Relativistic Positioning Systems in Flat Space-Time: The Location Problem

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Abstract The location problem in relativistic positioning is considered in flat space-time. When two formal solutions are possible for a user (receiver) of the system, its true location may be obtained from a standard set of emission data extended with an observational rule. The covariant expression giving the location of the user in inertial coordinates is decomposed with respect to an inertial observer.

1 Relativistic Positioning Systems

Basically, a *relativistic positioning system* (RPS) is a set of four clocks or emitters A ($A = 1, 2, 3, 4$), of world-lines $\gamma_A(\tau^A)$, broadcasting their respective proper times τ^A by means of electromagnetic signals. The set $\mathscr{R}$ of events reached by the broadcast signals is called the emission region of the RPS. The *characteristic emission function* Θ of the RPS assigns to each event P in $\mathscr{R}$ its four proper times received by it, $\{\tau^A\}$, $\Theta(P) = \{\tau^A\}$. The region $\mathscr{C}$ where Θ is invertible is called the *emission coordinate region* of the RPS, and the four proper times $\{\tau^A\}$ received at every event P are the *emission coordinates of* P.

The *orientation of a RPS* at the event P is the orientation of the emission coordinates at P, $\hat{\epsilon} \equiv sgn[*(d\tau^1 \wedge d\tau^2 \wedge d\tau^3 \wedge d\tau^4)]$, and it coincides with the sign of the Jacobian determinant of Θ, $\hat{\epsilon}(P) = sgn\, j_\Theta(P)$. The zero Jacobian hypersurface, $\mathscr{J} \equiv \{P \mid j_\Theta(P) = 0\}$, is of relevant interest in relativistic positioning, according to the following result by Coll and Pozo [3]: $\mathscr{J}$ *consists in those events for which any user at them can see the four emitters on a circle on its celestial sphere*.

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This result (being Lorentz invariant) suggests that at any event of the emission coordinate region $\mathscr{C} = \mathscr{R} - \mathscr{J}$, the orientation $\hat{\epsilon}$ could be obtained from the relative positions of the emitters on the celestial sphere of the user at this event. Let us denote by $\vec{v}_A$ the unit vectors ($\vec{v}_A^2 = 1$) along the line of sight from the user to the emitter A, so that the future pointing null vectors $m_A = (u \cdot m_A)(-u + \vec{v}_A)$ describe the propagation of the signals, u being the unit user velocity ($u^2 = -1$). The following *observational rule* to determine $\hat{\epsilon}$ has been proved in [5][1]:

▷ *Consider the circle of the celestial sphere defined by three emitters a (say* $a = 1, 2, 3$*). Their unit directions* $\vec{v}_1$*,* $\vec{v}_2$ *and* $\vec{v}_3$ *are contained in the cone confined by this circle. Accordingly, if the fourth emitter is in the interior of the circle,* $\vec{v}_4$ *is in the interior of the cone and* $\hat{\epsilon}$ *is given by* $\hat{\epsilon} = sgn[(\vec{v}_1, \vec{v}_2, \vec{v}_3)]$*. Otherwise,* $\hat{\epsilon} = -sgn[(\vec{v}_1, \vec{v}_2, \vec{v}_3)]$

Let $\{x^\alpha\}$ be any given specific coordinate system covering $\mathscr{R}$. In relativistic positioning, the *location problem* consists in determining the coordinates $\{x^\alpha\}$ of a user from its emission coordinates $\{\tau^A\}$ and a given set of suitable data. Equation (1) (see below) provides the solution in terms of the emitter trajectories and the orientation $\hat{\epsilon}$. The set of the emitter world-lines referred to the coordinates $\{x^\alpha\}$, and the values of the emission coordinates received by a user, $E \equiv \{\gamma_A(\tau^A), \{\tau^A\}\}$, is called the *standard emission data set*. This set E allows solving the location problem only in a part of the emission region $\mathscr{R}$, called the central region of the RPS (see [4] or footnote 3 below). Out of this region, the set E is unable to determine the orientation $\hat{\epsilon}$ so that it must be extended with, for example, the above observational rule in order to solve the location problem. This problem and the zero Jacobian points have been recently studied by using numerical codes (see [6,7]).

An extended version of this contribution has been presented at the workshop *Relativistic Positioning Systems and their Scientific Applications*.

2 The Location Problem in Minkowski Space-Time

In Minkowski space-time, the location problem is formally solved by finding the coordinate transformation, $x^\alpha(\tau^A)$, from emission $\{\tau^A\}$ to inertial $\{x^\alpha\}$ coordinates.[2] The *configuration of the emitters for an event* P is the set of the four events

[1]We use the following notation: $(-, +, +, +)$ is the signature of the Minkowski metric g; $i()$ denotes the interior product (if x is a vector and T a covariant 2-tensor, $[i(x)T]_\nu = x^\mu T_{\mu\nu}$); $\wedge$ stands for the exterior product; the asterisk $*$ denotes the Hodge dual operator associated to the metric volume element η, $\eta_{\alpha\beta\gamma\delta} = -\sqrt{-det\, g}\, \epsilon_{\alpha\beta\gamma\delta}$, where $\epsilon_{\alpha\beta\gamma\delta}$ is Levi–Civita permutation symbol, $\epsilon_{0123} = 1$. For a given inertial observer of unit velocity u, $u^2 = -1$, any vector x splits as $x = x^0 u + \vec{x}$ where $x^0 = -x \cdot u$ and $\vec{x} \in E_\perp$ with $E_\perp$ the three-space of u. For vectors $\vec{x}$, $\vec{y} \in E_\perp$, the vector product is given by $\vec{x} \times \vec{y} = *(u \wedge \vec{x} \wedge \vec{y})$ and $(\vec{x}, \vec{y}, \vec{z}) \equiv (\vec{x} \times \vec{y}) \cdot \vec{z}$ (with $\vec{z} \in E_\perp$) is the scalar triple product.

[2]It seems that Abel and Chaffee [1,2] were the first authors in considering the location problem in connection with Global Positioning System (GPS) by using Lorentzian algebra.

$\{\gamma_A(\tau^A)\}$ of the emitters at the emission times $\{\tau^A\}$ received at P. Let us denote by $x \equiv OP$ the position vector with respect to the origin O of a specific inertial coordinate system. If a user at P receives the broadcast times $\{\tau^A\}$, γ_A denote the position vectors of the emitters at the emission times, $\gamma_A \equiv O\gamma_A(\tau^A)$. Let us choose the fourth emitter as the *reference emitter* and name the other emitters the *referred emitters*, whose relative position vectors with respect the reference emitter are given by $e_a = \gamma_a - \gamma_4$ $(a = 1, 2, 3)$. The vectors $m_A \equiv x - \gamma_A$ represent the trajectories followed by the light signals from the emitters $\gamma_A(\tau^A)$ to the reception event P. The configuration of the emitters has associated the following quantities: the *configuration scalars* $\Omega_a = \frac{1}{2}(e_a)^2$, which are the world function of the pairs of emitters $\{\gamma_a, \gamma_4\}$, the *configuration vector* $\chi \equiv *(e_1 \wedge e_2 \wedge e_3)$ which is orthogonal to the hyperplane containing the four configuration events, and the *configuration bivector* $H \equiv *(\Omega_1\, e_2 \wedge e_3 + \Omega_2\, e_3 \wedge e_1 + \Omega_3\, e_1 \wedge e_2)$. All these quantities are computable from the sole standard data set E because they are defined from e_a. Here, we assume that $\chi \neq 0$, that is, the four emission events $\{\gamma_A(\tau^A)\}$ determine a hyperplane, the *configuration hyperplane* for P. Emitter configurations, with $\chi = 0$, can occur in current GPS as it was stressed in [1, 2].

2.1 Covariant Expression of the Solution

In flat space-time, the coordinate transformation $x^\alpha(\tau^A)$ is given by:

$$x = \gamma_4 + y_* - \lambda\chi, \quad y_* = \frac{1}{\xi \cdot \chi}\, i(\xi)H, \quad \lambda = \frac{y_*^2}{(y_* \cdot \chi) + \hat{\epsilon}\sqrt{\Delta}}, \tag{1}$$

ξ being any vector transversal to the configuration, $\xi \cdot \chi \neq 0$, and Δ being the following quadratic invariant of H, $\Delta \equiv -\frac{1}{2} H_{\mu\nu} H^{\mu\nu} = (y_* \cdot \chi)^2 - y_*^2 \chi^2$, which is non-negative, $\Delta \geq 0$ (see [4, 5]). Note that y_* and Δ are both computable from the sole standard data set E. The orientation $\hat{\epsilon}$ depends on x and it *is not always* computable from the sole set E. In fact, we have: (a) if $\chi^2 \leq 0$ there is a sole emission solution x, and there is no bifurcation. To obtain the solution, take $\hat{\epsilon} = sgn(u \cdot \chi)$, where u is any future pointing time-like vector. (b) If $\chi^2 > 0$ there are two emission solutions, x and x', which only differ by their orientation $\hat{\epsilon}$ (bifurcation problem). In this case, the sole standard data set E is insufficient to solve the location problem but our observational rule allows to determine $\hat{\epsilon}$ and to solve it.[3]

[3]The region $\mathscr{C}^C \equiv \{x \in \mathscr{C} \mid \chi^2 \leq 0\}$ is called the *central region* of the RPS. The orientation $\hat{\epsilon}$ is constant on $\mathscr{C}^C$, and may be evaluated from the sole standard data set E. The bifurcation problem always appears in the time-like configuration region $\mathscr{C}_t \equiv \{x \in \mathscr{C} \mid \chi^2 > 0\} = \mathscr{C} - \mathscr{C}^C$.

2.2 Splitting of the Solution for an Inertial Observer

Consider the inertial observer associated to the specific inertial coordinate system $\{x^\alpha\}$, of unit velocity u, $u^2 = -1$. Next, we decompose respect to this inertial observer the quantities appearing in the transformation (1) from emission to inertial coordinates. The position vector of the emitter A becomes $\gamma_A = t_A u + \bar{\gamma}_A$, $t_A \equiv \gamma_A^0$ being the value of the inertial time of the observer u at the event $\gamma_A(\tau^A)$; then $m_A = (x^0 - t_A)u + \vec{x} - \bar{\gamma}_A$ and $(x^0 - t_A)^2 = (\vec{x} - \bar{\gamma}_A)^2$ with $x^0 > t_A$, because each m_A is null and future pointing.

The position vector of the emitter a with respect to the reference emitter splits as $e_a = \sigma_a u + \vec{e}_a, a = 1, 2, 3$, with $\sigma_a = t_a - t_4$ and $\vec{e}_a = \bar{\gamma}_a - \bar{\gamma}_4$ and then the configuration scalars are given by $\Omega_a = \frac{1}{2}((\vec{e}_a)^2 - \sigma_a^2)$. The configuration vector splits as $\chi = \chi^0 u + \bar{\chi}$, with $\chi^0 = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $\bar{\chi} = \sigma_1 \vec{e}_2 \times \vec{e}_3 + \sigma_2 \vec{e}_3 \times \vec{e}_1 + \sigma_3 \vec{e}_1 \times \vec{e}_2$, and the configuration bivector is written as $H = u \wedge \vec{S} - *(u \wedge \vec{B})$, where the electric-like $\vec{S} \equiv -i(u)H$ and the magnetic-like $\vec{B} \equiv -i(u) * H$ parts of H are expressed as:

$$\vec{S} = \Omega_1 \vec{e}_2 \times \vec{e}_3 + \Omega_2 \vec{e}_3 \times \vec{e}_1 + \Omega_3 \vec{e}_1 \times \vec{e}_2, \tag{2}$$

$$\vec{B} = (\sigma_2 \Omega_3 - \sigma_3 \Omega_2)\vec{e}_1 + (\sigma_3 \Omega_1 - \sigma_1 \Omega_3)\vec{e}_2 + (\sigma_1 \Omega_2 - \sigma_2 \Omega_1)\vec{e}_3, \tag{3}$$

and satisfy $\vec{S}^2 \geq \vec{B}^2$ and $\vec{S} \cdot \vec{B} = 0$. By choosing $\xi^0 = 1$, $\xi = u + \bar{\xi}$, one has $i(\xi)H = -(\bar{\xi} \cdot \vec{S})u - \vec{S} - \bar{\xi} \times \vec{B}$, and $y_* = y_*^0 u + \vec{y}_*$ is provided by:

$$y_*^0 = -\frac{\bar{\xi} \cdot \vec{S}}{D}, \quad \vec{y}_* = -\frac{\vec{S} + \bar{\xi} \times \vec{B}}{D}, \quad D \equiv \bar{\xi} \cdot \bar{\chi} - (\vec{e}_1, \vec{e}_2, \vec{e}_3) \neq 0, \tag{4}$$

with $\vec{S}$ and $\vec{B}$ given by (2) and (3). Substituting λ in (1) by:

$$\lambda = \frac{-(y_*^0)^2 + \vec{y}_*^{\,2}}{-y_*^0 \chi^0 + \vec{y}_* \cdot \bar{\chi} + \hat{\epsilon}\sqrt{\vec{S}^2 - \vec{B}^2}}, \tag{5}$$

the user location is expressed in terms of the orientation $\hat{\epsilon}$ (which is obtainable from the observational rule) and $\gamma_4 = \{t_4, \bar{\gamma}_4\}$, $\chi = \{\chi^0, \bar{\chi}\}$, $H = \{\vec{S}, \vec{B}\}$ and $y_* = \{y_*^0, \vec{y}_*\}$ (which are obtainable from the sole standard data E). When $(\vec{e}_1, \vec{e}_2, \vec{e}_3) \neq 0$, one may take $\bar{\xi} = 0$ to simplify the above expressions.

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