

Chapter 2

Communication in Engineered Quantum Networks

Georgios M. Nikolopoulos, Thomas Brougham, Antonin Hoskovec,
and Igor Jex

Abstract We review the engineering of passive quantum networks for performing fundamental quantum communications tasks, such as the transfer, routing, and splitting of signals that are associated with quantum states. After an introduction to fundamental concepts and notions, the problem of quantum state transfer is discussed for networks of various physical and logical topologies. The discussion is cast in terms of a unified theoretical formalism, which is perfectly suited to addressing the problem in the context of various physical realizations.

2.1 Introduction

In this section we introduce the reader to fundamental concepts of quantum networks.

G.M. Nikolopoulos (✉)

Foundation for Research and Technology – Hellas, Institute of Electronic Structure and Laser,
P.O. Box 1385, GR-711 10 Heraklion, Greece
e-mail: nikolg@iesl.forth.gr

T. Brougham

Department of Physics, University of Strathclyde, 107 Rottenrow, Glasgow, G4 0NG,
Scotland, UK
e-mail: thomas.brougham@gmail.com

A. Hoskovec · I. Jex

Faculty of Nuclear Sciences and Physical Engineering, Department of Physics,
Czech Technical University in Prague, Břehová 7, Praha 1, Staré Město, 115 19,
Czech Republic
e-mail: a.hoskovec@gmail.com; igor.jex@fjfi.cvut.cz

2.1.1 Elements of Quantum Networks

A quantum network typically consists of a number of quantum objects (e.g., atoms, ions, quantum dots, cavities, etc.), to be referred to hereafter as the sites or the nodes of the network. In other words, quantum networks are discrete lattices of various geometric arrangements, and the interaction law underlying the coupling of the different sites depends on the details of the particular realization under consideration.

In the framework of quantum information processing (QIP), quantum networks may perform different communication tasks, such as transfer, routing, switching and splitting of quantum signals. The information to be communicated is encoded onto quantum states associated with excitations of the lattice. The elementary unit of quantum information is the quantum bit (qubit), which is represented by a physical system with well-defined characteristics allowing for meaningful separation and addressing of two orthogonal states making up the qubit basis, which can store the logical 0 and 1 as well as their arbitrary superpositions. In general, the encoding of a logical bit may involve one or more physical qubits (e.g., electrons, photons, atoms, ions, etc.). Theoretical studies have shown that the use of quantum states in various information-processing tasks may lead to remarkable qualitative as well as quantitative improvements [1].

Depending on whether external control is required for the transfer of the state (besides the initialization and the read-out processes which are always present), one can distinguish between active and passive networks. The operation of *active quantum networks* relies on some sort of external control. The most straightforward approach in this context pertains to a sequence of SWAP gates, which ensure the successive transfer of the state between neighbouring sites; an approach that requires the ability to modulate in time the strength and the nature of the interactions throughout the entire network. Another class of active networks relies on the encoding of information on wave-packet states that pertain to a large number of nodes and ensure low dispersion and transport at a definite group velocity. In either case, the operation of active networks relies on a judicious sequence of operations and/or measurements that are applied individually or collectively on the nodes. As a result, active networks are considered to be very susceptible to errors that accumulate in each operation applied during the transfer, as well as to decoherence and dissipation, since they have to be connected to the environment (e.g., to macroscopic devices), in order to allow for external control. Such a type of errors and problems can be avoided by resorting to networks with permanently coupled sites, the so-called *passive quantum networks*. These networks are engineered under the constraint of minimal external control, in the sense that they do not require any external control to perform their tasks (besides the initialization and read-out). Thus, they are also less susceptible to decoherence and dissipation, since they can be isolated from their environment to a large extent. The engineering typically pertains to the permanent adjustment of certain parameters across the network.

In the spirit of large-scale QIP, passive quantum networks may operate as black boxes that perform certain communication tasks, and they do not require

any external control, or prior knowledge on the information to be transferred. For instance, a passive quantum chain that ensures the faithful transfer of a quantum signal between its two ends, can be used to build a quantum chip by interconnecting different small components, such as processors and memories. Moreover, in this case there is no need for converting back and forth between different types of information carriers (e.g., photons and atoms), since passive quantum networks can be engineered with hardware that is fully compatible with the one used for the other components of the chip, including the information carriers.

In the following we focus on passive quantum networks, and the transfer of single- or multi-qubit states. Some of the results, however, are expected to be applicable to cases where the information is encoded in a higher-dimensional quantum system.¹

2.1.2 *Hamiltonians*

A physical implementation of a quantum network involves one way or another a discrete lattice, and there are different scenarios one may consider for the transfer of a state between two or more nodes. Schematic representations of two different scenarios are depicted in Fig. 2.1, for a one-dimensional (1D) network, and a single-qubit state. In the first scenario depicted in Fig. 2.1a, the transfer of the state is accompanied by the transfer of the physical qubit. The lattice is initially in its ground state (it could be also the vacuum), and the information is encoded onto the state of excess particle(s) that play the roles of information carriers. When the ground state is not the vacuum, but rather it pertains to a particular configuration of particles (not shown in Fig. 2.1a), it should remain frozen throughout the evolution of the system, so that only the excess information carrier(s) participate in the transfer. When the ground state is the vacuum, this condition is readily satisfied. In the second scenario depicted in Fig. 2.1b, each site of the network is occupied by a physical qubit, and the entire network is initially prepared in its ground state. The information is encoded in the state of some of the existing physical qubits, and this excitation (perturbation) propagates along the chain due to the presence of always-on couplings between adjacent qubits. In the following we refer to these two scenarios as A and B.

Whether a physical qubit is involved in the transfer of the state or not, the dynamics of the lattice can be analysed in the framework of second quantization. In general, each particle in the lattice, and thus the state of the network, is characterized by many degrees of freedom, such as its position on the lattice, the single-particle energy level (orbital) that it occupies, as well as additional internal degrees of freedom. Throughout this chapter we will focus on networks operating under the

¹For a d -dimensional system the unit of quantum information is the qudit.

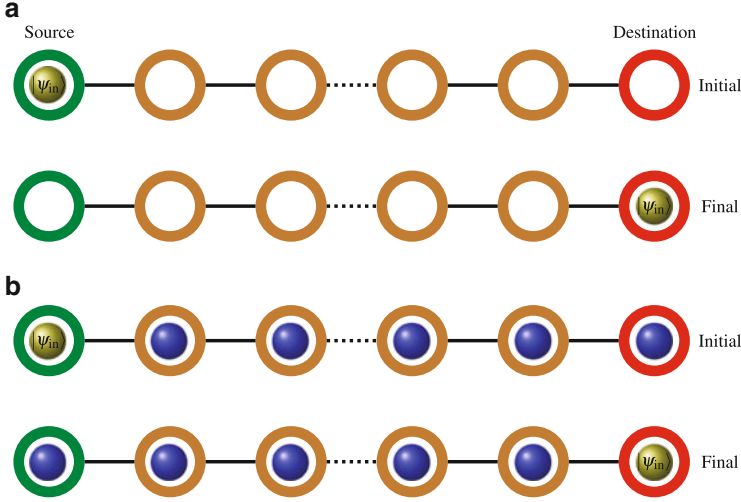


Fig. 2.1 Two different schemes for transfer of quantum states on a 1D lattice. The sites of the network are represented by *open circles*, and the physical qubits by *filled circles* (different colours denote different qubit states). **(a)** Scenario A: The transfer of the input, in this case single-qubit state $|\psi_{\text{in}}\rangle$, from the first to the last site is accompanied by the transfer of the associated physical qubit. **(b)** Scenario B: All the sites are occupied by physical qubits, and the state is transferred via exchange interactions, without altering the initial qubit configuration (i.e., one physical qubit per site with the states of the outermost qubits exchanged)

tight-binding approximation, with the dynamics restricted to the lowest available orbitals.² The Hamiltonians of interest are of the form

$$\begin{aligned} \hat{\mathcal{H}}_Q = & \sum_{i,\xi} \varepsilon_{i,\xi} \hat{n}_{i,\xi} + \frac{1}{2} \sum_{i,k} \sum_{\xi,\xi'} U_{i,\xi}^{k,\xi'} \hat{n}_{i,\xi} (\hat{n}_{k,\xi'} - \delta_{i,k} \delta_{\xi,\xi'}) \\ & + \sum_{i < k} \sum_{\xi} g_{i,k}^{(\xi)} (\hat{a}_{i,\xi}^\dagger \hat{a}_{k,\xi} + \hat{a}_{k,\xi}^\dagger \hat{a}_{i,\xi}), \end{aligned} \quad (2.1)$$

where the fermionic/bosonic operator $\hat{a}_{j,\xi}^\dagger$ creates a particle at the lowest available orbital of the j th site and in state $|\xi\rangle$, which accounts for internal degrees of freedom (e.g., spin, angular momentum, etc.). The number operator is $\hat{n}_{i,\xi} = \hat{a}_{i,\xi}^\dagger \hat{a}_{i,\xi}$, while typically the single-particle energy can be expressed as

$$\varepsilon_{i,\xi} = \varepsilon_i + \varepsilon_\xi, \quad (2.2)$$

²The energy separation of successive orbitals in a node is assumed to be much larger than the coupling constants between adjacent nodes, as well as than the typical energy scales of various excitation mechanisms.

where ε_i is associated with the occupied site and orbital, whereas ε_ξ refers to the internal state of the particles. Thus the first term of (2.1) can be split into two parts as follows

$$\sum_{i,\xi} \varepsilon_{i,\xi} \hat{n}_{i,\xi} = \sum_i \varepsilon_i \hat{n}_i + \sum_\xi \varepsilon_\xi \hat{n}_\xi, \quad (2.3)$$

where $\hat{n}_\xi \equiv \sum_j \hat{n}_{j,\xi}$ and $\hat{n}_j \equiv \sum_\xi \hat{n}_{j,\xi}$. The operator \hat{n}_j refers to the total number of particle at the lowest orbital of the j th site (irrespective of their internal state), and the operator \hat{n}_ξ gives the total number of particles with internal state $|\xi\rangle$ for all the sites. Clearly, the operator $\hat{n} = \sum_\xi \hat{n}_\xi = \sum_j \hat{n}_j$ refers to total number of particles in the system.

The second term of the Hamiltonian describes repulsive ($U_{i,\xi}^{k,\xi'} > 0$) or attractive ($U_{i,\xi}^{k,\xi'} < 0$) inter-particle interactions, whose magnitude may depend on the position of the two particles in the network as well as on ξ . The dependence on ξ , however, is usually so weak that can be neglected. Finally, the last term of the Hamiltonian describes the hopping of particles between the lowest orbitals of adjacent sites in the network, with the corresponding set of coupling constants denoted by $\{g_{i,k}^{(\xi)}\}$, and may in general depend on ξ . Such types of Hamiltonians appear in the context of various physical realizations of lattices (e.g., photonic lattices, optical lattices, electrons confined in coupled quantum dots, etc.).

Henceforth, the variable ξ will be associated with two orthogonal internal states that form our qubit basis

$$|\xi\rangle \in \{|u\rangle, |v\rangle\}. \quad (2.4)$$

In order for the states $\{|u\rangle, |v\rangle\}$ to form a qubit basis, they have to be well-separated, and thus the energy difference $|\varepsilon_{i,u} - \varepsilon_{i,v}|$ should be sufficiently large for all i .³ Let us define here the operator $\hat{\mathcal{E}} = \hat{n}_u - \hat{n}_v$, which is the total ξ -imbalance i.e., the difference of the total populations in the two qubit basis states. One can readily confirm the following observation.

Observation 1. Hamiltonian (2.1) commutes with the operators \hat{n} , \hat{n}_ξ and $\hat{\mathcal{E}}$. Thus, it preserves the total number of excitations, as well as the degrees of freedom ξ , and the system remains in the initially occupied sector of the Hilbert space.

By contrast to the conservation of degrees of freedom ξ , note that Hamiltonian (2.1) does not preserve the position of the excitations. Thus, it can be expressed in the form $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{V}}$, where $\hat{\mathcal{H}}_0$ and $\hat{\mathcal{V}}$ are the unperturbed and the interaction parts of the Hamiltonian, with respect to the position degree of freedom.

³It should be sufficiently large with respect to decoherence mechanisms, but at the same time sufficiently small with respect to other system parameters e.g., the energy separation between successive orbitals, so that the dynamics are restricted to equivalent orbitals only.

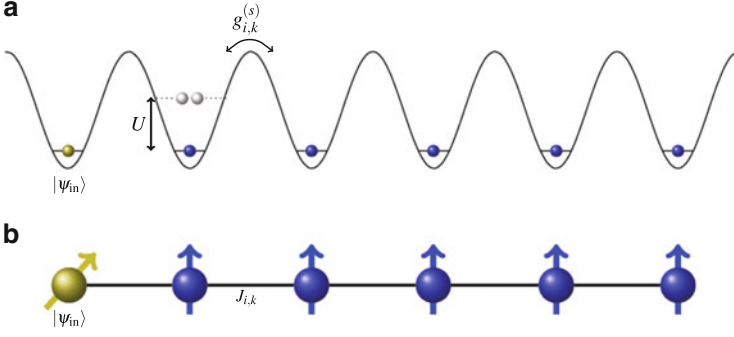


Fig. 2.2 (a) Schematic representation of a 1D lattice where the lowest orbital of each site is occupied by a fermionic particle with two internal states (the related energy difference is not shown). The input state is encoded in the state of the first particle. Single-particle hopping is energetically expensive, and double occupancy is absent. Exchange of particles, however, is possible, thus allowing for the transfer of the state. (b) The spin-chain equivalent of (a). Each site is associated with a spin, and the spins interact via spin-exchange interactions

The vast majority of papers on the problem of quantum state transfer have been formulated in the framework of spin-chain Hamiltonians. In many physical realizations of lattices, however, spin-chain Hamiltonians arise as effective Hamiltonians describing adequately the dynamics of the lattice only for a specific regime of parameters, although the real system does actually involve many degrees of freedom. To illustrate this fact, consider a deep 1D lattice with identical sites and all the interactions restricted to nearest-neighbours only (see Fig. 2.2a). The lowest orbital of each site is occupied by a single fermionic particle with two relevant orthogonal internal states $\{|u\rangle, |v\rangle\}$. This is a realistic scenario that can be implemented e.g., in the context of optical lattices or chains of quantum dots. The dynamics of the system are described by the Hamiltonian (2.1), which in view of the Pauli principle can be simplified to

$$\hat{\mathcal{H}}_Q = \sum_{i,\xi} \varepsilon_{i,\xi} \hat{n}_{i,\xi} + \sum_i U_{i,u}^{i,v} \hat{n}_{i,u} \hat{n}_{i,v} + \sum_{\langle i,k \rangle} \sum_{\xi} g_{i,k}^{(\xi)} (\hat{a}_{i,\xi}^{\dagger} \hat{a}_{k,\xi} + \hat{a}_{k,\xi}^{\dagger} \hat{a}_{i,\xi}), \quad (2.5)$$

where $\langle i,k \rangle$ denotes nearest neighbour sites. For a lattice with identical sites, we also have $\varepsilon_{i,\xi} = \varepsilon + \varepsilon_{\xi}$ and $U_{i,u}^{i,v} \equiv U_{u,v}$.

In this case, for $g_{i,k}^{(\xi)} \ll U_{u,v}$, the initial configuration of the particles remains practically frozen, because single-particle jumps cost a lot of energy, since they result in changes of the total on-site population. As shown in [2, 3], in this regime Eq. (2.5) is equivalent to the following effective spin-chain Hamiltonian

$$\hat{\mathcal{H}}_S = \sum_k B_k \hat{\sigma}_k^z - \sum_{\langle i,k \rangle} [J_{i,k} (\hat{\sigma}_i^x \hat{\sigma}_k^x + \hat{\sigma}_i^y \hat{\sigma}_k^y) + J'_{i,k} \hat{\sigma}_i^z \hat{\sigma}_k^z], \quad (2.6)$$

where $\hat{\sigma}_k^{x,y,z}$ are the Pauli spin operators at position k , B_k determines the energy separation between the spin-up and spin-down states playing the role of the local “magnetic field”, and $J_{i,k}$ is the nearest-neighbour spin-exchange interaction. This is the so-called anisotropic Heisenberg spin-chain Hamiltonian. One sees therefore that in this particular regime of parameters, the lattice with a single fermionic particle per site operates as a spin chain. Although single-particle jumps are practically absent, exchange of two particles in neighbouring sites is not so expensive energetically, and it can therefore take place. The Pauli operators can be expressed in terms of the annihilation/creation operators of (2.5) as follows,

$$\hat{\sigma}_k^z = \hat{n}_{k,u} - \hat{n}_{k,v}, \quad \hat{\sigma}_k^x = \hat{a}_{k,u}^\dagger \hat{a}_{k,v} + \hat{a}_{k,v}^\dagger \hat{a}_{k,u}, \quad \hat{\sigma}_k^y = -i(\hat{a}_{k,u}^\dagger \hat{a}_{k,v} - \hat{a}_{k,v}^\dagger \hat{a}_{k,u}). \quad (2.7)$$

Similarly, the parameters entering Hamiltonian (2.6) can be expressed in terms of $g_{i,k}^{(\xi)}$, $\varepsilon_{k,\xi}$ and $U_{u,v}$. Other effective spin-chain Hamiltonians can be also simulated in the framework of second quantization Hamiltonians e.g., by considering bosons instead of fermions, or by applying external fields (e.g., see [4]). In the spin-chain formalism the qubit basis states for the k th spin $\{|\uparrow\rangle, |\downarrow\rangle\}$, are typically represented by the eigenvectors of the operator $\hat{\sigma}_k^z$. In view of the correspondences (2.7), $\hat{\sigma}_k^z$ refers to the difference of populations of particles in states $|u\rangle$ and $|v\rangle$ at site k , and thus the eigenstate $|\uparrow\rangle$ with eigenvalue $+1$, corresponds to $|u\rangle$, while $|\downarrow\rangle$ with eigenvalue -1 , corresponds to $|v\rangle$.

Finally, employing the Jordan-Wigner transformation [5], the spin-chain Hamiltonian (2.6) can be mapped to a Hubbard Hamiltonian for non-interacting *spinless fermions* (or hard-core bosons). For instance, when $J'_{i,k} = 0$ we find

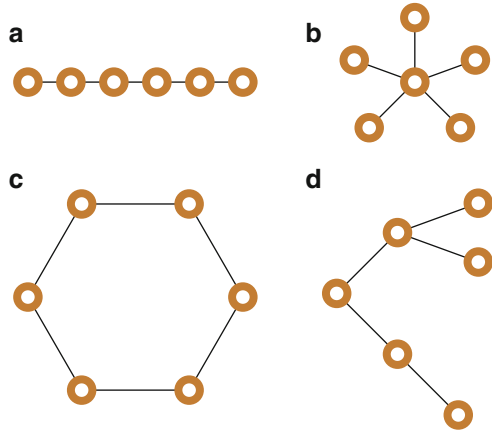
$$\mathcal{H}_Q' = 2 \sum_k B_k \hat{c}_k^\dagger \hat{c}_k - 2 \sum_{\langle i,k \rangle} J_{i,k} (\hat{c}_i^\dagger \hat{c}_k + \hat{c}_k^\dagger \hat{c}_i), \quad (2.8)$$

where

$$\hat{c}_k = \frac{1}{2} \left[\prod_{j < k} \hat{\sigma}_j^z \right] (\hat{\sigma}_k^x - i \hat{\sigma}_k^y), \quad \hat{c}_k^\dagger = \frac{1}{2} \left[\prod_{j < k} \hat{\sigma}_j^z \right] (\hat{\sigma}_k^x + i \hat{\sigma}_k^y), \quad (2.9)$$

and $\hat{c}_k^\dagger \hat{c}_k = (1 + \hat{\sigma}_k^z)/2$. One should not confuse here the new operators $\{\hat{c}_k, \hat{c}_k^\dagger\}$, with $\{\hat{a}_{k,\xi}, \hat{a}_{k,\xi}^\dagger\}$ in Eq. (2.5). The latter refer to the annihilation/creation of real particles, whereas the new operators $\{\hat{c}_k, \hat{c}_k^\dagger\}$ refer to the excitations of the lattice. Indeed, from Eqs. (2.7) and (2.9) one sees that $\hat{c}_k \sim \hat{a}_{k,v}^\dagger \hat{a}_{k,u}$. The Hamiltonian (2.8) is still an effective one that is equivalent to (2.5) only when $g_{i,k}^{(\xi)} \ll U_{i,v}^{i,u}$. The eigenvectors of $\hat{c}_k^\dagger \hat{c}_k$ are $|0\rangle$ and $|1\rangle$, with eigenvalues 0 and 1, and they correspond to the spin states $|\downarrow\rangle$ and $|\uparrow\rangle$, respectively. Hence, the vacuum state of Hamiltonian (2.8) refers

Fig. 2.3 Various network topologies: (a) line; (b) star; (c) ring; and (d) tree or hierarchical. The *circles* show physical nodes, and the *lines* represent the links between them. Configurations with couplings beyond nearest-neighbours are also possible but are not shown here



to the ground state of the spin chain where excitations are absent, whereas in the vacuum state of Hamiltonian (2.5) there are no particles.

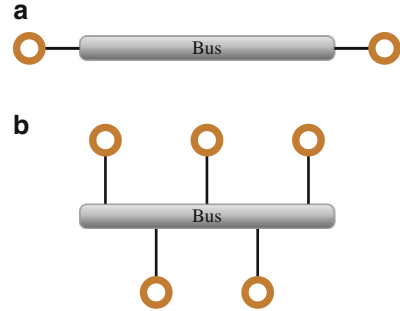
In closing this section, we keep in mind that discrete networks of various types can be described within the unified theoretical framework of second quantization. The state to be transferred is associated with excitations of the network, and is characterized by the position of the excitations on the lattice, and additional degrees of freedom ξ , that provide a good qubit basis for the encoding of information. The Hamiltonians under consideration preserve all the degrees of freedom, but the position of the excitations. Finally, it is worth emphasizing here that although all of the above Hamiltonians pertain to ξ -dependent coupling constants, the majority of the work on quantum networks and the faithful transfer of states has been performed in the framework of Hamiltonians with ξ -independent couplings.

2.1.3 Topologies

In general, networks of different configurations and geometries can be considered, depending on the particular tasks and physical realizations one aims at. For instance, the geometries depicted in Fig. 2.3, or combinations thereof, have been discussed in the context of quantum networks.

Physical topology describes how different sites are coupled to each other, whereas logical topology refers to the flow of information between the various sites in a particular network. These definitions are inspired by conventional network topologies (e.g., see [6, 7]) although, due to the discreteness of quantum networks, the analogies are not always so clear. For instance, in a classical setting two nodes are connected via wires, whereas as we will see later on, in the context of quantum networks under consideration a wire is represented by a 1D discrete lattice. The logical topology focuses on the flow of information from the source

Fig. 2.4 Point-to-point (a) and bus (b) logical topologies. The hosts (*open circles*) communicate via the bus that is provided by the remaining physical nodes



to the destination nodes in a particular configuration, and it is not necessarily the same as the physical topology. Furthermore, in a classical setting an outgoing signal originates from a particular source node, whereas in quantum networks the signal may involve many nodes in a superposition state. These are fundamental differences, which however, do not prevent us from defining logical topologies based on how signals move between the source and the destination nodes (also to be referred to hereafter as the *hosts*) [8]. *The set of hosts is thus a subset of the set of all the physical nodes in the network.* The nodes that are not hosts operate as gateways/routers, providing the channel (or the bus) for the communication. Throughout this chapter, the set of all the physical nodes will be denoted by \mathbb{P} , and the sets of source and the destination nodes by \mathbb{S} and \mathbb{D} , respectively.

The simplest logical topology one may consider is the so-called *point-to-point (PP) topology* pertaining to two hosts. Irrespective of any intermediate physical connections between the source and the destination nodes, any message originating from the source is intended for the destination node only (see Fig. 2.4a). A direct generalization of the PP logical topology is the so called *bus topology*, which enables a larger number of hosts to communicate using the same shared media. Physically speaking, the various hosts are connected to a common wire, known as backbone or bus (see Fig. 2.4b), which is provided by the other nodes of the network. A transmitted message is visible and, in principle, accessible by all the hosts, although the intended recipient is actually the one that accepts and processes the message.⁴ When the logical network encompasses the entire physical network (i.e., the sets of hosts and physical nodes coincide), we have the *universal bus topology*. Other types of logical topologies may also be defined in the same framework.

⁴In a classical setting, this can be achieved by means of the so-called internet-protocol (IP) addresses. As the message arrives at each node, the corresponding device checks the destination address contained in the message to see if it matches its own address. If the two addresses match the device processes the message, otherwise it does nothing. Clearly, no addressing is required in a PP topology since any data transmitted from one node is intended for the other node.

2.2 The Problem of Quantum State Transfer

Having introduced fundamental notions and concepts of quantum networks, in this section we define the problem of quantum state transfer (QST), which was introduced for the first time by Bose, in the framework of spin chains [9]. The present definitions and concepts, however, are provided in the general framework of quantum networks, without specifying any particular physical realization of the network.

The pure quantum state $|\psi_{\text{in}}\rangle$ to be transferred is initially prepared on one or more sites of the network, to be referred to hereafter as the *source* sites (or nodes). To this end, the quantum network has to be initially prepared in a state that remains invariant during the transfer, and this is usually taken to be the ground state of the network which, depending on the details of the system and the adopted theoretical formalism, may be the vacuum. Moreover, for the scenario of Fig. 2.1a, b, the encoding of the state requires the addition of excess physical qubits in the lattice, whereas this is not necessary for the setup of Fig. 2.1c, d. The preparation of the state in the source node(s) has to take place on a time scale much shorter than all the other characteristic time scales of the Hamiltonian. Hence, one typically assumes that the source nodes are initially decorrelated from the rest of the network i.e., the initial state of the entire network is

$$|\Psi(0)\rangle = |\psi_{\text{in}}\rangle_{\mathbb{S}} \otimes |\mathcal{O}\rangle_{\mathbb{R}}, \quad (2.10)$$

where the subscripts \mathbb{S} and \mathbb{R} refer to the source nodes and the rest of the network (i.e., $\mathbb{R} = \mathbb{P} \setminus \mathbb{S}$), whereas the ground state is denoted by $|\mathcal{O}\rangle$. This assumption allows for an unambiguous definition of the problem of QST. When \mathbb{R} is not initially in a pure state, we have

$$\hat{\rho}(0) = \hat{\rho}^{(\mathbb{S})}(0) \otimes \hat{\rho}^{(\mathbb{R})}(0), \quad (2.11)$$

where $\hat{\rho}^{(\mathbb{S})}(0) = (|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|)_{\mathbb{S}}$, and $\hat{\rho}^{(\mathbb{R})}$ refers to the mixed state of the rest of the network.

An excitation created somewhere in a quantum network will unavoidably propagate in all possible directions and, after some time, various sites of the network will be excited; albeit with different probabilities. In other words, even in the absence of imperfections (such as disorder, dissipation and decoherence), we have a dispersion (spreading) of the initially well-localized quantum information, all over the network (e.g., see Figs. 2.5a and 2.7a). Given the finite size of a network, the excitation will appear after some time at the destination nodes with high probability, but we do not know in advance when this will happen. In general the number of source nodes need not be equal to the number of destination nodes.

The problem of QST, refers to the engineering of Hamiltonians that govern the evolution of a quantum network and guarantee the transfer of an input quantum state from the source to the destination nodes, in a deterministic manner and after a prescribed time.

The presence of imperfections in a particular realization of a network is inevitable, and is expected to affect the transfer in addition to the aforementioned dispersion problems. A quantum network may be useful in practise only if it can be designed with sufficiently small imperfections, so that the associated disturbances are weak perturbations to the operation of an otherwise perfect network.⁵ In this spirit therefore dispersion (diffraction) effects are usually treated separately, by solving the problem of QST in the absence of any imperfections. One can thus derive Hamiltonians that ensure the faithful or even the perfect transfer of the state for the ideal scenario, and subsequently investigate their robustness against various types of imperfections in a particular physical realization. Moreover, a QST scheme is useful only if it operates reliably irrespective of the input qubit state. Hence, ideally we are interested in QST Hamiltonians $\hat{\mathcal{H}}$, for which the corresponding evolution operator

$$\mathcal{U}(t) \equiv e^{-i\hat{\mathcal{H}}t/\hbar}, \quad (2.12)$$

is such that the state of the entire network at a prescribed time τ is the target state $|\Psi_{\text{target}}\rangle$ (or $\hat{\rho}_{\text{target}}$) i.e.,

$$\mathcal{U}(\tau)|\Psi(0)\rangle = |\psi_{\text{in}}\rangle_{\mathbb{D}} \otimes |\diamond\rangle_{\mathbb{R}'} \equiv |\Psi_{\text{target}}\rangle, \quad (2.13)$$

or

$$\mathcal{U}(\tau)\hat{\rho}(0)\mathcal{U}^\dagger(\tau) = (|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|)_{\mathbb{D}} \otimes \hat{\rho}_{\mathbb{R}'}(\tau) = \hat{\rho}_{\text{target}}, \quad (2.14)$$

where $|\Psi(0)\rangle$ and $\hat{\rho}(0)$ are given by Eqs.(2.10) and (2.11), respectively. The subscript \mathbb{D} refers to the destination nodes, while \mathbb{R}' refers to the rest of the network (i.e., $\mathbb{R}' = \mathbb{P} \setminus \mathbb{D}$), with the corresponding pure state denoted by $|\diamond\rangle_{\mathbb{R}'}$. Note that we have allowed for arbitrary changes in the state of all the nodes, but the hosts, during the transfer i.e., the states of \mathbb{R} and \mathbb{R}' need not be the same. Our only requirement is that the input state has been transferred from the source to the destination nodes at time τ , perhaps up to an unimportant global phase φ that need not be considered here.

As soon as the state has been transferred to the destination nodes, it has to be read out and to this end, the destination nodes have to decouple from the rest of the network. Throughout this chapter we focus on the problem of QST and the engineering of related Hamiltonians, but it has to be emphasized that both the initialization and the read out processes are key components for the successful operation of a quantum network. In both cases, the coupling and decoupling of sites to the rest of the network have to be, on the one hand fast enough so that the state of the system remains practically frozen during the (de)coupling, and on the other hand adiabatic to prevent new non-resonant excitations. Keeping this in mind,

⁵For instance, if the network is viewed as part of a larger fault-tolerant quantum-information processor, the probability of errors at the end of the transfer must be below $\approx 10^{-4}$ [1].

from now on we focus on the engineering of time-independent QST Hamiltonians, and we begin with a necessary and sufficient condition for a Hamiltonian to satisfy Eqs. (2.13) and (2.14).

Theorem 1. *A Hamiltonian satisfies Eqs. (2.13) and (2.14) for given τ , if and only if its eigenenergies λ_j and eigenstates $|\lambda_j\rangle$ satisfy*

$$e^{-i\lambda_j\tau/\hbar}\langle\lambda_j|\Psi(0)\rangle = \langle\lambda_j|\Psi_{\text{target}}\rangle \quad (2.15)$$

and

$$e^{-i(\lambda_j-\lambda_k)\tau/\hbar}\langle\lambda_j|\hat{\rho}(0)|\lambda_k\rangle = \langle\lambda_j|\hat{\rho}_{\text{target}}|\lambda_k\rangle, \quad (2.16)$$

respectively.

Proof. Condition (2.15) is a special case of (2.16), and working along the lines of [10], we will prove the latter. Assume that a Hamiltonian $\hat{\mathcal{H}}$ has a spectral decomposition

$$\hat{\mathcal{H}} = \sum_{\lambda_j} \lambda_j |\lambda_j\rangle\langle\lambda_j|, \quad (2.17)$$

and satisfies

$$\hat{\mathcal{U}}(\tau)\hat{\rho}(0)\hat{\mathcal{U}}^\dagger(\tau) = \hat{\rho}_{\text{target}}. \quad (2.18)$$

Then condition (2.16) is readily obtained if we take the overlap with any two eigenstates. On the other hand, if condition (2.16) is satisfied, then using the spectral decomposition of the Hamiltonian we have

$$\begin{aligned} \hat{\mathcal{U}}(\tau)\hat{\rho}(0)\hat{\mathcal{U}}^\dagger(\tau) &= \sum_{\lambda_j, \lambda_k} e^{-i(\lambda_j-\lambda_k)\tau/\hbar} |\lambda_j\rangle\langle\lambda_j|\hat{\rho}(0)|\lambda_k\rangle\langle\lambda_k| \\ &= \sum_{\lambda_j, \lambda_k} |\lambda_j\rangle\langle\lambda_j|\hat{\rho}_{\text{target}}|\lambda_k\rangle\langle\lambda_k| = \hat{\rho}_{\text{target}}. \end{aligned} \quad (2.19)$$

□

The necessary and sufficient conditions of Theorem 1 can be also expressed in terms of the moduli and the phases of the overlaps.

Finally, in closing this section, we should emphasize that, in general, faithful transfer of excitation probabilities does not necessarily imply faithful transfer of the associated quantum state. The quality of the state transfer can be quantified by the fidelity

$$F = \langle\psi_{\text{in}}|\hat{\rho}^{(\mathbb{D})}(\tau)|\psi_{\text{in}}\rangle, \quad (2.20)$$

where $\hat{\rho}^{(\mathbb{D})}$ is the reduced density matrix describing the state of the destination nodes at the end of the transfer, and $|\psi_{\text{in}}\rangle$ is the input state. The reason is that the excitation probabilities do not carry any phase information, which is inherently present in a quantum state. Thus, even when the excitation has been transferred perfectly from the source to the destination nodes, the relative phase between the various components of the associated state may have changed, yielding thus small overlap with the input state $|\psi_{\text{in}}\rangle$. In general, dephasing is expected to be present, irrespective of the presence or the absence of imperfections. When imperfections are present, the dephasing will be random and the reduction of the fidelity is inevitable. On the contrary, in the absence of imperfections, which is the focus of the present chapter, the dephasing pertains to a fixed and known off-set of the phase and, in principle, it can be corrected. In this sense, the transfer of excitation probabilities does imply faithful transfer of the associated quantum state, assuming that we can always compensate for any known and fixed dephasing.

This point can be made clearer, if we recall here that the Hamiltonian under consideration can be written as $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{V}}$, with $\hat{\mathcal{V}}$ affecting the position of the excitations while preserving ξ . Typically the computational basis states are chosen according to the eigenstates of the unperturbed Hamiltonian $\hat{\mathcal{H}}_0$. For Hamiltonians of the form (2.1), a good orthonormal basis for expressing the state of the network consists of the vectors

$$|\mathbf{n}\rangle \equiv |n_{1,u}, n_{1,v}; n_{2,u}, n_{2,v}; \dots\rangle \equiv \bigotimes_{j,\xi} |n_{j,\xi}\rangle, \quad (2.21)$$

where $|n_{r,\xi}\rangle$ denotes the number of excitations or particles (depending on the scenario under consideration) in the r th site of the network and in state $|\xi\rangle$, while

$$\hat{\mathcal{H}}_0|\mathbf{n}\rangle = E(\mathbf{n})|\mathbf{n}\rangle. \quad (2.22)$$

When the state of \mathbb{R} is pure, the initial state of the network can be written as

$$|\Psi(0)\rangle = \left[\sum_{\mathbf{n}} A_{\mathbf{n}}(0) |\mathbf{n}\rangle_{\mathbb{S}} \right] \otimes \left[\sum_{\mathbf{m}} B_{\mathbf{m}}(0) |\mathbf{m}\rangle_{\mathbb{R}} \right] \quad (2.23)$$

for some complex amplitudes $A_{\mathbf{n}}$ and $B_{\mathbf{n}}$, whereas for a mixed state we have

$$\hat{\rho}(0) = \left[\sum_{\mathbf{n}, \mathbf{n}'} \rho_{\mathbf{n}, \mathbf{n}'}^{(\mathbb{S})}(0) |\mathbf{n}\rangle \langle \mathbf{n}'| \right] \otimes \left[\sum_{\mathbf{m}, \mathbf{m}'} \rho_{\mathbf{m}, \mathbf{m}'}^{(\mathbb{R})}(0) |\mathbf{m}\rangle \langle \mathbf{m}'| \right]. \quad (2.24)$$

In both cases, the first term in the tensor product refers to the pure state of the source, and the second term to the remaining sites.

In view of Eq. (2.22), different components in Eqs. (2.23) and (2.24) will, in general, experience different phase shifts even under the influence of the unperturbed Hamiltonian only, since

$$e^{-i\hat{\mathcal{H}}_0 t/\hbar} |\mathbf{n}\rangle_X |\mathbf{m}\rangle_Y = e^{-iE(\mathbf{n},\mathbf{m})t/\hbar} |\mathbf{n}\rangle_X |\mathbf{m}\rangle_Y, \quad (2.25)$$

where

$$\hat{\mathcal{H}}_0 |\mathbf{k}\rangle_X \otimes |\mathbf{q}\rangle_Y = E(\mathbf{k}, \mathbf{q}) |\mathbf{k}\rangle_X \otimes |\mathbf{q}\rangle_Y.$$

The perturbation $\hat{\mathcal{V}}$ is also expected, in general, to add additional relative phases. In principle, one can compensate for such phases when they are known and fixed (i.e., in the ideal scenario); a task that may require involved unitary rotations on the entire system, or individual subsystems. However, in the presence of imperfections that randomize the parameters entering $\hat{\mathcal{H}}$ (e.g., diagonal and off-diagonal disorder), there is no way to correct entirely the phases since they become random. In this case, one has to investigate the robustness of the scheme relative to different types and levels of imperfections. The reader may also refer to [11] for a thorough discussion on the quantification of state transfer by means of various measures.

2.3 Engineering of Quantum Networks

The design of a quantum network begins with the specification of the class of QST Hamiltonians that we are interested in, as well as the identification of all the relevant basis states [8, 12–14]. In most cases, the input state, the form of the Hamiltonians, and perhaps additional physical constraints, enable us to restrict the problem to a particular sector of the Hilbert space. Clearly, the initial state of the network, as well as the final state we are aiming at, have to belong to the same working subspace, and our task is the estimation of judicious parameters (energies and couplings) so that QST from the source to the destination nodes takes place in a deterministic manner, at a prescribed time τ . In this section we will discuss a particular approach to the design of QST Hamiltonians. Consider for the time being that the sets of source and destination nodes have the same cardinality (i.e., $|\mathbb{S}| = |\mathbb{D}|$), although the performance of certain QIP tasks may require relaxation of this constraint.

Let us consider first the case of a network initially prepared in any basis state

$$|\mathbf{n}\rangle_{\mathbb{S}} \otimes |\mathbf{m}\rangle_{\mathbb{R}} \equiv |n_{s_1,u}, n_{s_1,v}; n_{s_2,u}, n_{s_2,v}; \dots\rangle_{\mathbb{S}} \otimes |m_{r_1,u}, m_{r_1,v}; m_{r_2,u}, m_{r_2,v}; \dots\rangle_{\mathbb{R}} \quad (2.26)$$

where $s_j \in \mathbb{S}$, $r_j \in \mathbb{R}$, and $\mathbb{R} \equiv \mathbb{P} \setminus \mathbb{S}$. The problem of QST pertains to the quest for Hamiltonians, which perform the transformation

$$|\mathbf{n}\rangle_{\mathbb{S}} \otimes |\mathbf{m}\rangle_{\mathbb{R}} \rightarrow |\mathbf{n}\rangle_{\mathbb{D}} \otimes |\mathbf{m}'\rangle_{\mathbb{R}'} \quad (2.27)$$

where $|\mathbf{m}'\rangle_{\mathbb{R}'} = |m'_{r'_1,u}, m'_{r'_1,v}; m'_{r'_2,u}, m'_{r'_2,v}; \dots\rangle$, $r'_j \in \mathbb{R}'$ and $\mathbb{R}' \equiv \mathbb{P} \setminus \mathbb{D}$.

One way to achieve this transformation is to ask for the evolution operator to be a permutation on the basis states at time τ i.e., the evolution operator satisfies

$$\hat{\mathcal{U}}(\tau) = \hat{\mathcal{P}}, \quad (2.28)$$

where

$$\hat{\mathcal{P}} = \sum_{\mathbf{n}} |\bar{n}_{\mathcal{P}(1)}; \bar{n}_{\mathcal{P}(2)}; \bar{n}_{\mathcal{P}(3)}; \dots\rangle \langle \bar{n}_1; \bar{n}_2; \bar{n}_3; \dots|, \quad (2.29)$$

and $\mathbf{n} \equiv |\bar{n}_1; \bar{n}_2; \bar{n}_3; \dots\rangle$. For the sake of simplicity in Eq. (2.29) we have set $|\bar{n}_j\rangle \equiv |n_{j,u}, n_{j,v}\rangle$, and the action of the unitary on the nodes is determined by⁶

$$\hat{\mathcal{P}} = \begin{pmatrix} 1 & 2 & \dots & \mathbb{S} & \dots & \mathbb{D} & \dots \\ \mathcal{P}(1) & \mathcal{P}(2) & \dots & \mathbb{D} & \dots & \mathcal{P}(\mathbb{D}) & \dots \end{pmatrix}. \quad (2.30)$$

The only restriction imposed in these equations is that the permutation should map the set of source nodes to the set of destination nodes i.e., $\hat{\mathcal{P}} : \mathbb{S} \rightarrow \mathbb{D}$, which means that for each element $s_j \in \mathbb{S}$, we have $\mathcal{P}(s_j) = d_j$ with $d_j \in \mathbb{D}$. In view of the permutation, this mapping has to be one-to-one.⁷ Then, Eqs. (2.28)–(2.30) ensure that the initial state of the sources appears at the destination nodes at time τ . Indeed, we have

$$\hat{\mathcal{U}}(\tau)|\mathbf{n}\rangle_{\mathbb{S}} \otimes |\mathbf{m}\rangle_{\mathbb{R}} = \hat{\mathcal{P}}|\mathbf{n}\rangle_{\mathbb{S}} \otimes |\mathbf{m}\rangle_{\mathbb{R}} = |\mathbf{n}\rangle_{\mathbb{D}} \otimes |\mathbf{m}\rangle_{\mathbb{R}'}. \quad (2.31)$$

Using Eq. (2.31) in Eqs. (2.13) and (2.14), with $|\Psi(0)\rangle$ and $\hat{\rho}(0)$ given by Eqs. (2.23) and (2.24), one sees immediately what happens if the source nodes are initially prepared in a pure state $|\psi_{\text{in}}\rangle$, and the rest of the network is prepared in an arbitrary pure or mixed state. For instance, we find

$$|\Psi(\tau)\rangle = \left[\sum_{\mathbf{n}} A_{\mathbf{n}}(0) |\mathbf{n}\rangle_{\mathbb{D}} \right] \otimes \left[\sum_{\mathbf{m}} B_{\mathbf{m}}(0) |\mathbf{m}\rangle_{\mathbb{R}'} \right], \quad (2.32)$$

The evolution operator (2.28) has transferred the excitation from the source to the destination nodes, mixing up the state of the remaining nodes. An analogous result is obtained if the initial state of \mathbb{R} is a mixed state, which shows that our approach works efficiently irrespective of the actual state of \mathbb{R} .

⁶For the sake of simplicity in Eq. (2.30) we assume that the labels of the source and the destination nodes are successive. This is not important for the present formalism, but in general one can always relabel the sites in a given network so that the permutation is of the form (2.30).

⁷For more general mappings, one has to relax the constraint of the evolution operator being a permutation (e.g., see [13]).

Recall here that the information is encoded in the state ξ , which is preserved throughout the evolution of the system, whereas $\hat{\mathcal{V}}$ is responsible for the change of the position and thus for the transfer of the state. Asking for Hamiltonians that satisfy Eq. (2.28) one essentially engineers the network with respect to all of the degrees of freedom, ensuring thus that all of the components of the wavefunction (see the previous section) will experience the same phase shifts, so that there are no additional relative phases at the end of the transfer. This, however, may imply conditions on the parameters entering the Hamiltonian, which are hard to fulfil in practise. Hence, the majority of related work in the literature allows for relative phase shifts that are fixed and known and originate from $\hat{\mathcal{H}}_0$ and perhaps $\hat{\mathcal{V}}$ when the coupling constants are ξ -dependent. In this spirit, if various degrees of freedom can be separated in the Hamiltonian, we can exclude ξ from the engineering. The aforementioned engineering technique can be then applied on the part of the Hamiltonian that pertains to the orbital, position, and other degrees of freedom. A related example will be presented in the following section.

For a set of M basis states, there are $(M - 1)!$ different permutations, because one column and one row of the permutation are fixed by the imposed constraint $\mathcal{P}(s_j) = d_j$. The particular choice of the permutation is usually determined by additional physical constraints imposed e.g., by the geometry and the topology of the network. This will become clearer in the examples to follow. We turn now to discuss how one can obtain QST Hamiltonians that are of a particular form (e.g., tridiagonal, block-diagonal, etc.) and at the same time consistent with requirements (2.28)–(2.30).

In general, $\hat{\mathcal{P}}$ can be decomposed into disjoint cycles $\hat{\mathcal{P}}_j$, i.e., $\hat{\mathcal{P}} = \sum_j \hat{\mathcal{P}}_j$, and let \mathbb{B}_j denote the set of basis states permuted by $\hat{\mathcal{P}}_j$.⁸ The sets $\{\mathbb{B}_j\}$ are disjoint, while the dimensions of the support of each cycle $\hat{\mathcal{P}}_j$ is the cardinality of the corresponding set denoted by $|\mathbb{B}_j|$. The spectrum of each cycle $\hat{\mathcal{P}}_j$ is nondegenerate, with eigenvectors $|w_j^{(\lambda_n)}\rangle$ and corresponding eigenvalues λ_n , where $\hat{\mathcal{P}}_j |w_j^{(\lambda_n)}\rangle = \lambda_n |w_j^{(\lambda_n)}\rangle$, with

$$|w_j^{(\lambda_n)}\rangle = \frac{1}{\sqrt{|\mathbb{B}_j|}} \sum_{|b\rangle \in \mathbb{B}_j} \lambda_n^{\zeta_b} |b\rangle, \quad (2.33)$$

and

$$\lambda_n = \exp\left(i2\pi \frac{n}{|\mathbb{B}_j|}\right) \quad \text{for } n \in \mathbb{Z}_{|\mathbb{B}_j|} \equiv \{0, 1, \dots, |\mathbb{B}_j| - 1\}. \quad (2.34)$$

⁸A closed cycle is a permutation or sub-permutation, which cannot be decomposed further.

A label $\zeta_b \in \mathbb{Z}_{|\mathbb{B}_j|}$ can be assigned to every basis state $|b\rangle \in \mathbb{B}_j$, and thus the elements of \mathbb{B}_j are considered to be arranged in ascending order, with respect to their labels.⁹ Hence, for the permutation $\hat{\mathcal{P}}$ the eigenvalue λ_n corresponds to η_{λ_n} distinct eigenvectors $\{|w_j^{(\lambda_n)}\rangle\}$, with j running only on the various cycles having λ_n in common.

A class of Hamiltonians that satisfy Eq. (2.28) is of the form

$$\hat{\mathcal{H}}_{\mathbf{x}} = \frac{\hbar}{\tau} \sum_{\lambda_n} \sum_{a=1}^{\eta_{\lambda_n}} f_{\lambda_n}^{(a)} |y_{\lambda_n}^{(a)}\rangle \langle y_{\lambda_n}^{(a)}|, \quad (2.35)$$

where

$$f_{\lambda_n}^{(a)} = -\arg(\lambda_n) + 2\pi x_{\lambda_n}^{(a)} \quad (2.36)$$

and

$$\mathbf{x} \in \mathbb{Z}^d \equiv \{(x_{\lambda_0}^{(1)}, \dots, x_{\lambda_0}^{(\eta_{\lambda_0})}; x_{\lambda_1}^{(1)}, \dots, x_{\lambda_1}^{(\eta_{\lambda_1})}; \dots) \mid x_{\lambda_n}^{(a)} \in \mathbb{Z}\}.$$

Note here that $f_{\lambda_n}^{(a)}$ are associated with the eigenenergies of the Hamiltonian. They are basically equal to the phase of the eigenvalues λ_n of the chosen permutation, shifted by an arbitrary integer multiple of 2π .

For a given eigenvalue λ_n , the η_{λ_n} distinct vectors $\{|y_{\lambda_n}^{(a)}\rangle\}$ form an orthonormal basis for the corresponding subspace and are of the form

$$|y_{\lambda_n}^{(a)}\rangle = \sum_i \beta_{a,i}^{(\lambda_n)} |w_i^{(\lambda_n)}\rangle, \quad (2.37)$$

with $\beta_{a,j}^{(\lambda_n)} \in \mathbb{C}$ and $\sum_j \beta_{a,j}^{(\lambda_n)*} \beta_{a',j}^{(\lambda_n)} = \delta_{a,a'}$. In other words, the rows of the matrix \tilde{B}_{λ_n} with elements $\beta_{a,j}^{(\lambda_n)}$ are orthonormal, which implies unitarity of \tilde{B}_{λ_n} and thus orthonormality of its columns as well, i.e.,

$$\sum_a \beta_{a,j}^{(\lambda_n)} \beta_{a,j'}^{(\lambda_n)*} = \delta_{j,j'}. \quad (2.38)$$

So, starting from a particular permutation $\hat{\mathcal{P}}$, we have derived a class of Hamiltonians that is parametrized by the integer vector \mathbf{x} , and permutes the basis

⁹Clearly, the labelling of the basis states in \mathbb{B}_j is to some extent ambiguous. One may consider any ordering for which Eq. (2.33) yields a complete set of eigenstates of $\hat{\mathcal{P}}_j$. In this case, the ambiguity is unimportant, and does not affect our approach.

states at the prescribed time τ , as determined by the chosen $\hat{\mathcal{P}}$. Indeed, for any chosen \mathbf{x} , the evolution operator for the corresponding member of the class reads

$$\hat{\mathcal{U}}_{\mathbf{x}}(t) \equiv e^{-i\hat{\mathcal{H}}_{\mathbf{x}}t/\hbar} = \sum_{\lambda_n} \sum_{a=1}^{\eta_{\lambda_n}} \exp\left(-if_{\lambda_n}^{(a)}t/\tau\right) |y_{\lambda_n}^{(a)}\rangle \langle y_{\lambda_n}^{(a)}|, \quad (2.39)$$

with $f_{\lambda_n}^{(a)}$ and $|y_{\lambda_n}^{(a)}\rangle$ given by Eqs. (2.36) and (2.37), respectively. Hence, at $t = \tau$, this evolution operator reduces to (2.28), for the particular permutation under consideration.

The key point here is that the Hamiltonians (2.35) are diagonal in the eigenbasis of the permutation. This is possible only for Hamiltonians $\hat{\mathcal{H}}$ that commute with the chosen permutation i.e., for

$$[\hat{\mathcal{P}}, \hat{\mathcal{H}}] = 0. \quad (2.40)$$

This condition restricts our quest to QST Hamiltonians that have a particular form in the computation basis, so that they are compatible with the permutation. Finally, after expressing the eigenvectors of the permutation in terms of the computational basis states by means of Eqs. (2.33) and (2.37), one can estimate all of the open parameters entering the Hamiltonian, by solving a system of (non)linear equations.

In the following we apply the above design technique in the framework of networks of various topologies. Before this, however, it is worth emphasizing here that condition (2.28) is rather restrictive since, in general, one may derive QST Hamiltonians which do not lead to permutations at time τ , but rather to other unitary operations. Actually, there are tasks that cannot be described in the framework of permutations. For instance, this is the case of mapping different basis states onto the same basis state. Such a mapping requires inevitably, the extension of the present formalism to unitary transformations (e.g., see [13]).

Despite any weaknesses, it has been demonstrated that the present theoretical framework is suitable for engineering of passive networks [8, 12, 13], and actually many known perfect QST Hamiltonians satisfy condition (2.28). The present formalism has the advantage of being applicable to networks of various topologies, and it is perfectly suited to addressing the problem of QST in the context of various physical realizations, allowing for the inclusion of non-trivial interactions. Infinitely many QST Hamiltonians can be specified within the present theoretical framework, and this enables us to find the most suitable Hamiltonian for a given physical realization and topology. Finally, we have seen that there is no requirement for the initial overall state of the network (minus the source nodes) to have a specific form. It is crucial, however, for the source nodes to be initially decorrelated from the rest of the network, which can be prepared in any state.

2.4 Quantum State Transfer in the Single Excitation Subspace

The vast majority of papers on the problem on QST has been focused on the single-excitation subspace. For instance, in the first scenario discussed in Sect. 2.1.2, the excitation may correspond to the addition of a single excess particle in a network that is initially in the vacuum state, whereas for the second scenario the excitation refers to the perturbation of a single physical qubit in the initial ground state. We will demonstrate here how one can design perfect QST Hamiltonians that operate in the single-excitation subspace, by utilizing the above design technique. As showed in Sect. 2.3, the design works for any states but, for the sake of simplicity, in the following we assume that all the nodes, but the source, are initially in vacuum.

Based on the approach discussed in Sect. 2.2, we focus on the engineering of quantum networks that suppress (and if possible minimize) the dispersion (diffraction) effects, neglecting the effects of imperfections pertaining to disorder, decoherence, and dissipation mechanisms. The form of the QST Hamiltonians that we are interested in is

$$\hat{\mathcal{H}} = \sum_{i=1}^N \sum_{\xi} \varepsilon_{i,\xi} \hat{n}_{i,\xi} + \sum_{\substack{i,k=1 \\ i < k}}^N \sum_{\xi} g_{i,k} (\hat{a}_{i,\xi}^{\dagger} \hat{a}_{k,\xi} + \hat{a}_{k,\xi}^{\dagger} \hat{a}_{i,\xi}), \quad (2.41)$$

where the energies and the real couplings are open parameters to be determined, and N is the number of nodes in the network. Such Hamiltonians preserve the input state $|\psi_{\text{in}}\rangle$ and affect only its position on the network, ensuring thus the transfer irrespective of the details of the input state. If one neglects the variable ξ dropping the related summations in (2.41), then Hamiltonian (2.41) is of the form (2.8), with the corresponding spin-chain Hamiltonian given by Eq. (2.6) for $J'_{i,k} = 0$. In this case, we essentially focus on the transfer of the excitation, leaving aside the accompanied information encoded in the ξ degree of freedom.

Hamiltonian (2.41) preserves the total number of excitations in the network, which means that our theoretical analysis can be restricted to the single excitation sector only, spanned by the states $\{|j : \xi\rangle\}$, where for the sake of simplicity $|j : \xi\rangle \equiv a_{j,\xi}^{\dagger} |\mathbf{0}\rangle$ and $|\mathbf{0}\rangle$ denotes the vacuum state. In this representation, the single excitation is in the state $|\xi\rangle \in \{|u\rangle, |v\rangle\}$, and occupies the j th site. The initial state of the network is $|\Psi(0)\rangle = |\psi_{\text{in}}\rangle$, where

$$|\psi_{\text{in}}\rangle = \alpha |1 : u\rangle + \beta |1 : v\rangle = |1\rangle \otimes (\alpha |u\rangle + \beta |v\rangle), \quad (2.42)$$

and our task is to define $\{\varepsilon_{i,\xi}, g_{i,k}\}$ in (2.41), so that at time τ we have $|\Psi(\tau)\rangle = \alpha |N : u\rangle + \beta |N : v\rangle = |N\rangle \otimes (\alpha |u\rangle + \beta |v\rangle)$ i.e., the excitation is transferred to the N th site. One can follow the approach outlined in the previous section, to engineer Hamiltonians in the $2N$ -dimensional space spanned by the states $|j : \xi\rangle$. However, noting that the Hamiltonian preserves ξ , and affects only the position of

the excitation, we can restrict the engineering to smaller subspaces of the single-excitation sector.

The single excitation sector can be also decomposed into two subspaces $\mathbb{V}_\xi = \text{Span}\{|j : \xi\rangle\}$, with $\xi \in \{u, v\}$. The full single-excitation subspace will just be $\mathbb{V}_u \oplus \mathbb{V}_v$. Let $\hat{\Pi}_\xi$ be the projector onto the subspace \mathbb{V}_ξ . The Hamiltonian (2.41) contains no terms that would couple basis states with different internal degrees of freedom ξ , and thus the subspaces \mathbb{V}_ξ are invariant i.e., $[\hat{\Pi}_\xi, \hat{\mathcal{H}}] = 0$. One may thus express the Hamiltonian formally as $\hat{\mathcal{H}} = \sum_\xi \hat{\Pi}_\xi \hat{\mathcal{H}} \hat{\Pi}_\xi$, where the dynamics in the subspace \mathbb{V}_ξ are governed by the Hamiltonian $\hat{\Pi}_\xi \hat{\mathcal{H}} \hat{\Pi}_\xi$. All the invariant subspaces \mathbb{V}_ξ are equivalent, and the engineering of Hamiltonians can be restricted to anyone of them.

Alternatively, the problem can be solved in the N -dimensional subspace spanned by the positions $|j\rangle$ only. Using Eqs. (2.2) and (2.3), the Hamiltonian (2.41) can be written as

$$\hat{\mathcal{H}} = \hat{h}_{\text{pos}} \otimes \hat{\mathbf{1}}_\xi + \hat{\mathbf{1}}_{\text{pos}} \otimes \hat{h}_\xi, \quad (2.43a)$$

where

$$\hat{h}_{\text{pos}} = \sum_{i=1}^N \varepsilon_i |i\rangle\langle i| + \sum_{\substack{i,k=1 \\ i < k}}^N g_{i,k} (|i\rangle\langle k| + |k\rangle\langle i|), \quad (2.43b)$$

$$\hat{h}_\xi = \sum_{\xi} \varepsilon_\xi |\xi\rangle\langle \xi|, \quad (2.43c)$$

and the identity operators $\hat{\mathbf{1}}_{\text{pos}}$ and $\hat{\mathbf{1}}_\xi$ act on the subspaces spanned by $\{|j\rangle\}$ and $\{|\xi\rangle\}$. The engineering can be now restricted to \hat{h}_{pos} only and begins with the choice of a permutation of the form

$$\hat{\mathcal{P}} = |N\rangle\langle 1| + \hat{\mathcal{Q}} \quad (2.44)$$

where $\hat{\mathcal{Q}}$ acts on the nodes $2, 3, \dots, N$. There are $(N-1)!$ different permutations of this form, and our choice is dictated by the topology of the network under consideration, and thus by the form of the Hamiltonians that we are interested in. For instance, it turns out that a mirror-symmetric chain is compatible only with an antidiagonal permutation of the form

$$\hat{\mathcal{P}} = \sum_{j=1}^N |N+1-j\rangle\langle j|. \quad (2.45)$$

Indeed, from Eq. (2.40), we find that the Hamiltonians that are compatible with the permutation (2.45) satisfy

$$g_{i,k} = g_{N+1-i, N+1-k}, \quad \text{and} \quad \varepsilon_i = \varepsilon_{N+1-i}, \quad (2.46)$$

which refer to a centro-symmetric (or else mirror-symmetric) network. In the following we will outline the design of Hamiltonians in two special cases, by means of two examples.

2.4.1 Networks with Nearest-Neighbour Interactions

Assume that we are interested in QST Hamiltonians with nearest-neighbour (NN) interactions that operate on a 4-node centrosymmetric chain, and ensure the perfect transfer of a qubit state between its two ends [12]. In view of the previous discussion, we begin with the permutation

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.47)$$

This permutation involves two cycles of dimension two namely, $\mathcal{P}_1 = (1, 4)$ and $\mathcal{P}_2 = (2, 3)$.¹⁰ Hence, we have two eigenvalues $\lambda_0 = e^{i0} = +1$ and $\lambda_1 = e^{i\pi} = -1$. Both eigenvalues ± 1 are doubly degenerate (i.e., $\eta_{\pm} = 2$) and the corresponding eigenvectors are given by $|w_{\pm}^{(1)}\rangle = (|1\rangle \pm |4\rangle)/\sqrt{2}$ and $|w_{\pm}^{(2)}\rangle = (|2\rangle \pm |3\rangle)/\sqrt{2}$, for cycles \mathcal{P}_1 and \mathcal{P}_2 , respectively. Accordingly, the corresponding subspaces are

$$\Lambda_{\pm} = \{|w_{\pm}^{(1)}\rangle, |w_{\pm}^{(2)}\rangle\}.$$

Having defined the permutation matrix, the construction of perfect QST Hamiltonians now proceeds in two steps.

2.4.1.1 Parameterisation

For each one of the subspaces we can choose an orthonormal basis in many different ways. For example, we may choose for the two subspaces¹¹

¹⁰Thereby the notation $(2, 3)$ means that starting from the original site ordering of the sites $\{1, 2, 3, 4\}$, the second site is replaced by the third and the third site by the second.

¹¹We have simplified the notation relative to the previous section, but the overall approach remains the same.

$$\Lambda_+ : \begin{cases} |y_+^{(1)}\rangle = a_+^{(1)}|w_+^{(1)}\rangle + b_+^{(1)}|w_+^{(2)}\rangle \\ |y_+^{(2)}\rangle = a_+^{(2)}|w_+^{(1)}\rangle + b_+^{(2)}|w_+^{(2)}\rangle, \end{cases} \quad (2.48)$$

and

$$\Lambda_- : \begin{cases} |y_-^{(1)}\rangle = a_-^{(1)}|w_-^{(1)}\rangle + b_-^{(1)}|w_-^{(2)}\rangle \\ |y_-^{(2)}\rangle = a_-^{(2)}|w_-^{(1)}\rangle + b_-^{(2)}|w_-^{(2)}\rangle. \end{cases} \quad (2.49)$$

Thereby, $a_{\pm}^{(i)}, b_{\pm}^{(i)} \in \mathbb{C}$ with

$$|a_{\pm}^{(i)}|^2 + |b_{\pm}^{(i)}|^2 = 1, \quad i = 1, 2 \quad (2.50)$$

$$a_{\pm}^{(1)} a_{\pm}^{(2)*} + b_{\pm}^{(1)} b_{\pm}^{(2)*} = 0, \quad (2.51)$$

so that $|y_{\pm}^{(i)}\rangle$ are orthonormal.

According to the theory of the previous section, the corresponding eigenenergies can be chosen as

$$\Lambda_+ : \begin{cases} f_+^{(1)} = 0 + 2\pi x_+^{(1)} \\ f_+^{(2)} = 0 + 2\pi x_+^{(2)}, \end{cases} \quad (2.52)$$

and

$$\Lambda_- : \begin{cases} f_-^{(1)} = \pi + 2\pi x_-^{(1)} \\ f_-^{(2)} = \pi + 2\pi x_-^{(2)}, \end{cases} \quad (2.53)$$

for $x_j^{(k)} \in \mathbb{Z}$. Thereby, note that the eigenvalues of subspace Λ_+ (Λ_-) are even (odd) integer multiples of π .

Thus, the whole class of perfect QST Hamiltonians (2.35) reads

$$\hat{h}(\mathbf{x}) = \frac{\hbar}{\tau} \left[f_+^{(1)} |y_+^{(1)}\rangle \langle y_+^{(1)}| + f_+^{(2)} |y_+^{(2)}\rangle \langle y_+^{(2)}| + f_-^{(1)} |y_-^{(1)}\rangle \langle y_-^{(1)}| + f_-^{(2)} |y_-^{(2)}\rangle \langle y_-^{(2)}| \right], \quad (2.54)$$

with open parameters the spectrum $\{f_+^{(1)}, f_+^{(2)}, f_-^{(1)}, f_-^{(2)}\}$ (or equivalently the integers $x_{\pm}^{(1,2)}$) and two independent complex numbers (due to normalisation). As we discuss in the following subsection, all these parameters can be specified by imposing additional constraints on the form of the resulting Hamiltonian.

2.4.1.2 Parameter Estimation

Assume now that we are interested in the whole class of NN-type Hamiltonians leading to the antidiagonal permutation matrix (2.47) and are of the form

$$\hat{h}_{\text{pos}} = \begin{pmatrix} \varepsilon_1 & g_{1,2} & 0 & 0 \\ g_{1,2} & \varepsilon_2 & g_{2,3} & 0 \\ 0 & g_{2,3} & \varepsilon_2 & g_{1,2} \\ 0 & 0 & g_{1,2} & \varepsilon_1 \end{pmatrix}. \quad (2.55)$$

Since the class (2.54) has been derived within a rather general framework (limited only by the particular form of the permutation), our task reduces to the application of specific constraints on the parameters entering $\hat{h}(\mathbf{x})$, so that it acquires the form of Eq. (2.55).¹² In view of the symmetries in (2.55), we have the following constraints

$$\begin{aligned} \langle 1|\hat{h}_{\mathbf{x}}|1\rangle &= \varepsilon_1, & \langle 2|\hat{h}_{\mathbf{x}}|2\rangle &= \varepsilon_2, \\ \langle 1|\hat{h}_{\mathbf{x}}|2\rangle &= g_{1,2}, & \langle 2|\hat{h}_{\mathbf{x}}|3\rangle &= g_{2,3}, \\ \langle 1|\hat{h}_{\mathbf{x}}|3\rangle &= 0, & \langle 1|\hat{h}_{\mathbf{x}}|4\rangle &= 0, & \langle 2|\hat{h}_{\mathbf{x}}|4\rangle &= 0 \end{aligned}$$

which together with the 6 orthonormality conditions form a set of 13 equations that have to be solved simultaneously with respect to the 12 variables $\{\varepsilon_j, g_{i,j}, a_{\pm}^{(1,2)}, b_{\pm}^{(1,2)}\}$, for a particular fixed choice of $f_{\pm}^{(i)}$ or equivalently of the integer string \mathbf{x} . In general, changing \mathbf{x} one obtains new solutions for the 12 variables.

A solution that has been discussed thoroughly in the literature and can be obtained within the current theoretical approach, pertains to equidistant eigenenergies, and is of the form

$$\varepsilon_j = -\tilde{\varepsilon}(N-1), \quad \text{and} \quad g_{j,j+1} = \tilde{\varepsilon}\sqrt{j(N-j)} \quad (2.56)$$

with $j = 1, 2, \dots, N-1$ and the constant $\tilde{\varepsilon} = \hbar\pi/(2\tau)$, determined by the time at which the transfer has to take place. This solution involves resonant nodes and engineered couplings between nearest neighbours. It ensures the perfect transfer of the excitation from the j th to the $(N+1-j)$ th node at time τ (see Fig. 2.5). The total Hamiltonian is given by Eqs. (2.43) and (2.55), and the evolution operator reads

$$\hat{\mathcal{U}}(\tau) = \exp[-i(\hat{h}_{\text{pos}} \otimes \hat{\mathbf{1}}_{\xi} + \hat{\mathbf{1}}_{\text{pos}} \otimes \hat{h}_{\xi})\tau/\hbar] = \hat{\mathcal{P}} \otimes e^{-i\hat{h}_{\xi}\tau/\hbar}, \quad (2.57)$$

¹²We focus on the case of Hamiltonians with non-degenerate spectrum. The case of degenerate spectrum can be treated similarly and leads to NN-type Hamiltonians with vanishing couplings i.e., to a broken network and thus QST from the first to the last site is impossible.

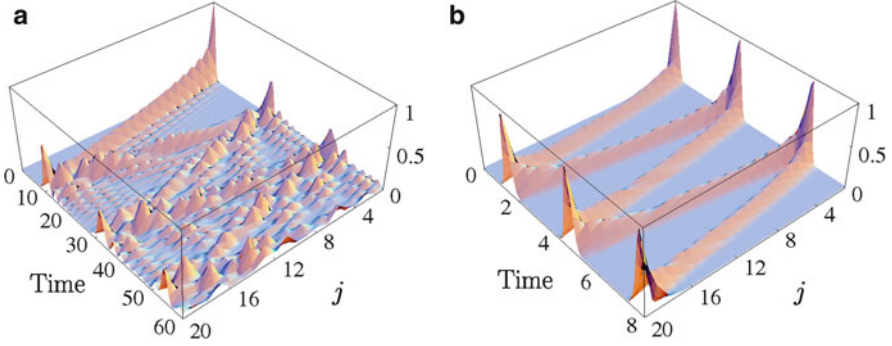


Fig. 2.5 Evolution of a single excitation in a chain of 20 resonant sites ($\varepsilon_j = 0$), in the absence of losses and disorder. The probability for the excitation to occupy the j th site is plotted as a function of time for (a) uniform couplings ($g_{j,j+1} = \tilde{\varepsilon}$) and (b) judiciously engineered couplings ($g_{j,j+1} = \tilde{\varepsilon}\sqrt{j(N-j)}$). Initially the excitation occupies the first site. Time is in units of $\tilde{\varepsilon}^{-1}$ (Adapted from [15])

where we have used the fact that, by construction, $e^{-i\hat{h}_{\text{pos}}\tau/\hbar} = \mathcal{P}$. Thus, for the input state (2.42) the output state at time τ , will be

$$|\Psi(\tau)\rangle = \alpha|N : u\rangle + e^{i\Delta_{u,v}\tau}\beta|N : v\rangle, \quad (2.58)$$

where $\Delta_{u,v} = \varepsilon_{j,u} - \varepsilon_{j,v}$, for all $j = 1, \dots, N$. As mentioned before, in the absence of disorder and imperfections, this relative phase is fixed and known, and therefore can be amended. It is worth noting here that perfect QST from the j th to the $(N + 1 - j)$ th site at time τ also occurs for $\varepsilon_j = \tilde{\varepsilon}$ in Eq. (2.56). In this case, one has

$$\hat{\mathcal{U}}(\tau) = e^{-i(\tilde{\varepsilon}\tau/\hbar)}(-i)^{N-1}\hat{\mathcal{P}},$$

i.e., condition (2.28) is satisfied up to an unimportant global phase. As depicted in Fig. 2.5b a chain with the couplings of Eq. (2.56) behaves as a two-level system i.e., the excitation oscillates back and forth between its two ends. Dispersion phenomena have been minimized as opposed to the case of a uniform chain (see Fig. 2.5a).

Surprisingly enough, the solution (2.56) has been known since the 1970s, albeit in a different context, namely the coherent dynamics of N -level atoms [16–18]. As shown in [17], the Hamiltonian (2.43b) with the parameters given by (2.56), is isomorphic to a Hamiltonian describing the rotation of a spin-system $J = (N - 1)/2$, around a fixed axis. In 2004, the scheme (2.56) was proposed almost simultaneously and independently, as a solution to the QST problem between the two ends of a mirror-symmetric chain, in four different works [15, 19–21]. Christandl et al., addressed the problem in the framework of spin chains and their formalism was based on graphs, whereas Plenio et al. considered entanglement dynamics in a system of coupled harmonic oscillators. The work of Nikolopoulos

et al., and Gordon on the other hand pertained to specific realizations of chains based on coupled quantum dots and coupled waveguides, respectively. Furthermore, Nikolopoulos et al. applied the scheme of Eqs. (2.41) and (2.56) to the transfer of states associated with two physical qubits and the controlled generation of entanglement between them [22], while they also studied its robustness against static disorder and dissipation [15]. It has been also demonstrated that, based on the coupling configuration (2.56), one can design networks where unitary rotations are performed while a qubit state is transferred, and build beam splitters and interferometric setups [23, 24]. Such networks, however, may require complex couplings between different sites and their realization in the context of many physical platforms is a rather difficult (if not impossible) task. Finally, the solution (2.56) has been also discussed in connection with state amplification [25] as well as with quantum walks [26, 27].

Besides solution (2.56), other perfect QST Hamiltonians have been also discussed in the literature, mainly in the context of mirror-symmetric chains, and most of them can be derived within the present theoretical framework. The commensurate spectrum seems to play a pivotal role in such perfect QST Hamiltonians, and this led other scientists to engineer perfect QST Hamiltonians starting from a specific spectrum and solving the inverse eigenvalue problem [10, 28–31]. The spectrum of Hamiltonian (2.43b) with the parameters given by (2.56) is integer, and other coupling schemes with commensurate spectra have been presented in the literature [19, 29, 32–34]. In many cases, however, the scheme of Eqs. (2.41) and (2.56), seems to be rather robust relative to other perfect QST schemes under various types of imperfections, whereas it seems to be the optimal coupling configuration with respect to quantum speed limit [31, 35]. For more information on the perfect QST scheme (2.56), the reader may refer to the review article by Kay [10].

Finally, although for the sake of simplicity throughout this section we have focused on networks with nearest-neighbour couplings, it is worth emphasizing that the present approach to network engineering has been shown capable of providing solutions even when interactions beyond nearest neighbours are present [12]. In the following we turn to review some of the main results on the design of networks with logical bus topology.

2.4.2 *Networks of Logical Bus Topology*

The problem of QST in networks of logical bus topologies, has been addressed by Brougham et al. [8], in the framework of the theoretical approach presented above. In the classical theory of networks [6, 7] one often assumes that the hosts have an equivalence property that is, no matter which host the signal starts at, it will be transferred around the predetermined hosts. Similarly, throughout this chapter we focus on passive quantum networks with equivalent hosts, and thus the QST (flow of information) is restricted within the predetermined hosts, irrespective of the source of the state. The authors in [8] have explored the conditions under which multiple PP

QST links are compatible (in the sense to be defined below) and can thus be used as building blocks for the design of passive networks with more involved logical topologies pertaining to a prescribed set of hosts. The following rather general theorem was proved.

Theorem 2. *The unitary operators of a given set $\{\hat{\mathcal{U}}_j \mid \hat{\mathcal{U}}_j \neq \hat{\mathcal{U}}_{j'}, \text{ for } j \neq j' \text{ and } j, j' \in \mathbb{N}\}$ are compatible iff any pair of these operators satisfy the following conditions*

- (i) *There exist $\tau_j, \tau_i \in (0, \infty) : \hat{\mathcal{U}}_i^{\tau_j} = \hat{\mathcal{U}}_j^{\tau_i}$ for $\tau_i \neq \tau_j$, and for all $t \in (0, \infty)$ $\hat{\mathcal{U}}_i^t \neq \hat{\mathcal{U}}_j^t$.*
- (ii) *The two unitary operators commute, i.e. $[\hat{\mathcal{U}}_1, \hat{\mathcal{U}}_2] = 0$.*

If the compatibility conditions of Theorem 2 are satisfied, there exists a single Hamiltonian to implement the unitary operators $\{\hat{\mathcal{U}}_j\}$ at well-defined distinct times $\{\tau_j \in (0, \infty) \mid \tau_j > \tau_{j'}, \text{ for } j > j'\}$. Theorem 2 is very general and applicable to various kinds of PP perfect QST links, including the schemes of Refs. [15, 19, 21, 29–31, 33]. If, however, one of the unitaries is a permutation one can show that *perfect transfer of a single excitation at several different destination nodes at different times, is possible only if all of these hosts belong to the same cycle of $\hat{\mathcal{P}}$* [8]. The choice of hosts thus places a restriction on our choice of $\hat{\mathcal{P}}$, and vice-versa. This in turn imposes physical constraints on the type of Hamiltonians that one can derive. From another point of view, we cannot achieve perfect QST between hosts that belong to different cycles of $\hat{\mathcal{P}}$. Hence, for the design of a *universal bus*, which transfers a single excitation successively to every node of the network within a well-defined time τ , $\hat{\mathcal{P}}$ must be a one-cycle permutation [8].

The design of quantum networks of logical bus topology always leads to Hamiltonians with complex couplings. In fact one can prove the following theorem.

Theorem 3. *Quantum networks of logical bus topology cannot be described by real perfect QST Hamiltonians.*

Proof. Working along the lines of the corresponding proof for a single excitation [10], we will provide here a more general proof pertaining to arbitrary numbers of source and destination nodes, as well as to multiple excitations. More precisely, the input state pertains to a set of source nodes S , and there are two different sets of destination nodes D_1 and D_2 .

Consider first the evolution operator corresponding to a *real* Hamiltonian

$$e^{-i\hat{\mathcal{H}}t/\hbar} = \hat{\mathcal{U}}(t). \quad (2.59)$$

Hermiticity of $\hat{\mathcal{H}}$ implies that

$$\langle \mathbf{k} | \hat{\mathcal{H}}(t) | \mathbf{l} \rangle = \langle \mathbf{l} | \hat{\mathcal{H}}(t) | \mathbf{k} \rangle^* = \langle \mathbf{l} | \hat{\mathcal{H}}(t) | \mathbf{k} \rangle,$$

for any two basis states $|\mathbf{k}\rangle$ and $|\mathbf{l}\rangle$, where for the last equality we have used the fact that $\hat{\mathcal{H}}$ is real. Thus one has that $\hat{\mathcal{H}}$ is symmetric, and by taking the Taylor

expansion in Eq. (2.59), one has that $\hat{\mathcal{U}}(t)$ is also symmetric. This means that $\hat{\mathcal{U}}(t) = [\hat{\mathcal{U}}(t)]^T$ and thus

$$[\hat{\mathcal{U}}(t)]^\dagger = [\hat{\mathcal{U}}(t)]^*. \quad (2.60)$$

Note here that this result applies to the evolution operator of any real Hamiltonian.

Now, assume that $\hat{\mathcal{H}}$ is a real perfect QST Hamiltonian, and let τ_2 be the *shortest time* for which one can achieve perfect QST between the set of source nodes S and a set of destination nodes D_2 . This means that $\hat{\mathcal{U}}(\tau_2)$ transforms the basis states¹³ as follows

$$\hat{\mathcal{U}}(\tau_2)|\mathbf{n}\rangle_S \otimes |\mathbf{m}\rangle_R = e^{i\theta_2}|\mathbf{n}\rangle_{D_2} \otimes |\mathbf{m}\rangle_{R_2}, \quad (2.61)$$

for some angle $\theta_2 \in [0, 2\pi]$, where $R = P \setminus S$ and $R_2 = P \setminus D_2$. Let us assume further that there exists $\tau_1 < \tau_2$, such that perfect QST is achieved between S and another set of destination nodes D_1 , with $D_1 \neq D_2$. That is, we have

$$\hat{\mathcal{U}}(\tau_1)|\mathbf{n}\rangle_S \otimes |\mathbf{m}\rangle_R = e^{i\theta_1}|\mathbf{n}\rangle_{D_1} \otimes |\mathbf{m}\rangle_{R_1}, \quad (2.62)$$

where $\theta_1 \in [0, 2\pi]$ and $R_1 = P \setminus D_1$.

Acting with $\hat{\mathcal{U}}(\tau_1)$ on Eq. (2.62), and using Eqs. (2.62) and (2.60), we obtain

$$[\hat{\mathcal{U}}(\tau_1)]^2|\mathbf{n}\rangle_S \otimes |\mathbf{m}\rangle_R = e^{i2\theta_1}|\mathbf{n}\rangle_S \otimes |\mathbf{m}\rangle_R.$$

Using Eqs. (2.61) and (2.60), we can express the $|\mathbf{n}\rangle_S \otimes |\mathbf{m}\rangle_R$ in terms of $|\mathbf{n}\rangle_{D_2} \otimes |\mathbf{m}\rangle_{R_2}$ obtaining

$$[\hat{\mathcal{U}}(\tau_1)]^2[\hat{\mathcal{U}}(\tau_2)]^*|\mathbf{n}\rangle_{D_2} \otimes |\mathbf{m}\rangle_{R_2} = e^{i(2\theta_1-\theta_2)}|\mathbf{n}\rangle_S \otimes |\mathbf{m}\rangle_R,$$

or equivalently, using Eq. (2.60),

$$\hat{\mathcal{U}}(2\tau_1 - \tau_2)|\mathbf{n}\rangle_S \otimes |\mathbf{m}\rangle_R = e^{i(2\theta_1-\theta_2)}|\mathbf{n}\rangle_{D_2} \otimes |\mathbf{m}\rangle_{R_2}.$$

This last equation shows that we can transfer between S and D_2 , at time $2\tau_1 - \tau_2 < \tau_2$, which contradicts our initial assumption that τ_2 is the shortest possible time at which perfect QST between these two particular sets of nodes can be achieved for the real perfect QST Hamiltonian $\hat{\mathcal{H}}$. Hence, we conclude that it is not possible to construct quantum networks with a logical bus topology that operate under real perfect QST Hamiltonians. \square

¹³Recall here that, in view of the previous discussion, it is sufficient to consider the transformation of the basis states for our purposes.

2.5 Perfect Transfer of States in Higher Excitation Subspaces

In the case of multiple excitations, one may consider various scenarios that cannot be described in the framework of Hamiltonians that have been derived in the single-excitation sector. As far as the problem of QST is concerned, when the state to be transferred pertains to a single quantum particle and no other particles are present, the particular character of the information carrier (boson or fermion) does not play any role, and is not reflected in the solutions one may derive. The situation, however, becomes substantially different when the state to be transferred is encoded on two or more particles. In this case, the bosonic or fermionic nature of the information carriers is expected to reveal itself, and together with interparticle interactions, will impose additional constraints on the physically acceptable solutions. Furthermore, one may also ask for more general solutions that cannot be described within the single-excitation formalism, such as the transfer of two or more excitations, from different nodes, onto the same node. As has been shown in [13], the approach presented in Sect. 2.3 is also applicable to the engineering of quantum networks for the transfer of states that involve multiple excitations.

For the sake of clarity, let us focus only on the case of two excitations; this will considerably simplify our notation. As shown before, the formalism of Sect. 2.3 is applicable to arbitrary states of the network. To keep the present formalism as simple as possible, we will consider a network that is initially in its vacuum state, and the information to be transferred is encoded in the state of two excess particles. The two-excitation Hilbert space \mathbb{H}_2 is spanned by the states $\{|i, \xi_1; j, \xi_2\rangle\}$, with $\xi_1, \xi_2 \in \{u, v\}$, where we have set $|i, \xi_1; j, \xi_2\rangle := |0, \dots, 0, 1_{i, \xi_1}, 0, \dots, 0, 1_{j, \xi_2}, 0, \dots, 0\rangle$. We can thus define three different subspaces of \mathbb{H}_2 , with respect to the initial ordering, that is

$$\mathbb{H}_2^{(<)}, \quad \text{spanned by } \{|i, \xi_1; j, \xi_2\rangle : i < j\}, \quad (2.63a)$$

$$\mathbb{H}_2^{(=)}, \quad \text{spanned by } \{|i, \xi_1; j, \xi_2\rangle : i = j\}, \quad (2.63b)$$

$$\mathbb{H}_2^{(>)}, \quad \text{spanned by } \{|i, \xi_1; j, \xi_2\rangle : i > j\}. \quad (2.63c)$$

The state to be transferred is, in general, a superposition of all the basis states $\{|i, \xi_1; j, \xi_2\rangle\}$ but, as discussed in Sect. 2.3, for our purposes it is sufficient to consider one of the basis states i.e., $|\Psi(0)\rangle = |s_1, \xi_1; s_2, \xi_2\rangle$, which automatically specifies a particular ordering of the states $\{\xi_1, \xi_2\}$, with respect to their position in the network. When the subspaces are decoupled, the system is restricted within the subspace that is specified by $|s_1, \xi_1; s_2, \xi_2\rangle$. Hence, the problem of QST is simplified further, as it can be solved separately within each subspace. In particular, one can consider permutations of the block-diagonal form depicted in Fig. 2.6. The block $\hat{\mathcal{P}}^{(i)}$ is a permutation that permutes all the basis states of the corresponding subspace $\mathbb{H}_2^{(i)}$. For networks with couplings that are insensitive to ξ , the subspaces $\mathbb{H}_2^{(<)}$ and $\mathbb{H}_2^{(>)}$ are basically equivalent and thus $\hat{\mathcal{P}}^{(<)} = \hat{\mathcal{P}}^{(>)}$. Starting from

Fig. 2.6 A permutation of block diagonal form (Adapted from [13])

$$\hat{\mathcal{P}} = \begin{pmatrix} \hat{\mathcal{P}}^{(1)} & & \\ & \hat{\mathcal{P}}^{(2)} & \\ & & \hat{\mathcal{P}}^{(3)} \end{pmatrix}$$

the eigenvalues of $\hat{\mathcal{P}}^{(\cdot)}$, one can work backwards along the lines of the Sect. 2.3, to design Hamiltonians that generate this permutation at time τ , i.e., they satisfy Eq. (2.28) with $\hat{\mathcal{P}} = \hat{\mathcal{P}}^{(\cdot)}$. In the most general scenarios, however, the subspaces are not decoupled and cannot be treated separately. In these cases, one may also have transfer between states that belong to different subspaces, and the QST problem is more involved.

Using the above methodology, one can design QST Hamiltonians in the case of two or more interacting as well as non-interacting excitations, which can be either bosons or fermions. In the following section we will demonstrate, using some simple examples, how various constraints regarding the nature of the excitations can be incorporated in the formalism. We restrict ourselves to networks with couplings that are insensitive to different values of $\xi \in \{u, v\}$, and their evolution is governed by time-independent Hamiltonians, that preserve the total number of excitations, as well as the initial choice of (ξ_1, ξ_2) on the time scales of interest, and are of the form (2.1) with $g_{i,k}^{(\xi)} = g_{i,k}$.

In the case of noninteracting excitations, the basis state $|i, \xi_1; j, \xi_2\rangle$ is characterized by the energy,

$$E_{i,\xi_1}^{j,\xi_2} = \varepsilon_{i,\xi_1} + \varepsilon_{j,\xi_2}.$$

In the case of interacting particles, however, one has

$$\tilde{E}_{i,\xi_1}^{j,\xi_2} = E_{i,\xi_1}^{j,\xi_2} + U_{i,\xi_1}^{j,\xi_2}.$$

The additional energy U_{i,ξ_1}^{j,ξ_2} , accounts for repulsive ($U_{i,\xi_1}^{j,\xi_2} > 0$) or attractive ($U_{i,\xi_1}^{j,\xi_2} < 0$) interactions, whose magnitude may depend on the position of the two excitations in the network, as well as on other degrees of freedom. When the excitations are associated with fermions and occupy the same site, the Pauli principle requires that $\xi_1 \neq \xi_2$, i.e., the excitations must pertain to different states. For instance, the ground state of a two-electron quantum dot in zero magnetic field is the spin singlet state, where the two electrons occupy the lowest orbital and they have antiparallel spins [36]. By contrast, in the case of bosons, there are no such restrictions, and both excitations can be in the same state, while occupying the same site. This is, for instance, the case of the so-called dimers, trimers, etc., in optical lattices [37–39]. Besides onsite interactions which pertain to particles occupying

the same site, one may also have inter-site interactions when the two interacting excitations occupy different sites, but their spatial separation is sufficiently small for interactions to occur. Certain implementations offer unprecedented controllability on both of these types of interactions (quantum dots, optical lattices, etc.), allowing thus for various scenarios.

2.5.1 Decoupled Subspaces

In various realistic scenarios, the onsite interaction U_{j,ξ_1}^{j,ξ_2} is much larger (in absolute value) than any other energy scale in the system.¹⁴ In such cases, the energy gap between the basis states $\{|i, \xi_1; j, \xi_2\rangle\}$ and $\{|j, \xi_1; j, \xi_2\rangle\}$ is of the order of U_{j,ξ_1}^{j,ξ_2} . Assuming further ξ -independent nearest-neighbour (NN) couplings in Eq. (2.1) (i.e., $g_{i,k}^{(\xi)} = g_{i,k} \neq 0$ for $i = k \pm 1$), transitions between states $\{|i, \xi_1; j, \xi_2\rangle\}$ with $i < j$ and $i > j$ are only possible via doubly occupied states. In the limit of $U_{j,\xi_1}^{j,\xi_2} \gg g_{i,k}$ double occupancy is practically forbidden throughout the evolution of the system. Reordering of the excitations is thus a second-order process, which may occur when neighbouring sites are occupied. E.g., consider the state $|j-1, \xi_1; j, \xi_2\rangle$ which is coupled to $\{|j, \xi_1; j-1, \xi_2\rangle, |j+1, \xi_1; j, \xi_2\rangle, |j-1, \xi_1; j-2, \xi_2\rangle\}$ via the doubly occupied states $|j, \xi_1; j, \xi_2\rangle$ and $|j-1, \xi_2; j-1, \xi_2\rangle$. Such a transition can be neglected if the time scale over which perfect QST is achieved, is sufficiently short relative to the time scales over which these second-order processes take place. Then, the subspaces of Eq. (2.63) can be considered, to a good approximation, decoupled throughout the evolution of the system.

Suppose that the system is initially prepared in the state $|\Psi(0)\rangle = |s_1, \xi_1; s_2, \xi_2\rangle$, with $s_1 < s_2$.¹⁵ Given that the system is restricted within the subspace $\mathbb{H}_2^{(<)}$ throughout its evolution, the initial ordering of $\{\xi_1, \xi_2\}$ will be preserved, whereas the states of the other subspaces, are practically forbidden. We will thus focus on the subspace $\mathbb{H}_2^{(<)}$, and onsite interparticle interactions can be neglected in Hamiltonian (2.1). The corresponding sub-permutation, $\hat{\mathcal{P}}^{(<)}$, will be determined by the definition of the destination nodes (d_1, d_2) as well as any additional physical restrictions imposed on the system. For instance, if one is interested in QST Hamiltonians with mirror symmetry, the permutation $\hat{\mathcal{P}}^{(<)}$ has to have the antidiagonal form

$$\hat{\mathcal{P}}^{(<)} = \sum_{i=1}^N \sum_{j>i} |i, \xi_1; j, \xi_2\rangle \langle (N+1-j), \xi_1; (N+1-i), \xi_2|, \quad (2.64)$$

¹⁴In the same fashion (i.e., by setting $U_{j,\xi}^{j,\xi} \rightarrow \infty$) one can formally describe the Pauli principle, which does not allow two electrons with the same state, to occupy the same site.

¹⁵As long as we are interested in networks that are insensitive to different degrees of freedom, the subspaces $\mathbb{H}_2^{(<)}$ and $\mathbb{H}_2^{(>)}$ are equivalent, and the case of $s_1 > s_2$ is covered by the present discussion on the case $s_1 < s_2$.

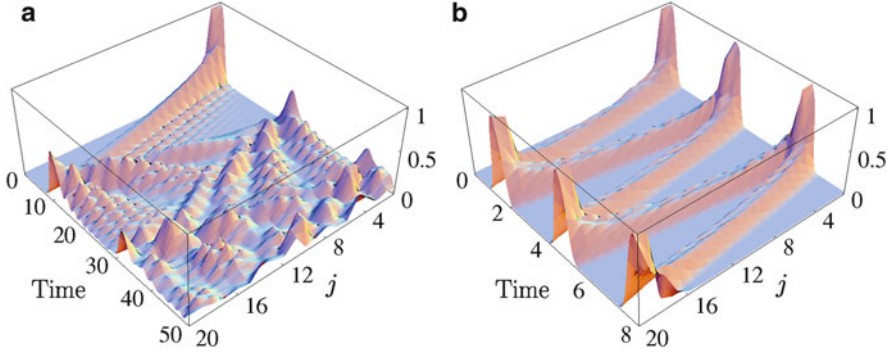


Fig. 2.7 As in Fig. 2.5, for two excitations initially occupying the first and the second site (Adapted from [15])

where $i, j \in \{1, 2, \dots, N\}$. Working along the lines of the Sect. 2.3, one may determine the parameters entering Hamiltonian (2.1), so that Eq. (2.28) is satisfied within the subspace $\mathbb{H}_2^{(<)}$.

In order for the Hamiltonian (2.1) to be compatible with the permutation (2.64), one has to ask for

$$[\hat{\mathcal{P}}^{(<)}, \hat{\mathcal{H}}] = 0, \quad (2.65)$$

within the subspace of interest spanned by the states $\{|i, \xi_1; j, \xi_2\rangle \mid i < j\}$. This requirement for real ξ -independent couplings imposes the first constraints on the choices of $\varepsilon_{i,\xi}$, $g_{i,k}$, and $U_{i,\xi}^{j,\xi'}$ namely,

$$g_{i,k} = g_{N+1-k, N+1-i}, \quad (2.66a)$$

$$\tilde{E}_{i,\xi_1}^{k,\xi_2} = \tilde{E}_{N+1-k,\xi_1}^{N+1-i,\xi_2}. \quad (2.66b)$$

Additional constraints, depend on the details of the network one is interested in.

In the absence of any intersite repulsive (or attractive) interactions (i.e., $U_{i,\xi}^{k,\xi'} = 0$ for $i \neq k$), within the subspace $\mathbb{H}_2^{(<)}$ the Hamiltonian (2.1) basically reduces to Eq. (2.43), and allows for the separation of different degrees of freedom. Given that ξ is preserved, in analogy to Sect. 2.4 one can restrict the engineering on the part of the Hamiltonian that acts on the position of the excitations. As depicted in Fig. 2.7, one of the solutions is given by Eq. (2.56). This was to be expected, since within $\mathbb{H}_2^{(<)}$ we essentially work with non-interacting particles that remain distinct as they never occupy the same site. This result has been also predicted in [15], and its robustness against various types of imperfections has been analysed [15, 40]. We should emphasize, however, that this is only one of the infinitely many solutions that one may obtain within the present theoretical framework, for the particular choice of $\hat{\mathcal{P}}^{(<)}$.

Relaxing the constraint $U_{i,\xi}^{k,\xi'} = 0$ for $i \neq k$, allows one to take account of repulsive(attractive) interactions between excitations. In this case, the derivation of analytic expressions is far from trivial, but one can always resort to numerical solutions. The allowed solutions are basically determined by the magnitude of the various $U_{i,\xi}^{k,\xi'}$ relative to the various couplings $g_{i,k}$ (e.g., see [13] for related examples).

2.5.2 Transitions Between Subspaces

Our examples up to now were for perfect QST Hamiltonians with NN couplings only. There are, however, realistic situations where the effects of couplings beyond NN cannot be ignored. Formally speaking, the presence of non-vanishing couplings beyond NN in Eq. (2.1), automatically enables transitions between different subspaces, and thus the permutations one may consider cannot be expressed in the block diagonal form of Fig. 2.6. The engineering of Hamiltonians proceeds along the lines of Sect. 2.3, and related examples can be found in [13].

From the perspective of quantum control a useful and important task would be to perfectly transfer two excitations that are in different sites onto the same site and vice-versa (i.e., $|i, \xi_1; j, \xi_2\rangle \leftrightarrow |j, \xi_1; i, \xi_2\rangle$), coupling thus the subspaces $\mathbb{H}_2^{(<)}$ and $\mathbb{H}_2^{(>)}$ to $\mathbb{H}_2^{(=)}$. This would allow one to perform simple logical operations within the quantum network. Such a class of problems cannot be described with the framework of permutations, since by definition they provide only a one-to-one mapping between the basis states. As discussed in [13], however, one can still work along the lines of Sect. 2.3, but starting from a judiciously chosen unitary rather than a permutation.

In closing, we would like to point out that in this section the formalism of Sect. 2.3 was used for the derivation of effective Hamiltonians, which act on a subspace of near-resonant states. An effective Hamiltonian is basically an approximation of the original Hamiltonian, and the state transfer is thus not perfect, but rather occurs with high fidelity. How large the fidelity can be made depends on to what extent the system remains in the particular subspace throughout its evolution on the time scale of interest. Ideally, for a network whose components allow for arbitrary adjustment of all the related parameters (such as couplings, detunings, interactions, etc.), the actual evolution of the system can be brought arbitrarily close to the effective one, and thus the fidelity can be arbitrarily close to 1 (in the absence of disorder and dissipation). In this direction, quantum dots, optical and photonic lattices, have been shown to offer unprecedented controllability in many respects, although deviations from the ideal case are to be expected.

2.6 The Quantum Directional Coupler (QDC)

A dual-channel directional coupler (DC) is a passive device with two input and two output ports. The ports are the ends of two waveguides, the so-called source and drain channels, which are brought in close proximity over a certain region. The division of a signal between the outputs of the two channels is achieved by varying a control parameter. DCs have many applications in (opto)electronics [41–44]. For instance, directional couplers (DCs) may perform a number of useful functions in thin-film devices such as power division, switching, frequency selection, and (de)multiplexing.

In 2008, a dual-channel DC for quantum computing and communication purposes was proposed [45]. In analogy to conventional DCs, a dual-channel quantum DC (QDC) can be defined as a device which allows the selective transfer of quantum signals between two adjacent quantum channels, and it is thus expected to be the key element for various useful quantum information processing tasks, such as quantum switching, (de)multiplexing, etc.

The main problem addressed in [45] was the following. Given two perfect quantum channels, is it possible to define inter-channel interactions, for which the entire system operates as a dual-channel QDC? Focusing on single-qubit states prepared initially in the first site of one of the channels, inter-channel interactions should allow for the transfer of the state to the last site of either of the two channels, in a controlled and deterministic manner. In the spirit of passive quantum networks, this study was focused on configurations with minimal external control, i.e., without elaborate sequences of time-dependent pulses and measurements, and here we will review the main results of this work.

2.6.1 Formalism

The dual-channel QDC under consideration involves two nearly identical channels, the source (s) and the drain (d). Each channel consists of $N > 2$ nearly identical sites denoted by $\mathbf{r} = (i, j)$, with $i \in \{s, d\}$ and $1 \leq j \leq N$. Accordingly, the computational basis of the system can be chosen as $\{|\mathbf{r} : \xi\rangle\}$, with $|\mathbf{r} : \xi\rangle \equiv a_{\mathbf{r}, \xi}^\dagger |\mathbf{0}\rangle$ denoting an excitation on the \mathbf{r} th site in state $|\xi\rangle$. The first two sites of the chains $\{(s, 1); (d, 1)\}$ play the role of the two input ports, whereas the output ports are represented by the last sites $\{(s, N); (d, N)\}$.

In the following we assume that the source and the drain chains have been independently engineered to allow for the perfect transfer of a qubit state between their two ends. Their dynamics are described by a perfect QST Hamiltonian of the form (2.43) with the parameters given in Eq. (2.56). Recall here Observation 1, together with the fact that the interaction part of the Hamiltonian that governs each channel cannot distinguish between different states $|\xi\rangle$. Without modifying all of these properties, our task is to define interactions between the channels so that an

excitation initially occupying one of the input ports, can be transferred to either of the two output ports in a controlled and deterministic way.¹⁶ More precisely, consider that the entire configuration is initially in the vacuum state, and the qubit state is prepared on the first site of the source channel. The initial state of the entire network is

$$|\Psi_C(0)\rangle = |s, 1 : \psi(\xi)\rangle, \quad (2.67)$$

where $|\psi(\xi)\rangle$ is the qubit state to be transferred.

In analogy to the previous sections, we will solve the problem in the ideal scenario first, i.e., in the absence of imperfections. Hence, it is sufficient to focus on the transfer of the excitation, ignoring degrees of freedom ξ . A dual-channel QDC, should be able to perform perfectly the following transformations

$$|s, 1\rangle \rightarrow |s, N\rangle, \quad (2.68a)$$

$$|s, 1\rangle \rightarrow |d, N\rangle, \quad (2.68b)$$

at well defined time instants, and irrespective of $|\xi\rangle$. The dynamics of the excitation in either of the two chains is governed by

$$\hat{\mathcal{H}}_{\text{QST}}^{(i)} = \sum_{j=1}^N \varepsilon_j |i, j\rangle \langle i, j| + \sum_{j=1}^{N-1} g_{j,j+1} (|i, j\rangle \langle i, j+1| + |i, j+1\rangle \langle i, j|), \quad (2.69)$$

with $i \in \{s, d\}$, $\varepsilon_j = -\tilde{\varepsilon}(N-1)$ and $g_{j,j+1} = \tilde{\varepsilon}\sqrt{j(N-j)}$.

2.6.2 Mathematical Analogy

Let us employee the mathematical analogy between Hamiltonian (2.69) and the rotation of a spin system around a fixed axis [16–18]. We introduce two angular-momentum operators $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ acting on different subspaces, with

$$J_1 = (M-1)/2, \quad m_1 = i - (M+1)/2; \quad (2.70a)$$

$$J_2 = (N-1)/2, \quad m_2 = j - (N+1)/2, \quad (2.70b)$$

¹⁶By contrast to passive networks of logical bus topology where the state is transferred successively to different destination nodes, here the transfer pertains to one of the two available output nodes, and the choice is performed in a controlled manner.

for a positive integer $M \geq 2$. An orthonormal basis for the state space of the spin- J_k system can be chosen as $\{|J_k, m_k\rangle\}$, where $|J_k, m_k\rangle$ are degenerate eigenvectors of the operator \hat{J}_k^2 . The basis states of the composite system are $\{|J_1, m_1; J_2, m_2\rangle\}$, where $|J_1, m_1; J_2, m_2\rangle \equiv |J_1, m_1\rangle \otimes |J_2, m_2\rangle$.

The dynamics of the excitation in either of the two nearly identical channels (source or drain) are described by the spin- J_2 system. Using Eqs. (2.70b) one may define a one-to-one correspondence between the basis states $\{|j\rangle\}$ and $\{|J_2, m_2\rangle\}$ for fixed channel parameters $\{N, \bar{\varepsilon}\}$ i.e., we have

$$|J_2, m_2\rangle \equiv |j\rangle. \quad (2.71a)$$

The spin- J_1 system describes the inter-channel dynamics, with the only convention being

$$|J_1, -J_1\rangle \equiv |s\rangle, \quad |J_1, J_1\rangle \equiv |d\rangle. \quad (2.71b)$$

Given the distinct roles of the two spin systems, in the angular-momentum representation the dynamics of a single excitation in a dual-channel QDC are described by a Hamiltonian of the form

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2, \quad (2.72)$$

where $\hat{\mathcal{H}}_k \equiv \hat{\mathcal{H}}_k^{(0)} + \hat{\mathcal{V}}_k$ refers to the spin- J_k system only. The basis states $\{|J_k, m_k\rangle\}$ are degenerate eigenstates of the corresponding unperturbed Hamiltonian $\hat{\mathcal{H}}_k^{(0)} \equiv \varepsilon_k \hat{J}_k^2$, while $\hat{\mathcal{V}}_k$ is the coupling between various states $\{|J_k, m_k\rangle\}$.

The part of the initial condition (2.67) pertaining to the position of the quantum state reads in angular-momentum representation

$$|\Psi_{\text{AM}}(0)\rangle = |J_1, -J_1; J_2, -J_2\rangle, \quad (2.73)$$

whereas for transformations (2.68) we have

$$|J_1, -J_1; J_2, -J_2\rangle \rightarrow |J_1, -J_1; J_2, J_2\rangle, \quad (2.74a)$$

$$|J_1, -J_1; J_2, -J_2\rangle \rightarrow |J_1, J_1; J_2, J_2\rangle. \quad (2.74b)$$

It is sufficient for our purposes therefore, to specify forms of the Hamiltonian (2.72) for which transformations (2.74) take place at well defined time instants, while one may switch between them in a controlled manner.

Transformation (2.74a) pertains to the evolution of the spin- J_2 system only, and is thus expected to be implementable by Hamiltonian (2.72) for $\hat{\mathcal{V}}_1 = 0$ (no inter-channel coupling). Recall now that the dynamics of the spin- J_2 system have to describe accurately the evolution of the excitation in either of the two identical channels. Given that both channels are described by a perfect QST Hamiltonian

of the form (2.69), with fixed parameters $\{N, \tilde{\varepsilon}\}$, using the correspondences (2.70b) we find

$$\hat{\mathcal{H}}_2 = \hat{\mathbf{1}}_1 \otimes (\varepsilon_2 \hat{J}_2^2 + 2\tilde{\varepsilon} \hat{J}_{2,x}/\hbar), \quad (2.75)$$

where $\varepsilon_2 = -\tilde{\varepsilon}/[\hbar^2 J_2(J_2 + 1)]$ and $\hat{J}_{2,x}$ is the x -component of the vector $\hat{\mathbf{J}}_2$. According to (2.75), the initial state of the isolated spin- J_2 system undergoes a rotation by an angle $\varphi_2 = 2\tilde{\varepsilon}t/\hbar$ around the x -axis, at time t . Hence, for the initial condition (2.73), the transformation (2.74a) takes place at time

$$\tau = \hbar\pi/(2\tilde{\varepsilon}).$$

2.6.3 A QDC Scheme Based on the Rotation of Independent Spins

We have to determine the inter-channel interaction $\hat{\mathcal{V}}_1$, for which transformation (2.74b) takes place at a well defined time instant. The transformation (2.74b) involves a simultaneous rotation of both spins. This leads us to introduce the total angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$, with $|J_1 - J_2| \leq J \leq J_1 + J_2$ and $|m| \leq J$, while the corresponding basis states $\{|J, m\rangle\}$ can be expanded on the basis $\{|J_1, m_1; J_2, m_2\rangle\}$ as follows

$$|J, m\rangle = \sum_{\substack{m_1, m_2 \\ m=m_1+m_2}} C_{m_1, m_2}^J |J_1, m_1; J_2, m_2\rangle,$$

with $|m_k| \leq J_k$. Using the properties of the Clebsch-Gordan coefficients C_{m_1, m_2}^J , initial condition (2.73) reads in the basis $\{|J, m\rangle\}$, $|\Psi_{\text{AM}}(0)\rangle = |J, -J\rangle$, while for the transformation (2.74b) we have

$$|J, -J\rangle \rightarrow |J, J\rangle. \quad (2.76)$$

This transformation, however, is similar to transformation (2.74a) which, as discussed earlier, can be achieved if the evolution of the corresponding spin system is governed by a Hamiltonian of the form (2.75). Hence, to perform transformation (2.76) it suffices to define the inter-channel coupling (which is represented by the Hamiltonian $\hat{\mathcal{H}}_1$), so that the total Hamiltonian (2.72) is of the form $\hat{\mathcal{H}} \sim \hat{J}_x$. Clearly, for fixed $\hat{\mathcal{H}}_2$ given by Eq. (2.75) and for $\hat{\mathcal{H}}_1^{(0)} = \varepsilon_1 \hat{J}_1^2$, this is possible only if we define interactions $\hat{\mathcal{V}}_1 \propto \hat{J}_{1,x}$, between the states $\{|J_1, m_1\rangle\}$. For instance, setting $\hat{\mathcal{V}}_1 = 2K \hat{J}_{1,x} \otimes \hat{\mathbf{1}}_2/\hbar$, with K denoting the inter-channel coupling strength, the total Hamiltonian (2.72) reads

$$\hat{\mathcal{H}} = (\varepsilon_1 \hat{J}_1^2 + 2K \hat{J}_{1,x}/\hbar) \otimes \hat{\mathbf{1}}_2 + \hat{\mathbf{1}}_1 \otimes (\varepsilon_2 \hat{J}_2^2 + 2\tilde{\varepsilon} \hat{J}_{2,x}/\hbar), \quad (2.77)$$

with $\varepsilon_1 = -K/[\hbar^2 J_1(J_1 + 1)]$. This Hamiltonian acquires the desired form for $K = \tilde{\varepsilon}$ i.e., we obtain $\hat{\mathcal{H}} = \varepsilon_1 \hat{J}_1^2 \otimes \hat{\mathbf{1}}_2 + \varepsilon_2 \hat{\mathbf{1}}_1 \otimes \hat{J}_2^2 + 2\tilde{\varepsilon} \hat{J}_x/\hbar$.

The Hamiltonian (2.77) describes the operation of a perfect dual-channel QDC in an angular-momentum representation. The corresponding evolution operator reads

$$\hat{\mathcal{U}}(t) = \left[e^{-i\varepsilon_1 \hat{J}_1^2 t/\hbar} \hat{\mathcal{R}}_1^{(x)}(\varphi_1) \right] \otimes \left[e^{-i\varepsilon_2 \hat{J}_2^2 t/\hbar} \hat{\mathcal{R}}_2^{(x)}(\varphi_2) \right] \quad (2.78)$$

where we have introduced the rotation operator $\hat{\mathcal{R}}_k^{(x)}(\varphi_k) = e^{-i\hat{J}_{k,x}\varphi_k/\hbar}$, with $\varphi_1(t) = 2Kt/\hbar$ and $\varphi_2(t) = 2\tilde{\varepsilon}t/\hbar$. According to Eq. (2.78) the two spins are rotated independently around the x axis, and after time t the initial state of the spin- J_k system has been rotated by an angle φ_k . Hence, the spin- J_k system is in a superposition state of the form $|\chi_k(t)\rangle = e^{i\varphi_k/2} \sum_{m_k} A_{m_k}^{(k)}(t) |J_k, m_k\rangle$, with $A_{m_k}^{(k)}(t) = \langle J_k, m_k | \hat{\mathcal{R}}_k^{(x)}(\varphi_k) | J_k, -J_k \rangle$. These amplitudes can be obtained from the expansion of the rotation operator, as follows $\hat{\mathcal{R}}_k^{(x)}(\varphi_k) = \sum_{l=0}^{2J_k} D_{k,l}(\varphi_k) \hat{J}_{k,x}^l$. For fixed \hat{J}_k , i.e., for fixed channels and coupler, the coefficients $D_{k,l}(\varphi_k)$ are expressed in closed form, in terms of trigonometric functions [46], which also determine the relative amplitudes in the superpositions $|\chi_k(t)\rangle$. We see therefore that for $t > 0$ the state of the composite spin system is, in general, a superposition of all the basis states $\{|J_1, m_1; J_2, m_2\rangle\}$. In the computational basis this situation corresponds to a completely delocalized excitation, distributed among all of the sites of both channels and the coupler.

Of particular interest, however, is the state of the system at time $t = \tau$, when the rotation angles of the two spins read: $\varphi_2(\tau) = \pi$; $\varphi_1(\tau) = \pi\mu$, with $\mu = K/\tilde{\varepsilon}$. Hence, with the system initially prepared in the state (2.73), the following transformation has occurred at $t = \tau$:

$$|J_1, -J_1; J_2, -J_2\rangle \rightarrow e^{i\mu\pi/2} \sum_{m_1=-J_1}^{J_1} A_{m_1}^{(1)}(\tau) |J_1, m_1; J_2, J_2\rangle. \quad (2.79)$$

The superposition state in this transformation involves only the spin- J_1 system, and can be controlled by adjusting the corresponding rotation angle $\varphi_1(\tau)$, by means of the ratio μ . Clearly, the transformations (2.74a) and (2.74b) are performed for $\mu = 0$ and $\mu = 1$, respectively.

A quantum network involving a number of coupled sites, may operate as a QDC if the Hamiltonian of the entire system in an angular momentum representation acquires the form (2.77). For instance, one can readily show, using Eqs. (2.70), that Hamiltonian (2.77) reduces to a Hamiltonian for an $M \times N$ grid of the form

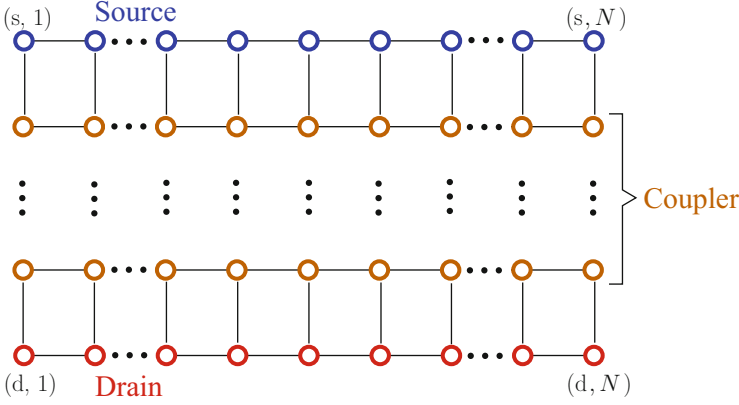


Fig. 2.8 A grid of $M \times N$ nodes operating as a dual-channel QDC with the source and drain channels represented by the two outermost chains, and the intermediate chains providing the coupling between them. The *horizontal* and the *vertical* couplings are modulated according to $\propto \sqrt{l(Z-l)}$, where $1 \leq l < Z$ and $Z \in \{N, M\}$

$$\begin{aligned}
 \hat{\mathcal{H}}_{M \times N} = & - \sum_{i=1}^M \sum_{j=1}^N [K(M-1) + \tilde{\varepsilon}(N-1)] \hat{a}_{i,j}^\dagger \hat{a}_{i,j} \\
 & + \sum_{i=1}^M \sum_{j=1}^{N-1} G_{i,j}^{i,j+1} (\hat{a}_{i,j}^\dagger \hat{a}_{i,j+1} + \hat{a}_{i,j+1}^\dagger \hat{a}_{i,j}) \\
 & + \sum_{i=1}^{M-1} \sum_{j=1}^N G_{i,j}^{i+1,j} (\hat{a}_{i,j}^\dagger \hat{a}_{i+1,j} + \hat{a}_{i+1,j}^\dagger \hat{a}_{i,j}), \quad (2.80)
 \end{aligned}$$

where adjacent sites are coupled with strengths $G_{i,j}^{i,j+1} = \tilde{\varepsilon} \sqrt{j(N-j)}$ and $G_{i,j}^{i+1,j} = K \sqrt{i(M-i)}$ (see Fig. 2.8). The operator $\hat{a}_{i,j}^\dagger$ creates an excitation on the j th site of the i th row, while the coupling strength between two different sites (i, j) and (i', j') , is denoted by $G_{i,j}^{i',j'}$. In this formalism, the two outermost chains represent the source and the drain channels (i.e., $s \equiv 1$ and $d \equiv M \geq 2$), while any intermediate sites (i, j) with $i \neq \{1, M\}$ pertain to the coupler. This 2D coupling configuration has also been investigated in [47], albeit in a different context. The present chapter, however, reveals another aspect of such a structure, namely its use as a QDC with source and drain channels the two outermost chains. In particular, a qubit state can be transferred selectively from the input port $(s, 1)$, to either of the two output ports $\{(s, N), (d, N)\}$ at time $t = \tau$ for $K = \{0, \tilde{\varepsilon}\}$, respectively. Moreover, we have seen the mathematical analogy between the 2D lattice Hamiltonian (2.80), and the rotation of independent spins.

The simplest configuration one may consider in the context of Hamiltonian (2.80), is a $2 \times N$ grid pertaining to two directly coupled quantum channels.

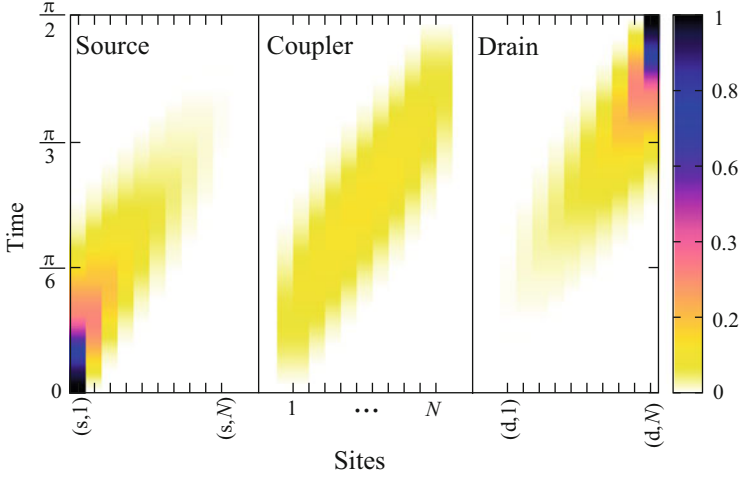


Fig. 2.9 Time evolution of an excitation in a dual-channel QDC, consisting of $M = 3$ nearly identical arrays, with $N = 10$ sites each. Setting the coupling strength to the coupler $K = \tilde{\varepsilon}$, an excitation initially occupying the input port of the source channel (s, 1), is transferred to the output port of the drain channel (d, N) at $\tau = \hbar\pi/(2\tilde{\varepsilon})$. The time is in units of $\tilde{\varepsilon}$

This system can be easily analysed in the angular-momentum representation (with $J_1 = 1/2$), obtaining an analytic expression for its wavefunction at any time $t > 0$. At time τ , the right-hand side of transformation (2.79) reads

$$e^{i\mu\pi/2} \left[\cos(\mu\pi/2) \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - i \sin(\mu\pi/2) \left| \frac{1}{2}, \frac{1}{2} \right\rangle \right] |J_2, J_2\rangle. \quad (2.81)$$

In the computational basis, this is equivalent to

$$e^{i\mu\pi/2} [\cos(\mu\pi/2)|s; N\rangle - i \sin(\mu\pi/2)|d; N\rangle].$$

So, in general, both output ports can be excited at $t = \tau$, with probabilities determined by μ ; the structure operates as a splitter. The transformations (2.68a) and (2.68b) are performed for $\mu = 0$ (i.e., decoupled chains) and $\mu = 1$ (i.e., $K = \tilde{\varepsilon}$), respectively. For the sake of illustration, in Fig. 2.9 we present results pertaining to directional coupling in a larger grid, which have been obtained through the solution of the Schrödinger equation in the computational basis. The depicted behaviour is in perfect agreement with the previous discussion.

2.6.4 A QDC Scheme Based on Coupled Spins

In the configuration depicted in Fig. 2.8, all the sites of the source channel have to be coupled (directly or indirectly) to the corresponding sites of the drain channel via

nearest-neighbour interactions. A question is whether one can achieve directional coupling between two chains by defining inter-channel interactions for a certain number of sites only.

This question has been addressed in [45], where it was shown that one of the solutions pertains to the coupling between a spin $J_1 = 1/2$ and an angular momentum $J_2 = 1$, such that the interaction part of the Hamiltonian (2.72) is of the form ($\hbar = 1$)¹⁷

$$\hat{\mathcal{V}} = \hat{\mathcal{V}}_1 + \hat{\mathcal{V}}_2 = 2\tilde{\varepsilon}\hat{\mathbf{1}}_1 \otimes \hat{J}_{2,x} + K\hat{J}_{1,y}\hat{J}_{2,y}, \quad (2.82)$$

and the transformation (2.74a) can be achieved for $K = 0$, when only the vector $\hat{\mathbf{J}}_2$ is rotated around the x -axis. Turning on the inter-channel interaction, i.e., setting $K \neq 0$, both vectors $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ can be rotated simultaneously around the y -axis. By contrast to Hamiltonian (2.77), we have two distinct evolution routes that may interfere either constructively or destructively, and the transformation (2.74b) can be achieved by choosing judiciously the ratio μ . Indeed, for the initial condition (2.73), one can show that the transformation (2.74b) occurs at $t = \tau/\sqrt{2}$ for $K = -4\tilde{\varepsilon}$.

The Hamiltonian (2.82), can be implemented in the 2×3 grid depicted in Fig. 2.10a, for $g = \sqrt{2}\tilde{\varepsilon}$ and $\kappa = K/(2\sqrt{2})$. This arrangement requires the adjustment of geometric phases which is possible e.g., by looping around magnetic or electric fields, depending on the nature of information carriers, while for optical networks one may use phase shifters. More interestingly, a configuration similar to Fig. 2.10a can be used as a coupler for selective transfer of an excitation between two chains involving an arbitrary number of sites $N > 3$ (see [45] for more information). For the sake of illustration, in Fig. 2.11 we present numerical results pertaining to the transfer of a single excitation between two chains of $N = 7$ sites each.

In closing we would like to emphasize that the directional coupling of quantum states between two lattices that operate as wires, is a rather difficult task. We have discussed the problem in the framework of a mathematical analogy between a particular perfect QST Hamiltonian and angular momenta rotations around a fixed axis. This analogy enabled us to find two solutions, which are by no means unique. Most probably there exist other schemes which are more efficient and flexible than the ones presented here. In any case, the performance of such schemes has to be investigated under realistic conditions, and perhaps in the framework of specific realizations.

¹⁷As we have seen, the main effect of the unperturbed Hamiltonians $\hat{\mathcal{H}}_k^{(0)}$ is the introduction of an accumulated phase at the end of the transfer which, however, is fixed and known in the absence of disorder and other imperfections. Hence, for the time being we focus on the interaction part of the Hamiltonian.

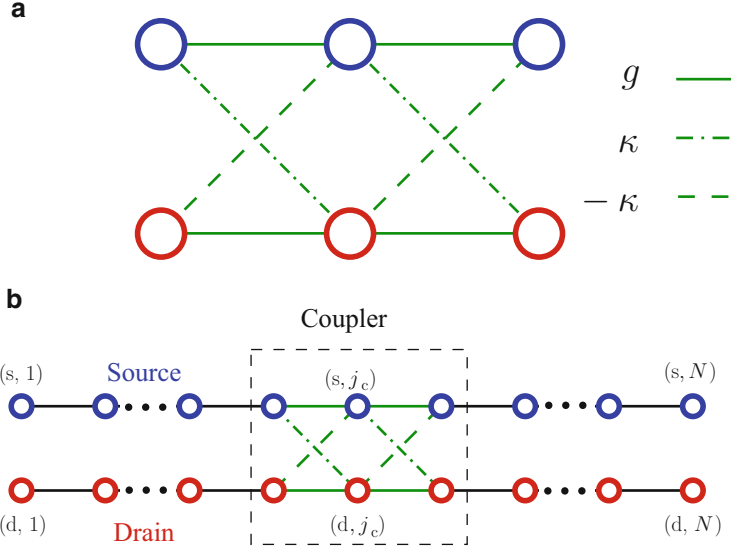


Fig. 2.10 (a) A QDC consisting of six sites, with inter-channel coupling constant $\kappa = -g$. (b) The coupler integrated in a dual-channel system (Adapted from [45])

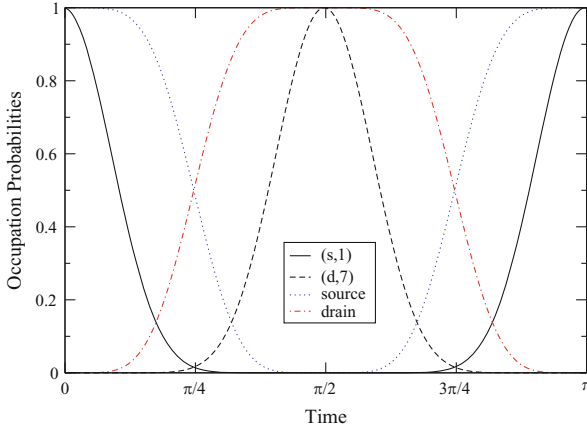


Fig. 2.11 Time evolution of an excitation in the QDC of Fig. 2.10b, with $N = 7$. The excitation is transferred from the input port $(s, 1)$, to the output port of the drain channel (d, N) at $t = \tau$. The time is in units of $\tilde{\epsilon}$ (Adapted from [45])

2.7 Effects of Bending in 2D and 3D Quantum Networks

In analogy to conventional networks [41], large-scale QIP and networking, irrespective of the physical platform, require efficient complex signal manipulations (such as routing, splitting, switching, etc.), which are possible only in higher-dimensional

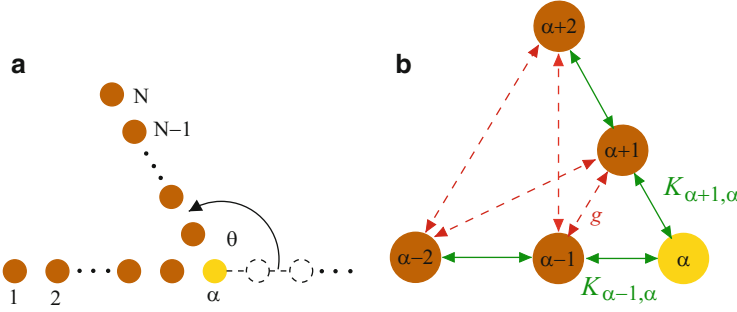


Fig. 2.12 (a) A bent quantum chain of N sites. (b) A closeup of the bend. Green (solid) arrows denote NN interactions of strength $K_{l,m}$. Red (dashed) arrows denote interactions beyond NNs, with the strongest one (of strength g) corresponding to the first neighbours of the corner site α (Adapted from [55])

geometric arrangements. This necessity has motivated studies on state transfer in various 2D arrangements [21, 23, 24, 45, 47–54], most of which rely on nearest-neighbour (NN) Hamiltonians. Bends are expected to be at the core of any 2D or 3D configuration, and their effects have been analysed recently by Nikolopoulos et al. [55]. Here we review some of the main results of [55].

2.7.1 Formalism

The 2D arrangement of [55] pertains to N identical sites and is depicted in Fig. 2.12. Assuming that the configuration is initially prepared in vacuum and the qubit state to be transferred is prepared on the first site, we work in the ideal scenario (i.e., in the absence of imperfections) for the reasons discussed above. We focus on the position of the state in the configuration, assuming that the remaining degrees of freedom ξ are preserved throughout the evolution of the system. The dynamics of the excitation in the bent chain is described by a Hamiltonian of the form ($\hbar = 1$)

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{V}}(\theta), \quad (2.83a)$$

where

$$\hat{\mathcal{H}}_0 = \sum_{j=1}^N \varepsilon_j \hat{a}_j^\dagger \hat{a}_j + \sum_{j=1}^{N-1} K_{j,j+1} (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j), \quad (2.83b)$$

is the unperturbed Hamiltonian corresponding to the unbent wire i.e., to $\theta = 0$, and $\hat{\mathcal{V}}(\theta)$ is the perturbation associated with the bending. We consider two different centrosymmetric faithful state-transfer Hamiltonians with NN interactions, which

ensure the faithful transfer of information between the two ends of the unbent chain, and we discuss their performance as we bend the chain.

Protocol 1. The first Hamiltonian was presented by Banchi et al. [56,57]. It involves $\varepsilon_j = \varepsilon$, $K_{j,j+1} = K_0 \forall j \neq 1, N-1$, and $K_{1,2} = K_{N-1,N} = K$. For a given N , the ratio K/K_0 is optimized so that faithful transfer of information between the two ends of the chain occurs at time $T_1^{(0)}$.

Protocol 2. The second Hamiltonian has been discussed in Sect. 2.4. Again, all the sites are on resonance, while the couplings along the entire chain are engineered according to $K_{j,j+1} = K_0 \sqrt{(N-j)j}$. In contrast to Protocol 1, this scheme promises ideally perfect transfer at time $T_2^{(0)} = \pi/(2K_0)$.

Both of these protocols are designed for chains with NN couplings. In many physical realizations of quantum networks, the coupling between two adjacent sites is directly related to their spatial separation (e.g., see [8, 12, 31, 58–62]). Hence, interactions beyond NNs are expected to get enhanced in the neighbourhood of bends (see Fig. 2.12), disturbing the communication. For not too sharp bends, the perturbation is dominated by the coupling between the first-order neighbours of the corner site α , and the perturbation in Eq. (2.83a) can be chosen as

$$\hat{\mathcal{V}}(\theta) = g(\theta)(\hat{a}_{\alpha-1}^\dagger \hat{a}_{\alpha+1} + \hat{a}_{\alpha+1}^\dagger \hat{a}_{\alpha-1}). \quad (2.83c)$$

In general, the dynamics of the chain are expected to be determined by the strength of the perturbation g , relative to the NN couplings around the corner, rather than the actual origin of the perturbation. Hence, for the following analysis we introduce the ratio $\kappa = g/K_{\max}$, where $K_{\max} \equiv \max\{K_{l,m}\}$.

The Hamiltonian (2.83) preserves the number of excitations and thus, the system is restricted to the one-excitation sector of the Hilbert space throughout its evolution. The computational basis can be chosen as $\{|j\rangle\}$, where $|j\rangle$ is the state with one excitation on the j th site. Initially $|\Psi(0)\rangle = |1\rangle$, and the probability for the excitation to occupy the N th site at time t when protocol i is used, is given by $|P_i(t)|^2 = |\langle N|\hat{\mathcal{U}}(t)|\Psi(0)\rangle|^2$, with $\hat{\mathcal{U}}(t) = e^{-i\hat{\mathcal{H}}t}$. The corresponding probabilities for the unbent chains are denoted by $P_i^{(0)}(t)$.

2.7.2 Analysis and Minimization of Bending Losses

The analysis of [55] pertained to $\kappa \in [0, 1]$ for various values of α , and focused on the ratios $Q_i \equiv P_i/P_i^{(0)}$ and $S_i \equiv T_i/T_i^{(0)}$.

The performance of Protocol 2 in the presence of a bend is summarized in Fig. 2.13. As depicted in the histograms, the ratio Q_i is close to 1 for relatively weak perturbations (up to $\kappa \approx 0.2$), and drops as we increase the strength of the perturbation, following a Gaussian law. The effect of the bend on the time of

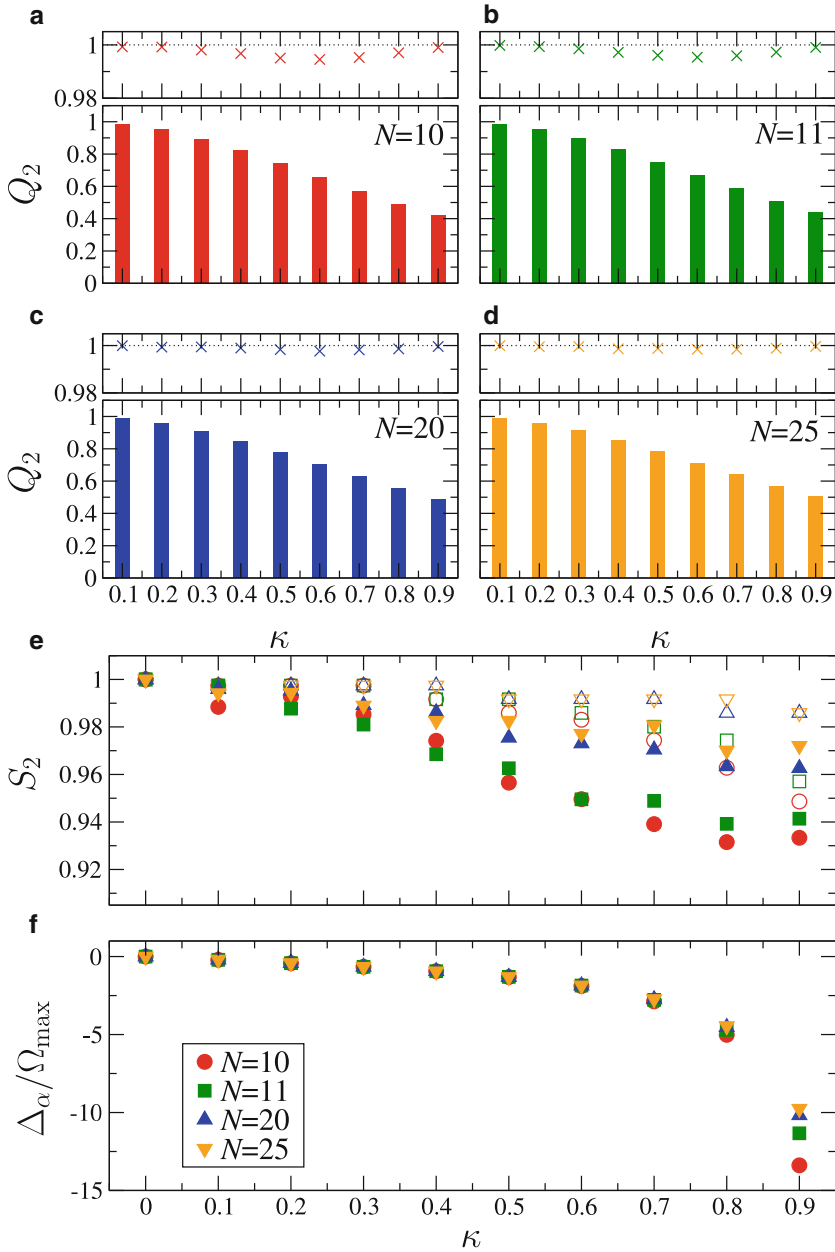


Fig. 2.13 Performance of the bent chain for protocol 2 and various N . **(a-d)** The normalized probability of transfer Q_i , for the bent chain without (*bars*), and with (\times) corner defect. Corner sites: **(a,b)** $\alpha = 6$; **(c)** $\alpha = 11$; **(d)** $\alpha = 13$. **(e)** Time of transfer through the bent chain without (*open symbols*) and with (*filled symbols*) corner defect. **(f)** Optimal detunings of the defect corner site relative to the other sites (Adapted from [55])

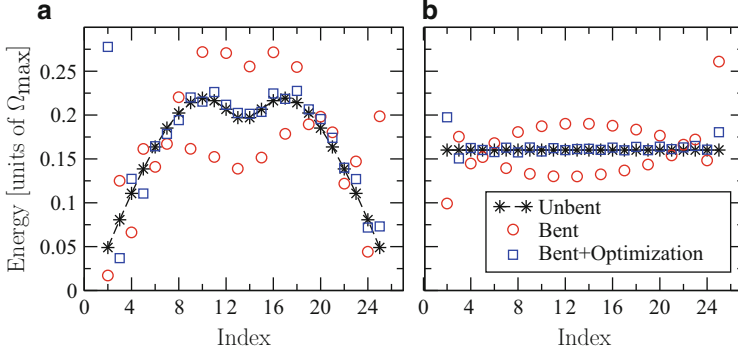


Fig. 2.14 Difference between successive eigenenergies of the Hamiltonians for $N = 25$ and $\alpha = 12$. (a) Protocol 1, $\kappa = 0.5$; (b) Protocol 2, $\kappa = 0.3$ (Adapted from [55])

transfer is not as disastrous as on the fidelity (see open symbols in Fig. 2.13e). The transfer is accelerated relative to the unbent chains, since the couplings around the corner increase whereas, as a result of the finite spatial extension of the perturbation, the acceleration becomes less pronounced as we increase N . The performance of Protocol 1 is very close to the performance of Protocol 2 and is not shown here (the reader may refer to [55] for detailed results).

In [55] it was shown that bending losses can be minimized, without deviating significantly from the protocol under consideration, as long as one has external control on the energy of the corner site only. The detuning Δ_α of the corner site relative to the others can be optimized so that the transfer from the first to the last site is maximized for a given strength of the perturbation κ and for times $t \in [0, T_i^{(0)}]$. As depicted in Fig. 2.13, for given N , κ and α , there is an optimal value of Δ_α for which the probability of transfer is above 99% of the corresponding probability for the unbent chain. The results are analogous for Protocol 1 and in fact, detunings of the same order are required for the minimization of losses in the two protocols against the same disturbance κ . The optimization procedure works efficiently irrespective of the position of the bend on the chain for both protocols.

The role of the bends and the minimization of the associated losses, can be understood in the framework of the spectrum of the Hamiltonians (one-excitation sector). The operation of many QST Hamiltonians (including the ones discussed here), relies on the details of the spectrum, and the overlap of the initial state $|\Psi(0)\rangle$ with the corresponding eigenvectors. As shown in Fig. 2.14 the presence of the bend tends to distort the commensurate spectrum of the unperturbed QST Hamiltonians, leading thus to the observed decrease of the probability of transfer. When the detuning of the corner site is optimized, however, the initial distribution is restored to a large extent, minimizing thus the losses.

In summary, the above analysis shows that the transfer of signals through bent chains that operate under NN QST Hamiltonians, is distorted significantly by interactions beyond nearest-neighbours that originate from the bend. The limited

spatial extent of such perturbations, allows one to minimize efficiently the associated losses by controlling the energy of the corner site. Bends play a pivotal role in the reliable and efficient navigation of quantum signals in higher-dimensional networks, which are essential for large-scale QIP. The present results shed light on the role of bends facilitating the engineering of reliable quantum networks in higher dimensions, including various 2D geometric arrangements that have been discussed in the literature [15, 19, 21, 23, 24, 45, 47–54, 56, 57].

2.8 Conclusions

We have reviewed many aspects of the problem of quantum state transfer in engineered quantum networks. The problem has been defined in a rather general theoretical framework, while fundamental aspects of quantum networks such as the form of the underlying Hamiltonians and the various topologies have been introduced. We have demonstrated a rather general and efficient theoretical framework that allows for the engineering of passive quantum networks of arbitrary topology and dimensionality, which perform various communication tasks (point-to-point transfer, routing, etc.) for states that involve multiple interacting or non-interacting excitations. The robustness of some of the state-transfer Hamiltonians discussed here, against various types of imperfections is discussed in the chapter by Stolze et al.

We have also reviewed the problem of the quantum directional coupler, which pertains to the directional coupling of a quantum state between two identical quantum chains. Such structures are expected to be of great importance for large-scale quantum-information processing, since they can perform a number of useful functions such as signal division, switching, (de) multiplexing, etc. To this end, however, one has to consider two-dimensional or even three-dimensional configurations, where the presence of bends is inevitable. We have discussed the effects of bends on known and rather promising state-transfer Hamiltonians, showing that they can be easily minimized by the inclusion of small defects.

Recently [62], one of the state-transfer Hamiltonians discussed here, has been demonstrated in the framework of photonic lattices, which offer a rather versatile tool for testing models that rely on tight-binding Hamiltonians (see the chapter by Bellec et al.). In view of these new developments, the ideas discussed in the present chapter, promise the systematic engineering of photonic architectures that operate in the linear regime and perform various communication tasks. On the other hand, for the realization of quantum state transfer in real spin chains, there are various candidates ranging from liquid- and solid-state NMR systems, to Nitrogen-Vacancy centers in diamond (see the chapter by Cappellaro). However, some of these systems do not allow for the degree of engineering that certain QST protocols require, not to speak about more complicated communication tasks. Hence, the field is certainly open to new challenging developments.

Acknowledgements I. J. received funding from MSM 6840770039, RVO 68407700 and GACR 13-33906S. A. H. was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS13/217/OHK4/3T/14.

References

1. M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000)
2. A.B. Kuklov, B.V. Svistunov, Phys. Rev. Lett. **90**, 100401 (2003)
3. L.M. Duan, E. Demler, M.D. Lukin, Phys. Rev. Lett. **91**, 090402 (2003)
4. S.R. Clark, C.M. Alves, D. Jaksch, New J. Phys. **7**, 124 (2005)
5. S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, UK, 1999)
6. P. Ciccarelli, P. Faulkner, *Networking Foundations: Technology Fundamentals for IT Success* (Sybex, San Francisco, 2004)
7. S. Steinke, *Network Tutorial* (CMP Books, San Francisco, 2003)
8. T. Brougham, G.M. Nikolopoulos, I. Jex, Phys. Rev. A **80**, 052325 (2009)
9. S. Bose, Phys. Rev. Lett. **91**, 207901 (2003)
10. A. Kay, Phys. Rev. A **84**, 022337 (2011)
11. G.M. Nikolopoulos, Phys. Rev. A **87**, 042311 (2013)
12. V. Kostak, G.M. Nikolopoulos, I. Jex, Phys. Rev. A **75**, 042319 (2007)
13. T. Brougham, G.M. Nikolopoulos, I. Jex, Phys. Rev. A **83**, 022323 (2011)
14. P. Törmä, S. Stenholm, I. Jex, Phys. Rev. A **52**, 4853 (1995)
15. G.M. Nikolopoulos, D. Petrosyan, P. Lambropoulos, J. Phys.: Condens. Matter **16**, 4991 (2004)
16. Z. Bialynicka-Birula, I. Bialynicka-Birula, J.H. Eberly, B.W. Shore, Phys. Rev. A **16**, 2048 (1977)
17. R.J. Cook, B.W. Shore, Phys. Rev. A **20**, 539 (1979)
18. J.H. Eberly, B.W. Shore, Z. Bialynicka-Birula, I. Bialynicka-Birula, Phys. Rev. A **16**, 2038 (1977)
19. M. Christandl, N. Datta, A. Ekert, A.J. Landahl, Phys. Rev. Lett. **92**, 187902 (2004)
20. R. Gordon, Opt. Lett. **29**, 2752 (2004)
21. M.B. Plenio, J. Hartley, J. Eisert, New J. Phys. **6**(1), 36 (2004)
22. G.M. Nikolopoulos, D. Petrosyan, P. Lambropoulos, Europhys. Lett. **65**, 297 (2004)
23. A. Kay, M. Ericsson, New J. Phys. **7**, 143 (2005)
24. S. Yang, Z. Song, C. Sun, Eur. Phys. J. B **52**, 377 (2006)
25. A. Kay, Phys. Rev. Lett. **98**, 010501 (2007)
26. H.L. Haselgrove, Phys. Rev. A **72**, 062326 (2005)
27. P. Kurzyński, A. Wójcik, Phys. Rev. A **83**, 062315 (2011)
28. S. Bose, Contemp Phys. **48**, 13 (2007)
29. M.H. Yung, S. Bose, Phys. Rev. A **71**, 032310 (2005)
30. P. Karbach, J. Stolze, Phys. Rev. A **72**, 030301 (2005)
31. A. Kay, Phys. Rev. A **73**, 032306 (2006)
32. C. Albanese, M. Christands, N. Datta, A. Ekert, Phys. Rev. Lett. **93**, 230502 (2004)
33. T. Shi, Y. Li, Z. Song, C.P. Sun, Phys. Rev. A **71**, 032309 (2005)
34. X.Q. Xi, J.B. Gong, T. Zhang, R.H. Yue, W.M. Liu, Eur. Phys. J. D **50**, 193 (2008)
35. M.H. Yung, Phys. Rev. A **74**, 030303(R) (2006)
36. R. Hanson, L.P. Kouwenhoven, J.R. Petta, S. Tarucha, L.M.K. Vandersypen, Rev. Mod. Phys. **79**, 1217 (2007)
37. D. Petrosyan, B. Schmidt, J.R. Anglin, M. Fleischhauer, Phys. Rev. A **76**, 033606 (2007)
38. A.J. Daley, A. Kantian, H.P. Büchler, P. Zoller, K. Winkler, G. Thalhammer, F. Lang, R. Grimm, J.H. Denschlag, in *AIP Conference Proceedings*, vol. 869, ed. by C. Roos, H. Häffner, R. Blatt (2006), vol. 869, p. 212

39. J.H. Denschlag, A.J. Daley, in *Proceedings of the International School of Physics "Enrico Fermi"*, vol. 164, ed. by M. Inguscio, W. Ketterle, C. Salomon (2007), vol. 164, p. 677
40. D. Petrosyan, G.M. Nikolopoulos, P. Lambropoulos, Phys. Rev. A **81**, 042307 (2010)
41. A. Yariv, P. Yue, *Photonics: Optical Electronics in Modern Communications* (Oxford University, New York, 2006)
42. N. Dagli, G. Snider, J. Waldman, E. Hu, J. Appl. Phys. **69**, 1047 (1991)
43. J.A. del Alamo, C.C. Eugster, Appl. Phys. Lett. **56**, 78 (1990)
44. S. Fan, P.R. Villeneuve, J.D. Joannopoulos, H.A. Haus, Phys. Rev. Lett. **80**, 960 (1998)
45. G.M. Nikolopoulos, Phys. Rev. Lett. **101**, 200502 (2008)
46. B.W. Shore, *The Theory of Coherent Atomic Excitation* (Wiley, New York, 1990)
47. Y. Li, Z. Song, C.P. Sun, Commun. Theor. Phys. **48**, 445 (2007)
48. P.J. Pemberton-Ross, A. Kay, Phys. Rev. Lett. **106**, 020503 (2011)
49. I. D'Amico, B.W. Lovett, T.P. Spiller, Phys. Rev. A **76**, 030302(R) (2007)
50. D. Zueco, F. Galve, S. Kohler, P. Hänggi, Phys. Rev. A **80**, 042303 (2009)
51. M.I. Makin, J.H. Cole, C.D. Hill, A.D. Greentree, Phys. Rev. Lett. **108**, 017207 (2012)
52. T. Tufarelli, V. Giovannetti, Phys. Rev. A **79**(2), 022313 (2009)
53. V. Karimipour, M.S. Rad, M. Asoudeh, Phys. Rev. A **85**, 010302 (2012)
54. F.A.A. El-Orany, M.R.B. Wahiddin, Journal of Physics B: Atomic Mol Opt Phys **43**, 085502 (2010)
55. G.M. Nikolopoulos, A. Hoskovec, I. Jex, Phys. Rev. A **85**, 062319 (2012)
56. L. Banchi, T.J.G. Apollaro, A. Cuccoli, R. Vaia, P. Verrucchi, Phys. Rev. A **82**, 052321 (2010)
57. L. Banchi, A. Bayat, P. Verrucchi, S. Bose, Phys. Rev. Lett. **106**, 140501 (2011)
58. M. Avellino, A.J. Fisher, S. Bose, Phys. Rev. A **74**, 012321 (2006)
59. G. Gualdi, V. Kostak, I. Marzoli, P. Tombesi, Phys. Rev. A **78**, 022325 (2008)
60. G. Burkard, D. Loss, D.P. DiVincenzo, Phys. Rev. B **59**, 2070 (1999)
61. S. Longhi, Phys. Rev. B **82**, 041106 (2010)
62. M. Bellec, G.M. Nikolopoulos, S. Tzortzakos, Opt. Lett. **37**, 4504 (2012)

Quantum State Transfer and Network Engineering

Nikolopoulos, G.M.; Jex, I. (Eds.)

2014, X, 250 p. 80 illus., 68 illus. in color., Hardcover

ISBN: 978-3-642-39936-7