

# Pattern-of-Zeros Approach to Fractional Quantum Hall States and a Classification of Symmetric Polynomial of Infinite Variables

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**Abstract** Some purely chiral fractional quantum Hall states are described by symmetric or anti-symmetric polynomials of infinite variables. In this article, we review a systematic construction and classification of those fractional quantum Hall states and the corresponding polynomials of infinite variables, using the pattern-of-zeros approach. We discuss how to use patterns of zeros to label different fractional quantum Hall states and the corresponding polynomials. We also discuss how to calculate various universal properties (i.e. the quantum topological invariants) from the pattern of zeros.

## 1 Introduction

To readers who are interested in physics, this is a review article on the pattern-of-zeros approach to fractional quantum Hall (FQH) states. To readers who are interested in mathematics, this is an attempt to classify symmetric polynomials of infinite variables and  $Z_n$  vertex algebra. To those interested in mathematical physics, this article tries to provide a way to systematically study pure chiral topological quantum field theories that can be realized by interacting bosons. In the next two subsections, we will review briefly the definition of quantum many-boson systems, and the definition of quantum phase for non-physicists. Then, we will give an introduction of the problems studied in this paper.

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## 1.1 What Is a Quantum Many-Boson System

The fermionic FQH states [1, 2] are described by anti-symmetric wave functions, while the bosonic FQH states are described by symmetric wave functions. Since there is an one-to-one correspondence between the anti-symmetric wave functions and the symmetric wave functions, in this article, we will only discuss bosonic FQH states and their symmetric wave functions.

Bosonic FQH systems are quantum many-boson systems. Let us first define mathematically what is a quantum many-boson system, using an  $N$ -boson system in two spatial dimensions as an example. A many-body state of  $N$  bosons is a symmetric complex function of  $N$  variables

$$\begin{aligned}\Psi(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_N) \\ = \Psi(\mathbf{r}_1, \dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots, \mathbf{r}_N)\end{aligned}\quad (1)$$

where the  $i^{\text{th}}$  variable  $\mathbf{r}_i = (x_i, y_i)$  describes the coordinates of the  $i^{\text{th}}$  boson. All such symmetric functions form a Hilbert space where the normal is defined as

$$\langle \Psi | \Psi \rangle = \int \prod_i dx_i dy_i \Psi^* \Psi \quad (2)$$

A quantum system of  $N$  bosons is described by a Hamiltonian, which is a Hermitian operator in the above Hilbert space. It may have a form

$$H(g_1, g_2) = \sum_{i=1}^N -\frac{1}{2}(\partial_{x_i}^2 + \partial_{y_i}^2) + \sum_{i < j} V_{g_1, g_2}(\mathbf{r}_i - \mathbf{r}_j) \quad (3)$$

Here  $V_{g_1, g_2}(\mathbf{r}_i - \mathbf{r}_j)$  is the interaction potential between two bosons. We require the interaction potential to be short ranged:

$$V_{g_1, g_2}(x, y) = 0, \quad \text{if } \sqrt{x^2 + y^2} > \xi, \quad (4)$$

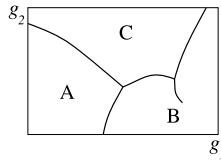
where  $\xi$  describes the interaction range. Hamiltonians with short-ranged interactions are called local Hamiltonians.

The ground state of the  $N$  boson system is an eigenvector of  $H$ :

$$H(g_1, g_2)\Psi_{g_1, g_2}(\mathbf{r}_1, \dots, \mathbf{r}_N) = E_{\text{grnd}}(g_1, g_2)\Psi_{g_1, g_2}(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad (5)$$

with the minimal eigenvalue  $E_{\text{grnd}}(g_1, g_2)$ . The eigenvalues of the Hamiltonian are called energies.

Here we assume that the interaction potential may depend on some parameters  $g_1, g_2$ . As we change  $g_1, g_2$ , the ground states  $\Psi_{g_1, g_2}$  for different  $g_1, g_2$ 's can some times have similar properties. We say that those states belong to the same phase. Some other times, they may have very different properties. Then we regard those states to belong to the different phases.



**Fig. 1** The curves mark the position of singularities in functions  $E_{\text{grnd}}(g_1, g_2)/N$  and  $\langle O \rangle(g_1, g_2)$ . They also represent phase transitions. The regions, A, B, and C, separated by phase transitions correspond to different phases

## 1.2 What Are Quantum Phases

More precisely, quantum phases are defined through quantum phase transitions. So we first need to define *what quantum phase transitions are*.

As we change the parameters  $g_1, g_2$  in the Hamiltonian  $H(g_1, g_2)$ , if the average of ground state energy per particle  $E_{\text{grnd}}(g_1, g_2)/N$  has a singularity in  $N \rightarrow \infty$  limit, then the system has a phase transition. More generally, if the average of any local operator  $O$  on the ground state

$$\langle O \rangle(g_1, g_2) = \int \prod_i dx_i dy_i \Psi_{g_1, g_2}^* O \Psi_{g_1, g_2} \quad (6)$$

has a singularity in  $N \rightarrow \infty$  limit as we change  $g_1, g_2$ , then the system has a phase transition (see Fig. 1).

Using the quantum phase transition, we can define an equivalence relation between quantum ground states  $\Psi_{g_1, g_2}$  in  $N \rightarrow \infty$  limit: Two quantum ground states  $\Psi_{g_1, g_2}$  and  $\Psi_{g'_1, g'_2}$  are equivalent if we can find a path that connect  $(g_1, g_2)$  and  $(g'_1, g'_2)$  such that we can change  $\Psi_{g_1, g_2}$  into  $\Psi_{g'_1, g'_2}$  without encountering a phase transition. The quantum phases are nothing but the equivalent classes of such an equivalence relation [3]. In short, the quantum phases are regions of  $(g_1, g_2)$  space which are separated by phase transitions (see Fig. 1).

## 1.3 How to Classify Quantum Phases of Matter

One of the most important questions in condensed matter physics is how to classify the many different quantum phases of matter. One attempt is the theory of symmetry breaking [4–6], which tells us that we should classify various phases based on the symmetries of the ground state wave function. Yet with the discovery of the FQH states [1, 2] came also the understanding that there are many distinct and fascinating quantum phases of matter, called topologically ordered phases [7, 8], whose characterization has nothing at all to do with symmetry. How should we systematically classify the different possible topological phases that may occur in a FQH system? In this paper, we will try to address this issue.

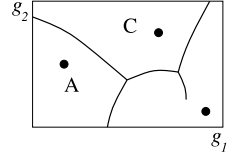
We know that the FQH states contain topology-dependent degenerate ground states, which are topologically stable (i.e. robust against any *local* perturbations of the Hamiltonians). This allows us to introduce the concept of topological order in FQH states [9, 10]. Such topology-dependent degenerate ground states suggest that the low energy theories describing the FQH states are topological quantum field theories [11–13], which take a form of pure Chern-Simons theory in  $2 + 1$  dimensions [14–19]. So one possibility is that we may try to classify the different FQH phases by classifying all of the different possible pure Chern-Simons theories. Although such a line of thinking leads to a classification of Abelian FQH states in terms of integer  $K$ -matrices [15–20], it is not a satisfactory approach for non-Abelian FQH states [21, 22] because we do not have a good way of knowing which pure Chern-Simons theories can possibly correspond to a physical system made of bosons and which cannot.

Another way to classify FQH states is through the connection between FQH wave functions and conformal field theory (CFT). It was discovered around 1990 that correlation functions in certain two-dimensional conformal field theories may serve as good model wave functions for FQH states [21, 23, 24]. Thus perhaps we may classify FQH states by classifying all of the different CFTs. However, the relation between CFTs and FQH states is not one-to-one. If a CFT produces a FQH wave function, then any other CFTs that contain the first CFT can also produce the FQH wave function [24].

Following the ideas of CFT and in an attempt to obtain a systematic classification of FQH states without using conformal invariance, it was shown recently that a wide class of FQH states and their topological excitations can be classified by their *patterns of zeros*, which describe the way ideal FQH wave functions go to zero when various clusters of particles are brought together [25–28]. (We would like to point out that the “1D charge-density-wave” characterization of FQH states [29–34] is closely related to the pattern-of-zeros approach.) This analysis led to the discovery of some new non-Abelian FQH states whose corresponding CFT has not yet been identified. It also helped to elucidate the role of CFT in constructing FQH wave functions: *The CFT encodes the way the wave function goes to zero as various clusters of bosons are brought together*. The order of these zeros must satisfy certain conditions and the solutions to these conditions correspond to particular CFTs. Thus in classifying and characterizing FQH states, one can bypass the CFT altogether and proceed directly to classifying the different allowed pattern of zeros and subsequently obtaining the topological properties of the quasiparticles from the pattern of zeros [26–28]. This construction can then even be thought of as a classification of the allowed CFTs that can be used to construct FQH states [35]. Furthermore, these considerations give a natural notion of which pattern of zeros solutions are simpler than other ones. In this sense, then, one can see that the Moore-Read Pfaffian quantum Hall state [21] is the “simplest” non-Abelian generalization of the Laughlin state.

We would like to point that in the pattern-of-zeros classification of FQH states, we do not try to study the phase transition and equivalence classes. Instead, we just try to classify some special complex functions of infinite variables. We hope those

**Fig. 2** The black dots represent the ideal wave functions that can represent each quantum phase



special complex functions can represent each equivalence class (i.e. represent each quantum phase) (see Fig. 2).

## 2 Examples of Fractional Quantum Hall States

Before trying to classify a type of quantum phases—FQH phases, let us study some examples of ideal FQH wave functions to gain some intuitions.

### 2.1 The Hamiltonian for FQH Systems

A FQH state of  $N$ -bosons is described by the following Hamiltonian:

$$H(g_1, g_2) = \sum_{i=1}^N \left( i\partial_{z_i} - i\frac{1}{4}z_i^* \right) \left( i\partial_{z_i^*} + i\frac{1}{4}z_i \right) + \sum_{i < j} V_{g_1, g_2}(z_i - z_j) \quad (7)$$

where the two dimensional plane is parametrized by  $z = x + iy$ . When  $V_{g_1, g_2} = 0$ , there are many wave functions

$$\Psi(z_1, \dots, z_N) = P(z_1, \dots, z_N) e^{-(1/4)\sum_{i=1}^N z_i z_i^*}, \quad (8)$$

$P = \text{a symmetric polynomial}$

that all have the minimal zero eigenvalue (or energy) for any  $P$ :

$$\left[ \sum_{i=1}^N \left( i\partial_{z_i} - i\frac{1}{4}z_i^* \right) \left( i\partial_{z_i^*} + i\frac{1}{4}z_i \right) \right] P(z_1, \dots, z_N) e^{-(1/4)\sum_{i=1}^N z_i z_i^*} = 0, \quad (9)$$

since

$$e^{(1/4)z z^*} \left( i\partial_z - i\frac{1}{4}z^* \right) \left( i\partial_{z^*} + i\frac{1}{4}z \right) e^{-(1/4)z z^*} = \left( i\partial_z - i\frac{1}{2}z^* \right) i\partial_{z^*} \quad (10)$$

For small non-zero  $V_{g_1, g_2}$ , there is only one minimal energy wave function described by a particular polynomial  $P$  whose form is determined by  $V_{g_1, g_2}$ . In general, it is very hard to calculate this unique ground state wave function. In the following, we will show that for some special interaction potential  $V_{g_1, g_2}$ , the ground state wave function can be obtained exactly.

## 2.2 Three Ideal FQH States: The Exact Zero-Energy Ground States

For interaction

$$V_{1/2}(z_1, z_2) = \delta(z_1 - z_2), \quad (11)$$

the wave function  $P_{1/2}(z_1, \dots, z_N) e^{-(1/4) \sum_{i=1}^N z_i z_i^*}$  with

$$P_{1/2} = \prod_{i < j} (z_i - z_j)^2 \quad (12)$$

is the only zero energy state with minimal total power of  $z_i$ 's. This is because

$$\int \prod_i d^2 z_i e^{-(1/4) \sum_i |z_i|^2} P_{1/2}^* \left[ \sum_{i < j} V_{1/2}(z_i, z_j) \right] P_{1/2} e^{-(1/4) \sum_i |z_i|^2} = 0. \quad (13)$$

Such a state is called  $\nu = 1/2$  Laughlin state.

For interaction

$$V_{1/4}(z_1, z_2) = v_0 \delta(z_1 - z_2) + v_2 \partial_{z_1}^2 \delta(z_1 - z_2) \partial_{z_1}^2, \quad (14)$$

the wave function  $P_{1/4}(z_1, \dots, z_N) e^{-(1/4) \sum_{i=1}^N z_i z_i^*}$  with

$$P_{1/4} = \prod_{i < j} (z_i - z_j)^4 \quad (15)$$

is the only zero energy state with minimal total power of  $z_i$ 's, since

$$\int \prod_i d^2 z_i e^{-(1/4) \sum_i |z_i|^2} P_{1/4}^* \left[ \sum_{i < j} V_{1/4}(z_i, z_j) \right] P_{1/4} e^{-(1/4) \sum_i |z_i|^2} = 0. \quad (16)$$

Such a state is called  $\nu = 1/4$  Laughlin state.

Now let us consider interaction [36, 37]

$$V_{\text{Pf}}(z_1, z_2, z_3) = \mathcal{S} \left[ v_0 \delta(z_1 - z_2) \delta(z_2 - z_3) - v_1 \delta(z_1 - z_2) \partial_{z_3}^* \delta(z_2 - z_3) \partial_{z_3} \right] \quad (17)$$

where  $\mathcal{S}$  symmetrizes among  $z_1, z_2, z_3$  to make  $V_{\text{Pf}}(z_1, z_2, z_3)$  a symmetric function.

Then the wave function  $P_{\text{Pf}}(z_1, \dots, z_N) e^{-(1/4) \sum_{i=1}^N z_i z_i^*}$  with

$$\begin{aligned} P_{\text{Pf}} &= \mathcal{A} \left( \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N} \right) \prod_{i < j} (z_i - z_j) \\ &= \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i < j} (z_i - z_j) \end{aligned} \quad (18)$$

is the only zero energy state with minimal total power of  $z_i$ 's, where  $\mathcal{A}$  anti-symmetrizes among  $z_1, \dots, z_N$ . This is because

$$\int \prod_i d^2 z_i e^{-(1/4) \sum_i |z_i|^2} P_{\text{Pf}}^* \left[ \sum_{i < j < k} V_{\text{Pf}}(z_i, z_j, z_k) \right] P_{\text{Pf}} e^{-(1/4) \sum_i |z_i|^2} = 0. \quad (19)$$

Such a state is called the Pfaffian state [21].

### 3 The Universal Properties of FQH Phases

The three many-body wave functions  $P_{1/2} e^{-(1/4) \sum_i |z_i|^2}$ ,  $P_{1/4} e^{-(1/4) \sum_i |z_i|^2}$ , and  $P_{\text{Pf}} e^{-(1/4) \sum_i |z_i|^2}$  have some amazing exact properties in  $N \rightarrow \infty$  limit. We believe that those properties do not depend on any local deformations of the wave functions.<sup>1</sup> In other words, those properties are shared by all the wave functions in the same phase. We call such kind of properties universal properties.

The universal properties can be viewed as quantum topological invariants in mathematics, since they do not change under any perturbations of the local Hamiltonian. Thus, from mathematical point of view, the symmetric polynomials of infinite variables, such as  $P_{1/2}$ ,  $P_{1/4}$ , and  $P_{\text{Pf}}$ , can have many quantum topological invariants (i.e. the universal properties) once we define their norm to be

$$\langle P | P \rangle = \int \prod_{i=1}^N d^2 z_i |P(z_1, \dots, z_N)|^2 e^{-(1/2) \sum |z_i|^2}. \quad (20)$$

Since the three wave functions have different universal properties, this implies that the three wave functions belong to three different quantum phases. In this section, we will discuss some of the universal properties, by first listing them in bold-face. Then we will give an understanding of them from physics point of view. Those conjectured universal properties are exact, but not rigorously proven to be true.

#### 3.1 The Filling Fractions of FQH Phases

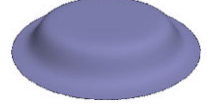
The density profile of a FQH wave function is given by

$$\rho(z) = \frac{\int d^2 z_2 \cdots d^2 z_N |P(z, z_2, \dots, z_N)|^2 e^{-(1/2) \sum |z_i|^2}}{\int d^2 z_1 d^2 z_2 \cdots d^2 z_N |P(z_1, z_2, \dots, z_N)|^2 e^{-(1/2) \sum |z_i|^2}} \quad (21)$$

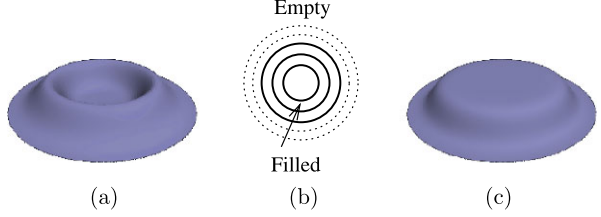
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<sup>1</sup>A local deformation of a many-body wave function  $\Psi$  is generated as  $\Psi \rightarrow \Psi' = e^{i\delta H} \Psi$  where  $\delta H$  is a hermitian operator that can be viewed as an local Hamiltonian.

**Fig. 3** The shape of the density profile  $\rho(z)$



**Fig. 4** **a** The density profile of the  $l^{\text{th}}$  orbital. **b** The filling of the orbitals gives rise to a disk-like density profile in **c**



We believe that

$$\nu \equiv 2\pi\rho(0) \quad (22)$$

is a rational number in  $N \rightarrow \infty$  limit.  $\nu$  is called the filling fraction of the corresponding FQH state. We find that

$$\begin{aligned} P_1 &= \prod (z_i - z_j) \rightarrow \nu = 1, & P_{1/2} &= \prod (z_i - z_j)^2 \rightarrow \nu = 1/2, \\ P_{1/4} &= \prod (z_i - z_j)^4 \rightarrow \nu = 1/4, \\ P_{\text{Pf}} &= \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod (z_i - z_j) \rightarrow \nu = 1. \end{aligned} \quad (23)$$

Note that  $P_1$  is anti-symmetric and describe a many-fermion state, while  $P_{1/2}$ ,  $P_{1/4}$ , and  $P_{\text{Pf}}$  are symmetric and describe many-boson states.

We also believe that the density profile  $\rho(z)$  has disk shape (see Fig. 3) in large  $N$  limit:  $\rho(z)$  is almost a constant  $\nu/2\pi$  for  $|z| < \sqrt{2N/\nu}$  and quickly drop to almost zero for  $|z| > \sqrt{2N/\nu}$ .

### 3.1.1 Why $\nu = 1$ for State $\Psi_1 = \prod_{i < j} (z_i - z_j) e^{-\sum |z_i|^2/4}$

We note that the one-particle eigenstates (the orbitals) for one-particle Hamiltonian  $H_0 = -\sum (\partial_z - \frac{1}{4}z^*)(\partial_{z^*} + (1/4)z)$  can be labeled by the angular momentum  $l$ , which is given by  $z^l e^{-(1/4)|z|^2}$ . The one-particle eigenstate has a ring-like shape with maximum at  $|z| = r_l = \sqrt{2l}$  (see Fig. 4a). The  $\nu = 1$  many-fermion state is obtained by filling the orbitals (see Fig. 4b):

$$\Psi = \prod_{i < j} (z_i - z_j) e^{-(1/4)\sum |z_i|^2} = \mathcal{A}[(z_1)^0 (z_2)^1 \dots] e^{-(1/4)\sum |z_i|^2} \quad (24)$$

We see that there are  $l$  fermions within radius  $r_l$ . So there is one fermion per  $\pi r_l^2/l = 2\pi$  area, and thus  $\nu = 1$  (see Fig. 4c).



### 3.1.2 Why $\nu = 1/m$ for the Laughlin State $\Psi_{1/m} = \prod_{i < j} (z_i - z_j)^m e^{-\sum |z_i|^2/4}$

Let us consider the joint probability distribution of boson positions, which is given by the absolute-value-square of the ground state wave function:

$$\begin{aligned} p(z_1 \cdots z_N) &\propto |\Psi_{1/m}(z_1 \cdots z_N)|^2 \\ &= e^{-2m \sum_{i < j} \ln |z_i - z_j| - (m/2) \sum_i |z_i|^2} = e^{-\beta V(z_1 \cdots z_N)} \end{aligned} \quad (25)$$

Choosing  $T = \frac{1}{\beta} = \frac{m}{2}$ , we can view  $e^{-\beta V(z_1 \cdots z_N)}$  as the probability distribution for  $N$  particles with potential energy  $V(z_1 \cdots z_N)$  at temperature  $T = \frac{m}{2}$ . The potential has a form

$$V = -m^2 \sum_{i < j} \ln |z_i - z_j| + \frac{m}{4} \sum_i |z_i|^2 \quad (26)$$

which is the potential for a two-dimensional plasma of ‘charge’  $m$  particles [2]. The two-body term  $-m^2 \ln |z - z'|$  represents the interaction between two particles and the one-body term  $\frac{m}{4} |z|^2$  represents the interaction of a particle with the background “charge”.

For a uniform background “charge” distribution with charge density  $\rho_\phi$ , a charge  $m$  particle at  $z$  feel a force,  $F = (\pi |z|^2 \rho_\phi)(m)/|z|$ . The corresponding background potential energy is  $-\rho_\phi m \frac{\pi}{2} |z|^2$ . We see that to produce the one-body potential energy  $\frac{m}{4} |z|^2$  we need to set  $\rho_\phi = -1/2\pi$ . Since the plasma must be “charge” neutral:  $m\rho + \rho_\phi = 0$ , we find that  $\rho = \frac{1}{m} \frac{1}{2\pi}$ . So  $\nu = 1/m$ .

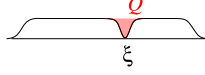
## 3.2 Quasiparticle and Fractional Charge in $\nu = 1/m$ Laughlin States

If we remove a boson at position  $\xi$  from the Laughlin wave function  $\prod_{i < j} (z_i - z_j)^m e^{-\sum |z_i|^2/4}$ , we create a hole-like excitation described by the wave function  $\Psi_\xi^{\text{hole}}(z_1, \dots, z_N)$ :

$$\Psi_\xi^{\text{hole}}(z_1, \dots, z_N) \propto \prod_i (\xi - z_i)^m \prod_{i < j} (z_i - z_j)^m e^{-\sum |z_i|^2/4} \quad (27)$$

Despite the hole-like excitation has a charge = 1, the minimal value for non-zero integers, it is not the minimally charged excitation. The minimally charged excitation corresponds to a quasi-hole excitation, which is described by the wave function

$$\Psi_\xi^{\text{quasi-hole}}(z_1, \dots, z_N) \propto \prod_i (\xi - z_i) \prod_{i < j} (z_i - z_j)^m e^{-\sum |z_i|^2/4} \quad (28)$$



**Fig. 5** The density profile of a many-boson wave function with a quasi-hole excitation at  $\xi$

The density profile for the quasi-hole wave function  $\Psi_{\xi}^{\text{quasi-hole}}(z_1, \dots, z_N)$  is given by

$$\rho_{\xi}(z) = \frac{\int \prod_{i=2}^N d^2 z_i |\Psi_{\xi}^{\text{quasi-hole}}(z, z_2, \dots, z_N)|^2}{\int \prod_{i=1}^N d^2 z_i |\Psi_{\xi}^{\text{quasi-hole}}(z_1, z_2, \dots, z_N)|^2} \quad (29)$$

$\rho_{\xi}(z)$  has a shape as in Fig. 5. The quasi-particle charge is defined as

$$Q = \int_{D_{\xi}} d^2 z \left( \frac{\nu}{2\pi} - \rho_{\xi}(z) \right) \quad (30)$$

in the  $N \rightarrow \infty$  limit, where  $D_{\xi}$  is a big disk covering  $\xi$ . (Note that, away from the quasi-hole,  $\rho_{\xi}(z) = \frac{\nu}{2\pi}$ .) We believe that *the quasi-hole charge is a rational number*  $Q = 1/m$  [2].

One way to understand the above result is to note that  $m$  quasi-holes correspond to a missing boson:  $[\prod_i (\xi - z_i)]^m = \prod_i (\xi - z_i)^m$ . So a quasi-hole excitation has a fractional charge  $1/m$  although the FQH state is formed by particles of charge 1!

We can also calculate the quasi-hole charge directly. Note that, for the Laughlin state  $\Psi_{\xi}^{\text{quasi-hole}}(z_1, \dots, z_N)$  with a quasi-hole at  $\xi$ , the corresponding joint probability distribution of boson positions is given by  $p(\{z_i\}) \propto |\Psi_{\xi}^{\text{quasi-hole}}(\{z_i\})|^2 = e^{-\beta V}$  with

$$V = -m^2 \sum_{i < j} \ln |z_i - z_j| - m \sum_i \ln |z_i - \xi| + \frac{m}{4} \sum_i |z_i|^2 \quad (31)$$

Now, the one-body potential term  $-m \ln |z - \xi| + \frac{m}{4} |z|^2$  is produced by background charge density:  $\rho_{\phi} = -\frac{1}{2\pi} + \delta(\xi)$ . The “charge” neutral condition  $m\rho_{\xi}(z) + \rho_{\phi}(z) \approx 0$  allows us to show that  $\rho_{\xi}(z)$  has a shape as in Fig. 5 and satisfies Eq. (30) with  $Q = 1/m$ .

### 3.3 The Concept of Quasiparticle Type

We would like to point out that the wave function  $\Psi_{\xi}^{\text{quasi-hole}}(z_1, \dots, z_N) \propto \prod_i (\xi - z_i) \prod_{i < j} (z_i - z_j)^m e^{-\sum |z_i|^2/4}$  just describes a particular kind of quasiparticle excitation. More general quasiparticle excitations can be constructed as

$$\Psi_{\xi}^{\text{quasi-hole-}k}(z_1, \dots, z_N) \propto \prod_i (\xi - z_i)^k \prod_{i < j} (z_i - z_j)^m e^{-\sum |z_i|^2/4} \quad (32)$$

which can be viewed as a bound state of  $k$  charge- $1/m$  quasi-holes. So it appears that different types of quasiparticles are labeled by integer  $k$ .

Here we would like to introduce a concept of quasiparticle type: *two quasiparticles belong to the same type if they only differ by a number of bosons that form the FQH state*. Since the quasiparticle labeled by  $k = m$  correspond to a boson, so the different types of quasiparticles in the  $\nu = 1/m$  Laughlin state are labeled by  $k \bmod m$ . *There are  $m$  types of quasiparticles in the  $\nu = 1/m$  Laughlin state (including the trivial type labeled by  $k = 0$ ).*

There is an amazing relation between the number of quasiparticle type and the ground state degeneracy of the FQH state on torus: *the number of quasiparticle type always equal to the ground state degeneracy on torus, in the  $N \rightarrow \infty$  limit.*

### 3.4 Fractional Statistics in Laughlin States

We note that the normalized state with a quasi-hole at  $\xi$  is described by an  $N$ -boson wave function parameterized by  $\xi$ :

$$\psi_{\xi}^{\text{quasi-hole}} = [N(\xi, \xi^*)]^{-1/2} \prod_i (\xi - z_i) \prod_{i < j} (z_i - z_j)^2 e^{-\sum |z_i|^2/4} \quad (33)$$

where  $N(\xi, \xi^*)$  is the normalization factor. The normalized two quasi-hole wave function is given by

$$\begin{aligned} \psi_{\xi, \xi'}^{\text{quasi-hole}} &= [N(\xi, \xi^*, \xi', \xi'^*)]^{-1/2} \\ &\times \prod_i (\xi - z_i) \prod_i (\xi' - z_i) \prod_{i < j} (z_i - z_j)^2 e^{-\sum |z_i|^2/4} \end{aligned} \quad (34)$$

We conjecture that the above two normalization factors are given by

$$N(\xi, \xi^*) = e^{(1/(2m))|\xi|^2} \times \text{Const.} \quad (35)$$

and

$$N(\xi, \xi^*, \xi', \xi'^*) = e^{(1/(2m))(|\xi|^2 + |\xi'|^2) + (1/m) \ln |\xi - \xi'|^2} \times \text{Const.} \quad (36)$$

in the  $N \rightarrow \infty$  limit, where  $\xi$  and  $\xi'$  are hold fixed in the limit.

The quasi-holes in the Laughlin states also have fractional statistics [38–41]. We can calculate the fractional statistics by calculating the Berry phase [42] of moving the quasi-holes. It turns out that the Berry phase of moving the quasi-holes can be calculated from the above normalization factors. Let us first calculate the Berry phase for one quasi-hole and the normalization factor  $N(\xi, \xi^*)$ . The Berry's phase  $\Delta\varphi$  induced by moving  $\xi$  is defined as  $e^{i\Delta\varphi} = \langle \psi_{\xi}^{\text{quasi-hole}} | \psi_{\xi+d\xi}^{\text{quasi-hole}} \rangle$ . It is given by

$$\Delta\varphi = a_{\xi} d\xi + a_{\xi^*} d\xi^*, \quad a_{\xi} = -i \langle \psi_{\xi} | \frac{\partial}{\partial \xi} | \psi_{\xi} \rangle, \quad a_{\xi^*} = -i \langle \psi_{\xi} | \frac{\partial}{\partial \xi^*} | \psi_{\xi} \rangle, \quad (37)$$

where  $a_\xi$  and  $a_{\xi^*}$  are Berry connections. Since the unnormalized state  $\prod_i (\xi - z_i) \prod_{i < j} (z_i - z_j)^2 e^{-\sum |z_i|^2/4}$  has a special property that it only depends only on  $\xi$  (holomorphic), the Berry connection ( $a_\xi, a_{\xi^*}$ ) can be calculated from the normalization  $N(\xi, \xi^*)$  of the holomorphic state:

$$a_\xi = -\frac{i}{2} \frac{\partial}{\partial \xi} \ln[N(\xi, \xi^*)], \quad a_{\xi^*} = \frac{i}{2} \frac{\partial}{\partial \xi^*} \ln[N(\xi, \xi^*)]. \quad (38)$$

Now let us calculate  $N(\xi, \xi^*)$ . Let us guess that  $N(\xi, \xi^*)$  is given by Eq. (35). To show the guess to be right, we need to show that the norm of  $|\Psi_\xi^{\text{quasi-hole}}\rangle$  does not depend on  $\xi$ . We note that  $|\Psi_\xi^{\text{quasi-hole}}|^2 = e^{-\beta V_\xi}$  with

$$\begin{aligned} V_\xi(z_1, \dots, z_N) &= -m^2 \sum_{i < j} \ln |z_i - z_j| - m \sum_i \ln |z_i - \xi| \\ &\quad + \frac{1}{4} |\xi|^2 + \frac{m}{4} \sum_i |z_i|^2. \end{aligned} \quad (39)$$

Here  $V_\xi$  can be viewed as the total energy of a plasma of  $N$  ‘charge’- $m$  particles at  $z_i$  and one ‘charge’-1 particle hold fixed at  $\xi$ . Both particles interact with the same background charge. Note that the norm  $\langle \Psi_\xi^{\text{quasi-hole}} | \Psi_\xi^{\text{quasi-hole}} \rangle$  is given by

$$\langle \Psi_\xi^{\text{quasi-hole}} | \Psi_\xi^{\text{quasi-hole}} \rangle = \int \prod d^2 z_i e^{-\beta V_\xi} \quad (40)$$

Due to the screening of the plasma, we argue that  $\int \prod d^2 z_i e^{-\beta V_\xi}$  does not depend on  $\xi$  in  $N \rightarrow \infty$  limit, which implies that  $\langle \Psi_\xi^{\text{quasi-hole}} | \Psi_\xi^{\text{quasi-hole}} \rangle$  does not depend on  $\xi$ . Thus  $N(\xi, \xi^*)$  is indeed given by Eq. (35).

This allows us to find

$$a_\xi = -i \frac{1}{4m} \xi^*, \quad a_{\xi^*} = i \frac{1}{4m} \xi \quad (41)$$

Using such a Berry connection, let us calculate the Berry’s phase for moving  $\xi$  around a circle  $C$  of radius  $r$  center at  $z = 0$ :

$$\begin{aligned} \Delta\varphi &= \oint_C (a_\xi d\xi + a_{\xi^*} d\xi^*) \\ &= 2\pi \frac{r^2}{4m} \times 2 = 2\pi \frac{\text{Area enclosed by } C}{2\pi m} \\ &= 2\pi \times \text{number of enclosed bosons by } C. \end{aligned} \quad (42)$$

We see that the Berry connection describes a uniform ‘magnetic’ field. The above result can also be understood directly from the wave function  $\prod_i (\xi - z_i) \prod_{i < j} (z_i - z_j)^2 e^{-\sum |z_i|^2/4}$ .

Similarly, we can calculate the Berry connection for two quasi-holes. Let us guess that  $N(\xi, \xi^*, \xi', \xi'^*)$  is given by Eq. (36). For such a normalization factor, we find that  $|\Psi_{\xi, \xi'}^{\text{quasi-hole}}|^2 = e^{-\beta V_{\xi, \xi'}}$  with

$$\begin{aligned} V_{\xi, \xi'}(z_1, \dots, z_N) = & -m \sum_i [\ln |z_i - \xi| + \ln |z_i - \xi'|] \\ & + \frac{1}{4} [|\xi|^2 + |\xi'|^2] - \ln |\xi - \xi'| \\ & - m^2 \sum_{i < j} \ln |z_i - z_j| + \frac{m}{4} \sum_i |z_i|^2 \end{aligned} \quad (43)$$

Such a  $V_{\xi, \xi'}$  can be viewed as the total energy of a plasma of  $N$  ‘charge’- $m$  particles at  $z_i$  and two ‘charge’-1 particles at  $\xi$  and  $\xi'$ . Due to the screening,  $\int \prod d^2 z_i e^{-\beta V_{\xi, \xi'}}$  does not depend on  $\xi$  and  $\xi'$  in  $N \rightarrow \infty$  limit, which implies that  $\langle \Psi_{\xi, \xi'}^{\text{quasi-hole}} | \Psi_{\xi, \xi'}^{\text{quasi-hole}} \rangle$  does not depend on  $\xi$  and  $\xi'$ . So our guess is correct. Using the normalization factor (36), we find the Berry connection to be

$$a_\xi = -i \frac{1}{4m} \xi^* + \frac{i}{2m} \frac{1}{\xi - \xi'}, \quad a_{\xi^*} = i \frac{1}{4m} \xi - \frac{i}{2m} \frac{1}{\xi^* - \xi'^*} \quad (44)$$

Using such a Berry connection, we can calculate the fractional statistics of the quasi-holes in the  $\nu = 1/m$  Laughlin state. Moving a quasi-hole around another, we find the Berry phase to be  $\Delta\varphi = \frac{\text{enclosed area}}{m} - \frac{2\pi}{m}$  (see Eq. (42) for comparison). If we only look at the sub-leading term  $-2\pi/m$ , we find that exchanging two quasi-holes give rise to phase  $\theta = -\pi/m$ , since exchanging two quasi-holes correspond to moving a quasi-hole half way around another and we get the half of  $-2\pi/m$ . We find that *quasi-holes in the  $\nu = 1/m$  Laughlin state have a fractional statistics described by the phase factor  $e^{-i\pi/m}$  [40, 41].*

The term  $\frac{\text{enclosed area}}{m}$  implies that the quasi-holes sees a uniform magnetic field. So the quasi-holes in the  $\nu = 1/m$  Laughlin state are anyons in magnetic field.

### 3.5 Quasi-holes in the $\nu = 1$ Pfaffian State

#### 3.5.1 Charge-1 and Charge-1/2 Quasi-holes

Ground state wave function for the  $\nu = 1$  Pfaffian state is given by

$$\Psi_{\text{Pf}} = \mathcal{A} \left( \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N} \right) \Psi_1 = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \Psi_1 \quad (45)$$

where  $\Psi_1$  is given by  $\prod_{i < j} (z_i - z_j) e^{-(1/4) \sum_i |z_i|^2}$ . A simple quasi-hole state is given by

$$\begin{aligned}
\psi_{\xi}^{\text{charge-1}} &= \prod (\xi - z_i) \psi_{\text{Pf}} \\
&= \mathcal{A} \left( \frac{(\xi - z_1)(\xi - z_2)}{z_1 - z_2} \frac{(\xi - z_3)(\xi - z_4)}{z_3 - z_4} \dots \right) \psi_1 \\
&= \text{Pf} \left( \frac{(\xi - z_i)(\xi - z_j)}{z_i - z_j} \right) \psi_1
\end{aligned} \tag{46}$$

which is created by multiplying the factor  $\prod (\xi - z_i)$  to the ground state wave function. *Such a quasi-hole has a charge 1.* The above quasi-hole can be splitted into two fractionalized quasi-holes. A state with two fractionalized quasi-holes at  $\xi$  and  $\xi'$  is given by

$$\begin{aligned}
\psi_{\xi, \xi'}^{\text{charge-1/2}} &= \mathcal{A} \left( \frac{(\xi - z_1)(\xi' - z_2) + (1 \leftrightarrow 2)}{z_1 - z_2} \frac{(\xi - z_3)(\xi' - z_4) + (3 \leftrightarrow 4)}{z_3 - z_4} \dots \right) \psi_1 \\
&= \text{Pf} \left( \frac{(\xi - z_i)(\xi' - z_j) + (\xi - z_j)(\xi' - z_i)}{z_i - z_j} \right) \psi_1
\end{aligned} \tag{47}$$

*Such a fractionalized quasi-hole has a charge 1/2.* We note that combining two charge-1/2 quasi-holes gives us one charge-1 quasi-hole:  $\psi_{\xi, \xi'}^{\text{charge-1/2}} \propto \psi_{\xi}^{\text{charge-1}}$ .

### 3.5.2 How Many States with Four Charge-1/2 Quasi-holes?

One of the state with four charge-1/2 quasi-holes at  $\xi_1, \xi_2, \xi_3$ , and  $\xi_4$  is given by

$$\begin{aligned}
P_{(12)(34)} &= \text{Pf} \left( \frac{(\xi_1 - z_i)(\xi_2 - z_i)(\xi_3 - z_j)(\xi_4 - z_j) + (i \leftrightarrow j)}{z_i - z_j} \right) \psi_1 \\
&= \text{Pf} \left( \frac{[12, 34]_{z_i z_j}}{z_i - z_j} \right) \psi_1
\end{aligned} \tag{48}$$

The other two are  $P_{(13)(14)}, P_{(14)(23)}$ . But only two of them are linearly independent [43]. Using the relation

$$[12, 34]_{z_i z_j} - [13, 24]_{z_i z_j} = (z_i - z_j)^2 (\xi_1 - \xi_4)(\xi_2 - \xi_3) = z_{ij}^2 \xi_{14} \xi_{23} \tag{49}$$

we find (with  $z_{12} = z_1 - z_2, \xi_{12} = \xi_1 - \xi_2$ , etc.)

$$\begin{aligned}
P_{(13)(24)} &= \mathcal{A} \left( \frac{[12, 34]_{z_1 z_2} - z_{12}^2 \xi_{14} \xi_{23}}{z_{12}} \frac{[12, 34]_{z_3 z_4} - z_{34}^2 \xi_{14} \xi_{23}}{z_{34}} \dots \right) \psi_1 \\
&= P_{(12)(34)} - N_{\text{pair}} \mathcal{A} \left( z_{12} \xi_{14} \xi_{23} \frac{[12, 34]_{z_3 z_4}}{z_{34}} \dots \right) \psi_1
\end{aligned} \tag{50}$$

So

$$P_{(12)(34)} - P_{(13)(24)} = N_{\text{pair}} \xi_{14} \xi_{23} \mathcal{A} \left( z_{12} \frac{[12, 34]_{z_3 z_4}}{z_{34}} \dots \right) \psi_1 \tag{51}$$

Similarly

$$P_{(12)(34)} - P_{(14)(23)} = N_{pair} \xi_{13} \xi_{24} \mathcal{A} \left( z_{12} \frac{[12, 34]_{z_3 z_4}}{z_{34}} \dots \right) \Psi_1 \quad (52)$$

Thus

$$\frac{P_{(12)(34)} - P_{(13)(24)}}{\xi_{14} \xi_{23}} = \frac{P_{(12)(34)} - P_{(14)(23)}}{\xi_{13} \xi_{24}} \quad (53)$$

We find that *there are two states for four charge-1/2 quasi-holes, even if we fixed their positions. The two states are topologically degenerate (have the same energy in  $N \rightarrow \infty$  limit) [43].* The appearance of the topological degeneracy even with fixed quasi-hole positions is a defining property of the non-Abelian statistics. In the presence of the topological degeneracy, as we exchange quasi-holes, we will generate non-Abelian Berry phases which also describe non-Abelian statistics.

More generally we find that there are  $D_n = \frac{1}{2}(\sqrt{2})^n$  topologically degenerate states for  $n$  charge-1/2 quasi-holes, even if we fixed their positions [43]. We see that there are  $\sqrt{2}$  states per charge-1/2 quasi-hole! The  $\sqrt{2}$  is called the quantum dimension for the charge-1/2 quasi-hole. We see that the charge-1/2 quasi-hole has a non-Abelian statistics, since for Abelian anyons, the quantum dimension is always 1.

### 3.6 Edge Excitations and Conformal Field Theory

Under the  $z \rightarrow e^{i\theta} z$  transformation, the  $N$ -particle  $\nu = 1/2$  Laughlin wave function  $\Psi_{1/2} = P_{1/2}(z_1, \dots, z_N) e^{-\sum |z_i|^2/4} = \prod_{1 \leq i < j \leq N} (z_i - z_j)^2 e^{-\sum |z_i|^2/4}$  transforms as  $\Psi_{1/2} \rightarrow e^{iS_N \theta} \Psi_{1/2}$ , with  $S_N = N(N-1)$ . We call  $S_N$  the angular momentum of the Laughlin wave function (which is also the total power of  $z_i$ 's of the polynomial  $P_{1/2}(z_1, \dots, z_N)$ ). For interaction  $V_{1/2} = \sum \delta(z_i - z_j)$ , the  $\nu = 1/2$  Laughlin wave function is the only zero energy state with angular momentum  $N(N-1)$  since  $\Psi_{1/2}(z_1, \dots, z_N)$  vanishes as  $z_i \rightarrow z_j$ . There are no zero energy states with angular momentum less than  $S_N$ . In fact, we believe that, *for wave functions  $\Psi$  with angular momentum less than  $S_N$ ,*

$$\langle V_{1/2} \rangle = \frac{\int \prod d^2 z_i V_{1/2} |\Psi(z_1, \dots, z_N)|^2}{\int \prod d^2 z_i |\Psi(z_1, \dots, z_N)|^2} \geq \Delta \quad (54)$$

for a positive  $\Delta$  and any  $N$ . The maximal  $\Delta$  is called the energy gap for the interaction  $V_{1/2}$ .

On the other hand, there are many zero energy states ( $\langle V_{1/2} \rangle = 0$ ) with angular momentum bigger than  $S_N$ . We call those zero energy states edge states, and denote them as  $\Psi_{\text{edge}}$ . We can introduce a sequence of integers  $D_L^{\text{edge}}$  to denote the number

of zero energy states with angular momentum  $S_N + L$ . We will call  $D_L^{\text{edge}}$  the edge spectrum.

To obtain the edge spectrum for the  $\nu = 1/2$  Laughlin state with interaction  $V_{1/2}$ , we note that the zero-energy edge states can be obtained by multiplying the Laughlin wave function by a symmetric polynomial which does not reduce the order of zeros:

$$\Psi_{\text{edge}} = P_{\text{sym}}(\{z_i\})\Psi_{1/2}. \quad (55)$$

Since the number of the symmetric polynomials with the total power of  $z_i$ 's equal to  $L$  is given by the partition number  $p_L$ , we find  $D_L^{\text{edge}} = p_L$ . Such an argument applies to any Laughlin states. So we believe that for  $\nu = 1/m$  Laughlin the edge spectrum is given by the partition numbers:  $D_L^{\text{edge}} = p_L$  [44]:

$L$	0	1	2	3	4	5	6
$D_L^{\text{edge}}$	1	1	2	3	5	7	11
$P_{\text{sym}}$	1	$\sum z_i$	$(\sum z_i)^2$	$\dots$	$\dots$	$\dots$	$\dots$
			$\sum z_i^2$	$\dots$	$\dots$	$\dots$	$\dots$

(56)

In large  $L$  limit,  $D_L^{\text{edge}} \approx \frac{1}{4\sqrt{3}L} e^{\pi\sqrt{2L/3}} \approx e^{\pi\sqrt{2L/3}}$ .

For the  $\nu = 1$  Pfaffian state with the ideal Hamiltonian  $\mathcal{S}[v_0\delta(z_1 - z_2)\delta(z_2 - z_3) - v_1\delta(z_1 - z_2)\partial_{z_3}\delta(z_2 - z_3)\partial_{z_3}]$ ,  $\Psi_{\text{Pf}} = \mathcal{A}(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \dots) \prod_{i < j} (z_i - z_j)$ , is the zero-energy state with the minimal total angular momentum  $S_N$ . Other zero-energy states with higher angular momenta are given by

$$\Psi_{\text{edge}} = \mathcal{A}\left(P_{\text{any}}(\{z_i\}) \frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \dots\right) \Psi_1, \quad (57)$$

where  $P_{\text{any}}$  is any polynomial. Now the counting is much more difficult, since linearly independent  $P_{\text{any}}$ 's may generate linearly dependent wave functions. We find, for large even total boson number  $N$ , the edge spectrum is given by [45]

$L$	0	1	2	3	4	5	6
$D_L^{\text{edge}}$	1	1	3	5	10	16	28

(58)

We believe that, for the  $\nu = 1$  Pfaffian state, the edge spectrum in large  $L$  limit is given by  $D_L^{\text{edge}} \approx e^{\pi\sqrt{2L/3}\sqrt{c}}$  with  $c = 3/2$ , if  $N \rightarrow \infty$  and  $L \ll N$ .

It turns out that the edge spectrum for  $\nu = 1/m$  Laughlin state can be produced by a central charge  $c = 1$  CFT and the edge spectrum for  $\nu = 1$  Pfaffian state can be produced by a central charge  $c = 3/2$  CFT [44, 45]. This allows us to connect the edge excitations of a FQH state to a CFT.

Using the quasi-hole wave function  $\Psi_{\xi}^{\text{quasi-hole}}(z_1, \dots, z_N)$  that describes a quasi-hole at  $\xi$ , we can even calculate the correlation function of the quasi-hole operator. We know that the circular quantum Hall droplet has a radius  $R = \sqrt{2N/\nu}$ . The



quasi-hole correlation function on the edge of the droplet is given by

$$\begin{aligned}
 G^{\text{quasi-hole}}(\theta' - \theta) \\
 \propto \int \prod d^2 z_i [\Psi_{\xi'}^{\text{quasi-hole}}(z_1, \dots, z_N)]^* \\
 \times \Psi_{\xi}^{\text{quasi-hole}}(z_1, \dots, z_N) \Big|_{\xi = R e^{i\theta}; \xi' = R e^{i\theta'}}.
 \end{aligned} \tag{59}$$

We find that  $G^{\text{quasi-hole}}(\theta - \theta')$  has a form

$$G^{\text{quasi-hole}}(\theta) \propto e^{iQv^{-1}N\theta} \left( \frac{1}{1 - e^{-i\theta}} \right)^{2h} \tag{60}$$

where  $Q$  is the quasi-hole charge and  $h$  is a rational number. We will call  $h$  the scaling dimension of the quasi-hole. For the  $\nu = 1/m$  Laughlin state, we find that  $h = \frac{1}{2m}$  for the charge  $Q = 1/m$  quasi-hole. For the  $\nu = 1$  Pfaffian state, we find that  $h = \frac{1}{2}$  for the charge-1 quasi-hole, and  $h = \frac{3}{16}$  for the charge-1/2 quasi-hole, all in  $N \rightarrow \infty$  limit [45, 46].

## 4 Pattern-of-Zeros Approach to FQH States and Symmetric Polynomials

Using  $P_{1/2}$ ,  $P_{1/4}$ , and  $P_{\text{Pf}}$  as examples, we have seen that symmetric polynomials with infinite variables can have some amazing universal properties, once we defined the norm of the infinite-variable polynomials to be

$$\langle P | P \rangle = \int \prod d^2 z_i |P|^2 e^{-(1/2) \sum |z_i|^2}. \tag{61}$$

This suggests that it may be possible to come up with a definition of “infinite-variable symmetric polynomials”. Such properly defined infinite-variable symmetric polynomials should have those amazing universal properties. The proper definition also allow us to classify infinite-variable symmetric polynomials, which will lead to a classification of FQH phases.

In this section, we will first discuss an attempt to define infinite-variable symmetric polynomials through pattern of zeros. Then, we will try to provide a classification of patterns of zeros. After that, we will use the patterns of zeros to calculate the universal properties of the corresponding infinite-variable symmetric polynomials.

### 4.1 What Is Infinite-Variable Symmetric Polynomial

The main difficulty to define symmetric polynomial with infinite variables is that the number of the variables is not fixed. To overcome this difficulty, we will characterize

the symmetric polynomials through their “local properties” that do not depend on the number of the variables. One such “local property” is pattern of zeros.

#### 4.1.1 What Is Pattern of Zeros?

We have seen that the different short-range interactions  $V(z_i - z_j)$  in Hamiltonian

$$H = \sum_{i=1}^N -\left(\partial_{z_i} - \frac{B}{4}z_i^*\right)\left(\partial_{z_i^*} + \frac{B}{4}z_i\right) + \sum_{i<j} V(z_i - z_j) \quad (62)$$

leads to different FQH states  $P(z_1, \dots, z_N)e^{-(1/4)\sum_{i=1}^N |z_i|^2}$ , which in turn leads to different symmetric polynomials  $P(z_1, \dots, z_N)$ .

One of the resulting polynomial  $P_{1/2} = \prod_{i<j} (z_i - z_j)^2$  has a property that as  $z_1 \approx z_2$ , it has a second-order zero  $P_{1/2} \propto (z_1 - z_2)^2$ . Another resulting polynomial  $P_{1/4} = \prod_{i<j} (z_i - z_j)^4$  has a property that as  $z_1 \approx z_2$ , it has a fourth-order zero  $P_{1/4} \propto (z_1 - z_2)^4$ . The third resulting polynomial

$$P_{\text{Pf}} = \mathcal{A}\left(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N}\right) \prod_{i<j} (z_i - z_j) \quad (63)$$

has a property that as  $z_1 \approx z_2$ ,  $P_{\text{Pf}}$  has no zero, while as  $z_1 \approx z_2 \approx z_3$ ,  $P_{\text{Pf}}$  has a second-order zero. We see that different polynomials can be characterized by different patterns of zeros.

The above examples suggest the following general definition of pattern of zeros for a symmetric polynomial  $P(\{z_i\})$ . Let  $z_i = \lambda\eta_i + z^{(a)}$ ,  $i = 1, 2, \dots, a$ . In the small  $\lambda$  limit, we have

$$P(\{z_i\}) = \lambda^{S_a} P(\eta_1, \dots, \eta_a; z^{(a)}, z_{a+1}, z_{a+2}, \dots) + O(\lambda^{S_a+1}) \quad (64)$$

The sequence of integers  $\{S_a\}$  characterizes the symmetric polynomial  $P(\{z_i\})$  and is called the *pattern of zeros* of  $P$ . We note that  $S_N$  happen to be the total power of  $z_i$  (or the total angular momentum) of  $P$  if the polynomial has  $N$  variables.

#### 4.1.2 The Unique Fusion Condition

If the above induced  $P(\{\eta_i\}; z^{(a)}, z_{a+1}, z_{a+2}, \dots)$ , does not depend on the “shape”  $\{\eta_i\}$

$$P(\{\eta_i\}; z^{(a)}, z_{a+1}, z_{a+2}, \dots) \propto P(z^{(a)}, z_{a+1}, z_{a+2}, \dots), \quad (65)$$

we then say that the symmetric polynomial  $P(\{z_i\})$  satisfy the *unique fusion condition*.

### 4.1.3 Different Encodings of Pattern of Zeros $S_a$

There are many different ways to encode the sequence of integers  $S_a$ . For example, we may use

$$l_a = S_a - S_{a-1}, \quad a = 1, 2, 3, \dots \quad (66)$$

to encode  $S_a$ ,  $a = 1, 2, 3, \dots$ :

$$S_a = \sum_{i=1}^a l_i. \quad (67)$$

Here we have assumed that  $S_0 = 0$ . It turns out that  $l_i \geq 0$  and  $l_i \leq l_{i+1}$ .

We may also use  $n_l$ ,  $l = 0, 1, 2, \dots$  to encode  $S_a$ . Here  $n_l$  is the number of times that the value  $l$  appears in the sequence  $l_i$ :

$$n_l = \sum_{i=1}^{\infty} \delta_{l, l_i}. \quad (68)$$

Let us list the pattern of zeros for some simple polynomials. For the  $\nu = 1$  integer quantum Hall state  $P_1 = \prod_{i < j} (z_i - z_j)$ , the pattern of zeros is given by

$$\begin{aligned} S_1, S_2, \dots &: 0, 1, 3, 6, 10, 15, \dots \\ l_1, l_2, \dots &: 0, 1, 2, 3, 4, 5, \dots \\ n_0 n_1 n_2 \dots &: 11111111 \dots \end{aligned} \quad (69)$$

We see that we can view  $l$  in  $n_l$  as the label for the orbital  $z^l e^{-(1/4)|z|^2}$ , and  $n_l$  as the occupation number on the  $l^{\text{th}}$  orbital (see Sect. 3.1.1 and Fig. 4b).

The pattern of zeros of  $\nu = 1/2$  Laughlin state  $P_{1/2}$  is described by

$$\begin{aligned} S_1, S_2, \dots &: 0, 2, 6, 12, 20, 30, \dots \\ l_1, l_2, \dots &: 0, 2, 4, 6, 8, 10, \dots \\ n_0 n_1 n_2 \dots &: 10101010101010 \dots \end{aligned} \quad (70)$$

We see that  $n_l$  has a periodic structure. Each unit cell (each cluster) has 1 particle and 2 orbitals.

The pattern of zeros of  $\nu = 1/4$  Laughlin state  $P_{1/4}$  is described by

$$\begin{aligned} S_1, S_2, \dots &: 0, 4, 12, 24, 40, 60, 84, \dots \\ l_1, l_2, \dots &: 0, 4, 8, 12, 16, 20, \dots \\ n_0 n_1 n_2 \dots &: 100010001000100010001 \dots \end{aligned} \quad (71)$$

Again,  $n_l$  has a periodic structure. Each unit cell (each cluster) has 1 particle and 4 orbitals.

For the  $\nu = 1$  Pfaffian state  $P_{\text{Pf}} = \mathcal{A}(\frac{1}{z_1 - z_2} \frac{1}{z_3 - z_4} \cdots) \prod_{i < j} (z_i - z_j)$ , the pattern of zeros is given by

$$\begin{aligned} S_1, S_2, \dots &: 0, 0, 2, 4, 8, 12, 18, 24, \dots \\ l_1, l_2, \dots &: 0, 0, 2, 2, 4, 4, 6, 6, \dots \\ n_0 n_1 n_2 \dots &: 202020202020202020 \dots \end{aligned} \quad (72)$$

Now a cluster (unit cell) has 2 particles and 2 orbitals.

#### 4.1.4 The Cluster Condition

Motivated by the above examples, here we would like to introduce a cluster condition for symmetric polynomials: *an symmetric polynomial satisfies a cluster condition if  $n_l$  is periodic*. Let each unit cell contains  $n$  particles and  $m$  orbitals. In this case,  $S_a$  has a form

$$S_{a+kn} = S_a + kS_n + \frac{k(k-1)nm}{2} + kma \quad (73)$$

Since  $S_1 = 0$ , we see that we can use a finite sequence  $(\frac{m}{n}; S_2, \dots, S_n)$  to describe the pattern of zeros for symmetric polynomial satisfying the cluster condition.

We note that the filling fraction  $\nu$  is given by the average number of particles per orbital. Thus  $\nu = n/m$ . We also call the cluster condition with  $n$  particles per unit cell an  $n$ -cluster condition.

#### 4.1.5 A Definition of Infinite-Variable Symmetric Polynomial

Now, we are ready to define *the infinite-variable symmetric polynomial as a symmetric polynomial of infinite variables that satisfy the unique fusion condition and the cluster condition*. The cluster condition makes the  $N \rightarrow \infty$  limit possible. [Or more precisely, the infinite-variable symmetric polynomial is a sequence of symmetric polynomials of  $N$  variables (with  $N \rightarrow \infty$ ), and those  $N$ -variable symmetric polynomials each has the minimal total power of the variables that satisfy the unique fusion condition and the cluster condition. We will loosely refer such a sequence of  $N$ -variable symmetric polynomials as an infinite-variable symmetric polynomial.]

From the above discussions, we see that an infinite-variable symmetric polynomial can be described by a finite amount of data  $(\frac{m}{n}; S_2, \dots, S_n)$ . The  $\nu = 1/2$  Laughlin state,  $P_{1/2}$ , satisfies the unique fusion condition and cluster condition. So  $P_{1/2}$  is an infinite-variable symmetric polynomial described by a pattern of zero:  $(\frac{m}{n}; S_2, \dots, S_n) = (\frac{2}{1}; )$ . Once we define the norm of those infinite-variable symmetric polynomials as Eq. (61), infinite-variable symmetric polynomials may have some very interesting universal properties discussed in Sect. 3. We like to mention

that the infinite-variable symmetric polynomials (also referred as symmetric functions) are studied in mathematics in various contexts, such as representation theory, combinatorics and algebraic topology [47, 48]. It is not clear if there is a relation between our pattern-of-zeros point of view and those previous studies. But we like to point out in our pattern-of-zeros approach, we only interested in symmetric polynomials of  $N \rightarrow \infty$  variables, and with the total power of the variables of order  $O(N^2)$ . We are not interested in the infinite-variable symmetric polynomials with all possible total power of the variables.

## 4.2 A Classification of Infinite-Variable Symmetric Polynomials

We have seen that each infinite-variable symmetric polynomial  $P(\{z_i\})$  has a sequence of integers  $\{S_a\}$ —a pattern of zeros. But each sequence of integers  $\{S_a\}$  may not correspond to an infinite-variable symmetric polynomial  $P(\{z_i\})$ . In this subsection, we will try to find all the conditions that a sequence  $\{S_a\}$  must satisfy, such that  $\{S_a\}$  describes a infinite-variable symmetric polynomial. This may lead to a classification of infinite-variable symmetric polynomials (or FQH states) through pattern of zeros.

### 4.2.1 Derived Polynomials

To find the conditions on  $\{S_a\}$ , it is very helpful to introduce the derived polynomials. Let  $z_1, \dots, z_a \rightarrow z^{(a)}$  in an infinite-variable symmetric polynomial  $P(\{z_i\})$  and use the unique fusion condition:

$$P(\{z_i\}) \rightarrow \lambda^{S_a} P_{\text{derived}}(z^{(a)}, z_{a+1}, z_{a+2}, \dots) + O(\lambda^{S_a+1}), \quad (74)$$

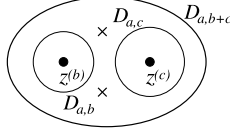
we obtain a derived polynomial  $P_{\text{derived}}(z^{(a)}, z_{a+1}, z_{a+2}, \dots)$  from the original polynomial  $P$ . Repeating the process on other variables, we get a more general derived polynomial  $P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \dots)$ , where  $z^{(a)}, z^{(b)}$ , etc. are fusions of  $a$  variables,  $b$  variables, etc.

The zeros in derived polynomials are described by  $D_{a,b}$ :

$$P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \dots) \sim (z^{(a)} - z^{(b)})^{D_{a,b}} P'_{\text{derived}}(z^{(a+b)}, \dots) + \dots \quad (75)$$

where  $z^{(a+b)} = (z^{(a)} + z^{(b)})/2$ .  $D_{a,b} = D_{b,a}$  also characterize the pattern of zeros. In effect,  $D_{a,b}$  and  $S_a$  encode the same information:

$$D_{a,b} = S_{a+b} - S_a - S_b, \quad S_a = \sum_{b=1}^{a-1} D_{b,1}. \quad (76)$$



**Fig. 6**  $W_{a,bc}$  obtained by moving  $z^{(a)}$  along a large loop around  $z^{(b)}$  and  $z^{(c)}$  counts the total numbers of zeros of  $f(z^{(a)})$  in the loop. The crosses mark the off-particle zeros of  $f(z^{(a)})$  not at  $z^{(b)}$  and  $z^{(c)}$

#### 4.2.2 The Concave Conditions on Pattern of Zeros

Since  $D_{a,b} \geq 0$ , we obtain the first concave condition:

$$\Delta_2(a, b) \equiv S_{a+b} - S_a - S_b \geq 0. \quad (77)$$

Such a condition comes from the fusion of two clusters. We also have a second concave condition:

$$\Delta_3(a, b, c) \equiv S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c \geq 0 \quad (78)$$

from the fusion of three clusters.

To derive the second concave condition, let us fix all variables  $z^{(b)}, z^{(c)}, \dots$  except  $z^{(a)}$  in the derived polynomial  $P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \dots)$ . Then the derived polynomial  $P_{\text{derived}}(z^{(a)}, z^{(b)}, z^{(c)}, \dots)$  can be viewed as a complex function  $f(z^{(a)})$ , which has isolated on-particle zeros at  $z^{(b)}, z^{(c)}, \dots$ , and possibly some other off-particle zeros.

Let us move  $z^{(a)}$  around both points  $z^{(b)}$  and  $z^{(c)}$ . The phase of the complex function  $f(z^{(a)})$  will change by  $2\pi W_{a,bc}$  where  $W_{a,bc}$  is an integer (see Fig. 6). Since  $f(z^{(a)})$  has an order  $D_{ab}$  zero at  $z^{(b)}$  and an order  $D_{ac}$  zero at  $z^{(c)}$ , the integer  $W_{a,bc}$  satisfy

$$W_{a,bc} \geq D_{ab} + D_{ac}$$

because  $f(z^{(a)})$  may also have off-particle zeros. Now let  $z^{(b)} \rightarrow z^{(c)}$  to fuse into  $z^{(b+c)}$ . In this limit  $W_{a,bc}$  becomes the order of zeros between  $z^{(a)}$  and  $z^{(b+c)}$ :  $W_{a,bc} = D_{a,b+c}$ . Thus we obtain the following condition on  $D_{ab}$ :  $D_{a,b+c} \geq D_{ab} + D_{ac}$ , which gives us the second concave condition (78).

We like to point out that the  $n$ -cluster condition has a very simple meaning in the derived polynomial:  $f(z^{(a)})$  has no off-particle zeros if  $a = 0 \bmod n$ . So  $D_{a+b,n} = D_{a,n} + D_{b,n}$  which leads to the cluster condition (73).

#### 4.2.3 Some Additional Conditions

The two concave conditions are the main conditions on  $\{S_a\}$ . We also have another condition

$$\Delta_2(a, a) = \text{even} \quad (79)$$

since the polynomial is a symmetric polynomial. It turns out that we need yet another a condition

$$\Delta_3(a, b, c) = \text{even}. \quad (80)$$

It is hard to prove this mysterious condition using elementary methods. Using the connection between the symmetry polynomial and CFT (or vertex algebra), we find that the condition  $\Delta_3(a, b, c) = \text{even}$  is directly related to the requirement that the fermionic operators have half-integer scaling dimensions and bosonic operators have integer scaling dimensions [35].

We conjecture that *the patterns of zeros  $(\frac{m}{n}; S_2, \dots, S_n)$  that satisfy the above conditions describe infinite-variable symmetric polynomials* [25]. Those  $(\frac{m}{n}; S_2, \dots, S_n)$  “classify” infinite-variable symmetric polynomials and FQH states with filling fraction  $\nu = n/m$ .

#### 4.2.4 Primitive Solutions for Pattern of Zeros

Let us list some patterns of zeros,  $(\frac{m}{n}; S_2, \dots, S_n)$ , that satisfy the above conditions. We note that the conditions are semi-linear in  $(\frac{m}{n}; S_2, \dots, S_n)$ . So, if  $(\frac{m}{n}; S_2, \dots, S_n)$  and  $(\frac{m'}{n'}; S'_2, \dots, S'_n)$  are solutions, then  $(\frac{m''}{n''}; S''_2, \dots, S''_n) = (\frac{m}{n}; S_2, \dots, S_n) + (\frac{m'}{n'}; S'_2, \dots, S'_n)$  is also a solution. Such a result has the following meaning: Let  $P(\{z_i\})$ ,  $P'(\{z_i\})$ , and  $P''(\{z_i\})$  are three symmetric polynomials described by pattern of zeros  $(\frac{m}{n}; S_2, \dots, S_n)$ ,  $(\frac{m'}{n'}; S'_2, \dots, S'_n)$ , and  $(\frac{m''}{n''}; S''_2, \dots, S''_n)$  respectively, we then have  $P''(\{z_i\}) = P(\{z_i\})P'(\{z_i\})$ . Such a property allow us to introduce the notion of primitive pattern of zeros as the patterns of zeros that cannot to written as the sum of two other patterns of zeros. In this section, we will only list the primitive patterns of zeros.

1-cluster state:  $\nu = 1/k$  Laughlin state

$$P_{1/k} : \left( \frac{m}{n}; \right) = \left( \frac{k}{1}; \right), \quad (81)$$

$$(n_0, \dots, n_{k-1}) = (1, 0, \dots, 0).$$

2-cluster state: Pfaffian state ( $Z_2$  parafermion state)

$$P_{2/2; Z_2} : \left( \frac{m}{n}; S_2 \right) = \left( \frac{2}{2}; 0 \right), \quad (82)$$

$$(n_0, \dots, n_{m-1}) = (2, 0)$$

3-cluster state:  $Z_3$  parafermion state

$$P_{3/2; Z_3} : \left( \frac{m}{n}; S_2, S_3 \right) = \left( \frac{2}{3}; 0, 0 \right), \quad (83)$$

$$(n_0, \dots, n_{m-1}) = (3, 0)$$

4-cluster state:  $Z_4$  parafermion state

$$P_{4/2; Z_4} : \left( \frac{m}{n}; S_2, \dots, S_4 \right) = \left( \frac{2}{4}; 0, 0, 0 \right), \quad (84)$$

$$(n_0, \dots, n_{m-1}) = (4, 0),$$

5-cluster states (we have two of them):  $Z_5$  (generalized) parafermion states

$$P_{5/2; Z_5} : \left( \frac{m}{n}; S_2, \dots, S_5 \right) = \left( \frac{2}{5}; 0, 0, 0, 0 \right), \quad (85)$$

$$(n_0, \dots, n_{m-1}) = (5, 0)$$

$$P_{5/8; Z_5^{(2)}} : \left( \frac{m}{n}; S_2, \dots, S_5 \right) = \left( \frac{8}{5}; 0, 2, 6, 10 \right), \quad (86)$$

$$(n_0, \dots, n_{m-1}) = (2, 0, 1, 0, 2, 0, 0, 0)$$

6-cluster state:

$$P_{6/2; Z_6} : \left( \frac{m}{n}; S_2, \dots, S_6 \right) = \left( \frac{2}{6}; 0, 0, 0, 0, 0 \right), \quad (87)$$

$$(n_0, \dots, n_{m-1}) = (6, 0)$$

7-cluster states (we have four of them):

$$P_{7/2; Z_7} : \left( \frac{m}{n}; S_2, \dots, S_7 \right) = \left( \frac{2}{7}; 0, 0, 0, 0, 0, 0 \right), \quad (88)$$

$$(n_0, \dots, n_{m-1}) = (7, 0)$$

$$P_{7/8; Z_7^{(2)}} : \left( \frac{m}{n}; S_2, \dots, S_7 \right) = \left( \frac{8}{7}; 0, 0, 2, 6, 10, 14 \right), \quad (89)$$

$$(n_0, \dots, n_{m-1}) = (3, 0, 1, 0, 3, 0, 0, 0)$$

$$P_{7/18; Z_7^{(3)}} : \left( \frac{m}{n}; S_2, \dots, S_7 \right) = \left( \frac{18}{7}; 0, 4, 10, 18, 30, 42 \right), \quad (90)$$

$$(n_0, \dots, n_{m-1}) = (2, 0, 0, 0, 0, 1, 0, 0, 0, 2, 0, 0, 0, 0, 0)$$

$$P_{7/14; C_7} : \left( \frac{m}{n}; S_2, \dots, S_7 \right) = \left( \frac{14}{7}; 0, 2, 6, 12, 20, 28 \right), \quad (91)$$

$$(n_0, \dots, n_{m-1}) = (2, 0, 1, 0, 1, 0, 1, 0, 2, 0, 0, 0, 0, 0, 0)$$

#### 4.2.5 How Good Is the Pattern-of-Zeros Classification?

How good is the pattern-of-zeros classification? Not so good, and not so bad.



Clearly, every symmetric polynomial  $P$  corresponds to a unique pattern of zeros  $\{S_a\}$ . But only some patterns of zeros correspond to a unique symmetric polynomial. So the pattern-of-zeros classification is not so good. It appears that all the primitive pattern of zeros correspond to a unique a unique symmetric polynomial. Therefore, the pattern-of-zeros classification is not so bad.

We also know that some composite patterns of zeros correspond a unique symmetric polynomial, while other composite patterns of zeros do not correspond a unique symmetric polynomial. Let  $P_{n_i}$  be a symmetric polynomial described by a primitive pattern of zeros with an  $n_i$ -cluster. It appear that  $P = \prod_i P_{n_i}$  will have a pattern of zeros that corresponds a unique symmetric polynomial if  $n_i$ 's has no common factor.

So only for certain patterns of zeros, the data  $\{\frac{m}{n}; S_2, \dots, S_n\}$  contain all the information to fix the symmetric polynomials. In general, we need more information than  $\{\frac{m}{n}; S_2, \dots, S_n\}$  to fully characterize symmetry polynomials of infinite variables.

### 4.3 Topological Properties from Pattern of Zeros

For those patterns of zeros that uniquely characterize the symmetry polynomials of infinite variables (or FQH wave functions), we should be able to calculate the universal properties of the FQH states from the data  $(\frac{m}{n}; S_2, \dots, S_n)$ . Those universal properties include:

- The filling fraction  $\nu$ .
- Topological degeneracy on torus and other Riemann surfaces
- Number of quasiparticle types
- Quasiparticle charges
- Quasiparticle scaling dimensions
- Quasiparticle fusion algebra
- Quasiparticle statistics (Abelian and non-Abelian)
- The counting of edge excitations (central charge  $c$  and spectrum)

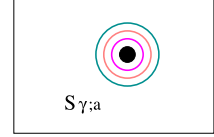
At moment, we can calculate many of the above universal properties from the pattern-of-zeros data  $(\frac{m}{n}; S_2, \dots, S_n)$ . For example, the filling fraction  $\nu$  is given by  $\nu = n/m$ . But we still do not know how to calculate scaling dimensions and statistics for some of the quasiparticles.

In this subsection, we develop a pattern-of-zeros description of the quasiparticle excitations in FQH states. This will allow us to calculate many universal properties from the pattern of zeros.

#### 4.3.1 Pattern of Zeros of Quasiparticle Excitations

A quasiparticle is a defect in the ground state wave function  $P(\{z_i\})$ . It is a place where we have more power of zeros. For example, the ground state wave function

**Fig. 7** The graphic picture of the pattern of zeros for a quasiparticle



of  $\nu = 1/2$  Laughlin state is given by  $\prod_{i < j} (z_i - z_j)^2$ . The state with a quasiparticle at  $\xi$  is given by  $\prod_i (z_i - \xi) \prod_{i < j} (z_i - z_j)^2$  (see Sect. 3.2). As we bring several  $z_i$ 's to  $\xi$ ,  $\prod_i (z_i - \xi) \prod_{i < j} (z_i - z_j)^2$  vanishes according to a pattern of zeros. In general, each quasiparticle labeled by  $\gamma$  in a FQH state can be quantitatively characterized by distinct pattern of zeros (see Fig. 7).

Let  $P_\gamma(\xi; \{z_i\})$  be the wave function with a quasiparticle  $\gamma$  at  $z = \xi$ . To describe the structure of the zeros as we bring bosons to the quasiparticle, we set  $z_i = \lambda \eta_i + \xi$ ,  $i = 1, 2, \dots, a$  and let  $\lambda \rightarrow 0$ :

$$P_\gamma(\xi; \{z_i\}) = \lambda^{S_{\gamma;a}} \tilde{P}_\gamma(z^{(a)} = \xi, z_{a+1}, z_{a+2}, \dots) + O(\lambda^{S_a+1}) \quad (92)$$

$S_{\gamma;a}$  is the order of zeros of  $P_\gamma(\xi; z_i)$  when we bring  $a$  bosons to  $\xi$ . The sequence of integers  $\{S_{\gamma;a}\}$  is the quasiparticle pattern of zeros that characterizes the quasiparticle  $\gamma$ . We note that the ground-state pattern of zeros  $\{S_a\}$  correspond to the trivial quasiparticle  $\gamma = 0$ :  $\{S_{0;a}\} = \{S_a\}$ .

To find the allowed quasiparticles, we simply need to find (i) the conditions that  $S_{\gamma;a}$  must satisfy and (ii) all the  $S_{\gamma;a}$  that satisfy those conditions.

### 4.3.2 Conditions on Quasiparticle Pattern of Zeros $S_{\gamma;a}$

The quasiparticle pattern of zeros also satisfy two concave conditions

$$S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0, \quad (93)$$

$$S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0 \quad (94)$$

and a cluster condition

$$S_{\gamma;a+kn} = S_{\gamma;a} + k(S_{\gamma;n} + ma) + mn \frac{k(k-1)}{2} \quad (95)$$

The cluster condition implies that a finite sequence  $(S_{\gamma;1}, \dots, S_{\gamma;n})$  determines the infinity sequence  $\{S_{\gamma;a}\}$ .

We can also use the sequence  $l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}$  or  $n_{\gamma;l} = \sum_{i=1}^l \delta_{l,l_{\gamma;i}}$  to describe the quasiparticle sequence  $S_{\gamma;a}$ . The  $n_{\gamma;l}$  description is simpler and reveals physical picture more clearly than  $S_{\gamma;a}$ .

### 4.3.3 The Solutions for the Quasiparticle Patterns of Zeros

We can find all  $(S_{\gamma;1}, \dots, S_{\gamma;n})$  that satisfy the above concave and cluster conditions through numerical calculations. This allow us to obtain all the quasiparticles.

For the  $\nu = 1$  Pfaffian state ( $n = 2$  and  $m = 2$ ) described by

$$\begin{aligned} S_1, S_2, \dots : 0, 0, 2, 4, 8, 12, 18, 24, \dots \\ n_0 n_1 n_2 \dots : 20202020202020202 \dots, \end{aligned} \quad (96)$$

we find that the quasiparticle patterns of zeros are given by (expressed in terms of  $n_{\gamma,l}$ )

$$\begin{aligned} n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \dots : 20202020202020202 \dots \quad Q_{\gamma} = 0 \\ n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \dots : 02020202020202020 \dots \quad Q_{\gamma} = 1 \\ n_{\gamma;0} n_{\gamma;1} n_{\gamma;2} \dots : 11111111111111111 \dots \quad Q_{\gamma} = 1/2 \end{aligned} \quad (97)$$

The above three pattern of zeros are not all the solutions of the quasiparticle conditions. However, all other quasiparticle solutions can be obtained from the above three by removing some bosons. Those quasiparticle solutions are equivalent to one of the above three solutions. For example  $n_{\gamma;0} n_{\gamma;1} \dots = 102020202 \dots$ ,  $n_{\gamma;0} n_{\gamma;1} \dots = 002020202 \dots$ , etc. are also quasiparticle solutions which are equivalent to  $n_{\gamma;0} n_{\gamma;1} \dots = 202020202 \dots$ . Therefore, we find that the  $\nu = 1$  Pfaffian state has three types of quasiparticles.

We note that the ground state degeneracy on torus is equal to the number of quasiparticle types. So the  $\nu = 1$  Pfaffian state has a three-fold degeneracy on a torus. The charge of quasiparticles can be also calculated from the quasiparticle pattern of zeros:

$$Q_{\gamma} = \frac{1}{m} \sum_{a=1}^n (l_{\gamma;a} - l_a) = \frac{1}{m} (S_{\gamma;n} - S_n). \quad (98)$$

Let us list the number of quasiparticle types calculated from pattern of zeros for various FQH states. For the parafermion states  $P_{\nu=n/2;Z_n}$  ( $m = 2$ ),

$P_{2/2;Z_2}$	$P_{3/2;Z_3}$	$P_{4/2;Z_4}$	$P_{5/2;Z_5}$	$P_{6/2;Z_6}$	$P_{7/2;Z_7}$	$P_{8/2;Z_8}$	$P_{9/2;Z_9}$	$P_{10/2;Z_{10}}$
3	4	5	6	7	8	9	10	11

For the parafermion states  $P_{\nu=n/(2+2n);Z_n}$  ( $m = 2 + 2n$ )

$P_{2/6;Z_2}$	$P_{3/8;Z_3}$	$P_{4/10;Z_4}$	$P_{5/12;Z_5}$	$P_{6/14;Z_6}$	$P_{7/16;Z_7}$	$P_{8/18;Z_8}$	$P_{9/20;Z_9}$	$P_{10/22;Z_{10}}$
9	16	25	36	49	64	81	100	121

For the generalized parafermion states  $P_{\nu=n/m;Z_n^{(k)}}$

$P_{5/8;Z_5^{(2)}}$	$P_{5/18;Z_5^{(2)}}$	$P_{7/8;Z_7^{(2)}}$	$P_{7/22;Z_7^{(2)}}$	$P_{7/18;Z_7^{(3)}}$	$P_{7/32;Z_7^{(3)}}$	$P_{8/18;Z_8^{(3)}}$	$P_{9/8;Z_9^{(2)}}$
24	54	32	88	72	128	81	40

where  $k$  and  $n$  are co-prime.

For the composite parafermion states  $P_{n_1/m_1; Z_{n_1}^{(k_2)}} P_{n_2/m_2; Z_{n_2}^{(k_2)}}$  obtained as products of two parafermion wave functions

$P_{2/2; Z_2} P_{3/2; Z_3}$	$P_{3/2; Z_3} P_{4/2; Z_4}$	$P_{2/2; Z_2} P_{5/2; Z_5}$	$P_{2/2; Z_2} P_{5/8; Z_5^{(2)}}$
30	70	63	117

where  $n_1$  and  $n_2$  are co-prime. The inverse filling fractions of the above composite states are  $\frac{1}{\nu} = \frac{1}{\nu_1} + \frac{1}{\nu_2} = \frac{m_1}{n_1} + \frac{m_2}{n_2}$ . More results can be found in [26].

All those results from the pattern of zeros agree with the results from parafermion CFT [27]:

$$\# \text{ of quasiparticles} = \frac{1}{\nu} \prod_i \frac{n_i(n_i + 1)}{2} \quad (99)$$

for the generalized composite parafermion state

$$P = \prod_i P_{n_i/m_i; Z_{n_i}^{(k_i)}}, \quad \{n_i\} \text{ co-prime}, (k_i, n_i) \text{ co-prime}. \quad (100)$$

The filling fraction for such generalized composite parafermion state is given by  $\nu = (\sum_i \frac{m_i}{n_i})^{-1}$ .

#### 4.3.4 Quasiparticle Fusion Algebra: $\gamma_1 \gamma_2 = \sum_{\gamma_3} N_{\gamma_1 \gamma_2}^{\gamma_3} \gamma_3$

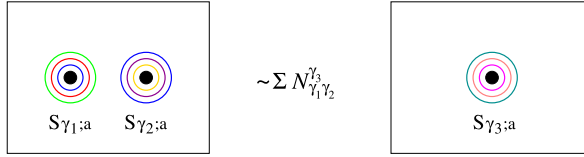
When we fuse quasiparticles  $\gamma_1$  and  $\gamma_2$  together, we can get a third quasiparticle  $\gamma_3$ . However, for non-Abelian quasiparticles, the fusion can be more complicated. Fusing  $\gamma_1$  and  $\gamma_2$  may produce several kind of quasiparticles. Such kind of fusion is described by quasiparticle fusion algebra (see Fig. 8):  $\gamma_1 \gamma_2 = \sum_{\gamma_3} N_{\gamma_1 \gamma_2}^{\gamma_3} \gamma_3$ , where  $N_{\gamma_1 \gamma_2}^{\gamma_3}$  are non-negative integers.

To calculate the fusion coefficients  $N_{\gamma_1 \gamma_2}^{\gamma_3}$  from the pattern of zeros, let us put the quasiparticle  $\gamma_1$  at  $z = 0$ . Far away from  $z = 0$ , such a quasiparticle has a pattern of zeros  $n_{\gamma_1; l}$  (in the occupation representation). We then insert a quasiparticle  $\gamma_2$  at  $z = R$  for a large  $R$ . At  $z = r \gg R$ , the occupation becomes the occupation of the quasiparticle  $\gamma_3$ :  $n_{\gamma_3; l}$ . We see that the fusion of  $\gamma_2$  changes the occupation pattern from  $n_{\gamma_1; l}$  to  $n_{\gamma_3; l}$ :

$$n_{\gamma_1; 0} n_{\gamma_1; 1} \cdots n_{\gamma_1; a} [\gamma_2] n_{\gamma_3; a+1} n_{\gamma_3; a+2} \cdots \quad (101)$$

So the quasiparticle  $\gamma_2$  becomes a “domain wall” between the  $\gamma_1$  occupation pattern and the  $\gamma_3$  occupation pattern [49].

From the above domain wall structure, we can see only  $n_{\gamma_1; l}$  and  $n_{\gamma_3; l}$ , but we cannot see  $n_{\gamma_2; l}$ . But this is enough for us. We are able to find a condition on  $n_{\gamma_2; l}$



**Fig. 8** The graphic picture of the fusion of two quasiparticles. *Each box* represent a many-boson wave function. *In the left box*, we have quasiparticle  $\gamma_1$  and  $\gamma_2$  described by patterns of zeros  $S_{\gamma_1;a}$  and  $S_{\gamma_2;a}$ . Far away from the two quasiparticles, the wave function may contain several different patterns of zeros  $S_{\gamma_3;a}$  that correspond to several different quasiparticle types  $\gamma_3$ . So we say that  $\gamma_1$  and  $\gamma_2$  may fuse into several different types of quasiparticles labeled by  $\gamma_3$

so that it can induce a domain wall between  $n_{\gamma_1;l}$  and  $n_{\gamma_3;l}$  [27]:

$$\sum_{j=1}^b (l_{\gamma_1;j+a}^{\text{sc}} + l_{\gamma_2;j+c}^{\text{sc}}) \leq \sum_{j=1}^b (l_{\gamma_3;j+a+c}^{\text{sc}} + l_j^{\text{sc}}) \quad (102)$$

for any  $a, b, c \in \mathbb{Z}_+$ , where  $l_{\gamma;a}^{\text{sc}} = l_{\gamma;a} - \frac{m(Q_{\gamma}+a-1)}{n}$ .

Solving the above equation allows us to determine when  $N_{\gamma_1\gamma_2}^{\gamma_3}$  can be non-zero. If we further assume that  $N_{\gamma_1\gamma_2}^{\gamma_3} = 0, 1$ , then the fusion algebra can be determined. Knowing  $N_{\gamma_1\gamma_2}^{\gamma_3}$  allows us to determine the ground state degeneracies of FQH state on any closed Riemann surfaces.

We like to mention that for the generalized composite parafermion states which have a CFT description, the pattern-of-zeros approach and the CFT approach give rise to the same fusion algebra. However, the pattern-of-zeros approach applies to other FQH states whose CFT may not be known.

## 5 The Vertex-Algebra + Pattern-of-Zeros Approach

### 5.1 Z-Graded Vertex Algebra

The symmetric polynomial  $P(\{z_i\})$  and the corresponding derived polynomial  $P_{\text{derived}}(\{z_i^{(a)}\})$  can be expressed as correlation functions in a vertex algebra:

$$P(\{z_i\}) = \left\langle \prod_i V(z_i) \right\rangle, \quad P_{\text{derived}}(\{z_i^{(a)}\}) = \left\langle \prod_{i,a} V_a(z_i^{(a)}) \right\rangle \quad (103)$$

$$V_a(z) = V^a, \quad V_a V_b = V_{a+b}.$$

The vertex algebra is generated by vertex operator  $V(z)$  and is described by the following operator product expansion:

$$V_a(z)V_b(w) = \frac{C_{ab}}{(z-w)^{h_a+h_b-h_{a+b}}} V_{a+b}(w) + \dots \quad (104)$$

where  $h_a$  is the scaling dimension of  $V_a$  and  $C_{ab}$  the structure constant of the vertex algebra. Such a vertex algebra is a  $Z$ -graded vertex algebra.

The pattern of zeros  $S_a$  discuss before is directly related to  $h_a$ :

$$h_{a+b} - h_a - h_b = D_{a,b} = S_{a+b} - S_a - S_b \quad (105)$$

The  $n$ -cluster condition implies that  $h_a \propto a^2$  if  $a = 0 \pmod n$ . This allows us to obtain

$$h_a = S_a - \frac{aS_n}{n} + \frac{am}{2} \quad (106)$$

We see that the pattern of zeros  $S_a$  only describe the scaling dimensions of the vertex operators. It does not describe the structure constants  $C_{a,b}$ . So a more complete characterization of FQH wave functions (symmetric polynomials) is given by  $(\frac{m}{n}; S_a; C_{ab}, \dots)$ . But  $(\frac{m}{n}; S_a; C_{ab}, \dots)$  may be an overkill. We like to find out what is the minimal set of data that can completely characterize the FQH wave functions (or the symmetric polynomials).

## 5.2 $Z_n$ -Vertex Algebra

If the above  $Z$ -graded vertex algebra satisfies the  $n$ -cluster condition, then it can be viewed a  $Z_n$ -vertex algebra  $\otimes$  a  $U(1)$  current algebra:

$$V_a(z) = \psi_a(z) e^{ia\phi(z)\sqrt{m/n}} \quad (107)$$

where  $j = \partial\phi$  generates the  $U(1)$  current algebra and  $\psi_a$  generates the  $Z_n$ -vertex algebra:

$$\psi_a(z)\psi_b(w) = \frac{C_{ab}}{(z-w)^{h_a^{\text{sc}}+h_b^{\text{sc}}-h_{a+b}^{\text{sc}}}} \psi_{a+b}(w) + \dots \quad (108)$$

where  $\psi_n = 1$  as the result of the  $n$ -cluster condition. The scaling dimension of  $\psi_a(z)$  is

$$h_a^{\text{sc}} = h_a - \frac{a^2m}{2n} = S_a - \frac{aS_n}{n} + \frac{am}{2} - \frac{a^2m}{2n}, \quad h_a^{\text{sc}} = h_{a+n}^{\text{sc}} \quad (109)$$

The two sets of data  $(\frac{m}{n}; S_2, \dots, S_n)$  and  $(\frac{m}{n}; h_1^{\text{sc}}, \dots, h_{n-1}^{\text{sc}})$  completely determine each other:

$$S_a = h_a^{\text{sc}} - ah_1^{\text{sc}} + \frac{a(a-1)m}{2n}. \quad (110)$$

So we can also use  $(\frac{m}{n}; h_1^{\text{sc}}, \dots, h_{n-1}^{\text{sc}})$  to describe the pattern of zeros.

From the pattern-of-zeros consideration, we find that  $h_a^{\text{sc}}$  must satisfy

$$S_a = h_a^{\text{sc}} - ah_1^{\text{sc}} + \frac{a(a-1)m}{2n} = \text{integer} \geq 0$$

$$h_{a+b}^{\text{sc}} - h_a^{\text{sc}} - h_b^{\text{sc}} + \frac{abm}{n} = D_{ab} = \text{integer} \geq 0 \quad (111)$$

$$h_{a+b+c}^{\text{sc}} - h_{a+b}^{\text{sc}} - h_{b+c}^{\text{sc}} - h_{a+c}^{\text{sc}} + h_a^{\text{sc}} + h_b^{\text{sc}} + h_c^{\text{sc}} \\ = \Delta_3(a, b, c) = \text{even integer} \geq 0 \quad (112)$$

But the above conditions are only on  $h_a^{\text{sc}}$ . To get the conditions on  $C_{ab}$ , we can use the generalized Jacobi identity [50] to obtain a set of non-linear equations for  $(h_a^{\text{sc}}, C_{ab}, \dots)$  [35]. Those conditions may be sufficient and necessary which may lead to a classification of  $Z_n$ -vertex algebra.

For some simple pattern of zeros  $h_a^{\text{sc}}$ , we are able to build a closed set of non-linear equations for  $(h_a^{\text{sc}}, C_{ab}, \dots)$ , which lead to a well defined  $Z_n$ -vertex algebra. This allows us to calculate quasiparticle scaling dimensions, quasiparticle statistics, central charge (edge spectrum) [35], ... We would like to point out that in [32] and [34], a very interesting approach based the pattern of zeros and modular transformation of torus is proposed, that allows us to calculate the fractional statistics of some quasiparticles directly from the pattern-of-zeros data. We also like to point out that finding valid  $(h_a^{\text{sc}}, C_{ab}, \dots)$  corresponds to finding a well defined  $Z_n$  vertex algebra. Finding the quasiparticle patterns of zeros corresponds to finding the representations of the  $Z_n$  vertex algebra.

But at moment, we cannot handle more general pattern of zeros  $h_a^{\text{sc}}$ , in the sense that we have some difficulties to obtain a closed set of non-linear algebraic equations for  $(h_a^{\text{sc}}, C_{ab}, \dots)$ . We hope that, after some further research, the pattern-of-zeros approach may lead to a classification of  $Z_n$ -vertex algebra, which in turn lead to a classification of symmetric polynomials and FQH states.

## 6 Summary

Although still incomplete, the pattern-of-zeros approach provides quite a powerful way to study symmetric polynomials with infinite variables and FQH states. It connects several very different fields, such as strongly correlated electron systems, topological quantum field theory, CFT (for the edge states), modular tensor category theory (for the quasiparticle statistics), and maybe a new field of infinite-variable symmetric polynomial. This article only reviews the first step in this very exciting direction. More exciting results are yet to come.

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