Chapter 1
Lévy Processes and Their Characteristics

Abstract We give an introductory review of Lévy processes and their properties with emphasis on subordinators and spectrally positive Lévy processes. The α-potentials of these processes are given. Results on the times of first exit of such processes are discussed. Several examples of such processes are given.

Keywords Lévy processes · The Lévy-Itô decomposition formula · Scale functions · Subordinators · Spectrally positive Lévy processes · Killed processes · First exit times · Brownian motion · Gamma processes · Stable processes

1.1 Lévy Processes

Definition 1.1 A stochastic process $X = \{X_t, t \geq 0\}$ is said to be a Lévy process if the following hold:

(i) It has right continuous sample paths with left limits.
(ii) X has *stationary increments*, i.e., for every $s, t \geq 0$, the distribution of $X_{t+s} - X_t$, is independent of t.
(iii) X has independent increments, i.e., for every $t, s \geq 0$, $X_{t+s} - X_t$ is independent of $(X_u, u \leq t)$.

That is to say, a Lévy process is a process with stationary and independent increments whose sample paths are right continuous with left-hand limits.

Any Lévy process X enjoys the following property: For all $t \geq 0$

$$E[e^{i\theta X_t}] = e^{t\Phi(\Theta)}.$$

The function Φ is known as the characteristic function of the process X, it has the form

$$i\theta d - \frac{\theta^2 b}{2} + \int_R [\exp(i\theta x) - 1 - i\theta x I_{\{|x|<1\}}]\nu(dx), \tag{1.1}$$

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where $d \in R$, $b \in R_+$ and ν is a measure on R satisfying $\nu(\{0\}) = 0$, $\int_R (x^2 \wedge 1)\nu(dx) < \infty$.

The measure ν is called the Lévy measure, it characterizes the size and frequency of the jumps. If this measure is infinite, then the process has an infinite number of jumps of every small sizes in any small interval.

Definition 1.2 A Lévy process is said to be of bounded variation if $b = 0$ and $\int_R (| x | \wedge 1)\nu(dx) < \infty$.

For such processes, the characteristic function is of the form

$$\Phi(\Theta) = ia\theta + \int_R [\exp(i\theta x) - 1]\nu(dx), \tag{1.2}$$

where $a = d - \int_{\{|x|<1\}} x\nu(dx)$.

1.2 The Lévy-Itô Decomposition

The Lévy-Itô decomposition identify any Lévy process as the sum of four independent processes, it is stated as follows:

Theorem 1.3 Given any $d \in R, b \in R_+$ and measure ν on R satisfying $\nu(\{0\}) = 0$, $\int_R (1 \wedge x^2)\nu(dx) < \infty$, there exists a probability space $(\Omega, \mathcal{F}, P)$ on which a *Lévy* process X is defined. Furthermore, for each $t \in R_+$

$$X_t = dt + B_t + \int_{[0,t]\times\{|x|>1\}} xM(ds, dx) + \int_{[0,t]\times\{|x|\leq 1\}} x(M - m)(ds, dx),$$

where M is a *Poisson random measure* on $R_+ \times R_0$ with mean measure $m(ds, dx) = ds\nu(dx)$ and $\int (x^2 \wedge 1)\nu(dx) < \infty$. Furthermore, $B = \{B_t : t \geq 0\}$ is a Brownian motion with zero mean and variance coefficient b, and d is called the drift term.

The above theorem implies that the jump random measure defined, for any $A \in \sigma(R_+ \times R_0)$, by

$$M(A) = \sum_{s\geq 0} \mathbf{I}_A(s, X_s - X_{s^-}),$$

is a Poisson random measure on $R_+ \times R_0$ with mean measure $m(ds, dx) = ds\nu(dx)$, and $\int (x^2 \wedge 1)\nu(dx) < \infty$. Furthermore, the process X is the sum of four independent processes $\overset{(1)}{X}$, $\overset{(2)}{X}$, $\overset{(3)}{X}$, and $\overset{(4)}{X}$, where $\overset{(1)}{X}$ is a constant drift, $\overset{(2)}{X}$ is a Brownian motion with zero mean and variance coefficient b, $\overset{(3)}{X}$ is a compound Poisson process with arrival rate equal to $\nu(| x |> 1)$, jump magnitude distribution function $F(dx) = \frac{\mathbf{I}_{\{|x|\geq 1\}}\nu(dx)}{\nu(|x|>1)}$, and $\overset{(4)}{X}$ is a pure jump martingale that has countably many jumps over every finite interval; these jumps are of magnitude less than one almost surely. The characteristic exponents of $\overset{(1)}{X}$, $\overset{(2)}{X}$, $\overset{(3)}{X}$, and $\overset{(4)}{X}$ (denoted by $\overset{(1)}{\varphi}, \overset{(2)}{\varphi}, \overset{(3)}{\varphi}$ and $\overset{(4)}{\varphi}$, respectively) are as follows:

$$\overset{(1)}{\varphi}(\theta) = i\theta d$$

$$\overset{(2)}{\varphi}(\theta) = -\frac{\theta^2 b}{2},$$

$$\overset{(3)}{\varphi}(\theta) = \int_{\{|x|\geq 1\}} (\exp(i\theta x) - 1)\nu(dx),$$

$$\overset{(4)}{\varphi}(z) = \int_{\{|x|<1\}} (\exp(i\theta x) - 1 - i\theta x)\nu(dx).$$

1.3 The Strong Markov Property for Lévy Processes

Definition 1.4 For any stopping time T with respect to $\mathcal{F}_\infty$, the sigma algebra generated by T is defined as follows:

$$\mathcal{F}_T = \{A \subset \Omega : A \cap \{T \leq t\} \in \mathcal{F}_t, t \in R_+\}.$$

The next theorem illustrates that the stationarity and the independence of the increments of Lévy processes hold even if the starting time in the increment is a stopping time, instead of being fixed.

Theorem 1.5 Let $L = \{L_t, t \in R_+\}$ be a Lévy processes. For any stopping time T with respect to $\mathcal{F}_\infty$ and for any $t \in R_+$, we define

$$Y_t = L_{T+t} - L_T\ .$$

Then, on the event $\{T < \infty\}$, the process Y has the same distribution as the process L and is independent of $\mathcal{F}_T$.

Proof Since the indicator function of any event can be approximated by a sequence of bounded continuous function, it suffices to show that, for $m = 1, 2, ..., t_1, ..., t_m \in R_+$, every bounded continuous function $f : R^m :\to R$, and every $A \in \mathcal{F}_T$

$$E[\mathbf{I}_{A\cap\{T<\infty\}} f(Y_{t_1}, .., Y_{t_m})] = P(A \cap \{T < \infty\})E[f(L_{t_1}, .., L_{t_m})].$$

From Theorem 10 in the Appendix, it suffices to show that the above identity holds for $m = 2$, and $f = f_1 f_2$ where f_1 , $f_2 : R :\to R$ are bounded continuous functions

For every $n = 1, 2, \ldots,$ we define

$$T^n = \sum_{k \geq 1} \frac{k}{2^n} \mathbf{I}_{\{\frac{k-1}{2^n} < T \leq \frac{k}{2^n}\}}.$$

For every $n = 1, 2, \ldots, t \in R_+$, we define $Y_t^n = L_{T^n+t} - L_{T^n}$. It is known that $T^n \downarrow T$, as $n \uparrow \infty$ almost surely. Since the process L is right continuous, f_1 and f_2 are continuous functions, it follows that $f_i(Y_{t_i}^n) \to f_i(Y_{t_i})$ almost surely, as $n \to \infty$, $i = 1, 2$. Using the *bounded convergence theorem* we have

$$\lim_{n\to\infty} E[\mathbf{I}_{A\cap\{T^n<\infty\}} f_1(Y_{t_1}^n) f_2(Y_{t_2}^n)] = E[\mathbf{I}_{A\cap\{T\ <\infty\}} f_1(Y_{t_1}) f_2(Y_{t_2})]$$

For simplicity, for k = 1,2,..., we will denote the set $\{\frac{k-1}{2^n} < T \leq \frac{k}{2^n}\}$ by A_k. For $A \in \mathcal{F}_T$, we write

$$\begin{aligned}
&E[\mathbf{I}_{A\cap\{T^n<\infty\}}\ f_1(Y_{t_1}^n) f_2(Y_{t_2}^n)] \\
&\qquad = E[\sum_{k\geq 1} \mathbf{I}_{A\cap A_k} f_1(L_{\frac{k}{2^n}+t_1} - L_{\frac{k}{2^n}}) f_2(L_{\frac{k}{2^n}+t_2} - L_{\frac{k}{2^n}})] \\
&\qquad = E[E[\sum_{k\geq 1} \mathbf{I}_{A\cap A_k} f_1(L_{\frac{k}{2^n}+t_1} - L_{\frac{k}{2^n}}) f_2(L_{\frac{k}{2^n}+t_2} - L_{\frac{k}{2^n}}) \mid \mathcal{F}_{\frac{k}{2^n}}]] \\
&\qquad = E[\sum_{k\geq 1} \mathbf{I}_{A\cap A_k} E[f_1(L_{t_1}) f_2(L_{t_2})]] \\
&\qquad = P(A \cap \{T^n\ < \infty\}) E[f_1(L_{t_1}) f_2(L_{t_2})],
\end{aligned}$$

where the third equation follows since the process L has stationary independent increments and since, for $k = 1, 2, \ldots,$ and $A \in \mathcal{F}_T$ the event $A \cap A_k \in \mathcal{F}_{\frac{k}{2^n}}$. Our assertion is proved by letting $n \to \infty$, in both sides of the last equation above. ■

The following shows that every Lévy process is a strong Markov process in the sense described in the Appendix.

Corollary 1.6 Let L be a Lévy process, then L is a strong Markov process.

Proof Let T be an arbitrary stopping time, we need to show that, for each $t \in R_+$ and every bounded function f,

$$E[f(L_{t+T}) \mid \mathcal{F}_T] = E[f(L_{t+T}) \mid L_T].$$

Let the process $Y = \{Y_t, t \in R_+\}$ be as defined in the previous theorem. Observe that $L_{t+T} = Y_t + L_T$, hence $E[f(L_{t+T}) \mid \mathcal{F}_T] = E[f(Y_t + L_T)) \mid \mathcal{F}_T]$. From the above theorem Y_t has the same distribution as the L_t and is independent of $\mathcal{F}_T$. Thus, $E[f(L_{t+T}) \mid \mathcal{F}_T] = E[f(Y_t + L_T)) \mid L_T] = E[f(L_t + L_T)) \mid L_T]$, L_t is independent fro L_T. But, given L_T, the random variable $L_t + L_T$ has the same distribution as L_{t+T} and our assertion is proven. ■

1.4 Subordinators

Definition 1.7 A Lévy process is called a *subordinator* if its sample paths are increasing.

From Theorem 1.3 it follows that, for each $t \in R_+$

$$X_t = \zeta t + \int_{[0,t)\times R_+} xM(ds, dx),$$

where $\zeta \geq 0$, and M is a Poisson random measure on $R_+ \times R_+$ with mean measure $m(ds, dx) = ds\upsilon(dx)$, and $\int_0^\infty (x \wedge 1)\upsilon(dx) < \infty$.

For such processes we have, for all $\theta \geq 0$,

$$E[e^{-\theta X_t}] = e^{-t\psi(\theta)}, \tag{1.3}$$

where

$$\psi(\theta) = \zeta\theta + \int_0^\infty (1 - e^{-\theta x})\upsilon(dx),$$

and $\zeta \geq 0$ is the drift term.

The function ψ is called the *Laplace exponent* of the subordinator. It follows that every subordinator is of bounded variation.

We now mention some examples of subordinators.

Example 1 *Compound Poisson processes.* A subordinator X with finite Lévy measure is called a *compound Poisson process with a positive drift.* In this case, $\upsilon(dx) = \lambda F(dx)$, where $\lambda > 0$, F is the distribution function with support R_+, and the corresponding Poisson random measure M is finite. Let $T = (T_n, n = 1, 2, ...)$ be a sequence of independent identically distributed exponential random variables, with mean $\frac{1}{\lambda}$. For $n \in N_+$, let $S_n = T_1 + \cdots + T_n$, $S_0 \equiv 0$, then, for every $t \in R_+$

$$X_t = \zeta t + \sum_n X_n \mathbf{I}_{(0,t]}(S_n)$$

where $\{X_n, n = 1, 2, ...\}$ is a sequence of independent positive identically distributed random variables with distribution function F, and independent of T.

In this case,

$$\psi(\theta) = \theta\zeta + \lambda \int_0^\infty (1 - e^{-\theta x})F(dx). \tag{1.4}$$

Assuming $\zeta = 0$, and that F has a density f, then the probability transition function of this process is as follows:

$$p(t, x, y) = \begin{cases} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f^{(n)}(y-x), & y > x \\ 0, & y \leq x, \end{cases}$$

where $f^{(n)}$ is the nth convolution of f with itself.

Example 2 *Inverse Brownian motion.* A subordinator X with Lévy measure of the form

$$\upsilon(dx) = \frac{1}{\sqrt{2\pi\sigma^2 x^3}} \exp\left(\frac{-x\mu^2}{2\sigma^2}\right).$$

is called *inverse Brownian process.*

In this case,

$$\psi(\theta) = \theta\zeta + \frac{1}{\sigma^2}(\sqrt{2\theta\sigma^2 + \mu^2} - \mu). \tag{1.5}$$

If $\zeta = 0$, then, *for* $t,x,y \in R_+$, the probability transition function of the process X is as follows:

$$p(t, x, y) = \begin{cases} \frac{t}{\sigma\sqrt{2\pi(y-x)^3}} \exp\{-\frac{[\mu(y-x)-t]^2}{2(y-x)\sigma^2}\}, & y > x \\ 0, & y \leq x. \end{cases}$$

In this case, $EX_1 = \frac{1}{\mu}$, and $Var(X_1) = \frac{\sigma^2}{\mu^3}$.

Example 3 *Gamma processes.* A subordinator X with Lévy measure

$$\nu(dx) = \frac{\alpha}{x} \exp(-x\beta)dx, \qquad x > 0$$

where $\alpha, \beta > 0$, is called a *gamma process*.

It follows that

$$\psi(\theta) = \theta\zeta + \alpha \ln(1 + \theta/\beta). \tag{1.6}$$

Furthermore, if $\zeta = 0$, its probability transition function is of the form

$$p(t, x, y) = \begin{cases} \frac{\beta^{at}}{\Gamma(at)} e^{-(y-x)\beta} (y-x)^{at-1}, & y > x \\ 0, & y \leq x. \end{cases}$$

In this case, the mean term ($E(X_1)$) and the variance term ($V(X_1)$) are equal to α/β and α/β^2, respectively.

Example 4 *Stable processes with stability parameter* β, $\beta \in (0, 1)$. A subordinator X with Lévy measure

$$\nu(dx) = \frac{\beta}{\Gamma(1-\beta)x^{\beta+1}} \qquad x > 0$$

is called a stable process with stability parameter β.

In this case

$$\psi(\theta) = \theta\zeta + \theta^{\beta}, \tag{1.7}$$

and $E[X_t] = \infty$, for all $t \geq 0$.

We will discuss stable processes in general and the case when the index $\beta \in (1, 2)$ in Sect. 1.6 of this chapter.

Example 5 *Generalized stable processes with stability parameter β, $\beta \in (0, 1)$.* A subordinator X with Lévy measure

$$\nu(dx) = \frac{\beta e^{-\lambda x}}{\Gamma(1-\beta)x^{\beta+1}} dx \qquad x, \lambda > 0$$

is called a generalized stable subordinator.

It is easily seen that

$$\psi(\theta) = \theta\zeta + (\theta+\lambda)^{\beta} - \lambda^{\beta}, \tag{1.8}$$

and $E[X_1] = \zeta + \beta\lambda^{\beta-1}$.

1.5 Spectrally Positive Processes

Definition 1.8 A non-subordinator is said to be spectrally positive (negative) if it has no negative (positive) jumps.

For any spectrally positive process L, we let $\hat{L} = -L$, throughout. It is clear that L is spectrally positive if and only if the process $\hat{L}$ is spectrally negative.

From (1.1), it follows that, for each $\theta \in R_+$, the Laplace transform $E[e^{-\theta L_t})]$ exists, furthermore,

$$E[e^{-\theta L_t})] = e^{t\phi(\theta)},$$

where

$$\phi(\theta) = -d\theta + \frac{\theta^2\sigma^2}{2} - \int_0^{\infty} (1 - e^{-\theta x} - \theta x 1_{\{x<1\}})\upsilon(dx). \tag{1.9}$$

The term $d \in R$ is the drift term, $\sigma^2 \in R_+$ is the variance of the Brownian motion and υ is a positive measure on $[0, \infty)$, $\upsilon(\{0\}) = 0$, and $\int_0^{\infty} (x^2 \wedge 1)\nu(dx) < \infty$.

The function ϕ is known as the *Laplace exponent* of the spectrally positive process. The following gives some properties of the Laplace component above:

Lemma 1.9 Let ϕ be as defined in (1.9). Then

(i) $\phi(0) = 0$.
(ii) ϕ is a convex function in its argument.
(iii) If $\phi^{'}(0+) > 0$, then ϕ is strictly increasing on R_+.
(iv) If $\phi^{'}(0+) \leq 0$, then there exists $\theta^* > 0$ such that $\phi(\theta) < 0$ if $\theta < \theta^*$, and $\phi(\theta) \geq 0$ and increasing if $\theta \geq \theta^*$.
(vi) $\lim\limits_{\theta \to \infty} \phi(\theta) = \infty$.

Proof (i) This follows immediately from the definition of ϕ.
(ii) From the definition of ϕ, it follows that $\phi^{''}(\theta)$ has the same sign as $E[e^{-\theta L_1}]$ $E[L_1^2\, e^{-\theta L_1}] - (E[L_1 e^{-\theta L_1}])^2$. The fact that this term is positive, is easily seen from Hölder inequality. This establishes the assertion.
(iii) This assertion also follows from (ii) above.
(iv) This assertion also follows from (i) and (ii) above.
(vi) Since the process L is spectrally positive, then there exists a $t \in (0, \infty)$ such that $P\{L_t < 0\} > 0$. For such a t, $e^{t\phi(\theta)} = E[e^{-\theta L_t}] \geq E[e^{-\theta L_t}, L_t < 0]$. The assertion follows by letting $\theta \to \infty$, in the last inequality. ■

It is clear that, $\phi^{'}(0+) = -E[L_1\,]$. For $\alpha \in R_+$, we define $\eta(\alpha) = \phi^{-1}(\alpha)$, i.e,

$$\eta(\alpha) = \sup\{\theta : \phi(\theta) = \alpha\}, \tag{1.10}$$

It is seen that $\eta(0) = 0$ if and only $E[L_1\,] \leq 0$. Note that, $E(L_1\,) = \int\limits_1^{\infty} x\upsilon(dx) + d$. Furthermore, $\lim\limits_{t\to\infty} L_t = \infty$ if and only if $E(L_1\,) > 0$, and $\lim\limits_{t\to\infty} L_t = -\infty$ if and only if $E[L_1\,] < 0$. Also, if $E(L_1\,) = 0$, then L oscillates from $-\infty$ to ∞.

A version of the following theorem is given in Theorem 1 of [1], it is also included in Theorem 8 p. 194 of [2].

Theorem 1.10 Let X be a spectrally positive process, with Laplace exponent ϕ, and η is as defined in (1.10). Then, there exists an absolutely continuous increasing function W such that,

$$\int_0^{\infty} e^{-\theta x} W(x)dx = \frac{1}{\phi(\theta)}, \theta > \eta(0). \tag{1.11}$$

Definition 1.11 For any spectrally positive process with Laplace component ϕ and for $\alpha \geq 0$, the α-scale function W^{α}: $R \twoheadrightarrow R_+$, $W^{\alpha}(x) = 0$ for every $x < 0$, and on $[0, \infty)$ it is defined as the unique continuous increasing function such that

$$\int_0^{\infty} e^{-\theta x} W^{(\alpha)}(x)dx = \frac{1}{\phi(\theta) - \alpha}, \theta > \eta(\alpha). \tag{1.12}$$

The existence of $W^{(\alpha)}$ and its relation to W above is established as follows. Since $\theta > \eta(\alpha)$ if and only if $\phi(\theta) > \alpha$, then we have

$$\begin{aligned}\frac{1}{\phi(\theta)-\alpha} &= \frac{1}{\phi(\theta)}[\frac{1}{1-\alpha/\phi(\theta)}] \\ &= \sum_{k\geq 0}\alpha^k[\frac{1}{\phi(\theta)}]^{k+1} \\ &= \sum_{k\geq 0}\alpha^k[\int_0^\infty e^{-\theta x}W(x)]^{k+1}dx \\ &= \sum_{k\geq 0}\alpha^k\int_0^\infty e^{-\theta x}W^{*(k+1)}(x)dx,\end{aligned}$$

where for k = 1, 2, ..., $W^{*(k)}$ is the kth convolution of W with itself. Note that, since W is increasing

$$\begin{aligned}W^{*(2)}(x) &= \int_0^x W(x-y)W(y)dy \\ &\leq \frac{x}{1!}W(x)^2.\end{aligned}$$

By induction on k, it follows that for $k \geq 1$,

$$W^{*(k+1)}(x) \leq \frac{x^k}{k!}(W\ (x))^{k+1}.$$

Hence, for each $x \in R_+$, the series $\sum_{k\geq 0}\alpha^k W^{*(k+1)}(x)$ converges. Using Fubini's Theorem we have

$$\sum_{k\geq 0}\alpha^k\int_0^\infty e^{-\theta x}W^{*(k+1)}(x)dx = \int_0^\infty e^{-\theta x}\sum_{k\geq 0}\alpha^k W^{*(k+1)}(x)dx.$$

From the uniqueness of the Laplace transform, we have, for $\alpha > 0$

$$W^{(\alpha)}(x) = \sum_{k=0}^\infty \alpha^k W^{*(k+1)}(x). \tag{1.13}$$

If a spectrally positive Lévy process has bounded variation, then using (1.2) it follows that

$$\phi(\theta) = \zeta\theta - \int_0^\infty (1-e^{-\theta x})\upsilon(dx). \tag{1.14}$$

where

$$\zeta = \int_{\{|x|<1\}} x\nu(dx) - d > 0. \tag{1.15}$$

In this case, we can write, for each $t \geq 0$

$$X_t = Y_t - \zeta t$$

where the process Y is a subordinator with drift term equal to zero,

Lemma 1.12 Let X be a spectrally positive process. Then, for each $\alpha > 0$

(a) $W^{(\alpha)}(0) = \frac{1}{\zeta}$ if and only if X is of bounded variation, where ζ is given in (1.15).
(b) $W^{(\alpha)}(0) = 0$ if and only if X is of unbounded variation.

Proof (a) From the *initial value theorem* for the Laplace transform, and (1.12) we have

$$\begin{aligned}
W^{(\alpha)}(0) &= \lim_{\theta\to\infty} \int_0^\infty \theta e^{-\theta x} W^{(\alpha)}(x)dx \\
&= \lim_{\theta\to\infty} \frac{\theta}{\phi(\theta) - \alpha} \\
&= (\lim_{\theta\to\infty} \frac{\phi(\theta)}{\theta})^{-1}.
\end{aligned}$$

Since, for $x, \theta \in R_+$ and θ large enough, $(1 - e^{-\theta x}) \leq (\theta x \wedge 1) < \theta(x \wedge 1)$, using the fact that $\int_0^\infty (x \wedge 1)\upsilon(dx) < \infty$, (1.14) and the *Lebesgue dominated convergence theorem*, we have

$$W^{(\alpha)}(0) = \frac{1}{\zeta}$$

if and only if X is of bounded variation.

(b) The assertion that $W^{(\alpha)}(0) = 0$ if the process L is of unbounded variation follows, since in this case and from the definition of ϕ, $\lim_{\theta\to\infty} \frac{\theta}{\phi(\theta)-\alpha} = 0$. ∎

Furthermore, (see Lemma 8.2 of [3]), $W^{(\alpha)}$ is right and left differentiable on $(0, \infty)$. By $W_+^{(\alpha)'}(x)$, we will denote the right derivative of $W^{(\alpha)}$ in x.

The adjoint α-scale function associated with $W^{(\alpha)}$ (denoted by $Z^{(\alpha)}$) is defined as follows:

Definition 1.13 For $\alpha \geq 0$, the *adjoint* α-scale function $Z^{(\alpha)} : R_+ \twoheadrightarrow [1, \infty)$ is defined as

$$Z^{(\alpha)}(x) = 1 + \alpha \int_0^x W^{(\alpha)}(y)dy. \tag{1.16}$$

Lemma 1.14 For $\alpha > 0$

(a)

$$W^{(\alpha)}(x) \sim \frac{e^{\eta(\alpha)\, x}}{\phi'(\eta(\alpha)\,)}\text{, as } x \to \infty. \tag{1.17}$$

(b)

$$Z^{(\alpha)}(x) \sim \frac{\alpha e^{\eta(\alpha)\, x}}{\eta(\alpha)\phi'(\eta(\alpha)\,)}\text{, as } x \to \infty. \tag{1.18}$$

Proof (a) Let $\overset{*}{W}{}^{(\alpha)}(x) = e^{-\eta(\alpha)x}W^{(\alpha)}(x)$, then from (1.12) we have, for $\theta \in R_+$

$$\int_0^\infty e^{-\theta x}\overset{*}{W}{}^{(\alpha)}(x)dx = \frac{1}{\phi(\theta + \eta(\alpha)) - \alpha}.$$

From the *final-value theorem* of the Laplace transform we have

$$\begin{aligned}
\lim_{x\to\infty}\overset{*}{W}{}^{(\alpha)}(x) &= \lim_{\theta\to 0}\int_0^\infty \theta e^{-\theta x}\overset{*}{W}{}^{(\alpha)}(x)dx \\
&= \lim_{\theta\to 0}\frac{\theta}{\phi(\theta + \eta(\alpha)) - \alpha} \\
&= \lim_{\theta\to 0}\frac{\theta}{\phi(\theta + \eta(\alpha)) - \phi(\eta(\alpha))} \\
&= \frac{1}{\phi'(\eta(\alpha))}.
\end{aligned}$$

Hence,

$$W^{(\alpha)}(x) \sim \frac{e^{\eta(\alpha)\, x}}{\phi'(\eta(\alpha)\,)}, \text{ as } x \to \infty.$$

(b) From (1.16) and (1.17), it follows that as $x \to \infty$, for $\alpha > 0$, $\frac{Z^{(\alpha)}(x)}{W^{(\alpha)}(x)} \sim \frac{\alpha W^{(\alpha)}(x)}{W'^{(\alpha)}(x)} = \frac{\alpha}{\eta(\alpha)}$, hence

$$Z^{(\alpha)}(x) \sim \frac{\alpha e^{\eta(\alpha)\, x}}{\eta(\alpha)\phi'(\eta(\alpha)\,)}, \text{ as } x \to \infty.$$

1.6 Examples of Spectrally Positive Processes

Example 1 *Brownian Motion*. The Brownian motion with mean $\mu \in R$, variance term σ^2, is an example of spectrally positive Lévy processes, where $\nu(R_+) = 0$. From (1.1) we have, that for $\theta \geq 0$, $\phi(\theta) = -\mu\theta + \frac{\theta^2\sigma^2}{2}$. It follows that, for $\alpha \geq 0$, $\eta(\alpha) = \frac{\sqrt{2\alpha\sigma^2+\mu^2}+\mu}{\sigma^2}$. For each $t \in R_+, x, y \in R$, the transition probability function of this process is given as follows:

$$p(t, x, y) = \frac{1}{\sqrt{2\pi b^2 t}} \exp\{\frac{(y - x - \mu t)^2}{2\sigma^2}\}.$$

Let $\delta = \sqrt{2\alpha\sigma^2 + \mu^2}$, then

$$\begin{aligned} W^{(\alpha)}(x) &= \frac{2}{\delta} e^{\mu x/\sigma^2} \sinh(x\delta/\sigma^2), \\ Z^{(\alpha)}(x) &= e^{\mu x/\sigma^2}(\cosh(x\delta/\sigma^2) - \frac{\mu}{\delta}\sinh(x\delta/\sigma^2)) \end{aligned} \tag{1.19}$$

Example 2 *Stable processes with stability parameter* $\beta \in (1, 2)$. A Lévy process X is called stable process with stability parameter $\beta > 0$, if its Lévy measure has support $[0, \infty)$ and for each $t \geq 0$, X_t has the same distribution as $t^{(1/\beta)} X_1$. When $\beta \in (0, 1)$, the process X is a subordinator with no drift, as discussed in Example 4 of Sect. 1.4. Here we will deal with the case where $\beta \in (1, 2)$, in this case the process is spectrally positive. Let X be such a process, it follows that for $t, \theta \geq 0$,

$$E[e^{-\theta X_t}] = E[e^{-\theta t^{(1/\beta)} X_1}].$$

Since the left-hand side of the above equation is equal to $e^{t\phi(\theta)}$, then we must have

$$E[e^{-\theta t^{(1/\beta)} X_1}] = e^{t\phi(\theta)}.$$

Clearly $\phi(\theta) = C\theta^\beta$, is the solution of the last equation. Since $\underset{\infty\to\infty}{lim} \phi(\theta) = \infty$, (Lemma 1.9 (vi)), the constant C must be greater than zero. In summary

$$\phi(\theta) = C\theta^\beta, \tag{1.20}$$

$C > 0$. In this case, the Lévy measure is of the form

$$\nu(dx) = \frac{a}{x^{\beta+1}}, \tag{1.21}$$

where a is a positive real number.

It follows that, for all $t \geq 0$, $E[X_t] = 0$ and the value of the term d in (1.9) is equal $-\int\limits_1^\infty x\upsilon(dx)$. Furthermore,

$$\phi(\theta) = \int_0^\infty (e^{-\theta x} - 1 + \theta x)\upsilon(dx). \tag{1.22}$$

If C in (1.20) is taken to be 1, then the constant a in (1.21) is found to be $\frac{1}{\Gamma(-\beta)}$. In this case,

$$Z^{(\alpha)}(x) = E_\beta(\alpha x^\beta) \tag{1.23}$$

$$W^{(\alpha)}(x) = \beta x^{\beta-1} E_\beta^{'}(\alpha x^\beta), \tag{1.24}$$

where, for $\upsilon > 0$, $E_\upsilon(x) = \sum\limits_{k\geq 0} x^k/\Gamma(1+\upsilon k)$ is the Mittag–Leffler function. (see [3], p. 233)

The process X jumps upwards only and creeps downwards (in the sense that, for every negative x, $P\{X_{T_x^-} = x\} = 1$, where T_x^- is the first time the process X hits x from above). Furthermore, $\sigma = 0$, and $\int_0^\infty (x \wedge 1)\upsilon(dx) = \infty$, thus X is of unbounded variation.

Example 3 *Spectrally positive processes of bounded variation*. Assume that X is a spectrally positive process of bounded variation, with Laplace exponent given in (1.14). Let $\mu = \int_0^\infty x\upsilon(dx)$ and assume that $\mu < \infty$. For every $x \in R_+$, we let $\bar{\upsilon}(x) = \upsilon((x,\infty))$, define the probability density function $f(x) = \frac{\bar{\upsilon}(x))}{\mu}$, $F(x)$ as the distribution function corresponding to f, and $\rho = \frac{\mu}{\varsigma}$ which is assumed to be ρ less than one. From (1.14), we have

$$\begin{aligned}\phi(\theta) &= \zeta\theta - \int_0^\infty (1 - e^{-\theta x})\upsilon(dx)\\ &= \zeta\theta - \theta\int_0^\infty e^{-\theta x}\bar{\upsilon}(x)dx.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{\phi(\theta)} &= \frac{1}{\varsigma\theta[1 - \rho\int_0^\infty e^{-\theta x} f(x)dx]}\\ &= \frac{1}{\varsigma\theta}\int_0^\infty e^{-\theta x}\sum_{n=0}^\infty \rho^n f^{(n)}(x)dx\\ &= \frac{1}{\varsigma}\int_0^\infty e^{-\theta x}\sum_{n=0}^\infty \rho^n F^{(n)}(x)dx.\end{aligned}$$

Since, $\rho < 1$, then $\eta(0) = 0$, thus

$$\int_0^\infty e^{-\theta x} W(x)dx = \frac{1}{\phi(\theta)}, \theta > 0.$$

Thus, we must have

$$W(x) = \frac{1}{\varsigma} \sum_{n=0}^{\infty} \rho^n F^{(n)}(x). \tag{1.25}$$

The following are three examples of spectrally positive processes of bounded variation.

Example 4 *Spectrally positive processes of bounded variation with a gamma subordinator.* If X is a spectrally positive process with gamma subordinator, then from (1.14), for each $t \geq 0$,

$$X_t = Y_t - \zeta t,$$

$\zeta > 0$, and the process Y is a gamma process with drift term equal to zero, and parameters $\alpha, \beta > 0$, in the sense described in Example 3 of Sect. 1.4.

In this case the Laplace exponent of the process X is given as follows:

$$\phi(\theta) = \zeta\theta - \alpha \ln (1 + \theta\beta). \tag{1.26}$$

Note that, $\mu = E[Y_1] = \alpha\beta < \infty$. Then, assuming that $\alpha\beta < \zeta$, the scale function W is computed using (1.25), where $\rho = \frac{\alpha\beta}{\varsigma}$, and, for $x > 0$, $\overline{\upsilon}(x) = \int_x^\infty \frac{\alpha}{y} exp(-y/\beta)dy$, $F(x) = \int_{[0,x)} \overline{\upsilon}(y)dy/\alpha\beta$.

Example 5 *Spectrally positive processes of bounded variation with a stable subordinator.* From Example 4 of Sect. 1.4, for $\beta \in (0, 1)$, it follows that

$$\phi(\theta) = \theta\zeta - \theta^{\beta}, \tag{1.27}$$

$\zeta > 0$.

In this case $\eta(0) = \zeta^{(\frac{1}{\beta-1})}$, $\mu = \infty$. Thus, we cannot apply (1.25) to compute the scale function. However, when $\zeta = 1$, then from (1.11) and (1.27), we have

$$\int_0^\infty e^{-\theta x} W(x)dx = \frac{1}{\zeta\theta - \theta^{\beta}}, \quad \theta > \zeta^{(\frac{1}{\beta-1})}.$$

It can be shown that the solution of the last equation above is

$$W(x) = \frac{1}{\zeta} E_{1-\beta}(\frac{x^{1-\beta}}{\zeta}), \tag{1.28}$$

where, for $v > 0$, $E_v(x)$ is the Mittag–Leffler function with parameter v, which is defined in Example 2 of this section.

Example 6 *Spectrally positive processes of bounded variation with a generalized stable subordinator. Let* X be a spectrally positive process, with generalized stable subordinator. From Example 5 of Sect. 1.4 and (1.14), the Laplace exponent of X is of the form

$$\phi(\theta) = \theta\zeta - (\theta + \lambda)^{\beta} + \lambda^{\beta}, \tag{1.29}$$

where $\beta \in (0, 1)$, $\zeta > 0$, and $\lambda > 0$.

In this case, $\mu = \lambda^{\beta-1} < \infty$. Assuming that $\lambda^{\beta-1} < \zeta$, we can use (1.25) to compute the scale function W, with the following ingredients: $\rho = \frac{\lambda^{\beta-1}}{\zeta}$ and $F'(x) = \frac{\lambda^{1-\beta}}{\Gamma(1-\beta)} \int_x^{\infty} \frac{e^{-\lambda y}}{y^{\beta+1}} dy$.

1.7 The Compensation Formula

For a proof of the following theorem the reader should consult [4], also see Chapter II of [5].

Theorem 1.15 Let X be a Lévy process, defined on a probability space (Ω, P). Let M be a random measure on $(R_+ \times R_0)$. Then, M is a Poisson random measure with mean measure $ds\nu(dx)$ if and only if

$$E[\int_{[0,t)\times R_0} G_s f(x) M(ds, dx)] = E[\int_{[0,t)\times R_0} G_s f(x) ds\nu(dx)],$$

for each $t \in R_+$, for every positive measurable function f on R_0, and every $\mathcal{F}_t$ predictable process (G_s).

The following is an extension of the above theorem.

Theorem 1.16 Let $g(t, x, \omega)$ be such that
(i) $x \to g(t, x, \omega)$ is a positive bounded measurable function, and
(ii) $t \to g(t, x, \omega)$ is predictable with respect to $\mathcal{F}_t$.
Then, For each $t \in R_+$, we have

$$E\left(\int_{[0,t)\times R_0} g(s,x)M(ds,dx)\right) = E\left(\int_{[0,t)\times R_0} g(s,x)ds\nu(dx)\right)$$

Proof We use the monotone class theorem. Take $\mathcal{F}$ to be the class of functions for which the above equation holds. Let Ⱡ $= \{g(s,x) : g(s,x) = G_s f,\ \text{where} f : R \to R_+$ is a measurable function, and (G_s) is $\mathcal{F}_t$ $-$predictable$\}$. From Theorem 1.15, we have $\mathcal{F} \supset$ Ⱡ. It is clear that $\mathcal{F}$ is a vector space that contains the constant functions and, by the *monotone convergence theorem,* is closed under taking monotone limits of functions. From Theorem 10 of the appendix $\mathcal{F}$ contains every bounded $\sigma($Ⱡ$)$ measurable function. But $\sigma($Ⱡ$)$ is nothing but the sigma algebra generated by functions satisfying conditions (i) and (ii) of this theorem. Thus the class of all functions g satisfying the assumptions of this theorem are in $\mathcal{F}$, this finishes the proof. ■

1.8 Non-homogeneous Lévy Processes

The classes of Lévy processes dealt with thus far are known as "homogeneous Lévy processes". Nonhomogeneous Lévy processes are encountered in practice. More than one definition of such processes are found in the literature. The following definition of such processes is suitable for our purposes.

A nonhomogeneous subordinator has the same properties as the homogeneous subordinator with the exception that the increments are not stationary. In this case, we have that, for each $t, \theta \geq 0$

$$E[e^{-\theta X_t}] = \exp\big(-\theta\Lambda(t) - \int_{[0,t]\times R_+} (1-e^{-\theta x})n(ds,dx)\big), \tag{1.30}$$

where $n(ds,dx) = \Lambda(ds)\nu(dx)$, $\int_0^\infty (x \wedge 1)\upsilon(dx) < \infty$, Λ is an arbitrary positive measure on R_+ with $0 \leq \Lambda[0,t] < \infty$ for every $t \geq 0$, and $\Lambda(0) = 0$. We assume that the function $t \to \Lambda(t) \equiv \Lambda[0,t]$ is continuous. It follows that a stochastic process X is a nonhomogeneous subordinator, if and only if, for every $t \in R_+$, $X_t = Y_{\Lambda(t)}$, where the process Y is a homogeneous subordinator, and Λ is as defined above.

In the same manner we define a nonhomogeneous Lévy process as a stochastic process L, for every $t \in R_+$, $L_t = Y_{\Lambda(t)}$, where Y is homogeneous Lévy process, and Λ is as defined above. In this case we have

$$E[e^{i\theta L_t}] = \exp\begin{pmatrix} \int_{[0,t]\times R}[\exp(i\theta x) - 1 - i\theta x I_{\{|x|<1\}}]\Lambda(ds)\upsilon(dx) \\ +i\theta a\Lambda(t) - \frac{\theta^2 b}{2}\Lambda(t) \end{pmatrix}, \tag{1.31}$$

where $\int_{R_0}(x^2 \wedge 1)\upsilon(dx) < \infty$.

1.9 Potentials

We begin by defining the α-potential measure $\mathbf{R}^\alpha(x, .)$.

Definition 1.17 Let X be a stochastic process with state space S. For $x \in S$, any Borel set $A \subset S$, and $\alpha \geq 0$

$$\mathbf{R}^\alpha(x, A) = E_x \int_0^\infty e^{-\alpha t} \mathbf{I}_{\{X_t \in A\}} dt = \int_0^\infty P_t(x, A) e^{-\alpha t} dt. \tag{1.32}$$

Since every bounded measurable function can be approximated by a sequence of simple functions, from the *bounded convergence theorem,* and (1.32) it follows that for every bounded measurable function f on S

$$\mathbf{R}^\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) dt = \int_0^\infty f(y) \mathbf{R}^\alpha(x, dy). \tag{1.33}$$

We note that if X is a Lėvy process, then $\mathbf{R}^\alpha(x, dy) = \mathbf{R}^\alpha(0, dy - x)$. We will denote $\mathbf{R}^\alpha(0, dy)$ by $\mathbf{R}^\alpha(dy)$ throughout.

Lemma 1.18 Let X be a subordinator, with Laplace exponent ψ given in (1.3), then $\mathbf{R}^\alpha(dy)$ is obtained by inverting the function $\frac{1}{\alpha + \phi(\theta)}$ with respect to θ.

Proof For $\theta > 0$, let $f(x) = e^{-\theta x}$, $x > 0$, in (1.33). Then,

$$\begin{aligned}
\mathbf{R}^\alpha f(0) &= E \int_0^\infty e^{-\alpha t} e^{-\theta X_t} dt \\
&= \int_0^\infty e^{-\alpha t} E[e^{-\theta X_t}] dt \\
&= \int_0^\infty e^{-\alpha t} e^{-t\phi(\theta)} dt \\
&= \frac{1}{\alpha + \psi(\theta)},
\end{aligned}$$

where the second equation above follows from *Fubini's theorem*. The assertion follows, since for $f(x) = e^{-\theta x}$, $\mathbf{R}^\alpha f(0) = \int_0^\infty e^{-\theta y} \mathbf{R}^\alpha(dy)$. ■

Corollary 1.19 Let X be a compound Poisson process with no drift, rate λ, and jump distribution function F whose support is R_+. For $\alpha \geq 0$, let $F_\alpha = \frac{\lambda}{\lambda + \alpha} F$, for $n =, 1, \ldots,$ $F_\alpha^{(n)}$ is the nth convolution of F_α, $F^{(0)}$ is the Dirac measure $\delta_0(x)$, and we write $F_\alpha^{(n)}(dy)$ instead of $dF_\alpha^{(n)}(y)$. Then, for each $y \geq 0$,

$$\mathbf{R}^\alpha(dy) = \frac{1}{(\alpha + \lambda)} \sum_{n \geq 0} F_\alpha^{(n)}(dy). \tag{1.34}$$

Proof Let the function f be as defined in the proof of Lemma 1.18. Note that $\alpha - \psi(\theta) = \alpha + \lambda \int_0^\infty (1 - e^{-\theta x})F(dx) = \alpha + \lambda - \lambda \int_0^\infty e^{-\theta x} F(dx) = (\alpha + \lambda)(1 - \int_0^\infty e^{-\theta x} F_\alpha(dx))$. Thus,

$$\begin{aligned}\mathbf{R}^\alpha f(0) &= \frac{1}{\alpha + \psi(\theta)} \\ &= \frac{1}{(\alpha + \lambda)} \frac{1}{(1 - \int_0^\infty e^{-\theta x} F_\alpha(dx))}\end{aligned}$$

The result is immediate from Lemma 1.18 upon inverting the right-hand side of the last equation with respect to θ. ■

Corollary 1.20 Assume that X is an inverse Gaussian process, as defined in Example 2 of Sect. 1.4. Let φ be the density function of the standard normal random variable, and erfc be the well-known complimentary error functions. Then R^α is absolutely continuous with respect to the Lebesgue measure on R_+, for $y \in R_+$

$$\mathbf{R}^\alpha(dy) = r^\alpha(y)dy,$$

where

$$r^\alpha(y) = \frac{\sigma}{\sqrt{y}}\varphi(\sqrt{y}\mu/\sigma) + (\frac{\mu - \alpha\sigma^2}{2})e^{\alpha y(\frac{\alpha\sigma^2}{2} - \mu)} \operatorname{erf} c(\sqrt{y}\frac{\alpha\sigma^2 - \mu}{\sqrt{2\sigma^2}}). \tag{1.35}$$

Proof Let f be as defined in the proof of Lemma 1.18, then from (1.5) we have

$$\mathbf{R}^\alpha f(0) = \frac{\sigma^2}{\alpha\sigma^2 + \{\sqrt{2\theta\sigma^2 + \mu^2} - \mu\}}. \tag{1.36}$$

Our assertion is proven using Lemma 1.18 and inverting the right-hand side of (1.36) with respect to θ. ■

We now introduce the so-called *killed process*.

Definition 1.21 Let L be Lévy process and τ be a stopping time. For $t \geq 0$, let

$$X_t = \{L_t, t < \tau\}. \tag{1.37}$$

The process X is obtained by killing the process L at time τ.

Let X be the process defined in (1.37) then, for every Borel set A contained in the state space of X, $t \in R_+$, the probability transition function of this process is given as follows:

$$P_t(x, A) = P_x(L_t \in A, t < \tau\}$$

and for each $\alpha \in R_+$ its α-potential is defined as follows:

$$U^{\alpha}(x, A) = \int_0^{\infty} P_t(x, A)e^{-\alpha t}dt = E_x \int_0^{\tau} e^{-\alpha t}\mathbf{I}_{\{L_t \in A\}}dt. \tag{1.38}$$

For $\lambda \in R_+$, we define

$$T_{\lambda}^{+} = \inf\{t : L_t \geq \lambda\}. \tag{1.39}$$

If the stopping time τ in (1.37) is taken to T_{λ}^{+}, then the state space of the process X is $[0, \lambda)$ if it is a subordinate and $(-\infty, \lambda)$ if it is spectrally positive.

Lemma 1.22 Assume that the process L is a subordinator, and the process X is obtained by killing L at T_{λ}^{+}. For any Borel set $A \subset [0, \lambda)$, let $\mathbf{R}^{\alpha}(x, A)$ be as defined in (1.32), and $U^{\alpha}(x, A)$ be as defined in (1.38). Then, for $x \in [0, \lambda)$

$$U^{\alpha}(x, A) = \mathbf{R}^{\alpha}(x, A). \tag{1.40}$$

Proof Write

$$\begin{aligned} U^{\alpha}(x, A) &= E_x \int_0^{\infty} e^{-\alpha t}\mathbf{I}_{\{L_t \in A, t < T_{\lambda}^{+}\}}dt \\ &= E_x \int_0^{\infty} e^{-\alpha t}\mathbf{I}_{\{L_t \in A,, \bar{L}_t < \lambda\}}dt \\ &= E_x \int_0^{\infty} e^{-\alpha t}\mathbf{I}_{\{L_t \in A\}}dt \\ &= \mathbf{R}^{\alpha}(x, A), \end{aligned}$$

where the second equation above follows from the definition of T_{λ}^{+}, the third equation follows since, for each $t \geq 0$, $\bar{L}_t = L_t$ almost everywhere and $A \subset [0, \lambda)$. Furthermore, the last equation follows from (1.32). ■

As an application of the above result we have the following.

Theorem 1.23 Let X be a positive compound Poisson process as defined in Example 1 of Sect. 1.4. For $\alpha \geq 0$, let $\mathbf{R}^{\alpha}$ be as given in (1.34). For $x \geq 0$, let $\bar{\upsilon}(x) = \upsilon([x, \infty)) = \lambda\bar{F}(x)$, where $\bar{F} \equiv 1 - F$. Then for any $\lambda \geq 0$, $v \geq \lambda$, $u < \lambda$

$$E\{e^{-\alpha T_{\lambda}^{+}}, X_{T_{\lambda}^{+}} > v, X_{T_{\lambda}^{+}-} \leq u\} = \int_{(0,u]} \bar{\upsilon}(v - y)\mathbf{R}^{\alpha}(dy). \tag{1.41}$$

Proof For $n = 1, 2, ..$, let $Y_n = X_1 + \cdots + X_n$, $Y_0 = 0$. Note that, for $n = 0, 1, ..$, Y_n is the value of the compound Poison process at time S_n = time of the nth jump of the process X. Let N be the renewal process associated with $\{Y_n, n = 0, 1, .\}$, i.e. $N_x = \sup\{n : Y_n \leq x\}$. Furthermore, for $n = 1, 2, ...$, S_n is a gamma random variable with mean n/λ. We write

$$
\begin{aligned}
E\{e^{-\alpha T_\lambda^+}, X_{T_\lambda^+} > v, X_{T_\lambda^+ -} \le u\} \\
&= \sum_{k=0}^{\infty} E[e^{-\alpha S_{k+1}}, Y_{k+1} > v, Y_k \le u, N_\lambda = k\} \\
&= \sum_{k=0}^{\infty} E[e^{-\alpha S_{k+1}}, Y_{k+1} > v, Y_k \le u\} \\
&= \sum_{k=0}^{\infty} E[e^{-\alpha S_{k+1}}, Y_k + X_{k+1} > v, Y_k \le u] \\
&= \sum_{k=0}^{\infty} E[e^{-\alpha S_{k+1}}, X_{k+1} > v - Y_k, Y_k \le u] \\
&= \sum_{n=0}^{\infty} E[e^{-\alpha S_{k+1}}] \int_{[0,u]} P\{X_{k+1} > v - y\} P\{Y_k \in dy\} \\
&= \sum_{k=0}^{\infty} (\frac{\lambda}{\lambda + \alpha})^{k+1} \int_{[0,u]} \bar{F}(v - y) P\{Y_k \in dy\} \\
&= \frac{\lambda}{\lambda + \alpha} \int_{[0,v]} \bar{F}(v - y) \sum_{nk=0}^{\infty} F_{\alpha}^{(k)}(dy) \\
&= \lambda \int_{[0,v]} \bar{F}(v - y) \mathbf{R}^\alpha(dy) \\
&= \int_{[0,u]} \bar{\upsilon}(v - y) \mathbf{R}^\alpha(dy).
\end{aligned}
$$

where the second equation follows since for every $k = 0, 1, ..., v > \lambda, u \le \lambda$, $\{Y_{k+1} > v, Y_k \le u\} \subset \{N_\lambda = k\}$, and the fifth equation follows since for $k = 0, 1, ..,$ the random variable S_{k+1} is independent of X_{k+1} and Y_k. ■

We conclude this section by computing the potential for spectrally positive processes. We start by computing the potential of a spectrally positive process killed at time T_λ^+. First, we let X be a spectrally positive Lévy process, and as usual we define $\hat{X} = -X$. For any $a \in R$, we let $T_a^- = \inf\{t \ge 0 : X_t \le a\}$, $\top_a^+ = \inf\{t \ge 0 : \hat{X}_t \ge a\}$, and $\top_a^- = \inf\{t \ge 0 : \hat{X}_t \le a\}$.

Lemma 1.24 Let X be a spectrally positive process, with α-scale function $W^{(\alpha)}$. For $\alpha \ge 0, a \le \lambda$ the α-potential $(\overset{(1)}{U^\alpha})$ of the process X killed at time $T = T_\lambda^+ \wedge T_a^-$ is absolutely continuous with respect to the Lebesgue measure on (a, λ) and a version of its density is given by

$$\overset{(1)}{u^{\alpha}}(x,y) = W^{(\alpha)}(\lambda - x)\frac{W^{(\alpha)}(y-a)}{W^{(\alpha)}(\lambda - a)} - W^{(\alpha)}(y-x), \quad x, y \in (a, \lambda). \tag{1.42}$$

Proof For any Borel set $A \subset (a, \lambda)$

$$\begin{aligned}
\overset{(1)}{U^{\alpha}}(x, A) &= E_x \int_0^{T_\lambda^+ \wedge T_a^-} e^{-\alpha t} \mathbf{I}_{\{I_t \in A\}} dt \\
&= E_{-x} \int_0^{\mathsf{T}^-_{-\lambda} \wedge \mathsf{T}^+_{-a}} e^{-\alpha t} \mathbf{I}_{\{\hat{I}_t \in -A\}} dt \\
&= E_{\lambda - x} \int_0^{\mathsf{T}^-_0 \wedge \mathsf{T}^+_{\lambda - a}} e^{-\alpha t} \mathbf{I}_{\{\hat{I}_t \in \lambda - A\}} dt \\
&= \int_{(\lambda - A)} [W^{(\alpha)}(\lambda - x)\frac{W^{(\alpha)}(\lambda - a - y)}{W^{(\alpha)}(\lambda - a)} - W^{(\alpha)}(y - x)] dy,
\end{aligned}$$

where the last equation follows from Theorem 8.7 of [3], this establishes our assertion. ■

Corollary 1.25 Let X be a spectrally positive Lévy process, with α-scale function $W^{(\alpha)}$. For $\alpha \geq 0$ the α-potential (U^{α}) of the process killed at time T_λ^+ is absolutely continuous with respect to the Lebesgue measure on $(-\infty, \lambda)$ and a version of its density is given by

$$u^{\alpha}(x, y) = W^{\alpha}(\lambda - x)e^{-(\lambda - y)\eta(\alpha)} - W^{\alpha}(y - x), x, y \in (-\infty, \lambda). \tag{1.43}$$

Proof The proof follows from (1.42) by letting $a \to -\infty$ and since from (1.18), for $\alpha \geq 0$, $W^{(\alpha)}(x) \sim \frac{e^{\eta(\alpha)\, x}}{\phi'(\eta(\alpha)\,)}$ as $x \to \infty$. ■

Corollary 1.26 Let X be a spectrally positive Lévy process, with α-scale function $W^{(\alpha)}$. For $\alpha \geq 0$ the α-potential ($\mathbf{R}^{\alpha}$) is absolutely continuous with respect to the Lebesgue measure on $(-\infty, \infty)$ and a version of its density is given by

$$r^{\alpha}(x, y) = \frac{e^{-(x-y)\eta(\alpha)}}{\phi'(\eta(\alpha)\,)} - W^{\alpha}(y - x), x, y \in (-\infty, \infty). \tag{1.44}$$

Proof The proof follows from (1.43) by letting $\lambda \to \infty$ and since, for $\alpha \geq 0$, $W^{(\alpha)}(\lambda - x) \sim \frac{e^{\eta(\alpha)(\lambda - x)}}{\phi'(\eta(\alpha)\,)}$ as $\lambda \to \infty$. ■

The following is well known (see (8.8) of [3]), whose proof is outside the scope of this book and is omitted.

Lemma 1.27 Let $\hat{X}$ be a spectrally negative Lévy process, $\top_a^-$ and $\top_a^+$ be the times of first hitting level a from above and below, respectively. Then, for $x \leq a$ and $\alpha \in R_+$,

$$E_x[e^{-\alpha \top_a^+}, \top_0^- > \top_a^+] = \frac{W^{(\alpha)}(x)}{W^{(\alpha)}(a)}. \tag{1.45}$$

References

1. Takács L (1968) On dams with finite capacity. J Aust Math Soc 8:161–170
2. Bertoin J (1996) Lévy processes. Cambridge University Press, Cambridge
3. Kyprianou AE (2006) Introductory lectures on fluctuations of Lévy processes and their applications. Springer, Berlin
4. Jacod J (1975) Multivariate point processes: predictable projection, Randon-Nikodym derivatives, representation of martingales. Z Wahrscheinlichkeitstheorie Verw Gebiete 31:235–253
5. Jacod J, Shiryaev A (2003) Limit theorems for stochastic processes. Springer, Berlin

Further Reading

6. Bernyk V, Danlang RC, Beskir G (2008) The law of the supremum of a stable Lévy process with no negative jump. Ann Probab 36:1777–1789
7. Feller W (1971) An Introduction to Probability Theory and its Applications, vol 2. Wiley, New York
8. Takács L (1967) Combinatorial methods in the theory of stochastic processes. Wiley, New York

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