shape;

- (ii) $h(t) \rightarrow \infty$ for $t \rightarrow 0$ or $t \rightarrow \infty$;
- (iii) The change point t^* increases as the shape parameter α decreases.

The hazard rate plots of the distribution are given in Fig. 2.17.

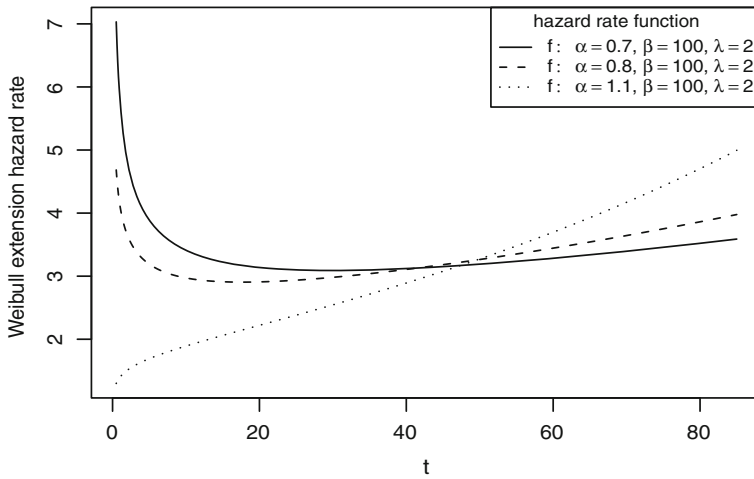


Fig. 2.17 Weibull extension hazard rate functions

2.21.3 Mean Residual Life

For this model, the mean residual life function $\mu(t)$ does not have a closed form although its turning point μ_0 can be obtained numerically. In fact, $\mu(t)$ attains its maximum value at $t = \mu_0$. Xie et al. (2004) have given plots of $\mu(t)$ and $h(t)$ for $\alpha = 0.45, 0.5$, and 0.6 with $\beta = 100, \lambda = 2$.

2.21.4 Other Properties

Quantile function, skewness and kurtosis, estimation of parameters, etc., were given in Tang et al. (2003).

Nadarajah (2005) derived explicit algebraic formulae for the k th moment (about zero) of the distribution when $1/\alpha$ is a non-negative integer.

2.21.5 Model Adequacy

Complete Bayesian analysis of the model has been provided by Gupta et al. (2008) using Markov chain Monte Carlo simulations. A thorough study has also been conducted for checking adequacy of the model for a given data using some of the graphical and numerical methods based on predictive simulation ideas. A real data set was considered for an illustration of their methods.

2.21.6 Applications

The model has been fitted to:

- the failure times of Aarset data (Aarset 1987)
- the failure times of a sample of devices in a large system (Gupta et al. 2008). The detailed description of the data set can be found in Meeker and Escobar (1998).

2.22 The Odd Weibull Distribution

Cooray (2006) has constructed a generalized Weibull family called the odd Weibull family based on the idea of evaluating the distribution of the ‘odd of death’ of a lifetime variable.

2.22.1 Distribution Function

The distribution function of the odd Weibull is given as

$$F(t) = 1 - \left(1 + (e^{(t/\beta)^\alpha} - 1)^\theta\right)^{-1}, \quad (2.85)$$

with $\beta > 0$ be the scale parameter and $\alpha\theta > 0$ as the shape parameter.

When $\theta = 1$, $F(t)$ is the cdf of the Weibull distribution. On the other hand, when $\theta = -1$, $F(t)$ has the inverse Weibull distribution given by $F(t) = e^{-(t/\beta)^\alpha}$.

Jiang et al. (2008) showed that the logit function, the logarithm of the odds, of the odd Weibull distribution can be written as the product of the logit function of the standard Weibull and θ . Similarly, it can also be written as the product of the logit function of the inverse Weibull and $-\theta$.

The density function is

$$f(t) = \left(\frac{\alpha\theta}{t}\right) \left(\frac{t}{\beta}\right) e^{(t/\beta)^\alpha} \left(e^{(t/\beta)^\alpha} - 1\right)^{\theta-1} \left[1 + \left(e^{(t/\beta)^\alpha} - 1\right)^\theta\right]^{-2}. \quad (2.86)$$

2.22.2 Quantile Function

The quantile function can be shown to be

$$Q(u) = \beta \left\{ \log \left[1 + \left(\frac{u}{1-u} \right)^{1/\theta} \right] \right\}. \quad (2.87)$$

2.22.3 Hazard Rate Function

The hazard rate function of the odd Weibull distribution is

$$h(t) = \left(\frac{\alpha\theta}{t}\right) \left(\frac{t}{\beta}\right) e^{(t/\beta)^\alpha} \left(e^{(t/\beta)^\alpha} - 1\right)^{\theta-1} \left[1 + \left(e^{(t/\beta)^\alpha} - 1\right)^\theta\right]^{-1}. \quad (2.88)$$

Cooray (2006) has shown that the odd Weibull family can model various hazard shapes (increasing, decreasing, bathtub, and unimodal); thus the family is proved to be flexible for fitting reliability and survival data. More precisely, he established that

- (1) for $\alpha < 0, \theta < 0$ or $\alpha < 1, \alpha\theta \geq 1$, $h(t)$ is unimodal,
- (2) for $\alpha > 1, \alpha\theta > 1$, $h(t)$ is increasing,
- (3) for $\alpha < 1, \alpha\theta < 1$, $h(t)$ is decreasing, and
- (4) for $\alpha > 1, \alpha\theta \leq 1$, $h(t)$ has a bathtub shape.

Cooray (2006) also indicated that in the regions where $(\alpha > 1, \alpha\theta > 1)$ and $(\alpha < 1, \alpha\theta < 1)$, $h(t)$ may have some other shapes. Jiang et al. (2008) found, using numerical analysis, that the ‘other shapes’ are N (modified bathtub) and reflected N shapes when the model parameters are near the boundary line $\alpha\theta = 1$. The latter authors also studied the tail behaviours of $h(t)$ in detail.

2.22.4 Weibull Probability Plot Parameter Estimation

The shapes of the Weibull probability plot with different parameters were presented in Jiang et al. (2008) and the steps of the graphical estimation were also iterated. In estimating parameters of the odd Weibull model by a graphical approach, it is necessary to determine if the shape parameters are positive or negative. Cooray (2006) suggested to employ the total-time-test (TTT) for this purpose whereas Jiang et al. (2008) concluded that it is easy to check the sign of the shape parameters using a Weibull probability plot.

2.22.5 Optimal Burn-In Time and Useful Period

For a product lifetime exhibiting a bathtub shaped hazard rate, an important issue is to determine the optimal burn-in time. As seen from an earlier subsection, the odd Weibull model can indeed display such a shape. Jiang et al. (2008) observed that the second portion (stable period) of the odd Weibull model, when exhibiting a bathtub shape, could be quite long and flat which is a good property in application. They discussed those issues concerning the optimal burn-in time and the useful period for the model.

2.22.6 Application

The model has been fitted to a sample of 208 data points, which represent the ages at death in weeks for male mice exposed to 240r of gamma radiation (Kimball 1960).

2.23 Generalized Logistic Frailty Model

Vaupel (1990) proposed and examined a logistic frailty model to correct an inherent deficiency in the Gompertz-Makeham law often used in a mortality study. The Vaupel model has incorporated a deceleration parameter $s \geq 0$ in such a way that the resulting hazard is a logistic function given by

$$h(t) = \frac{Ae^{t/\beta}}{1 + sA\beta(e^{t/\beta} - 1)}. \quad (2.89)$$

The survival function corresponding to (2.89) is

$$\bar{F}(t) = \left[1 + sA\beta(e^{t/\beta} - 1)\right]^{-\frac{1}{s}} \quad (2.90)$$

which is relatively simple. On differentiation of (2.89), we have

$$h'(t) = \frac{Ae^{(t/\beta)}(1/\beta - sA)}{[1 + sA\beta(e^{t/\beta} - 1)]^2}. \quad (2.91)$$

From the preceding equation, we see that $h(t)$ is increasing (decreasing) for $\beta s A < 1$ ($sA\beta > 1$) and it converges to a constant as $t \rightarrow \infty$. For human mortality, the case ($sA\beta > 1$) is obviously unrealistic as it would imply the immortality of man.

2.23.1 Generalized Logistic Frailty Distribution

In view of its hazard rate function being monotonic, the logistic frailty model has limited applications in reliability and survival analysis. To provide a more flexible model that is applicable to various disciplines associated with lifetime data, Lai and Izadi (2012) generalized the logistic frailty model in (2.90) by incorporating a shape parameter:

$$\bar{F}(t) = \left[1 + sA\beta(e^{(t/\beta)^\alpha} - 1)\right]^{-\frac{1}{s}} \quad (2.92)$$

where $A, \alpha, \beta > 0$ and $s \geq 0$. Here α and β are clearly the shape and scale parameter, respectively. As mentioned earlier, s is a deceleration parameter whereas A is some kind of ‘normalizing constant’, particularly if $s = 0$.

We will see how this distribution is related to the Weibull in the next subsection.

2.23.2 Limiting Case

As $s \rightarrow 0$, (2.92) is reduced to

$$\begin{aligned} \lim_{s \rightarrow 0} \bar{F}(t) &= \lim_{s \rightarrow 0} \left[1 + sA\beta(e^{(t/\beta)^\alpha} - 1) \right]^{-\frac{1}{s}} \\ &= \exp \left\{ -A\beta(e^{(t/\beta)^\alpha} - 1) \right\}. \end{aligned} \quad (2.93)$$

which is the Weibull extension model of Chen (2000) and Xie et al. (2002). It has been shown in Xie et al. (2002) that (2.93) is either IFR (increasing failure rate), DFR (decreasing failure rate) or a bathtub shaped hazard rate distribution.

2.23.3 Probability Density Function

The density function of the generalized logistic frailty model can be obtained easily by differentiating (2.92) with respect to t and it is given by

$$f(t) = A\alpha(t/\beta)^{\alpha-1} e^{(t/\beta)^\alpha} \left[1 + sA\beta(e^{(t/\beta)^\alpha} - 1) \right]^{-\frac{s+1}{s}}. \quad (2.94)$$

2.23.4 Hazard Rate Function

The hazard rate function that corresponds to (2.92) is

$$h(t) = \frac{A\beta(t/\beta)^{\alpha-1} e^{(t/\beta)^\alpha}}{1 + sA\beta(e^{(t/\beta)^\alpha} - 1)}. \quad (2.95)$$

Rewriting the above in a slightly different form,

$$h(t) = \frac{A\alpha(t/\beta)^{\alpha-1} e^{(t/\beta)^\alpha} / (sA\beta)}{e^{(t/\beta)^\alpha} + (1 - sA\beta) / (sA\beta)}. \quad (2.96)$$

From the preceding equation, it is clear that $h(t)$ is increasing if $sA\beta < 1$ and $\alpha > 1$. We have also mentioned that for $s = 0$, (2.96) can yield IFR (increasing failure rate), DFR (decreasing failure rate) or a bathtub shape.

Other possible shapes are displayed in Fig. 2.18.

Note that the one of the above curves almost has an S shape which may be useful for describing human late life mortality deceleration phenomenon as observed in Greenwood and Irwin (1931).

The shapes of the hazard rate of the generalized logistic frailty model are summarized in terms β and c ($c = aA\alpha$) by Lai and Izadi (2012).

2.23.5 Applications

The model has been fitted to:

- the data set given in Wang (2000) which consists of the lifetime failures of 18 electronic devices, and
- the data of active repair times (in hours) for an airborne communication transceiver which was reported in Von Alven (1964).

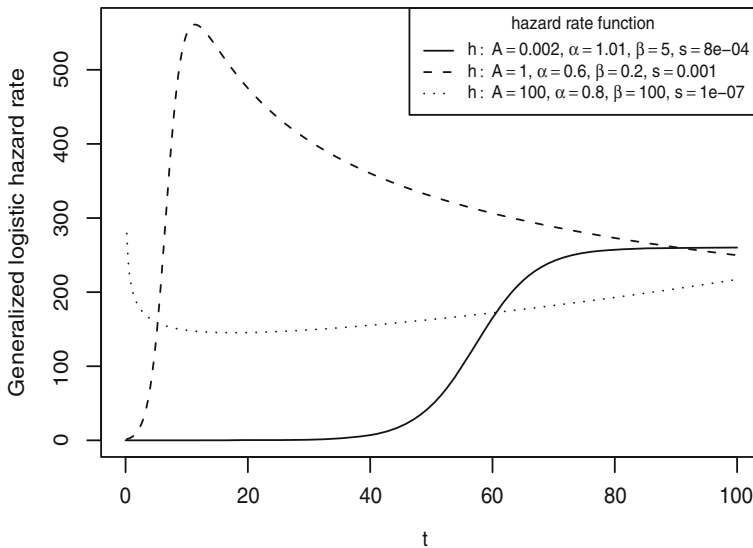


Fig. 2.18 Generalized logistic hazard rate functions

2.24 Generalized Weibull-Gompertz Distribution

Nadarajah and Kotz (2005) proposed a generalization of the standard Weibull model with four parameters having survival function given as below:

$$R(t) = \exp \left\{ -at^b \left(e^{ct^d} - 1 \right) \right\}, \quad a, d > 0; \quad b, c \geq 0; \quad t \geq 0. \quad (2.97)$$

Since (2.97) includes the Gompertz (1825) distribution as its special case when $b = 0$ and $d = 1$. For this reason we may refer it as the generalized Weibull-Gompertz distribution. For $b = 0$ it contains the Weibull extension model of Xie et al. (2002) as given in Sect. 2.21 above. Since the models have four parameters, it may be considered as an over-parameterized model.

2.25 Generalized Weibull Distribution of Gurvich et al.

It has been pointed by Nadarajah and Kotz (2005) that several of the modifications of the Weibull distribution discussed in this section can arise from a representation suggested by Gurvich et al. (1997). This distribution did not arise from a reliability perspective but from the context of modeling random length of brittle materials. The distribution function of this class is given by

$$F(t) = 1 - \exp \{ -aG(t) \}, \quad (2.98)$$

where $G(t)$ is a monotonically increasing function in t such that $G(t) \geq 0$.

Several distributions we presented in this chapter can be derived by assigning appropriate expressions to $G(t)$. For example,

- the modified Weibull model of Lai et al. (2003) is obtained by setting $G(t) = t^\alpha \exp(\lambda t)$,
- the Weibull extension model of Xie et al. (2002) is obtained by letting $G(t) = \alpha \exp(t/\beta)^\alpha - 1$,
- the log Weibull model is derived by assigning $G(t) = \exp((t - a)/b)$, and
- the model considered by Nadarajah and Kotz (2005) follows from taking $G(t) = t^b (e^{ct^d} - 1)$.

Several other Weibull related distributions are also contained in this family.

Pham and Lai (2007) have noted that Gurvich's model is so general in which $G(t)$ is in fact the cumulative hazard function of an arbitrary lifetime distribution. We note that any survival function $\bar{F}$ can be expressed as $\bar{F}(t) = \exp\{-\int_0^t h(t)dt\} = \exp\{-H(t)\}$. Letting $aG(t) = H(t)$, we see that the Gurvich's model includes all the continuous lifetime distributions.

2.26 Weibull Models with Varying Parameters

The parameters of the models discussed so far are held constants. This section deals with models where some of parameters are

- (i) functions of the variable t ,
- (ii) function of some supplementary variables (denoted by S), or
- (iii) random variables (covariates).

2.26.1 Time Varying Parameters

In these models the scale parameter (β) and/or the shape parameter (α) of the standard Weibull model given by (1.4) are function of the variable t .

Examples

- (i) Only one of the two parameters is varying with time.
- (ii) Both α and β are functions of time. For example,

$$\alpha(t) = a \left(1 + \frac{1}{t}\right)^b e^{c/t}, \quad \beta(t) = a' t^{b'} e^{c'/t}. \quad (2.99)$$

- (iii) $F(t) = 1 - \exp\{-\Lambda(t)\}$ where $\Lambda(t)$ is a nondecreasing function with $\Lambda(0)=0$ and $\Lambda(\infty) = \infty$. For example,

$$\Lambda(t) = \sum_{i=1}^m \lambda_i \phi_i(t), \quad 1 \leq i \leq m. \quad (2.100)$$

2.26.2 Weibull Accelerated Models

In these models the scale parameter β is a function of some supplementary variable (covariate) S . In reliability applications S represents the stress on the item so the life of the item (a random variable with distribution F) is a function of S . The shape parameter is unaffected by S and hence a constant. See Sect. 2.6.2 of Murthy et al. (2004).

Arrhenius Model

The relationship is given by

$$\beta(S) = \exp(\gamma_0 + \gamma_1 S). \quad (2.101)$$

(See Jensen (1995) for a summary this model). We see that $\log(\beta(S))$ is linear in S .

Power Model

The relationship is given by

$$\beta(S) = \frac{e^{\gamma_0}}{S^{\gamma_1}}. \quad (2.102)$$

The power model is also briefly in Jensen (1995).

A more general formulation is one where the scale parameter is expressed as

$$\beta(S) = \exp\left(b_0 + \sum_{i=1}^k b_i s_i\right) \quad (2.103)$$

where $S = (s_1, s_2, \dots, s_k)'$ is a k -dimensional vector of supplementary variables.

One may note that these types of models have been used extensively in accelerated life testing in reliability, see for example, Nelson (1990) or Chap. 6 of Lawless (2003).

2.26.3 Weibull Proportional Hazard Models

An alternative approach to modeling the effect of the supplementary variables (covariates) on the survival function $\bar{F}(t)$ is through the hazard rate function. The relationship is given by

$$h(t|S) = \psi(S)h_0(t), \quad (2.104)$$

where $h_0(t)$ is called the baseline hazard for a two-parameter Weibull distribution. The only restriction on the scalar function $\psi(\cdot)$ is that it be positive. The last equation tells us $h(t|S)$ is proportional to the baseline hazard rate h_0 and thus the name ‘proportional hazard’ model.

Many different forms for $\psi(\cdot)$ have been proposed in the literature. One such is the following:

$$\psi(S) = \exp\left(\sum_{i=1}^k b_i s_i\right). \quad (2.105)$$

(See for e.g., Kalbfleisch and Prentice 2002 and Lawless 2003). It now follows from (2.104) that the survival function

$$\bar{F}(t|S) = \bar{F}(t)^{\psi(S)}. \quad (2.106)$$

(Note that in the current context, $\bar{F}(t)$ is the Weibull survival function, but it could denote any other survival functions in a general setting).

When ψ is given by (2.105) and (2.104) becomes

$$h(t|S) = h_0(t) \exp\left(\sum_{i=1}^k b_i s_i\right) \quad (2.107)$$

The hazard model above is generally known as the Cox proportional hazard model. Taking the natural logarithm of both sides of the preceding equation, we have

$$\log(h(t|S)/h_0(t)) = \sum_{i=1}^k b_i s_i \quad (2.108)$$

We now have a fairly ‘simple’ linear model where the parameters can be readily estimated.

2.26.4 Applications

In the recent years, the Cox proportional model is a popular survival model when covariates are involved. Examples of applications include:

- epidemiology and biostatistics, e.g., multiple infection data,
- financial analysis, e.g., stock exchange market,
- clinical trials, and
- survival analysis, e.g., cancer relapse data.

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