

Chapter 4

Interpolation

Interpolation by polynomials is a field in which stability issues have been addressed quite early. Section 4.5 will list a number of classical results.

4.1 Interpolation Problem

The usual *linear*¹ *interpolation problem* is characterised by a subspace V_n of the Banach space $C([0, 1])$ (with norm $\|\cdot\|_\infty$; cf. §3.4.7.1) and a set

$$\{x_{i,n} \in [0, 1] : 0 \leq i \leq n\}$$

of $n + 1$ different² interpolation points, also called nodal points. Given a tuple $\{y_i : 0 \leq i \leq n\}$ of ‘function values’, an interpolant $\Phi \in V_n$ with the property

$$\Phi(x_{i,n}) = y_i \quad (0 \leq i \leq n) \quad (4.1)$$

has to be determined.

Exercise 4.1. (a) *The interpolation problem is solvable for all tuple $\{y_i : 0 \leq i \leq n\}$, if and only if the linear space*

$$V_n := \{(\Phi(x_{i,n}))_{i=0}^n \in \mathbb{R}^{n+1} : \Phi \in V_n\}$$

has dimension $n + 1$.

(b) *If $\dim V_n = n + 1$, the interpolation problem is uniquely solvable.*

The interpolation problem (4.1) can be reduced to a system of $n + 1$ linear equations. As is well known, there are two alternatives for linear systems:

¹ The term ‘linear’ refers to the underlying *linear* space V_n , not to linear functions.

² In the case of the more general Hermite interpolation, a p -fold interpolation point ξ corresponds to prescribed values of the derivatives $f^{(m)}(\xi)$ for $0 \leq m \leq p - 1$.

- (a) either the interpolation problem is uniquely solvable for *arbitrary* values y_i or
 (b) the interpolant either does not exist for certain y_i or is not unique.

The *polynomial interpolation* is characterised by

$$V_n = \{\text{polynomials of degree } \leq n\}$$

and is always solvable. In the case of general vector spaces V_n , we always assume that the interpolation problem is uniquely solvable.

For the special values $y_i = \delta_{ij}$ (j fixed, δ_{ij} Kronecker symbol), one obtains an interpolant $\Phi_{j,n} \in V_n$, which we call the *j-th Lagrange function* (analogous to the Lagrange polynomials in the special case of polynomial interpolation).

Exercise 4.2. (a) The interpolant for arbitrary y_i ($0 \leq i \leq n$) is given by

$$\Phi = \sum_{i=0}^n y_i \Phi_{i,n} \in V_n. \quad (4.2)$$

(b) In the case of polynomial interpolation, the Lagrange polynomial is defined by

$$L_{i,n}(x) := \Phi_{i,n}(x) := \prod_{j \in \{0, \dots, n\} \setminus \{i\}} \frac{x - x_j}{x_i - x_j}. \quad (4.3)$$

For continuous functions f we define

$$I_n(f) := \sum_{i=0}^n f(x_{i,n}) \Phi_{i,n} \quad (4.4)$$

as interpolant of $y_i = f(x_{i,n})$. Hence

$$I_n : C([0, 1]) \rightarrow C([0, 1])$$

is a linear mapping from the continuous functions into itself.

Exercise 4.3. (a) The interpolation $I_n : X = C([0, 1]) \rightarrow C([0, 1])$ is continuous and linear; i.e., $I_n \in \mathcal{L}(X, X)$.

(b) I_n is a projection; i.e., $I_n I_n = I_n$.

The terms ‘convergence’, ‘consistency’ and ‘stability’ of the previous chapter can easily be adapted to the interpolation problem. Note that we have not only one interpolation I_n , but a family $\{I_n : n \in \mathbb{N}_0\}$ of interpolations.

The interval $[0, 1]$ is chosen without loss of generality. The following results can immediately be transferred to a general interval $[a, b]$ by means of the affine mapping $\phi(t) = (t - a)/(b - a)$. The Lagrange functions $\Phi_{i,n} \in C([0, 1])$ become $\hat{\Phi}_{i,n} := \Phi_{i,n} \circ \phi \in C([a, b])$. Note that in the case of polynomials, $\Phi_{i,n}$ and $\hat{\Phi}_{i,n}$ have the same polynomial degree n . The norms $\|I_n\|$ and the stability constant C_{stab} from §4.3 do not change! Also the error estimate (4.8) remains valid.

Another subject are interpolations on higher-dimensional domains $D \subset \mathbb{R}^d$. The general concept is still true, but the concrete one-dimensional interpolation methods do not necessarily have a counterpart in d dimensions. An exception are domains which are Cartesian products. Then one can apply the tensor product interpolation discussed in §4.7.

4.2 Convergence and Consistency

Definition 4.4 (convergence). A family $\{I_n : n \in \mathbb{N}_0\}$ of interpolations is called convergent if

$$\lim_{n \rightarrow \infty} I_n(f) \text{ exists for all } f \in C([0, 1]).$$

Of course, we intend that $I_n(f) \rightarrow f$, but here convergence can be defined without fixing the limit, since $\lim I_n(f) = f$ will come for free due to consistency.

Concerning consistency, we follow the model of (3.27).

Definition 4.5 (consistency). A family $\{I_n : n \in \mathbb{N}_0\}$ of interpolations is called consistent if there is a dense subset $X_0 \subset C([0, 1])$ such that

$$I_n(g) \rightarrow g \quad \text{for all } g \in X_0.$$

Exercise 4.6. Let $\{I_n\}$ be the interpolation by polynomials of degree $\leq n$. Show that a possible choice of the dense set in Definition 4.5 is $X_0 := \{\text{polynomials}\}$.

4.3 Stability

First, we characterise the operator norm $\|I_n\|$ (cf. (3.23)).

Lemma 4.7. $\|I_n\| = \|\sum_{i=0}^n |\Phi_{i,n}(\cdot)|\|_\infty$ holds with $\Phi_{i,n}$ from (4.4).

Proof. (i) Set $C_n := \|\sum_{i=0}^n |\Phi_{i,n}(\cdot)|\|_\infty$. For arbitrary $f \in C([0, 1])$ we conclude that

$$\begin{aligned} |I_n(f)(x)| &= \left| \sum_{i=0}^n f(x_{i,n}) \Phi_{i,n}(x) \right| \leq \sum_{i=0}^n \underbrace{|f(x_{i,n})|}_{\leq \|f\|_\infty} |\Phi_{i,n}(x)| \leq \|f\|_\infty \sum_{i=0}^n |\Phi_{i,n}(x)| \\ &\leq \|f\|_\infty C_n. \end{aligned}$$

Since this estimate holds for all $x \in [0, 1]$, it follows that $\|I_n(f)\| \leq C_n \|f\|_\infty$. Because f is arbitrary, $\|I_n\| \leq C_n$ is proved.

(ii) Let the function $\sum_{i=0}^n |\Phi_{i,n}(\cdot)|$ be maximal at x_0 : $\sum_{i=0}^n |\Phi_{i,n}(x_0)| = C_n$. Choose $f \in C([0, 1])$ with $\|f\|_\infty = 1$ and $f(x_{i,n}) = \text{sign}(\Phi_{i,n}(x_0))$. Then

$$|I_n(f)(x_0)| = \left| \sum_{i=0}^n f(x_{i,n}) \Phi_{i,n}(x_0) \right| = \sum_{i=0}^n |f(x_{i,n}) \Phi_{i,n}(x_0)| = C_n \|f\|_\infty$$

holds; i.e., $\|I_n(f)\|_\infty = C_n \|f\|_\infty$ for this f . Hence the operator norm

$$\|I_n\| = \sup \{ \|I_n(f)\|_\infty / \|f\|_\infty : f \in C([0, 1]) \setminus \{0\} \}$$

is bounded from below by $\|I_n\| \geq C_n$. Together with (i), the equality $\|I_n\| = C_n$ is proved. \square

Again, stability expresses the boundedness of the sequence of norms $\|I_n\|$.

Definition 4.8 (stability). A family $\{I_n : n \in \mathbb{N}_0\}$ of interpolations is called stable if

$$C_{\text{stab}} := \sup_{n \in \mathbb{N}_0} \|I_n\| < \infty \quad \text{for} \quad \|I_n\| = \left\| \sum_{i=0}^n |\Phi_{i,n}(\cdot)| \right\|_\infty. \quad (4.5)$$

In the context of interpolation, the stability constant C_{stab} is called *Lebesgue constant*.

Polynomial interpolation is a particular way to approximate a continuous function by a polynomial. Note that the more general approximation due to Weierstrass is convergent. The relation between the best possible polynomial approximation and the polynomial interpolation is considered next.

Remark 4.9. Given $f \in C([0, 1])$, let p_n^* be the best approximation to f by a polynomial³ of degree $\leq n$, while p_n is its interpolant. Then the following estimate holds:

$$\|f - p_n\| \leq (1 + C_n) \|f - p_n^*\| \quad \text{with } C_n = \|I_n\|. \quad (4.6)$$

Proof. Any polynomial of degree $\leq n$ is reproduced by interpolation, in particular, $I_n p_n^* = p_n^*$. Hence,

$$f - p_n = f - I_n f = f - [I_n(f - p_n^*) + I_n p_n^*] = f - p_n^* + I_n(f - p_n^*)$$

can be estimated as claimed above. \square

Note that by the Weierstrass approximation theorem 3.28,

$$\|f - p_n^*\| \rightarrow 0$$

holds. An obvious conclusion from (4.6) is the following: If stability would hold (i.e., $C_n \leq C_{\text{stab}}$), also $\|f - p_n\| \rightarrow 0$ follows. Instead, we shall show instability, and the asymptotic behaviour on the right-hand side in (4.6) depends on which process is faster: $\|f - p_n^*\| \rightarrow 0$ or $C_n \rightarrow \infty$.

³ The space of polynomials can be replaced by any other interpolation subspace V_n .

4.4 Equivalence Theorem

Following the scheme (3.22), we obtain the next statement.

Theorem 4.10 (convergence theorem). *Assume that the family $\{I_n : n \in \mathbb{N}_0\}$ of interpolations is consistent and stable. Then it is also convergent, and furthermore, $I_n(f) \rightarrow f$ holds.*

Proof. Let $f \in C([0, 1])$ and $\varepsilon > 0$ be given. There is some $g \in X_0$ with

$$\|f - g\|_\infty \leq \frac{\varepsilon}{2(1 + C_{\text{stab}})},$$

where C_{stab} is the stability constant. According to Definition 4.5, there is an n_0 such that $\|I_n(g) - g\|_\infty \leq \frac{\varepsilon}{2}$ for all $n \geq n_0$. The triangle inequality yields the desired estimate:

$$\begin{aligned} \|I_n(f) - f\|_\infty &\leq \|I_n(f) - I_n(g)\|_\infty + \|I_n(g) - g\|_\infty + \|g - f\|_\infty \\ &\leq C_{\text{stab}} \|f - g\|_\infty + \frac{\varepsilon}{2} + \|f - g\|_\infty \leq \frac{\varepsilon}{2(1 + C_{\text{stab}})} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2(1 + C_{\text{stab}})} = \varepsilon. \quad \square \end{aligned}$$

Again, the stability condition turns out to be necessary.

Lemma 4.11. *A convergent family $\{I_n : n \in \mathbb{N}_0\}$ of interpolations is stable.*

Proof. Since $\{I_n(f)\}$ converges, the I_n are uniformly bounded. Apply Corollary 3.39 with $X = Y = C([0, 1])$ and $T_n := I_n \in \mathcal{L}(X; Y)$. \square

Theorem 4.10 and Lemma 4.11 yield the following equivalence theorem.

Theorem 4.12. *Let the family $\{I_n : n \in \mathbb{N}_0\}$ of interpolations be consistent. Then convergence and stability are equivalent.*

4.5 Instability of Polynomial Interpolation

We choose the equidistant interpolation points $x_{i,n} = i/n$ and restrict ourselves to even n . The Lagrange polynomial $L_{\frac{n}{2},n}$ is particularly large in the subinterval $(0, 1/n)$. In its midpoint we observe the value

$$\begin{aligned} |L_{\frac{n}{2},n}(\tfrac{1}{2n})| &= \left| \prod_{\substack{j=0 \\ j \neq \frac{n}{2}}}^n \frac{\frac{1}{2n} - \frac{j}{n}}{\frac{1}{2} - \frac{j}{n}} \right| = \left| \prod_{\substack{j=0 \\ j \neq \frac{n}{2}}}^n \frac{\frac{1}{2} - j}{\frac{n}{2} - j} \right| \\ &= \frac{\frac{1}{2} \times \frac{1}{2} \times \frac{3}{2} \times \dots \times (\frac{n}{2} - \frac{3}{2}) \times (\frac{n}{2} + \frac{1}{2}) \times \dots \times (n - \frac{1}{2})}{[(\frac{n}{2})!]^2}. \end{aligned}$$

Exercise 4.13. *Show that the expression from above diverges exponentially.*

Because of $C_n = \|\sum_{i=0}^n |L_{i,n}(\cdot)|\|_\infty \geq \|L_{\frac{n}{2},n}\|_\infty \geq |L_{\frac{n}{2},n}(\frac{1}{2n})|$, interpolation (at equidistant interpolation points) cannot be stable. The true behaviour of C_n has first⁴ been described by Turetskii [21]:

$$C_n \approx \frac{2^{n+1}}{e n \log n}.$$

The asymptotic is improved by Schönhage [18, Satz 2] to⁵

$$C_n \approx 2^{n+1}/[e n (\gamma + \log n)],$$

where γ is Euler's constant.⁶ Even more asymptotic terms are determined in [11].

One may ask whether the situation improves for another choice of interpolation points. In fact, an asymptotically optimal choice are the so-called *Chebyshev points*:

$$x_{i,n} = \frac{1}{2} \left(1 + \cos \left(\frac{i+1/2}{n+1} \pi \right) \right)$$

(these are the zeros of the Chebyshev polynomial⁷ $T_{n+1} \circ \phi$, where $\phi(\xi) = 2\xi + 1$ is the affine transformation from $[0, 1]$ onto $[-1, 1]$). In this case, one can prove that⁸

$$\|I_n\| \leq 1 + \frac{2}{\pi} \log(n+1) \quad (4.7)$$

(cf. Rivlin [17, Theorem 1.2]), which is asymptotically the best bound, as the next result shows.

Theorem 4.14. *There is some $c > 0$ such that*

$$\|I_n\| > \frac{2}{\pi} \log(n+1) - c$$

holds for any choice of interpolation points.

In 1914, Faber [6] proved

$$\|I_n\| > \frac{1}{12} \log(n+1),$$

while, in 1931, Bernstein [1] showed the asymptotic estimate

⁴ For historical comments see [20].

⁵ The function $\varphi = \sum_{i=0}^n |L_{i,n}(\cdot)|$ attains its maximum C_n in the first and last interval. As pointed out by Schönhage [18, §4], φ is of similar size as in (4.7) for the middle interval.

⁶ The value $\gamma = 0.5772\dots$ is already given in Euler's first article [5]. Later, Euler computed 15 exact decimals places of γ .

⁷ The Chebyshev polynomial $T_n(x) := \cos(n \arccos(x))$, $n \in \mathbb{N}_0$, satisfies the three-term recursion $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ ($n \geq 1$), starting from $T_0(x) = 1$ and $T_1(x) = x$.

⁸ A lower bound is $\|I_n\| > \frac{2}{\pi} \log(n+1) + \frac{2}{\pi} (\gamma + \log \frac{8}{\pi}) = \lim_{n \rightarrow \infty} \|I_n\|$, where $\frac{2}{\pi} (\gamma + \log \frac{8}{\pi}) = 0.96252\dots$

$$\|I_n\| > \frac{2-\varepsilon}{\pi} \log(n+1) \quad \text{for all } \varepsilon > 0.$$

The estimate of Theorem 4.14 originates from Erdős [4]. The bound

$$\|I_n\| > \frac{1}{8\sqrt{\pi}} \log(n+1)$$

can be found in Natanson [12, p. 370f].

The idea of the proof is as follows. Given $x_{i,n} \in [0, 1]$, $0 \leq i \leq n$, construct a polynomial P of degree $\leq n$ (concrete construction, e.g., in [12, p. 370f], [13]) such that $|P(x_{i,n})| \leq 1$, but $P(\xi) > M_n$ for at least one point $\xi \in [0, 1]$. Since the interpolation of P is exact, i.e., $I_n(P) = P$, the evaluation at ξ yields

$$\begin{aligned} \|I_n\| &= \left\| \sum_{i=0}^n |L_{i,n}(\cdot)| \right\|_{\infty} \geq \sum_{i=0}^n |L_{i,n}(\xi)| \\ &\geq \sum_{i=0}^n |P(x_{i,n}) L_{i,n}(\xi)| \geq \left| \sum_{i=0}^n P(x_{i,n}) L_{i,n}(\xi) \right| = |P(\xi)| > M_n, \end{aligned}$$

proving $\|I_n\| > M_n$.

We conclude that any sequence of polynomial interpolations I_n is unstable.

4.6 Is Stability Important for Practical Computations?

Does the instability of polynomial interpolation mean that one should avoid polynomial interpolation altogether? Practically, one may be interested in an interpolation I_{n^*} for a *fixed* n^* . In this case, the theoretically correct answer is: the property of I_{n^*} has nothing to do with convergence and stability of $\{I_n\}_{n \in \mathbb{N}}$. The reason is that convergence and stability are asymptotic properties of the sequence $\{I_n\}_{n \in \mathbb{N}}$ and are in no way related to the properties of a particular member I_{n^*} of the sequence. One can construct two different sequences $\{I'_n\}_{n \in \mathbb{N}}$ and $\{I''_n\}_{n \in \mathbb{N}}$ —one stable, the other unstable—such that $I'_{n^*} = I''_{n^*}$ belongs to both sequences. This argument also holds for the quadrature discussed in the previous chapter.

On the other hand, we may expect that instability expressed by $C_n \rightarrow \infty$ may lead to large values of C_n , unless n is very small. We return to this aspect later.

The convergence statement from Definition 4.4 is, in practice, of no help. The reason is that the convergence from Definition 4.4 can be arbitrarily slow, so that for a fixed n , it yields no hint concerning the error $I_n(f) - f$. Reasonable error estimates can only be given if f has a certain smoothness, e.g., $f \in C^{n+1}([0, 1])$. Then the standard error estimate of polynomial interpolation states that

$$\|f - I_n(f)\|_{\infty} \leq \frac{1}{(n+1)!} C_{\omega}(I_n) \|f^{(n+1)}\|_{\infty}, \quad (4.8)$$

where

$$C_\omega(I_n) := \|\omega\|_\infty \quad \text{for } \omega(x) := \prod_{i=0}^n (x - x_{i,n})$$

(cf. [14, §1.5], [19], [15, §8.1.1], [8, §B.3]). The quantity $C_\omega(I_n)$ depends on the location of the interpolation points. It is minimal for the Chebyshev points, where

$$C_\omega(I_n) = 4^{-(n+1)}.$$

In spite of the instability of polynomial interpolation, we conclude from estimate (4.8) that convergence holds, provided that $\|f^{(n+1)}\|_\infty$ does not grow too much as $n \rightarrow \infty$ (of course, this requires that f be analytic). However, in this analysis we have overlooked the numerical rounding errors of the input data.⁹ When we evaluate the function values $f(x_{i,n})$, a perturbed result $f(x_{i,n}) + \delta_{i,n}$ is returned with an error $|\delta_{i,n}| \leq \eta \|f\|_\infty$. Therefore, the true interpolant is $I_n(f) + \delta I_n$ with $\delta I_n = \sum_{i=0}^n \delta_{i,n} \Phi_{i,n}$. An estimate of δI_n is given by $\eta \|I_n\| \|f\|_\infty$. This yields the error estimate

$$\begin{aligned} \|f - I_n(f) - \delta I_n\|_\infty &\leq \varepsilon_n^{\text{int}} + \varepsilon_n^{\text{per}} \quad \text{with} \\ \varepsilon_n^{\text{int}} &= \frac{1}{(n+1)!} C_\omega(I_n) \|f^{(n+1)}\|_\infty \quad \text{and} \quad \varepsilon_n^{\text{per}} = \eta \|I_n\| \|f\|_\infty. \end{aligned}$$

Since η is small (maybe of the size of machine precision), the contribution $\varepsilon_n^{\text{int}}$ is not seen in the beginning. However, with increasing n , the part $\varepsilon_n^{\text{int}}$ is assumed to tend to zero, while $\varepsilon_n^{\text{per}}$ increases to infinity because of the instability of I_n .

We illustrate this situation in two different scenarios. In both cases we assume that the analytic function f is such that the exact interpolation error (4.8) decays like $\varepsilon_n^{\text{int}} = e^{-n}$.

(1) Assume a perturbation error $\varepsilon_n^{\text{per}} = \eta e^n$ due to an exponential increase of $\|I_n\|$. The resulting error is

$$e^{-n} + \eta e^n.$$

Regarding n as a real variable, we find a minimum at $n = \frac{1}{2} \log \frac{1}{\eta}$ with the value $2\sqrt{\eta}$. Hence, we cannot achieve better accuracy than half the mantissa length.

(2) According to (4.7), we assume that $\varepsilon_n^{\text{int}} = \eta(1 + \frac{2}{\pi} \log(n+1))$, so that the sum

$$e^{-n} + \eta(1 + \frac{2}{\pi} \log(n+1))$$

is the total error. Here, minimising n is the solution to the fixed-point equation $n = \log(n+1) - \log(2\eta/\pi)$. For $\eta = 10^{-16}$ the minimal value 3.4η of the total error is taken at the integer value $n = 41$. The precision corresponds to almost the full mantissa length. Hence, in this case the instability $\|I_n\| \rightarrow \infty$ is completely harmless.¹⁰

⁹ There are further rounding errors, which we ignore to simplify the analysis.

¹⁰ To construct an example, where even for (4.7) the instability becomes obvious, one has to assume that the interpolation error decreases very slowly like $\varepsilon_n^{\text{int}} = 1/\log(n)$.

4.7 Tensor Product Interpolation

Finally, we give an example where the norm $\|I_n\|$ is required for the analysis of the interpolation error, even if we ignore input errors and rounding errors. Consider the function $f(x, y)$ in two variables $(x, y) \in [0, 1] \times [0, 1]$. The two-dimensional polynomial interpolation can easily be constructed from the previous I_n . The tensor product¹¹ $I_n^2 := I_n \otimes I_n$ can be applied as follows. First, we apply the interpolation with respect to x . For any $y \in [0, 1]$ we have

$$F(x, y) := I_n(f(\cdot, y))(x) = \sum_{i=0}^n f(x_{i,n}, y) \Phi_{i,n}(x).$$

In a second step, we apply I_n with respect to y :

$$I_n^2(f)(x, y) = I_n(F(x, \cdot))(y) = \sum_{i=0}^n \sum_{j=0}^n f(x_{i,n}, x_{j,n}) \Phi_{i,n}(x) \Phi_{j,n}(y).$$

Inequality (4.8) yields a first error¹²

$$|f(x, y) - F(x, y)| \leq \frac{1}{(n+1)!} C_\omega(I_n) \left\| \frac{\partial^{n+1}}{\partial x^{n+1}} f \right\|_\infty \quad \text{for all } x, y \in [0, 1].$$

The second one is

$$F(x, y) - I_n^2(f)(x, y) = \sum_{i=0}^n |f(x_{i,n}, y) - I_n(f(x_{i,n}, \cdot))(y)| \Phi_{i,n}(x).$$

Again

$$|f(x_{i,n}, y) - I_n(f(x_{i,n}, \cdot))(y)| \leq \frac{1}{(n+1)!} C_\omega(I_n) \left\| \frac{\partial^{n+1}}{\partial y^{n+1}} f \right\|_\infty$$

holds and leads us to the estimate

$$\|F - I_n^2(f)\|_\infty \leq \|I_n\| \frac{1}{(n+1)!} C_\omega(I_n) \left\| \frac{\partial^{n+1}}{\partial y^{n+1}} f \right\|_\infty.$$

The previous estimates and the triangle inequality yield the final estimate

$$\|f - I_n^2(f)\|_\infty \leq \frac{1}{(n+1)!} C_\omega(I_n) \left[\|I_n\| \left\| \frac{\partial^{n+1}}{\partial y^{n+1}} f \right\|_\infty + \left\| \frac{\partial^{n+1}}{\partial x^{n+1}} f \right\|_\infty \right].$$

Note that the divergence of $\|I_n\|$ can be compensated by $\frac{1}{(n+1)!}$.

¹¹ Concerning the tensor notation see [9].

¹² Here, $\|\cdot\|_\infty$ is the maximum norm over $[0, 1]^2$.

4.8 Stability of Piecewise Polynomial Interpolation

One possibility to obtain stable interpolations is by constructing a piecewise polynomial interpolation. Here, the degree of the piecewise polynomials is fixed, while the size of the subintervals approaches zero as $n \rightarrow \infty$. Let $J = [0, 1]$ be the underlying interval. The subdivision is defined by $\Delta_n := \{x_0, x_1, \dots, x_n\} \subset J$ containing points satisfying

$$0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1.$$

This defines the subintervals $J_k := [x_{k-1}, x_k]$ of length $h_k := x_k - x_{k-1}$ and $\delta_n := \max_{1 \leq k \leq n} h_k$. In principle, all quantities x_k , J_k , h_k should carry an additional index n , since each subdivision of the sequence $(\Delta_n)_{n \in \mathbb{N}}$ has different $x_k = x_k^{(n)}$. For the sake of simplicity we omit this index, except for the grid size δ_n , which has to satisfy $\delta_n \rightarrow 0$.

Among the class of piecewise polynomial interpolations, we can distinguish two types depending on the support¹³ of the Lagrange functions $\Phi_{j,n}$. In case of Type I, $\Phi_{j,n}$ has a local support, whereas $\text{supp}(\Phi_{j,n}) = J$ for Type II. The precise definition of a local support is: there are $\alpha, \beta \in \mathbb{N}_0$ independent of n such that

$$\text{supp}(\Phi_{j,n}) \subset \bigcup_{k=\max\{1, j-\alpha\}}^{\min\{n, j+\beta\}} J_k. \quad (4.9)$$

4.8.1 Case of Local Support

The simplest example is the linear interpolation where $I_n(f)(x_k) = f(x_k)$ and $f|_{J_k}$ (i.e., f restricted to J_k) is a linear polynomial. The corresponding Lagrange function $\Phi_{j,n}$ is called the *hat function* and has the support¹⁴ $\text{supp}(\Phi_{j,n}) = J_j \cup J_{j+1}$.

We may fix another polynomial degree d and fix points $0 = \xi_0 < \xi_1 < \dots < \xi_d = 1$. In each subinterval $J_k = [x_{k-1}, x_k]$ we define interpolation nodes $\zeta_\ell := x_{k-1} + (x_k - x_{k-1}) \xi_\ell$. Interpolating f by a polynomial of degree d at these nodes, we obtain $I_n(f)|_{J_k}$. Altogether, $I_n(f)$ is a continuous¹⁵ and piecewise polynomial function on J . Again, $\text{supp}(\Phi_{j,n}) = J_j \cup J_{j+1}$ holds.

A larger but still local support occurs in the following construction of piecewise cubic functions. Define $I_n(f)|_{J_k}$ by cubic interpolation at the nodes¹⁴ $x_{k-2}, x_{k-1}, x_k, x_{k+1}$. Then the support $\text{supp}(\Phi_{j,n}) = J_{j-1} \cup J_j \cup J_{j+1} \cup J_{j+2}$ is larger than before.

¹³ The support of a function f defined on I is the closed set $\text{supp}(f) := \overline{\{x \in I : f(x) \neq 0\}}$.

¹⁴ The expression has to be modified for the indices 1 and n at the end points.

¹⁵ If $I_n(f) \in C^1(I)$ is desired, one may use Hermite interpolation; i.e., also $dI_n(f)/dx = f'$ at $x = x_{k-1}$ and $x = x_k$. This requires a degree $d \geq 3$.

The error estimates can be performed for each subinterval separately. Transformation of inequality (4.8) to J_k yields $\|I_n(f) - f\|_{\infty, J_k} \leq Ch_k^{-(d+1)} \|f^{(d+1)}\|_{\infty}$, where d is the (fixed) degree of the local interpolation polynomial. The overall estimate is

$$\|I_n(f) - f\|_{\infty} \leq C\delta_n^{-(d+1)} \|f^{(d+1)}\|_{\infty} \rightarrow 0, \quad (4.10)$$

where we use the condition $\delta_n \rightarrow 0$.

Stability is controlled by the maximum norm of $\Phi_n := \sum_{i=1}^n |\Phi_{i,n}(\cdot)|$. For the examples from above it is easy to verify that $\|\Phi_{i,n}\| \leq K$ independently of i and n . Fix an argument $x \in I$. The local support property (4.9) implies that $\Phi_{i,n}(x) \neq 0$ holds for at most $\alpha + \beta + 1$ indices i . Hence $\sum_{i=1}^n |\Phi_{i,n}(x)| \leq \overline{C}_{\text{stab}} := (\alpha + \beta + 1)K$ holds and implies $\sup_n \|I_n\| \leq \overline{C}_{\text{stab}}$ (cf. (4.5)).

4.8.2 Spline Interpolation as an Example for Global Support

The space V_n of the *natural cubic splines* is defined by

$$V_n = \{f \in C^2(I) : f''(0) = f''(1) = 0, f|_{J_k} \text{ cubic polynomial for } 1 \leq k \leq n\}.$$

The interpolating spline function $S \in V_n$ has to satisfy $S(x_k) = f(x_k)$ for $0 \leq k \leq n$. We remark that S is also the minimiser of

$$\min \left\{ \int_J |g''(x)|^2 dx : g \in C^2(J) : S(x_k) = f(x_k) \text{ for } 0 \leq k \leq n \right\}.$$

In this case the support of a Lagrange function $\Phi_{j,n}$, which now is called a *cardinal spline*, has global support:¹⁶ $\text{supp}(\Phi_{j,n}) = J$. Interestingly, there is another basis of V_n consisting of so-called *B-splines* B_j , whose support is local:¹⁴ $\text{supp}(B_j) = J_{j-1} \cup J_j \cup J_{j+1} \cup J_{j+2}$. Furthermore, they are non-negative and sum up to

$$\sum_{j=0}^n B_j = 1. \quad (4.11)$$

We choose an equidistant¹⁷ grid; i.e., $J_i = [(i-1)h, ih]$ with $h := 1/n$. The stability estimate $\|I_n\| = \|\sum_{i=0}^n |\Phi_{i,n}(\cdot)|\|_{\infty} \leq C_{\text{stab}}$ (cf. (4.5)) is equivalent to

$$\|S\|_{\infty} \leq C_{\text{stab}} \|y\|_{\infty}, \quad \text{where } S = \sum_{i=0}^n y_i \Phi_{i,n} \in V_n$$

is the spline function interpolating $y_i = S(x_i)$. In the following, we make use of the

¹⁶ $\Phi_{j,n}$ is non-negative in $J_j \cup J_{j+1}$ and has oscillating signs in neighbouring intervals. One can prove that the maxima of $\Phi_{j,n}$ in J_k are exponentially decreasing with $|j - k|$. This fact can already be used for a stability proof.

¹⁷ For the general case compare [14, §2], [15, §8.7], [19, §2.4].

B-splines, which easily can be described for the equidistant case.¹⁸ The evaluation at the grid points yields

$$\begin{aligned} B_0(0) &= B_n(1) = 1, & B_0(h) &= B_n(1-h) = 1/6, \\ B_1(0) &= B_{n-1}(1) = 0, \\ B_1(h) &= B_{n-1}(1-h) = 2/3, & B_1(2h) &= B_{n-1}(1-2h) = 1/6, \\ B_j(x_j) &= 2/3, \quad B_j(x_{j\pm 1}) = 1/6 & \text{for } 2 \leq j \leq n-2. \end{aligned} \quad (4.12)$$

One verifies that $y_j := \sum_{j=0}^n B_j(x_j) = 1$. Since the constant function $S = 1 \in V_n$ is interpolating, the unique solvability of the spline interpolation proves (4.11). Now we return to a general spline function $S = \sum_{i=0}^n y_i \Phi_{i,n}$. A representation by B-splines reads $S = \sum_{j=0}^n b_j B_j$. Note that $y_i = S(x_i) = \sum_{j=0}^n b_j B_j(x_i)$. Inserting the values from (4.12), we obtain

$$y = Ab \quad \text{with } A = \frac{1}{6} \begin{bmatrix} 6 & & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & & 6 \end{bmatrix} b$$

for the vectors $y = (y_i)_{i=0}^n$ and $b = (b_i)_{i=0}^n$. A can be written as $A = \frac{2}{3}[I + \frac{1}{2}C]$ with $\|C\|_\infty = 1$; i.e., A is strongly diagonal dominant and the inverse satisfies $\|A^{-1}\|_\infty \leq 3$ because of

$$A^{-1} = \frac{3}{2}[I + \frac{1}{2}C]^{-1} = \frac{3}{2} \sum_{\nu=0}^{\infty} 2^{-\nu} C^\nu.$$

Using $b = A^{-1}y$, we derive from $S = \sum_{j=0}^n b_j B_j$ that

$$|S(x)| = \left| \sum_{j=0}^n b_j B_j(x) \right| \underset{B_j \geq 0}{\leq} \sum_{j=0}^n |b_j| B_j(x) \leq \|b\|_\infty \sum_{j=0}^n B_j(x) \underset{(4.11)}{=} \|b\|_\infty$$

for all $x \in J$, so that the stability estimate $\|S\|_\infty \leq C_{\text{stab}} \|y\|_\infty$ is proved with $C_{\text{stab}} := 3$.

¹⁸ The explicit polynomials are

$$\begin{aligned} B_j &= \frac{1}{6h^3} \begin{cases} \xi^3, & \xi = x - x_{j-2}, \quad x \in J_{j-1} \\ h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3, & \xi = x - x_{j-1}, \quad x \in J_j, \\ h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3, & \xi = x_{j+1} - x, \quad x \in J_{j+1}, \\ \xi^3, & \xi = x_{j+2} - x, \quad x \in J_{j+2}, \end{cases} \quad \text{for } 2 \leq j \leq n-2, \\ B_1 &= \frac{1}{6h^3} \begin{cases} 6h^2x - 2x^3, & x \in J_1, \\ h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3, & \xi = 2h - x, \quad x \in J_2, \\ \xi^3, & \xi = 3h - x, \quad x \in J_3, \end{cases}, \quad B_{n-1}(x) = B_1(1-x), \\ B_0 &= \frac{1}{6h^3} \begin{cases} h^3 + 3h^2\xi + 3h\xi^2 - \xi^3, & \xi = h - x, \quad x \in J_1, \\ \xi^3, & \xi = 2h - x, \quad x \in J_2, \end{cases}, \quad B_n(x) = B_0(1-x). \end{aligned}$$

Remark 4.15. *The previous results show that consistency is in conflict with stability. Polynomial interpolation has an increasing order of consistency, but suffers from instability (cf. Theorem 4.14). On the other hand, piecewise polynomial interpolation of bounded order is stable.*

4.9 From Point-wise Convergence to Operator-Norm Convergence

As already mentioned in §3.5 in the context of quadrature, only *point-wise convergence* $I_n(f) \rightarrow f$ ($f \in X$) can be expected, but not operator-norm convergence $\|I_n - id\| \rightarrow 0$. However, there are situations in which point-wise convergence can be converted into operator-norm convergence.

An operator $K : X \rightarrow Y$ is called *compact* if the image $B := \{Kf : \|f\|_X \leq 1\}$ is precompact (cf. page 44). The following theorem is formulated for an arbitrary, point-wise convergent sequence of operators $A_n : Y \rightarrow Z$.

Theorem 4.16. *Let X, Y, Z be Banach spaces, and $A, A_n \in \mathcal{L}(Y, Z)$. Suppose that point-wise convergence $A_n y \rightarrow Ay$ holds for all $y \in Y$. Furthermore, let $K : X \rightarrow Y$ be compact. Then the products $P_n := A_n K$ converge with respect to the operator norm to $P := AK$; i.e., $\|P_n - P\| \rightarrow 0$.*

Proof. $M := \{Kx : \|x\|_X \leq 1\} \subset Y$ is precompact because of the compactness of K , so that we can apply Lemma 3.49:

$$\begin{aligned} \|P_n - P\| &= \sup\{\|P_n x - Px\|_Z : x \in X, \|x\| \leq 1\} \\ &= \sup\{\|A_n(Kx) - A(Kx)\|_Z : x \in X, \|x\| \leq 1\} \\ &= \sup\{\|A_n y - Ay\|_Z : y \in M\} \xrightarrow{(3.29)} 0. \end{aligned} \quad \square$$

A typical example of a compact operator is the embedding

$$E : \left(C^\lambda([0, 1]), \|\cdot\|_{C^\lambda([0, 1])} \right) \rightarrow (C([0, 1]), \|\cdot\|_\infty).$$

For integer $\lambda \in \mathbb{N}$, $C^\lambda([0, 1])$ is the space of λ -times continuously differentiable functions, where the norm $\|\cdot\|_{C^\lambda([0, 1])}$ is the maximum of all derivatives up to order λ . For $0 < \lambda < 1$, $C^\lambda([0, 1])$ are the Hölder continuous functions with $\|\cdot\|_{C^\lambda([0, 1])}$ explained in Footnote 9 on page 44. The embedding is the identity mapping: $E(f) = f$; however, the argument f and the image $E(f)$ are associated with different norms. As mentioned in the proof of Theorem 3.50, $E \in \mathcal{L}(C^\lambda([0, 1]), C([0, 1]))$ is compact.

In the case of $\lambda = 4$, estimate (4.10) already yields the operator-norm convergence $\|I_n - id\|_{C([0, 1]) \leftarrow C^4([0, 1])} \leq C/n^4 \rightarrow 0$. To show a similar operator-norm convergence for $0 < \lambda < 1$, interpret $I_n - id$ as $(I_n - id)E : C^\lambda([0, 1]) \rightarrow C([0, 1])$. Applying Theorem 4.16 with $A = id$, $A_n = I_n$, and $K = E$, we obtain

$$\|I_n - id\|_{C([0, 1]) \leftarrow C^\lambda([0, 1])} \rightarrow 0.$$

4.10 Approximation

Often, interpolation is used as a simple tool to obtain an approximation; i.e., the interpolation condition (4.1) is not essential. Instead, we can directly ask for a best approximation $\Phi_n \in V_n$ of $f \in B$, where $V_n \subset B$ is an $(n + 1)$ -dimensional subspace of a Banach space B with norm $\|\cdot\|$:

$$\|f - \Phi_n\| = \inf \{\|f - g\| : g \in V_n\}. \quad (4.13)$$

Using compactness arguments one obtains the existence of a minimiser Φ . If the space B is strictly convex,¹⁹ the minimiser is unique (cf. [10]).

A prominent choice of V_n are the polynomials of degree $\leq n$, while $B = C([a, b])$ is equipped with the maximum norm. Polynomials satisfy the following *Haar condition*: any $0 \neq f \in V_n$ has at most n zeros (cf. Haar [7]). As a consequence, also in this case, the best approximation problem (4.13) has a unique solution. For the numerical solution of the best approximation the following equi-oscillation property is essential (cf. Chebyshev [2]):

Theorem 4.17. *Let $\varepsilon := f - \Phi_n$ be the error of the best approximation in (4.13). Then there are $n + 2$ points x_μ with $a \leq x_0 < x_1 < \dots < x_{n+1} \leq b$ such that*

$$|\varepsilon(x_\mu)| = \|f - \Phi_n\| \quad \text{and} \quad \varepsilon(x_\mu) = -\varepsilon(x_{\mu+1}) \quad \text{for } 0 \leq \mu \leq n. \quad (4.14)$$

The second part of (4.14) describes $n + 1 = \dim(V_n)$ equations, which are used by the Remez algorithm to determine $\Phi_n \in V_n$ (cf. Remez [16]).

From (4.14) one concludes that there are n zeros $\xi_1 < \dots < \xi_n$ of $\varepsilon = f - \Phi_n$; i.e., Φ_n can be regarded as an interpolation polynomial with these interpolation points. However note that the ξ_μ depend on the function f .

The mapping $f \mapsto \Phi_n$ is in general nonlinear. Below, when we consider Hilbert spaces, it will become a linear projection.

Since the set of polynomials is dense in $C([a, b])$ (cf. Theorem 3.28), the condition

$$V_0 \subset V_1 \subset \dots \subset V_n \subset V_{n+1} \subset \dots \quad \text{and} \quad \overline{\bigcup_{n \in \mathbb{N}_0} V_n} = C([a, b]) \quad (4.15)$$

is satisfied. Condition (4.15) implies

$$\|f - \Phi_n\| \searrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for } \Phi_n \text{ from (4.13)}. \quad (4.16)$$

Stability issues do not appear in this setting. One may consider the sequence $\{\|\Phi_n\| : n \in \mathbb{N}_0\}$, but (4.16) proves convergence $\|\Phi_n\| \rightarrow \|f\|$; i.e., the sequence must be uniformly bounded.

The approximation is simpler if B is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Then the best approximation from (4.13) is obtained by means of the orthogonal

¹⁹ B is strictly convex if $\|f\| = \|g\| = 1$ and $f \neq g$ imply $\|f + g\| < 2$.

projection²⁰ $\Pi_n \in \mathcal{L}(B, B)$ onto V_n :

$$\Phi_n = \Pi_n f.$$

Given any orthonormal basis $\{\phi_\mu : 0 \leq \mu \leq n\}$ of V_n , the solution has the explicit representation

$$\Phi_n = \sum_{\mu=0}^n \langle f, \phi_\mu \rangle \phi_\mu. \quad (4.17)$$

The standard example is the Fourier approximation of 2π periodic real functions in $[-\pi, \pi]$. The L^2 scalar product is $\langle f, g \rangle = \int_{-\pi}^{\pi} f g dx$. Let n be even. V_n is spanned by the orthonormal basis functions

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(mx)}{\sqrt{\pi}}, \frac{\sin(mx)}{\sqrt{\pi}} : 1 \leq m \leq n/2 \right\}.$$

At first glance there is no stability problem to be discussed, since the operator norm of orthogonal projections equals one: $\|\Pi_n\|_{L^2 \leftarrow L^2} = 1$. However, if we consider the operator norm $\|\Pi_n\|_{B \leftarrow B}$ for another Banach space, (in)stability comes into play.

Let Π_n be the Fourier projection from above and choose the Banach space $B = C_{2\pi} := \{f \in C([-\pi, \pi]) : f(-\pi) = f(\pi)\}$ equipped with the maximum norm $\|\cdot\|_\infty$. We ask for the behaviour of $\|\Pi_n\|_\infty$, where now $\|\cdot\|_\infty = \|\cdot\|_{C_{2\pi} \leftarrow C_{2\pi}}$ denotes the operator norm. The mapping (4.17) can be reformulated by means of the Dirichlet kernel,

$$(\Phi_n f)(x) = \frac{1}{\pi} \int_0^\pi \frac{\sin(2n+1)y}{\sin(y)} [f(x+2y) + f(x-2y)] dy.$$

From this representation we infer that

$$\|\Pi_n\|_\infty = \frac{1}{\pi} \int_0^\pi \left| \frac{\sin(2n+1)y}{\sin(y)} \right| dy.$$

Lower and upper bounds of this integral are

$$\frac{4}{\pi^2} \log(n+1) \leq \|\Pi_n\|_\infty \leq 1 + \log(2n+1).$$

This shows that the Fourier projection is unstable with respect to the maximum norm. The negation of the uniform boundedness theorem 3.38 together with $\|\Pi_n\|_\infty \rightarrow \infty$ implies the well-known fact that uniform convergence $\Pi_n f \rightarrow f$ cannot hold for any $f \in C_{2\pi}$.

The orthogonal Fourier projection Π_n is the best choice for the Hilbert space $L^2([-\pi, \pi])$. For $C_{2\pi}$ one may choose another projection P_n from $C_{2\pi}$ onto V_n .

²⁰ That means (i) $\Pi_n \Pi_n = \Pi_n$ (projection property) and (ii) Π_n is selfadjoint: $\langle \Pi_n f, g \rangle = \langle f, \Pi_n g \rangle$ for all $f, g \in B$.

This, however, can only lead to larger norms $\|P_n\|_\infty$ due to the following result of Cheney et al. [3].

Theorem 4.18. *The Fourier projection Π_n is the unique minimiser of*

$$\min\{\|P_n\|_\infty : P_n \in \mathcal{L}(C_{2\pi}, C_{2\pi}) \text{ projection onto } V_n\}.$$

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