

Chapter 2

Intuitionistic Fuzzy Aggregation Operators and Multiattribute Decision-Making Methods with Intuitionistic Fuzzy Sets

2.1 Introduction

How to aggregate information and preference is an important problem in real management and decision process [1–7]. Due to the complexity of management environments and decision problems themselves, decision makers may provide their ratings or judgments to some certain degree, but it is possible that they are not so sure about their judgments. Namely, there may exist some hesitancy degree, which is a very important factor to be taken into account when trying to construct really adequate models and solutions of decision problems. Such a kind of hesitancy degrees is suitably expressed with intuitionistic fuzzy sets rather than exact numerical values. Thus, how to aggregate intuitionistic fuzzy information becomes an important part of multiattribute decision-making with intuitionistic fuzzy sets [8–13].

The classical weighted aggregation is usually known in the literature by the linear (or simple additive) weighted averaging method [14–16]. Another important aggregation operator within the class of weighted averaging operators is the ordered weighted averaging (OWA) operator introduced by Yager [4], which has been used in many management applications. In 2004, Yager [5] further introduced the generalized ordered weighted averaging (GOWA) operator, which is a generalization of the OWA operator and the generalized mean operator through adding an additional parameter as the power of the OWA operator.

This chapter mainly discusses extension forms of these aggregation operators with intuitionistic fuzzy sets, including the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy OWA operator, intuitionistic fuzzy hybrid weighted averaging operator, intuitionistic fuzzy GOWA operator, intuitionistic fuzzy generalized hybrid weighted averaging operator and their applications to multiattribute decision-making with intuitionistic fuzzy sets [9–13].

2.2 Intuitionistic Fuzzy Aggregation Operators and Properties

For the sake of convenience, as stated earlier, if an intuitionistic fuzzy set $\tilde{A} = \{\langle x_j, \mu_{\tilde{A}}(x_j), v_{\tilde{A}}(x_j) \rangle \mid x_j \in X\}$ on the finite universal set $X = \{x_1, x_2, \dots, x_n\}$ has only one element, i.e., $|\tilde{A}| = 1$, then usually $\tilde{A}$ is denoted by $A_j = \langle \mu_j, v_j \rangle$ ($j = 1, 2, \dots, n$) for short,¹ where $\mu_j = \mu_{\tilde{A}}(x_j)$ and $v_j = v_{\tilde{A}}(x_j)$ satisfy the conditions: $\mu_j \in [0, 1]$, $v_j \in [0, 1]$, and $0 \leq \mu_j + v_j \leq 1$. The set of these singleton intuitionistic fuzzy sets is denoted by F .²

In the sequent, to investigate on aggregation problems of imprecise and uncertain information in decision-making with intuitionistic fuzzy sets, we discuss several commonly-used aggregation operators such as the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy OWA operator, intuitionistic fuzzy hybrid weighted averaging operator and intuitionistic fuzzy GOWA operator as well as intuitionistic fuzzy generalized hybrid weighted averaging operator.

2.2.1 The Intuitionistic Fuzzy Weighted Averaging Operator

Definition 2.1 Let $A_j = \langle \mu_j, v_j \rangle$ ($j = 1, 2, \dots, n$) be intuitionistic fuzzy sets. A mapping $f_{\omega}^A : F^n \rightarrow F$ is called an intuitionistic fuzzy weighted averaging operator if it satisfies

$$f_{\omega}^A(A_1, A_2, \dots, A_n) = \sum_{j=1}^n \omega_j A_j,$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is a weight vector of $A_j = \langle \mu_j, v_j \rangle$ ($j = 1, 2, \dots, n$), which should satisfy the normalized conditions: $\omega_j \in [0, 1]$ ($j = 1, 2, \dots, n$) and $\sum_{j=1}^n \omega_j = 1$.

In particular, when $\omega = (1/n, 1/n, \dots, 1/n)^T$, the above intuitionistic fuzzy weighted averaging operator f_{ω}^A can be rewritten as follows:

$$f_{\omega}^A(A_1, A_2, \dots, A_n) = \frac{1}{n} \sum_{j=1}^n A_j.$$

In this case, f_{ω}^A is called an intuitionistic fuzzy arithmetic mean operator, denoted by f^A for short.

¹ For convenience, unless otherwise stated, the symbol “ $\sim$ ” is not added to the top of any singleton intuitionistic fuzzy set.

² For conciseness, unless otherwise stated, we do not explicitly write out the universal set X of the set F which consists of singleton intuitionistic fuzzy sets.

The intuitionistic fuzzy weighted averaging operator f_{ω}^A has the following remarkable characteristic: each intuitionistic fuzzy set A_j ($j = 1, 2, \dots, n$) is firstly weighted with ω_j and then the products $\omega_j A_j$ are summed.

Theorem 2.1 Assume that $A_j = \langle \mu_j, v_j \rangle$ ($j = 1, 2, \dots, n$) are intuitionistic fuzzy sets. Then, the aggregation result through using the intuitionistic fuzzy weighted averaging operator f_{ω}^A is an intuitionistic fuzzy set and

$$f_{\omega}^A(A_1, A_2, \dots, A_n) = \left\langle 1 - \prod_{j=1}^n (1 - \mu_j)^{\omega_j}, \prod_{j=1}^n v_j^{\omega_j} \right\rangle. \quad (2.1)$$

Proof According to Definition 2.1 and the operations (6) and (8) of Definition 1.2, Theorem 2.1 can be proven by using the mathematical induction (omitted).

Example 2.1 There are four experts who are invited to evaluate some decision alternative. Their evaluations are expressed with the intuitionistic fuzzy sets $A_1 = \langle 0.3, 0.2 \rangle$, $A_2 = \langle 0.1, 0.5 \rangle$, $A_3 = \langle 0.7, 0.1 \rangle$, and $A_4 = \langle 0.4, 0.2 \rangle$, respectively. $\omega = (0.1, 0.4, 0.3, 0.2)^T$ is the weight vector of the four experts. Compute the comprehensive evaluation of the four experts on the decision alternative through using the intuitionistic fuzzy weighted averaging operator.

Solving Using the intuitionistic fuzzy weighted averaging operator f_{ω}^A [i.e., Eq. (2.1)], we obtain the comprehensive evaluation of the four experts on the decision alternative as follows:

$$\begin{aligned} f_{\omega}^A(A_1, A_2, A_3, A_4) &= \left\langle 1 - \prod_{j=1}^4 (1 - \mu_j)^{\omega_j}, \prod_{j=1}^4 v_j^{\omega_j} \right\rangle \\ &= \left\langle 1 - (1 - 0.3)^{0.1} (1 - 0.1)^{0.4} (1 - 0.7)^{0.3} (1 - 0.4)^{0.2}, 0.2^{0.1} \right. \\ &\quad \times 0.5^{0.4} \times 0.1^{0.3} \times 0.2^{0.2} \rangle \\ &= \langle 1 - 0.9650 \times 0.9587 \times 0.6968 \times 0.9029, 0.8513 \times 0.7579 \\ &\quad \times 0.5012 \times 0.7248 \rangle \\ &= \langle 0.4180, 0.2344 \rangle, \end{aligned}$$

which means that the satisfaction degree and dissatisfaction degree of the four experts on the decision alternative are 0.4180 and 0.2344, respectively, and hereby the hesitancy degree is 0.3476.

2.2.2 The Intuitionistic Fuzzy Hybrid Weighted Averaging Operator

For the sake of the sequent discussions, firstly we introduce a ranking method of intuitionistic fuzzy sets, which is simply called the scoring function ranking method of intuitionistic fuzzy sets.

Let $A = \langle \mu, \nu \rangle$ be an intuitionistic fuzzy set. A score function M of an intuitionistic fuzzy set is defined as follows [17]:

$$M(A) = \mu - \nu. \quad (2.2)$$

Obviously, $M(A) \in [-1, 1]$. The larger the score $M(A)$ the greater the intuitionistic fuzzy set A .

An accuracy function Δ of an intuitionistic fuzzy set is defined as follows [18]:

$$\Delta(A) = \mu + \nu. \quad (2.3)$$

Obviously, $\Delta(A) \in [0, 1]$. The larger the accuracy $\Delta(A)$ (i.e., the more the degree of accuracy of the intuitionistic fuzzy set A) the greater A .

It is easy to see from Eqs. (2.2) and (2.3) that the score function and accuracy function are the difference and sum of the membership and nonmembership degrees of the element belonging to the intuitionistic fuzzy set A , respectively. The meanings of the score function and accuracy function are similar to those of the mean and variance in statistics, respectively. Hereby, we develop the scoring function ranking method of intuitionistic fuzzy sets based on the score function and accuracy function as follows [17, 18].

Let A and B be any two intuitionistic fuzzy sets. According to their scores and accuracies, the ranking order of A and B is stipulated as follows:

- (1) If $M(A) > M(B)$, then A is greater than B , denoted by $A > B$;
- (2) If $M(A) < M(B)$, then A is smaller than B , denoted by $A < B$;
- (3) If $M(A) = M(B)$, then
 - (3a) If $\Delta(A) = \Delta(B)$, then A is equal to B , denoted by $A = B$;
 - (3b) If $\Delta(A) < \Delta(B)$, then A is smaller than B , denoted by $A < B$;
 - (3c) If $\Delta(A) > \Delta(B)$, then A is greater than B , denoted by $A > B$.

Example 2.2 Let us consider the ranking order of the intuitionistic fuzzy sets $A_0 = \langle 0.5, 0.2 \rangle$ and $B_0 = \langle 0.6, 0.35 \rangle$.

Solving Using Eq. (2.2), we obtain the scores of the intuitionistic fuzzy sets A_0 and B_0 as follows:

$$M(A_0) = 0.5 - 0.2 = 0.3$$

and

$$M(B_0) = 0.6 - 0.35 = 0.25,$$

respectively.

It is obvious that $M(A_0) > M(B_0)$. Thus, according to the above scoring function ranking method, we believe that A_0 is bigger than B_0 , i.e., $A_0 > B_0$.

It is worthwhile to point out that the aforementioned intuitionistic fuzzy sets A_0 and B_0 do not possess the inclusion relation as given by the operation (1) of Definition 1.2. Namely, A_0 does not include B_0 . Conversely, B_0 does not include A_0 also.

Example 2.3 Assume that the intuitionistic fuzzy set B_0 given in Example 2.2 is changed to $B'_0 = \langle 0.6, 0.3 \rangle$. Let us consider the ranking order of the intuitionistic fuzzy sets A_0 in Example 2.2 and B'_0 .

Solving Using Eq. (2.2), we obtain the score of the intuitionistic fuzzy set B'_0 as follows:

$$M(B'_0) = 0.6 - 0.3 = 0.3.$$

Combining with the computational results of Example 2.2, it follows that $M(A_0) = M(B'_0)$. Thus, the ranking order of the intuitionistic fuzzy sets A_0 and B'_0 can not be determined.

Further, according to Eq. (2.3), we can obtain the accuracies of the intuitionistic fuzzy sets A_0 and B'_0 as follows:

$$\Delta(A_0) = 0.5 + 0.2 = 0.7$$

and

$$\Delta(B'_0) = 0.6 + 0.3 = 0.9,$$

respectively. Obviously, $\Delta(A_0) < \Delta(B'_0)$. Hence, according to the above scoring function ranking method, we believe that A_0 is smaller than B'_0 , i.e., $A_0 < B'_0$.

Definition 2.2 Let $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, \dots, n$) be intuitionistic fuzzy sets. A mapping $f_w^O : F^n \rightarrow F$ is called an intuitionistic fuzzy OWA operator if it satisfies

$$f_w^O(A_1, A_2, \dots, A_n) = \sum_{k=1}^n w_k B_k, \quad (2.4)$$

where $w = (w_1, w_2, \dots, w_n)^T$ is a weight vector associated with the mapping f_w^O , which satisfies the normalized conditions: $w_k \in [0, 1]$ and $\sum_{k=1}^n w_k = 1$; $B_k = \langle \hat{\mu}_k, \hat{\nu}_k \rangle$ is the k -th largest of the n intuitionistic fuzzy sets A_j ($j = 1, 2, \dots, n$), which is determined through using some ranking method such as the above scoring function ranking method.

Specially, if $\mathbf{w} = (1/n, 1/n, \dots, 1/n)^T$, then the intuitionistic fuzzy OWA operator f_w^O degenerates to the intuitionistic fuzzy arithmetic mean operator f^A .

The intuitionistic fuzzy OWA operator f_w^O has the following characteristic: the nonincreasing order of the intuitionistic fuzzy sets A_j ($j = 1, 2, \dots, n$) is firstly generated and then the re-ranked intuitionistic fuzzy sets B_k ($k = 1, 2, \dots, n$) are aggregated through using the intuitionistic fuzzy weighted averaging operator. The weight w_j ($j = 1, 2, \dots, n$) has nothing to do with the intuitionistic fuzzy set A_j . It only concerns with the j -th-position of the ranking order in aggregation process. Thus, sometimes the weight vector $\mathbf{w}$ is called a position weight vector. In addition, it is easy to see from Definition 2.2 that the intuitionistic fuzzy OWA operator f_w^O is essentially nonlinear.

Theoretically, determination methods of position weights and attribute weights are pretty much the same thing. Thus, we may appropriately choose some effective methods to determine position weights according to need in real management situations. For example, to eliminate effect of the maximum and minimum on aggregation result, we may take the position weights $w_1 = w_n = 0$. In this case, the maximum and minimum are taken out. In reality, this aggregation process is just the commonly-used counting method, which is also the simple additive (or linear) weighted averaging operator through taking out the highest score and the lowest score.

Theorem 2.2 Assume that $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, \dots, n$) are intuitionistic fuzzy sets. Then, the aggregation result through using the intuitionistic fuzzy OWA operator f_w^O [i.e., Eq. (2.4)] is an intuitionistic fuzzy set and

$$f_w^O(A_1, A_2, \dots, A_n) = \left\langle 1 - \prod_{k=1}^n (1 - \hat{\mu}_k)^{w_k}, \prod_{k=1}^n \hat{\nu}_k^{w_k} \right\rangle, \quad (2.5)$$

where $B_k = \langle \hat{\mu}_k, \hat{\nu}_k \rangle$ is the k -th largest of the n intuitionistic fuzzy sets A_j ($j = 1, 2, \dots, n$), which is determined through using some ranking method of intuitionistic fuzzy sets.

Proof Theorem 2.2 can be proven in a similar way to that of Theorem 2.1 (omitted).

Example 2.2 There are four experts who are invited to evaluate some enterprise. Their evaluations are expressed with the intuitionistic fuzzy sets $A_1 = \langle 0.5, 0.1 \rangle$, $A_2 = \langle 0.1, 0.2 \rangle$, $A_3 = \langle 0.2, 0.4 \rangle$, and $A_4 = \langle 0.3, 0.2 \rangle$, respectively. To eliminate effect of individual bias on comprehensive evaluation, the unduly high evaluation and the unduly low evaluation are punished through giving a smaller weight. Assume that the position weight vector is $\mathbf{w} = (0.155, 0.345, 0.345, 0.155)^T$. Compute the comprehensive evaluation of the four experts on the enterprise through using the intuitionistic fuzzy OWA operator.

Solving According to Eq. (2.2), the scores of the intuitionistic fuzzy sets $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, 3, 4$) are obtained as follows:

$$\begin{aligned}
M(A_1) &= 0.5 - 0.1 = 0.4, \\
M(A_2) &= 0.1 - 0.2 = -0.1, \\
M(A_3) &= 0.2 - 0.4 = -0.2
\end{aligned}$$

and

$$M(A_4) = 0.3 - 0.2 = 0.1,$$

respectively. It is obvious that $M(A_1) > M(A_4) > M(A_2) > M(A_3)$. Hence, according to the above scoring function ranking method, it follows that $A_1 > A_4 > A_2 > A_3$. Thus, we have:

$$\begin{aligned}
B_1 &= A_1 = \langle 0.5, 0.1 \rangle, \\
B_2 &= A_4 = \langle 0.3, 0.2 \rangle, \\
B_3 &= A_2 = \langle 0.1, 0.2 \rangle
\end{aligned}$$

and

$$B_4 = A_3 = \langle 0.2, 0.4 \rangle.$$

It follows from Eq. (2.5) that

$$\begin{aligned}
f_w^O(A_1, A_2, A_3, A_4) &= \left\langle 1 - (1 - 0.5)^{0.155} (1 - 0.3)^{0.345} (1 - 0.1)^{0.345} (1 - 0.2)^{0.155}, \right. \\
&\quad \left. 0.1^{0.155} \times 0.2^{0.345} \times 0.2^{0.345} \times 0.4^{0.155} \right\rangle \\
&= \langle 1 - 0.8981 \times 0.8842 \times 0.9643 \times 0.9660, 0.6998 \\
&\quad \times 0.5739 \times 0.5739 \times 0.8676 \rangle \\
&= \langle 0.2603, 0.20 \rangle,
\end{aligned}$$

which means that the satisfaction degree and dissatisfaction degree of the four experts on the enterprise are 0.2603 and 0.20, respectively, and hereby the hesitancy degree is 0.5397.

Obviously, the intuitionistic fuzzy weighted averaging operator f_w^A only considers importance of the aggregated intuitionistic fuzzy sets themselves. The intuitionistic fuzzy OWA operator f_w^O only concerns with position importance of the ranking order of the aggregated intuitionistic fuzzy sets. To overcome the disadvantages of the aforementioned two intuitionistic fuzzy aggregation operators, we may define the following intuitionistic fuzzy hybrid weighted averaging operator.

Definition 2.3 Let $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, \dots, n$) be intuitionistic fuzzy sets. A mapping $f_{\omega, w}^H : F^n \rightarrow F$ is called an intuitionistic fuzzy hybrid weighted averaging operator if it satisfies

$$f_{\omega, w}^H(A_1, A_2, \dots, A_n) = \sum_{k=1}^n w_k \widehat{B}_k, \quad (2.6)$$

where $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ is a (position) weight vector associated with the mapping $f_{\omega, \mathbf{w}}^H$; the intuitionistic fuzzy set of A_j weighted with $n\omega_j$ ($j = 1, 2, \dots, n$) is denoted by $\hat{A}_j$, i.e., $\hat{A}_j = n\omega_j A_j$, here n is regarded as a balance factor; $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is a weight vector of the intuitionistic fuzzy sets A_j ($j = 1, 2, \dots, n$); $\hat{B}_k$ is the k -th largest of the n intuitionistic fuzzy sets $\hat{A}_j$ ($j = 1, 2, \dots, n$), which is determined through using some ranking method such as the above scoring function ranking method.

Particularly, if $\mathbf{w} = (1/n, 1/n, \dots, 1/n)^T$, then the intuitionistic fuzzy hybrid weighted averaging operator $f_{\omega, \mathbf{w}}^H$ degenerates to the intuitionistic fuzzy weighted averaging operator f_{ω}^A . If $\omega = (1/n, 1/n, \dots, 1/n)^T$, then $f_{\omega, \mathbf{w}}^H$ degenerates to the intuitionistic fuzzy OWA operator $f_{\mathbf{w}}^O$. Thus, the intuitionistic fuzzy hybrid weighted averaging operator $f_{\omega, \mathbf{w}}^H$ is a generalization of the intuitionistic fuzzy weighted averaging operator f_{ω}^A and the intuitionistic fuzzy OWA operator $f_{\mathbf{w}}^O$.

Theorem 2.3 Assume that $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, \dots, n$) are intuitionistic fuzzy sets. Then, the aggregation result through using the intuitionistic fuzzy hybrid weighted averaging operator $f_{\omega, \mathbf{w}}^H$ [i.e., Eq. (2.6)] is an intuitionistic fuzzy set and

$$f_{\omega, \mathbf{w}}^H(A_1, A_2, \dots, A_n) = \left\langle 1 - \prod_{k=1}^n (1 - \hat{\mu}_k)^{w_k}, \prod_{k=1}^n \hat{\nu}_k^{w_k} \right\rangle, \quad (2.7)$$

where $\hat{B}_k = \langle \hat{\mu}_k, \hat{\nu}_k \rangle$ is the k th largest of the n intuitionistic fuzzy sets $\hat{A}_j = n\omega_j A_j$ ($j = 1, 2, \dots, n$), which is determined through using some ranking method of intuitionistic fuzzy sets.

Proof Theorem 2.3 can be proven in a similar way to that of Theorems 2.1 and 2.2 (omitted).

Example 2.3 There are five experts who are invited to evaluate some decision alternative. Their evaluations are expressed with the intuitionistic fuzzy sets $A_1 = \langle 0.2, 0.5 \rangle$, $A_2 = \langle 0.7, 0.1 \rangle$, $A_3 = \langle 0.5, 0.2 \rangle$, $A_4 = \langle 0.3, 0.4 \rangle$, and $A_5 = \langle 0.6, 0.2 \rangle$, respectively. Assume that the weight vector of the five experts is $\omega = (0.25, 0.20, 0.15, 0.18, 0.22)^T$ and the position weight vector is $\mathbf{w} = (0.112, 0.236, 0.304, 0.236, 0.112)^T$. Compute the comprehensive evaluation of the five experts on the decision alternative through using the intuitionistic fuzzy hybrid weighted averaging operator.

Solving According to the operation (8) of Definition 1.2, we have:

$$\hat{A}_1 = 5 \times 0.25 A_1 = \left\langle 1 - (1 - 0.2)^{5 \times 0.25}, 0.5^{5 \times 0.25} \right\rangle = \langle 0.2434, 0.4204 \rangle.$$

Likewise, we obtain:

$$\begin{aligned}\hat{A}_2 &= 5 \times 0.2A_2 = \langle 1 - (1 - 0.7)^{5 \times 0.2}, 0.1^{5 \times 0.2} \rangle = \langle 0.7, 0.1 \rangle, \\ \hat{A}_3 &= 5 \times 0.15A_3 = \langle 1 - (1 - 0.5)^{5 \times 0.15}, 0.2^{5 \times 0.15} \rangle = \langle 0.4054, 0.2991 \rangle, \\ \hat{A}_4 &= 5 \times 0.18A_4 = \langle 1 - (1 - 0.3)^{5 \times 0.18}, 0.4^{5 \times 0.18} \rangle = \langle 0.2746, 0.4384 \rangle\end{aligned}$$

and

$$\hat{A}_5 = 5 \times 0.22A_5 = \langle 1 - (1 - 0.6)^{5 \times 0.22}, 0.2^{5 \times 0.22} \rangle = \langle 0.6350, 0.1703 \rangle.$$

Using Eq. (2.2), we obtain the scores of the intuitionistic fuzzy sets $\hat{A}_j$ ($j = 1, 2, \dots, 5$) as follows:

$$\begin{aligned}M(\hat{A}_1) &= 0.2434 - 0.4204 = -0.1770, \\ M(\hat{A}_2) &= 0.7 - 0.1 = 0.6, \\ M(\hat{A}_3) &= 0.4054 - 0.2991 = 0.1063, \\ M(\hat{A}_4) &= 0.2746 - 0.4384 = -0.1638\end{aligned}$$

and

$$M(\hat{A}_5) = 0.6350 - 0.1703 = 0.4647,$$

respectively. Obviously, $M(\hat{A}_2) > M(\hat{A}_5) > M(\hat{A}_3) > M(\hat{A}_4) > M(\hat{A}_1)$. Thereby, according to the above scoring function ranking method, we have:

$$\begin{aligned}\hat{B}_1 &= \hat{A}_2 = \langle 0.7, 0.1 \rangle, \\ \hat{B}_2 &= \hat{A}_5 = \langle 0.6350, 0.1703 \rangle, \\ \hat{B}_3 &= \hat{A}_3 = \langle 0.4054, 0.2991 \rangle, \\ \hat{B}_4 &= \hat{A}_4 = \langle 0.2746, 0.4384 \rangle\end{aligned}$$

and

$$\hat{B}_5 = \hat{A}_1 = \langle 0.2434, 0.4204 \rangle.$$

It follows from Eq. (2.7) that

$$\begin{aligned}
f_{\omega, \mathbf{w}}^H(A_1, A_2, A_3, A_4, A_5) &= \left\langle 1 - (1 - 0.7)^{0.112} \times (1 - 0.6350)^{0.236} \right. \\
&\quad \times (1 - 0.4054)^{0.304} \times (1 - 0.2746)^{0.236} \times (1 - 0.2434)^{0.112}, \\
&\quad 0.1^{0.112} \times 0.1703^{0.236} \times 0.2991^{0.304} \times 0.4384^{0.236} \\
&\quad \left. \times 0.4204^{0.112} \right\rangle \\
&= \langle 1 - 0.8739 \times 0.7883 \times 0.8538 \times 0.9270 \\
&\quad \times 0.9692, 0.7727 \times 0.6585 \times 0.6929 \times 0.8232 \times 0.9075 \rangle \\
&= \langle 0.4716, 0.2634 \rangle,
\end{aligned}$$

which means that the satisfaction degree and dissatisfaction degree of the five experts on the decision alternative are 0.4716 and 0.2634, respectively, and hereby the hesitancy degree is 0.2650.

2.2.3 The Intuitionistic Fuzzy Generalized Hybrid Weighted Averaging Operator

Definition 2.4 Let $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, \dots, n$) be intuitionistic fuzzy sets. A mapping $f_{\mathbf{w}}^{\text{GO}} : F^n \rightarrow F$ is called an intuitionistic fuzzy GOWA operator if it satisfies

$$f_{\mathbf{w}}^{\text{GO}}(A_1, A_2, \dots, A_n) = \sqrt[q]{\sum_{k=1}^n w_k B_k^q}, \quad (2.8)$$

where $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ is a (position) weight vector associated with the mapping $f_{\mathbf{w}}^{\text{GO}}$; $q > 0$ is a control parameter, which may be chosen according to need in real management situations; $B_k = \langle \hat{\mu}_k, \hat{\nu}_k \rangle$ is the k -th largest of the n intuitionistic fuzzy sets A_j ($j = 1, 2, \dots, n$), which is determined through using some ranking method such as the above scoring function ranking method.

In particular, if $q = 1$, then Eq. (2.8) may be simply written as Eq. (2.5), i.e., the intuitionistic fuzzy GOWA operator $f_{\mathbf{w}}^{\text{GO}}$ degenerates to the intuitionistic fuzzy OWA operator f_{ω}^{O} .

It is not difficult to follow from Definition 2.4 and the operations (6), (8) and (9) of Definition 1.2 that

$$\begin{aligned}
f_w^{\text{GO}}(A_1, A_2, \dots, A_n) &= \sqrt[q]{\sum_{k=1}^n w_k B_k^q} \\
&= \sqrt[q]{\sum_{k=1}^n w_k (\langle \hat{\mu}_k^q, 1 - (1 - \hat{\nu}_k)^q \rangle)} \\
&= \sqrt[q]{\sum_{k=1}^n \langle 1 - (1 - \hat{\mu}_k^q)^{w_k}, [1 - (1 - \hat{\nu}_k)^q]^{w_k} \rangle} \\
&= \sqrt[q]{\left\langle 1 - \prod_{k=1}^n (1 - \hat{\mu}_k^q)^{w_k}, \prod_{k=1}^n [1 - (1 - \hat{\nu}_k)^q]^{w_k} \right\rangle} \\
&= \left\langle \sqrt[q]{1 - \prod_{k=1}^n (1 - \hat{\mu}_k^q)^{w_k}}, 1 - \sqrt[q]{1 - \prod_{k=1}^n [1 - (1 - \hat{\nu}_k)^q]^{w_k}} \right\rangle,
\end{aligned}$$

which is summarized as in Theorem 2.4.

Theorem 2.4 Assume that $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, \dots, n$) are intuitionistic fuzzy sets. Then, the aggregation result through using the intuitionistic fuzzy GOWA operator f_w^{GO} [i.e., Eq. (2.8)] is an intuitionistic fuzzy set and

$$f_w^{\text{GO}}(A_1, A_2, \dots, A_n) = \left\langle \sqrt[q]{1 - \prod_{k=1}^n (1 - \hat{\mu}_k^q)^{w_k}}, 1 - \sqrt[q]{1 - \prod_{k=1}^n [1 - (1 - \hat{\nu}_k)^q]^{w_k}} \right\rangle, \quad (2.9)$$

where $B_k = \langle \hat{\mu}_k, \hat{\nu}_k \rangle$ is the k -th largest of the n intuitionistic fuzzy sets A_j ($j = 1, 2, \dots, n$), which is determined through using some ranking method of intuitionistic fuzzy sets.

Obviously, the several useful conclusions are easily drawn from Theorem 2.4 [i.e., Eq. (2.9)] as follows.

Corollary 2.1 If $q \rightarrow 0$, then $f_w^{\text{GO}}(A_1, A_2, \dots, A_n) = \prod_{k=1}^n B_k^{w_k} = \langle \prod_{k=1}^n \hat{\mu}_k^{w_k}, 1 - \prod_{k=1}^n (1 - \hat{\nu}_k)^{w_k} \rangle$, i.e., the intuitionistic fuzzy GOWA operator f_w^{GO} degenerates to the intuitionistic fuzzy ordered weighted geometric (OWG) operator.

Corollary 2.2 If $q = 1$, then $f_w^{\text{GO}}(A_1, A_2, \dots, A_n) = \sum_{k=1}^n w_k B_k$, i.e., the intuitionistic fuzzy GOWA operator f_w^{GO} degenerates to the intuitionistic fuzzy OWA operator f_w^{O} .

Corollary 2.3 If $q \rightarrow +\infty$ and all weights $w_k \neq 0$, then $f_w^{\text{GO}}(A_1, A_2, \dots, A_n) = B_1 = \max_{1 \leq j \leq n} \{A_j\}$, i.e., the intuitionistic fuzzy GOWA operator f_w^{GO} degenerates to the intuitionistic fuzzy max operator.

Example 2.4 Let the four intuitionistic fuzzy sets and the position weight vector be given as in Example 2.2. Compute the comprehensive evaluation of the four

experts on the decision alternative through using the intuitionistic fuzzy GOWA operator.

Solving It is easy to see from Example 2.2 that $A_1 > A_4 > A_2 > A_3$. Hence, we have:

$$\begin{aligned} B_1 &= A_1 = \langle 0.5, 0.1 \rangle, \\ B_2 &= A_4 = \langle 0.3, 0.2 \rangle, \\ B_3 &= A_2 = \langle 0.1, 0.2 \rangle \end{aligned}$$

and

$$B_4 = A_3 = \langle 0.2, 0.4 \rangle.$$

Using Eq. (2.9), we obtain:

$$\begin{aligned} f_w^{\text{GO}}(A_1, A_2, A_3, A_4) &= \left\langle \sqrt[q]{1 - (1 - 0.5^q)^{0.155} (1 - 0.3^q)^{0.345} (1 - 0.1^q)^{0.345} (1 - 0.2^q)^{0.155}}, \right. \\ &\quad \left. 1 - \sqrt[q]{1 - [1 - (1 - 0.1)^q]^{0.155} [1 - (1 - 0.2)^q]^{0.345} [1 - (1 - 0.2)^q]^{0.345} [1 - (1 - 0.4)^q]^{0.155}} \right\rangle \\ &= \left\langle \sqrt[q]{1 - (1 - 0.5^q)^{0.155} (1 - 0.3^q)^{0.345} (1 - 0.1^q)^{0.345} (1 - 0.2^q)^{0.155}}, \right. \\ &\quad \left. 1 - \sqrt[q]{1 - (1 - 0.9^q)^{0.155} (1 - 0.8^q)^{0.345} (1 - 0.8^q)^{0.345} (1 - 0.6^q)^{0.155}} \right\rangle. \end{aligned} \quad (2.10)$$

For some specific values of the parameter q , corresponding aggregation results (i.e., comprehensive evaluations of the four experts on the decision alternative) can be obtained. For example, taking $q = 2$, it follows from Eq. (2.10) that

$$\begin{aligned} f_w^{\text{GO}}(A_1, A_2, A_3, A_4) &= \left\langle \sqrt[2]{1 - (1 - 0.5^2)^{0.155} (1 - 0.3^2)^{0.345} (1 - 0.1^2)^{0.345} (1 - 0.2^2)^{0.155}}, \right. \\ &\quad \left. 1 - \sqrt[2]{1 - (1 - 0.9^2)^{0.155} (1 - 0.8^2)^{0.345} (1 - 0.8^2)^{0.345} (1 - 0.6^2)^{0.155}} \right\rangle \\ &= \langle 0.2886, 0.1978 \rangle, \end{aligned}$$

which means that the satisfaction degree and dissatisfaction degree of the four experts on the decision alternative are 0.2886 and 0.1978, respectively, and hereby the hesitancy degree is 0.5136.

In the same way, taking $q = 1$, it is directly derived from Corollary 2.2 and Example 2.2 that

$$f_w^{\text{GO}}(A_1, A_2, A_3, A_4) = f_w^{\text{O}}(A_1, A_2, A_3, A_4) = \langle 0.2603, 0.20 \rangle,$$

which means that the satisfaction degree and dissatisfaction degree of the four experts on the decision alternative are 0.2603 and 0.20, respectively, and hereby the hesitancy degree is 0.5379.

When $q \rightarrow +\infty$, it is easy to see from Corollary 2.3 that

$$f_w^{\text{GO}}(A_1, A_2, A_3, A_4) = B_1 = A_1 = \langle 0.5, 0.1 \rangle,$$

which means that the satisfaction degree and dissatisfaction degree of the four experts on the decision alternative are 0.5 and 0.1, respectively, and hereby the hesitancy degree is 0.4. In fact, the comprehensive evaluation of the four experts on the decision alternative is just the evaluation of Expert 1 who expressed his/her opinion with the intuitionistic fuzzy set A_1 .

When $q \rightarrow 0$, it follows from Corollary 2.1 that

$$\begin{aligned} f_w^{\text{GO}}(A_1, A_2, A_3, A_4) &= \left\langle 0.5^{0.155} \times 0.3^{0.345} \times 0.1^{0.345} \times 0.2^{0.155}, 1 - (1 - 0.1)^{0.155} \right. \\ &\quad \left. \times (1 - 0.2)^{0.345} \times (1 - 0.2)^{0.345} (1 - 0.4)^{0.155} \right\rangle \\ &= \left\langle 0.5^{0.155} \times 0.3^{0.345} \times 0.1^{0.345} \times 0.2^{0.155}, 1 - 0.9^{0.155} \right. \\ &\quad \left. \times 0.8^{0.345} \times 0.8^{0.345} \times 0.6^{0.155} \right\rangle \\ &= \langle 0.8981 \times 0.6601 \times 0.4519 \times 0.7792, 1 - 0.9838 \times 0.9259 \\ &\quad \times 0.9259 \times 0.9239 \rangle \\ &= \langle 0.2087, 0.2208 \rangle, \end{aligned}$$

which implies that the satisfaction degree and dissatisfaction degree of the four experts on the decision alternative are 0.2087 and 0.2208, respectively, and hereby the hesitancy degree is 0.5705.

Analogously, we can define the following intuitionistic fuzzy generalized hybrid weighted averaging operator.

Definition 2.5 Let $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, \dots, n$) be intuitionistic fuzzy sets. A mapping $f_{\omega, w}^{\text{GH}} : F^n \rightarrow F$ is called an intuitionistic fuzzy generalized hybrid weighted averaging operator if it satisfies

$$f_{\omega, w}^{\text{GH}}(A_1, A_2, \dots, A_n) = \sqrt[q]{\sum_{k=1}^n w_k \hat{B}_k^q}, \quad (2.11)$$

where $w = (w_1, w_2, \dots, w_n)^T$ is a (position) weight vector associated with the mapping $f_{\omega, w}^{\text{GH}}$; $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is a weight vector of the intuitionistic fuzzy sets A_j ($j = 1, 2, \dots, n$); $\hat{B}_k$ is the k -th largest of the n intuitionistic fuzzy sets $\hat{A}_j = n\omega_j A_j$ ($j = 1, 2, \dots, n$), which is determined through using some ranking method such as the above scoring function ranking method; $q > 0$ is a control parameter.

In particular, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the intuitionistic fuzzy generalized hybrid weighted averaging operator $f_{\omega, w}^{\text{GH}}$ degenerates to the intuitionistic fuzzy GOWA operator f_w^{GO} .

Obviously, the intuitionistic fuzzy generalized hybrid weighted averaging operator $f_{\omega, w}^{\text{GH}}$ takes into consideration not only importance of the aggregated

intuitionistic fuzzy sets themselves but also position importance of the ranking order of the aggregated intuitionistic fuzzy sets.

Theorem 2.5 Assume that $A_j = \langle \mu_j, \nu_j \rangle$ ($j = 1, 2, \dots, n$) are intuitionistic fuzzy sets. Then, the aggregation result through using the intuitionistic fuzzy generalized hybrid weighted averaging operator $f_{\omega, w}^{GH}$ [i.e., Eq. (2.11)] is an intuitionistic fuzzy set and

$$f_{\omega, w}^{GH}(A_1, A_2, \dots, A_n) = \left\langle \sqrt[q]{1 - \prod_{k=1}^n (1 - \hat{\mu}_k^q)^{w_k}}, 1 - \sqrt[q]{1 - \prod_{k=1}^n [1 - (1 - \hat{\nu}_k)^q]^{w_k}} \right\rangle, \quad (2.12)$$

where $\hat{B}_k = \langle \hat{\mu}_k, \hat{\nu}_k \rangle$ is the k -th largest of the n intuitionistic fuzzy sets $\hat{A}_j = n\omega_j A_j$ ($j = 1, 2, \dots, n$), which is determined through using some ranking method of intuitionistic fuzzy sets.

Proof Theorem 2.5 can be proven in a similar way to that of Theorem 2.4 (omitted).

The following several useful conclusions are easily drawn from Theorem 2.5.

Corollary 2.4 If $q \rightarrow 0$, then $f_{\omega, w}^{GH}(A_1, A_2, \dots, A_n) = \prod_{k=1}^n \hat{B}_k^{w_k} = \langle \prod_{k=1}^n \hat{\mu}_k^{w_k}, 1 - \prod_{k=1}^n (1 - \hat{\nu}_k)^{w_k} \rangle$, i.e., the intuitionistic fuzzy generalized hybrid weighted averaging operator $f_{\omega, w}^{GH}$ degenerates to the intuitionistic fuzzy hybrid weighted geometric operator.

Corollary 2.5 If $q = 1$, then $f_{\omega, w}^{GH}(A_1, A_2, \dots, A_n) = \sum_{k=1}^n w_k \hat{B}_k$, i.e., the intuitionistic fuzzy generalized hybrid weighted averaging operator $f_{\omega, w}^{GH}$ degenerates to the intuitionistic fuzzy hybrid weighted averaging operator $f_{\omega, w}^H$.

Corollary 2.6 If $q \rightarrow +\infty$ and all weights $w_k \neq 0$ ($k = 1, 2, \dots, n$), then $f_{\omega, w}^{GH}(A_1, A_2, \dots, A_n) = \hat{B}_1 = \max_{1 \leq j \leq n} \{\hat{A}_j\}$, i.e., the intuitionistic fuzzy generalized hybrid weighted averaging operator $f_{\omega, w}^{GH}$ degenerates to the intuitionistic fuzzy weighted max operator.

Example 2.5 Let the five intuitionistic fuzzy sets and the position weight vector as well as the weight vector of the five experts be given as in Example 2.3. Compute the comprehensive evaluation of the five experts on the decision alternative through using the intuitionistic fuzzy generalized hybrid weighted averaging operator.

Solving It easily follows from Example 2.3 that

$$\begin{aligned}
\widehat{B}_1 &= \widehat{A}_2 = \langle 0.7, 0.1 \rangle, \\
\widehat{B}_2 &= \widehat{A}_5 = \langle 0.6350, 0.1703 \rangle, \\
\widehat{B}_3 &= \widehat{A}_3 = \langle 0.4054, 0.2991 \rangle, \\
\widehat{B}_4 &= \widehat{A}_4 = \langle 0.2746, 0.4384 \rangle
\end{aligned}$$

and

$$\widehat{B}_5 = \widehat{A}_1 = \langle 0.2434, 0.4204 \rangle.$$

Using Eq. (2.12), we obtain:

$$\begin{aligned}
f_{\omega, w}^{GH}(A_1, A_2, A_3, A_4, A_5) &= \left\langle \sqrt[q]{\frac{1 - (1 - 0.7^q)^{0.112} (1 - 0.6350^q)^{0.236} (1 - 0.4054^q)^{0.304} (1 - 0.2746^q)^{0.236} (1 - 0.2434^q)^{0.112}}{1 - \sqrt[q]{1 - [1 - (1 - 0.1)^q]^{0.112} [1 - (1 - 0.1703)^q]^{0.236} [1 - (1 - 0.2991)^q]^{0.304} [1 - (1 - 0.4384)^q]^{0.236} [1 - (1 - 0.4204)^q]^{0.112}}}} \right\rangle \\
&= \left\langle \sqrt[q]{\frac{1 - (1 - 0.7^q)^{0.112} (1 - 0.6350^q)^{0.236} (1 - 0.4054^q)^{0.304} (1 - 0.2746^q)^{0.236} (1 - 0.2434^q)^{0.112}}{1 - \sqrt[q]{1 - (1 - 0.9^q)^{0.112} (1 - 0.8297^q)^{0.236} (1 - 0.7009^q)^{0.304} (1 - 0.5616^q)^{0.236} (1 - 0.5796^q)^{0.112}}}} \right\rangle.
\end{aligned} \tag{2.13}$$

For some specific values of the parameter q , we can compute the corresponding aggregation results. For instance, when $q = 2$, it is derived from Eq. (2.13) that

$$\begin{aligned}
f_{\omega, w}^{GH}(A_1, A_2, A_3, A_4, A_5) &= \left\langle \sqrt[2]{\frac{1 - (1 - 0.7^2)^{0.112} (1 - 0.6350^2)^{0.236} (1 - 0.4054^2)^{0.304} (1 - 0.2746^2)^{0.236} (1 - 0.2434^2)^{0.112}}{1 - \sqrt[2]{1 - (1 - 0.9^2)^{0.112} (1 - 0.8297^2)^{0.236} (1 - 0.7009^2)^{0.304} (1 - 0.5616^2)^{0.236} (1 - 0.5796^2)^{0.112}}}} \right\rangle \\
&= \left\langle \sqrt{1 - 0.9274 \times 0.8853 \times 0.9469 \times 0.9817 \times 0.9932}, \right. \\
&\quad \left. 1 - \sqrt{1 - 0.8303 \times 0.7594 \times 0.8143 \times 0.9145 \times 0.9552} \right\rangle \\
&= \langle 0.4919, 0.2574 \rangle,
\end{aligned}$$

which means that the satisfaction degree and dissatisfaction degree of the five experts on the decision alternative are 0.4919 and 0.2574, respectively, and hereby the hesitancy degree is 0.2507.

Analogously, when $q = 1$, it is easy to follow from Corollary 2.5 and Example 2.3 that

$$\begin{aligned}
f_{\omega, w}^{GH}(A_1, A_2, A_3, A_4, A_5) &= f_{\omega, w}^H(A_1, A_2, A_3, A_4, A_5) \\
&= \langle 0.4716, 0.2634 \rangle,
\end{aligned}$$

which means that the satisfaction degree and dissatisfaction degree of the five experts on the decision alternative are 0.4716 and 0.2634, respectively, and hereby the hesitancy degree is 0.2650.

When $q \rightarrow 0$, it follows from Corollary 2.4 that

$$\begin{aligned}
 f_{\omega, w}^{GH}(A_1, A_2, A_3, A_4, A_5) &= \langle 0.7^{0.112} \times 0.6350^{0.236} \times 0.4054^{0.304} \times 0.2746^{0.236} \times 0.2434^{0.112}, \\
 &\quad 1 - (1 - 0.1)^{0.112} \times (1 - 0.1703)^{0.236} \times (1 - 0.2991)^{0.304} \\
 &\quad \times (1 - 0.4384)^{0.236} \times (1 - 0.4204)^{0.112} \rangle \\
 &= \langle 0.7^{0.112} \times 0.6350^{0.236} \times 0.4054^{0.304} \times 0.2746^{0.236} \times 0.2434^{0.112}, \\
 &\quad 1 - 0.9^{0.112} \times 0.8297^{0.236} \times 0.7009^{0.304} \times 0.5616^{0.236} \times 0.5796^{0.112} \rangle \\
 &= \langle 0.9608 \times 0.8984 \times 0.7600 \times 0.7371 \times 0.8536, \\
 &\quad 1 - 0.9883 \times 0.9569 \times 0.8976 \times 0.8727 \times 0.9407 \rangle \\
 &= \langle 0.4128, 0.3031 \rangle,
 \end{aligned}$$

which implies that the satisfaction degree and dissatisfaction degree of the five experts on the decision alternative are 0.4128 and 0.3031, respectively, and hereby the hesitancy degree is 0.2841.

When $q \rightarrow +\infty$, it is obvious from Corollary 2.6 that

$$f_{\omega, w}^{GH}(A_1, A_2, A_3, A_4, A_5) = \hat{B}_1 = \hat{A}_2 = \langle 0.7, 0.1 \rangle,$$

which means that the satisfaction degree and dissatisfaction degree of the five experts on the decision alternative are 0.7 and 0.1, respectively, and hereby the hesitancy degree is 0.2. Obviously, the comprehensive evaluation of the five experts on the decision alternative only reflects the opinion of Expert 2 whose evaluation is expressed with the intuitionistic fuzzy set A_2 .

2.3 The Intuitionistic Fuzzy Generalized Hybrid Weighted Averaging Method of Multiattribute Decision-Making with Intuitionistic Fuzzy Sets

2.3.1 Formal Representation of Multiattribute Decision-Making with Intuitionistic Fuzzy Sets

Suppose that there are n alternatives x_j ($j = 1, 2, \dots, n$) evaluated with respect to m attributes o_i ($i = 1, 2, \dots, m$). The sets of the alternatives and attributes are denoted by $X = \{x_1, x_2, \dots, x_n\}$ and $O = \{o_1, o_2, \dots, o_m\}$, respectively. The rating (or evaluation) of any alternative $x_j \in X$ ($j = 1, 2, \dots, n$) on each attribute $o_i \in O$ ($i = 1, 2, \dots, m$) is expressed with an intuitionistic fuzzy set $F_{ij} = \{\langle (o_i, x_j), \mu_{ij}, \nu_{ij} \rangle\}$, where $\mu_{ij} \in [0, 1]$ and $\nu_{ij} \in [0, 1]$ are respectively the satisfaction (or

membership) degree and dissatisfaction (or nonmembership) degree of the alternative $x_j \in X$ ($j = 1, 2, \dots, n$) on the attribute $o_i \in O$ ($i = 1, 2, \dots, m$) so that they satisfy the condition: $0 \leq \mu_{ij} + v_{ij} \leq 1$. Usually, $F_{ij} = \{(\langle o_i, x_j \rangle, \mu_{ij}, v_{ij})\}$ is denoted by $F_{ij} = \langle \mu_{ij}, v_{ij} \rangle$ for short. Thus, the ratings of any alternative $x_j \in X$ on all m attributes are concisely expressed with the vector $A_j = (F_{1j}, F_{2j}, \dots, F_{mj})^T = (\langle \mu_{1j}, v_{1j} \rangle, \langle \mu_{2j}, v_{2j} \rangle, \dots, \langle \mu_{mj}, v_{mj} \rangle)^T$. A multiattribute decision-making problem with intuitionistic fuzzy sets is concisely expressed in matrix format as follows:

$$F = \begin{matrix} & \begin{matrix} x_1 & x_2 & \dots & x_n \end{matrix} \\ \begin{matrix} o_1 \\ o_2 \\ \vdots \\ o_m \end{matrix} & \begin{pmatrix} \langle \mu_{11}, v_{11} \rangle & \langle \mu_{12}, v_{12} \rangle & \dots & \langle \mu_{1n}, v_{1n} \rangle \\ \langle \mu_{21}, v_{21} \rangle & \langle \mu_{22}, v_{22} \rangle & \dots & \langle \mu_{2n}, v_{2n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mu_{m1}, v_{m1} \rangle & \langle \mu_{m2}, v_{m2} \rangle & \dots & \langle \mu_{mn}, v_{mn} \rangle \end{pmatrix} \end{matrix}, \quad (2.14)$$

denoted by $F = (\langle \mu_{ij}, v_{ij} \rangle)_{m \times n}$ for short, which is usually referred to an intuitionistic fuzzy decision matrix usually represented a multiattribute decision making problem with intuitionistic fuzzy sets.

Attributes may be of different importance. Assume that the weight of each attribute $o_i \in O$ ($i = 1, 2, \dots, m$) is ω_i , which should satisfy the normalized conditions: $\omega_i \in [0, 1]$ ($i = 1, 2, \dots, m$) and $\sum_{i=1}^m \omega_i = 1$. The weight vector of the m attributes is denoted by $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$.

2.3.2 Process of the Intuitionistic Fuzzy Generalized Hybrid Weighted Averaging Method for Multiattribute Decision-Making with Intuitionistic Fuzzy Sets and Real Example Analysis

According to the previous discussions, the algorithm and process of the intuitionistic fuzzy generalized hybrid weighted averaging method for multiattribute decision-making with intuitionistic fuzzy sets are summarized as follows.

- Step 1: Identify and determine the attributes and alternatives, denoted the sets of the attributes and alternatives by $X = \{x_1, x_2, \dots, x_n\}$ and $O = \{o_1, o_2, \dots, o_m\}$, respectively;
- Step 2: Pool the decision maker's opinion to get ratings (or evaluations) of the alternatives on the attributes, i.e., construct the intuitionistic fuzzy decision matrix $F = (F_{ij})_{m \times n} = (\langle \mu_{ij}, v_{ij} \rangle)_{m \times n}$;
- Step 3: Determine weights of the attributes through using some existing methods, i.e., construct the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$;
- Step 4: Determine the (position) weight vector $w = (w_1, w_2, \dots, w_m)^T$ associated with the mapping $f_{\omega, w}^{GH}$;

- Step 5: Compute the intuitionistic fuzzy sets $\hat{F}_{ij} = m\omega_i F_{ij}$ ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$) according to the operation (8) of Definition 1.2;
- Step 6: Generate the nonincreasing order of the m intuitionistic fuzzy sets $\hat{F}_{ij}$ ($i = 1, 2, \dots, m$) using some ranking method of intuitionistic fuzzy sets such as the aforementioned scoring function ranking method [i.e., Eqs. (2.2) and (2.3)], i.e., determine the k -th largest $\hat{B}_{kj} = \langle \hat{\mu}_{kj}, \hat{\nu}_{kj} \rangle$ ($k = 1, 2, \dots, m$) of the m intuitionistic fuzzy sets $\hat{F}_{ij}$ ($i = 1, 2, \dots, m$);
- Step 7: Compute the aggregation result (or comprehensive evaluation) of each alternative $x_j \in X$ ($j = 1, 2, \dots, n$) through using the intuitionistic fuzzy generalized hybrid weighted averaging operator $f_{\omega, w}^{\text{GH}}$ [i.e., Eq. (2.12)], namely,

$$f_{\omega, w}^{\text{GH}}(A_j) = f_{\omega, w}^{\text{GH}}(F_{1j}, F_{2j}, \dots, F_{mj}) \\ = \left\langle \sqrt[q]{1 - \prod_{k=1}^m (1 - \hat{\mu}_{kj}^q)^{w_k}}, 1 - \sqrt[q]{1 - \prod_{k=1}^m [1 - (1 - \hat{\nu}_{kj})^q]^{w_k}} \right\rangle; \quad (2.15)$$

- Step 8: Compute specific aggregation results $f_{\omega, w}^{\text{GH}}(A_j)$ of the alternatives $x_j \in X$ ($j = 1, 2, \dots, n$) according to Eq. (2.15) with the specifically adequate value of the parameter q ;
- Step 9: Generate the nonincreasing order of the n intuitionistic fuzzy sets $f_{\omega, w}^{\text{GH}}(A_j)$ ($j = 1, 2, \dots, n$) using some ranking method of intuitionistic fuzzy sets and hereby determine the best alternative and the ranking order of the alternatives x_j ($j = 1, 2, \dots, n$).

Example 2.6 An enterprise plans to seek an adequate supplier for purchasing parts of equipments. The purchasing manager in the enterprise takes into consideration the four attributes (or criteria, factors) as follows: performance (e.g., delivery, quality, price) o_1 , technology (e.g., manufacturing capability, design capability, ability to cope with technology changes) o_2 , finance (e.g., economic performance, financial stability) o_3 , organizational culture and strategy (e.g., feeling of trust, internal and external integration of suppliers, compatibility across levels and functions of the buyer and supplier) o_4 . The set of the four attributes is denoted by $O = \{o_1, o_2, o_3, o_4\}$. After firstly screening, there are four suppliers to be further evaluated and selected. The set of the four suppliers is denoted by $X = \{x_1, x_2, x_3, x_4\}$. Ratings (or evaluations) F_{0ij} of the suppliers x_j ($j = 1, 2, 3, 4$) on the above four attributes o_i ($i = 1, 2, 3, 4$) can be obtained through analyzing historical data and using some methods such as statistics and the case study. All the ratings F_{0ij} ($i = 1, 2, 3, 4$; $j = 1, 2, 3, 4$) are concisely expressed with the intuitionistic fuzzy decision matrix as follows:

$$\hat{F}_0 = (F_{0ij})_{4 \times 4}$$

	x_1	x_2	x_3	x_4
o_1	$\langle 0.5, 0.44 \rangle$	$\langle 0.46, 0.5 \rangle$	$\langle 0.6, 0.34 \rangle$	$\langle 0.56, 0.3 \rangle$
o_2	$\langle 0.45, 0.3 \rangle$	$\langle 0.36, 0.5 \rangle$	$\langle 0.56, 0.4 \rangle$	$\langle 0.72, 0.18 \rangle$
o_3	$\langle 0.38, 0.6 \rangle$	$\langle 0.42, 0.45 \rangle$	$\langle 0.5, 0.42 \rangle$	$\langle 0.6, 0.3 \rangle$
o_4	$\langle 0.35, 0.5 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.42, 0.54 \rangle$	$\langle 0.5, 0.35 \rangle$

Determine the best supplier and the ranking order of the four suppliers for the enterprise through using the intuitionistic fuzzy generalized hybrid weighted averaging method of multiattribute decision-making with intuitionistic fuzzy sets.

Solving According to the ordered chain pairwise comparison method of weight determination [19], the weight vector of the attributes is obtained as follows:

$$\omega = (0.35, 0.3, 0.25, 0.10)^T.$$

Using the weight determination based on the normal distribution, we determine the position weight vector as follows:

$$w = (0.24, 0.26, 0.26, 0.24)^T.$$

According to the operation (8) of Definition 1.2 and the intuitionistic fuzzy decision matrix F_0 , we can obtain the intuitionistic fuzzy set $\hat{F}_{011}$ as follows:

$$\begin{aligned} \hat{F}_{011} &= 4 \times 0.35 \langle 0.5, 0.44 \rangle \\ &= \langle 1 - (1 - 0.5)^{4 \times 0.35}, 0.44^{4 \times 0.35} \rangle \\ &= \langle 1 - 0.5^{1.4}, 0.44^{1.4} \rangle \\ &= \langle 0.6211, 0.3168 \rangle. \end{aligned}$$

Likewise, we can obtain other intuitionistic fuzzy sets $\hat{F}_{0ij} = m\omega_i F_{0ij}$ ($i = 1, 2, 3, 4; j = 1, 2, 3, 4$), which are given by the intuitionistic fuzzy decision matrix $\hat{F}_0 = (\hat{F}_{0ij})_{4 \times 4}$ as follows:

$$\hat{F}_0 =$$

	x_1	x_2	x_3	x_4
o_1	$\langle 1 - 0.5^{1.4}, 0.44^{1.4} \rangle$	$\langle 1 - 0.54^{1.4}, 0.5^{1.4} \rangle$	$\langle 1 - 0.4^{1.4}, 0.34^{1.4} \rangle$	$\langle 1 - 0.44^{1.4}, 0.3^{1.4} \rangle$
o_2	$\langle 1 - 0.55^{1.2}, 0.3^{1.2} \rangle$	$\langle 1 - 0.64^{1.2}, 0.5^{1.2} \rangle$	$\langle 1 - 0.44^{1.2}, 0.4^{1.2} \rangle$	$\langle 1 - 0.28^{1.2}, 0.18^{1.2} \rangle$
o_3	$\langle 1 - (1 - 0.38), 0.6 \rangle$	$\langle 1 - (1 - 0.42), 0.45 \rangle$	$\langle 1 - (1 - 0.5), 0.42 \rangle$	$\langle 1 - (1 - 0.6), 0.3 \rangle$
o_4	$\langle 1 - 0.65^{0.4}, 0.5^{0.4} \rangle$	$\langle 1 - 0.3^{0.4}, 0.2^{0.4} \rangle$	$\langle 1 - 0.58^{0.4}, 0.54^{0.4} \rangle$	$\langle 1 - 0.5^{0.4}, 0.35^{0.4} \rangle$

i.e.,

$$\hat{F}_0 = \begin{matrix} & x_1 & x_2 & x_3 & x_4 \\ \begin{matrix} o_1 \\ o_2 \\ o_3 \\ o_4 \end{matrix} & \begin{pmatrix} \langle 0.6211, 0.3168 \rangle \\ \langle 0.5120, 0.2358 \rangle \\ \langle 0.38, 0.6 \rangle \\ \langle 0.1583, 0.7579 \rangle \end{pmatrix} & \begin{pmatrix} \langle 0.5780, 0.3789 \rangle \\ \langle 0.4146, 0.4353 \rangle \\ \langle 0.42, 0.45 \rangle \\ \langle 0.3822, 0.5253 \rangle \end{pmatrix} & \begin{pmatrix} \langle 0.7227, 0.2208 \rangle \\ \langle 0.6266, 0.3330 \rangle \\ \langle 0.5, 0.42 \rangle \\ \langle 0.1958, 0.7816 \rangle \end{pmatrix} & \begin{pmatrix} \langle 0.6832, 0.1853 \rangle \\ \langle 0.7829, 0.1277 \rangle \\ \langle 0.6, 0.3 \rangle \\ \langle 0.2421, 0.6571 \rangle \end{pmatrix} \end{matrix}.$$

Using Eq. (2.2), we can obtain the scores of the intuitionistic fuzzy sets $\hat{F}_{0ij}$ of the alternatives (i.e., suppliers) x_j ($j = 1, 2, 3, 4$) on the four attributes o_i ($i = 1, 2, 3, 4$) as follows:

$$\begin{aligned} M(\hat{F}_{011}) &= 0.6211 - 0.3168 = 0.3043, \\ M(\hat{F}_{021}) &= 0.5120 - 0.2358 = 0.2762, \\ M(\hat{F}_{031}) &= 0.38 - 0.6 = -0.22, \\ M(\hat{F}_{041}) &= 0.1583 - 0.7579 = -0.5996, \\ M(\hat{F}_{012}) &= 0.5780 - 0.3789 = 0.1991, \\ M(\hat{F}_{022}) &= 0.4146 - 0.4353 = -0.0207, \\ M(\hat{F}_{032}) &= 0.42 - 0.45 = -0.03, \\ M(\hat{F}_{042}) &= 0.3822 - 0.5253 = -0.1431, \\ M(\hat{F}_{013}) &= 0.7227 - 0.2208 = 0.5019, \\ M(\hat{F}_{023}) &= 0.6266 - 0.3330 = 0.2936, \\ M(\hat{F}_{033}) &= 0.5 - 0.42 = 0.08, \\ M(\hat{F}_{043}) &= 0.1958 - 0.7816 = -0.5858, \\ M(\hat{F}_{014}) &= 0.6832 - 0.1853 = 0.4979, \\ M(\hat{F}_{024}) &= 0.7829 - 0.1277 = 0.6552, \\ M(\hat{F}_{034}) &= 0.6 - 0.3 = 0.3, \end{aligned}$$

and

$$M(\hat{F}_{044}) = 0.2421 - 0.6571 = -0.4150,$$

respectively. Thereby, according to the above scoring function ranking method of intuitionistic fuzzy sets, it follows that

$$\begin{aligned} \hat{F}_{011} &> \hat{F}_{021} > \hat{F}_{031} > \hat{F}_{041}, \\ \hat{F}_{012} &> \hat{F}_{022} > \hat{F}_{032} > \hat{F}_{042}, \\ \hat{F}_{013} &> \hat{F}_{023} > \hat{F}_{033} > \hat{F}_{043} \end{aligned}$$

and

$$\hat{F}_{024} > \hat{F}_{014} > \hat{F}_{034} > \hat{F}_{044}.$$

Thus, we have:

$$\begin{aligned}
\widehat{B}_{11} &= \widehat{F}_{011}, \widehat{B}_{21} = \widehat{F}_{021}, \widehat{B}_{31} = \widehat{F}_{031}, \widehat{B}_{41} = \widehat{F}_{041}, \\
\widehat{B}_{12} &= \widehat{F}_{012}, \widehat{B}_{22} = \widehat{F}_{022}, \widehat{B}_{32} = \widehat{F}_{032}, \widehat{B}_{42} = \widehat{F}_{042}, \\
\widehat{B}_{13} &= \widehat{F}_{013}, \widehat{B}_{23} = \widehat{F}_{023}, \widehat{B}_{33} = \widehat{F}_{033}, \widehat{B}_{43} = \widehat{F}_{043}
\end{aligned}$$

and

$$\widehat{B}_{14} = \widehat{F}_{024}, \widehat{B}_{24} = \widehat{F}_{014}, \widehat{B}_{34} = \widehat{F}_{034}, \widehat{B}_{44} = \widehat{F}_{044}.$$

Using Eq. (2.15), the comprehensive evaluations of the alternatives (i.e., suppliers) x_j ($j = 1, 2, 3, 4$) can be obtained as follows:

$$\begin{aligned}
f_{\omega, w}^{GH}(A_1) &= f_{\omega, w}^{GH}(F_{11}, F_{21}, F_{31}, F_{41}) = \left\langle \sqrt[q]{1 - \prod_{k=1}^4 (1 - \widehat{\mu}_{k1}^q)^{w_k}}, 1 - \sqrt[q]{1 - \prod_{k=1}^4 [1 - (1 - \widehat{\nu}_{k1})^q]^{w_k}} \right\rangle \\
&= \left\langle \sqrt[q]{1 - (1 - 0.6211q)^{0.24} (1 - 0.5120q)^{0.26} (1 - 0.38q)^{0.26} (1 - 0.1583q)^{0.24}}, \right. \\
&\quad \left. 1 - \sqrt[q]{1 - [1 - (1 - 0.3168)q]^{0.24} [1 - (1 - 0.2358)q]^{0.26} [1 - (1 - 0.6)q]^{0.26} [1 - (1 - 0.7579)q]^{0.24}} \right\rangle,
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
f_{\omega, w}^{GH}(A_2) &= f_{\omega, w}^{GH}(F_{12}, F_{22}, F_{32}, F_{42}) = \left\langle \sqrt[q]{1 - \prod_{k=1}^4 (1 - \widehat{\mu}_{k2}^q)^{w_k}}, 1 - \sqrt[q]{1 - \prod_{k=1}^4 [1 - (1 - \widehat{\nu}_{k2})^q]^{w_k}} \right\rangle \\
&= \left\langle \sqrt[q]{1 - (1 - 0.5780q)^{0.24} (1 - 0.4146q)^{0.26} (1 - 0.42q)^{0.26} (1 - 0.3822q)^{0.24}}, \right. \\
&\quad \left. 1 - \sqrt[q]{1 - [1 - (1 - 0.3789)q]^{0.24} [1 - (1 - 0.4353)q]^{0.26} [1 - (1 - 0.45)q]^{0.26} [1 - (1 - 0.5253)q]^{0.24}} \right\rangle,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
f_{\omega, w}^{GH}(A_3) &= f_{\omega, w}^{GH}(F_{13}, F_{23}, F_{33}, F_{43}) = \left\langle \sqrt[q]{1 - \prod_{k=1}^4 (1 - \widehat{\mu}_{k3}^q)^{w_k}}, 1 - \sqrt[q]{1 - \prod_{k=1}^4 [1 - (1 - \widehat{\nu}_{k3})^q]^{w_k}} \right\rangle \\
&= \left\langle \sqrt[q]{1 - (1 - 0.7227q)^{0.24} (1 - 0.6266q)^{0.26} (1 - 0.5q)^{0.26} (1 - 0.1958q)^{0.24}}, \right. \\
&\quad \left. 1 - \sqrt[q]{1 - [1 - (1 - 0.2208)q]^{0.24} [1 - (1 - 0.3330)q]^{0.26} [1 - (1 - 0.42)q]^{0.26} [1 - (1 - 0.7816)q]^{0.24}} \right\rangle
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
f_{\omega, w}^{GH}(A_4) &= f_{\omega, w}^{GH}(F_{14}, F_{24}, F_{34}, F_{44}) = \left\langle \sqrt[q]{1 - \prod_{k=1}^4 (1 - \widehat{\mu}_{k4}^q)^{w_k}}, 1 - \sqrt[q]{1 - \prod_{k=1}^4 [1 - (1 - \widehat{\nu}_{k4})^q]^{w_k}} \right\rangle \\
&= \left\langle \sqrt[q]{1 - (1 - 0.7829q)^{0.24} (1 - 0.6832)q^{0.26} (1 - 0.6q)^{0.26} (1 - 0.2421q)^{0.24}}, \right. \\
&\quad \left. 1 - \sqrt[q]{1 - [1 - (1 - 0.1277)q]^{0.24} [1 - (1 - 0.1853)q]^{0.26} [1 - (1 - 0.3)q]^{0.26} [1 - (1 - 0.6571)q]^{0.24}} \right\rangle,
\end{aligned} \tag{2.19}$$

respectively.

Taking $q = 2$, it is easily derived from Eqs. (2.16)–(2.19) that

$$\begin{aligned} f_{\omega, w}^{\text{GH}}(A_1) &= \left\langle \sqrt{1 - (1 - 0.6211^2)^{0.24}(1 - 0.5120^2)^{0.26}(1 - 0.38^2)^{0.26}(1 - 0.1583^2)^{0.24}} 1 -, \right. \\ &\quad \left. \sqrt{1 - [1 - (1 - 0.3168)^2]^{0.24}[1 - (1 - 0.2358)^2]^{0.26}[1 - (1 - 0.6)^2]^{0.26}[1 - (1 - 0.7579)^2]^{0.24}} \right\rangle \\ &= \langle 0.4642, 0.4040 \rangle, \end{aligned}$$

$$\begin{aligned} f_{\omega, w}^{\text{GH}}(A_2) &= \left\langle \sqrt{1 - (1 - 0.5780^2)^{0.24}(1 - 0.4146^2)^{0.26}(1 - 0.42^2)^{0.26}(1 - 0.3822^2)^{0.24}}, \right. \\ &\quad \left. 1 - \sqrt{1 - [1 - (1 - 0.3789)^2]^{0.24}[1 - (1 - 0.4353)^2]^{0.26}[1 - (1 - 0.45)^2]^{0.26}[1 - (1 - 0.5253)^2]^{0.24}} \right\rangle \\ &= \langle 0.4577, 0.4428 \rangle, \end{aligned}$$

$$\begin{aligned} f_{\omega, w}^{\text{GH}}(A_3) &= \left\langle \sqrt{1 - (1 - 0.7227^2)^{0.24}(1 - 0.6266^2)^{0.26}(1 - 0.5^2)^{0.26}(1 - 0.1958^2)^{0.24}}, \right. \\ &\quad \left. 1 - \sqrt{1 - [1 - (1 - 0.2208)^2]^{0.24}[1 - (1 - 0.3330)^2]^{0.26}[1 - (1 - 0.42)^2]^{0.26}[1 - (1 - 0.7816)^2]^{0.24}} \right\rangle \\ &= \langle 0.5690, 0.3748 \rangle \end{aligned}$$

and

$$\begin{aligned} f_{\omega, w}^{\text{GH}}(A_4) &= \left\langle \sqrt{1 - (1 - 0.7829^2)^{0.24}(1 - 0.6832^2)^{0.26}(1 - 0.6^2)^{0.26}(1 - 0.2421^2)^{0.24}}, \right. \\ &\quad \left. 1 - \sqrt{1 - [1 - (1 - 0.1277)^2]^{0.24}[1 - (1 - 0.1853)^2]^{0.26}[1 - (1 - 0.3)^2]^{0.26}[1 - (1 - 0.6571)^2]^{0.24}} \right\rangle \\ &= \langle 0.6376, 0.2485 \rangle, \end{aligned}$$

respectively.

According to Eq. (2.2), the scores of $f_{\omega, w}^{\text{GH}}(A_j)$ ($j = 1, 2, 3, 4$) can be obtained as follows:

$$M(f_{\omega, w}^{\text{GH}}(A_1)) = 0.4642 - 0.4040 = 0.0602,$$

$$M(f_{\omega, w}^{\text{GH}}(A_2)) = 0.4577 - 0.4428 = 0.0149,$$

$$M(f_{\omega, w}^{\text{GH}}(A_3)) = 0.5690 - 0.3748 = 0.1942$$

and

$$M(f_{\omega, w}^{\text{GH}}(A_4)) = 0.6376 - 0.2485 = 0.3891,$$

respectively. It is obvious that

$$M(f_{\omega, w}^{\text{GH}}(A_4)) > M(f_{\omega, w}^{\text{GH}}(A_3)) > M(f_{\omega, w}^{\text{GH}}(A_1)) > M(f_{\omega, w}^{\text{GH}}(A_2)).$$

Therefore, according to the scoring function ranking method of intuitionistic fuzzy sets, the ranking order of the suppliers x_j ($j = 1, 2, 3, 4$) is generated as follows:

$x_4 \succ x_3 \succ x_1 \succ x_2$. The best supplier for the enterprise is x_4 .

In a similar way, taking $q = 1$, it easily follows from Corollary 2.5 that $f_{\omega,w}^{\text{GH}}(\mathbf{A}_1) = f_{\omega,w}^{\text{H}}(\mathbf{A}_1)$, $f_{\omega,w}^{\text{GH}}(\mathbf{A}_2) = f_{\omega,w}^{\text{H}}(\mathbf{A}_2)$, $f_{\omega,w}^{\text{GH}}(\mathbf{A}_3) = f_{\omega,w}^{\text{H}}(\mathbf{A}_3)$, and $f_{\omega,w}^{\text{GH}}(\mathbf{A}_4) = f_{\omega,w}^{\text{H}}(\mathbf{A}_4)$. According to Eq. (2.7), we can obtain $f_{\omega,w}^{\text{GH}}(\mathbf{A}_j)$ ($j = 1, 2, 3, 4$) as follows:

$$\begin{aligned} f_{\omega,w}^{\text{GH}}(\mathbf{A}_1) &= f_{\omega,w}^{\text{H}}(F_{11}, F_{21}, F_{31}, F_{41}) = \left\langle 1 - \prod_{k=1}^4 (1 - \widehat{\mu}_{k1})^{w_k}, \prod_{k=1}^4 (\widehat{\nu}_{k1})^{w_k} \right\rangle \\ &= \left\langle 1 - (1 - 0.6211)^{0.24} (1 - 0.5120)^{0.26} (1 - 0.38)^{0.26} (1 - 0.1583)^{0.24}, \right. \\ &\quad \left. 0.3168^{0.24} \times 0.2358^{0.26} \times 0.6^{0.26} \times 0.7579^{0.24} \right\rangle \\ &= \langle 1 - 0.7922 \times 0.8298 \times 0.8831 \times 0.9595, 0.7589 \times 0.6868 \\ &\quad \times 0.8756 \times 0.9356 \rangle \\ &= \langle 0.4430, 0.4270 \rangle, \end{aligned}$$

$$\begin{aligned} f_{\omega,w}^{\text{GH}}(\mathbf{A}_2) &= f_{\omega,w}^{\text{H}}(F_{12}, F_{22}, F_{32}, F_{42}) = \left\langle 1 - \prod_{k=1}^4 (1 - \widehat{\mu}_{k2})^{w_k}, \prod_{k=1}^4 (\widehat{\nu}_{k2})^{w_k} \right\rangle \\ &= \left\langle 1 - (1 - 0.5780)^{0.24} (1 - 0.4146)^{0.26} (1 - 0.42)^{0.26} (1 - 0.3822)^{0.24}, \right. \\ &\quad \left. 0.3789^{0.24} \times 0.4353^{0.26} \times 0.45^{0.26} \times 0.5253^{0.24} \right\rangle \\ &= \langle 1 - 0.8130 \times 0.8700 \times 0.8679 \times 0.8908, 0.7922 \times 0.8055 \\ &\quad \times 0.8125 \times 0.8568 \rangle \\ &= \langle 0.4532, 0.4442 \rangle, \end{aligned}$$

$$\begin{aligned} f_{\omega,w}^{\text{GH}}(\mathbf{A}_3) &= f_{\omega,w}^{\text{H}}(F_{13}, F_{23}, F_{33}, F_{43}) = \left\langle 1 - \prod_{k=1}^4 (1 - \widehat{\mu}_{k3})^{w_k}, \prod_{k=1}^4 (\widehat{\nu}_{k3})^{w_k} \right\rangle \\ &= \left\langle 1 - (1 - 0.7227)^{0.24} (1 - 0.6266)^{0.26} (1 - 0.5)^{0.26} (1 - 0.1958)^{0.24}, \right. \\ &\quad \left. 0.2208^{0.24} \times 0.3330^{0.26} \times 0.42^{0.26} \times 0.7816^{0.24} \right\rangle \\ &= \langle 1 - 0.7350 \times 0.7740 \times 0.8351 \times 0.9490, 0.6959 \times 0.7513 \\ &\quad \times 0.7981 \times 0.9426 \rangle \\ &= \langle 0.5491, 0.3933 \rangle \end{aligned}$$

and

$$\begin{aligned}
f_{\omega,w}^{\text{GH}}(\mathbf{A}_4) &= f_{\omega,w}^{\text{H}}(F_{14}, F_{24}, F_{34}, F_{44}) = \left\langle 1 - \prod_{k=1}^4 (1 - \widehat{\mu}_{k4})^{w_k}, \prod_{k=1}^4 (\widehat{\nu}_{k4})^{w_k} \right\rangle \\
&= \left\langle 1 - (1 - 0.7829)^{0.24} (1 - 0.6832)^{0.26} (1 - 0.6)^{0.26} (1 - 0.2421)^{0.24}, \right. \\
&\quad \left. 0.1277^{0.24} \times 0.1853^{0.26} \times 0.3^{0.26} \times 0.6571^{0.24} \right\rangle \\
&= \langle 1 - 0.6931 \times 0.7417 \times 0.7880 \times 0.9356, 0.6102 \times 0.6451 \times 0.7312 \times 0.9041 \rangle \\
&= \langle 0.6210, 0.2602 \rangle,
\end{aligned}$$

respectively.

Using Eq. (2.2), we can obtain the scores of $f_{\omega,w}^{\text{GH}}(\mathbf{A}_j)$ ($j = 1, 2, 3, 4$) as follows:

$$\begin{aligned}
M(f_{\omega,w}^{\text{GH}}(\mathbf{A}_1)) &= 0.4430 - 0.4270 = 0.016, \\
M(f_{\omega,w}^{\text{GH}}(\mathbf{A}_2)) &= 0.4532 - 0.4442 = 0.009, \\
M(f_{\omega,w}^{\text{GH}}(\mathbf{A}_3)) &= 0.5491 - 0.3933 = 0.1558
\end{aligned}$$

and

$$M(f_{\omega,w}^{\text{GH}}(\mathbf{A}_4)) = 0.6210 - 0.2602 = 0.3608,$$

respectively. Obviously, we have

$$M(f_{\omega,w}^{\text{GH}}(\mathbf{A}_4)) > M(f_{\omega,w}^{\text{GH}}(\mathbf{A}_3)) > M(f_{\omega,w}^{\text{GH}}(\mathbf{A}_1)) > M(f_{\omega,w}^{\text{GH}}(\mathbf{A}_2)).$$

Thereby, according to the scoring function ranking method of intuitionistic fuzzy sets, the ranking order of the suppliers x_j ($j = 1, 2, 3, 4$) is generated as follows: $x_4 \succ x_3 \succ x_1 \succ x_2$. The best supplier for the enterprise is x_4 .

When $q \rightarrow 0$, it follows from Corollary 2.4 that

$$\begin{aligned}
f_{\omega,w}^{\text{GH}}(\mathbf{A}_1) &= f_{\omega,w}^{\text{GH}}(F_{11}, F_{21}, F_{31}, F_{41}) = \prod_{k=1}^4 \widehat{B}_{k1}^{w_k} = \left\langle \prod_{k=1}^4 \widehat{\mu}_{k1}^{w_k}, 1 - \prod_{k=1}^4 (1 - \widehat{\nu}_{k1})^{w_k} \right\rangle \\
&= \left\langle 0.6211^{0.24} \times 0.5120^{0.26} \times 0.38^{0.26} \times 0.1583^{0.24}, 1 - (1 - 0.3168)^{0.24} \right. \\
&\quad \left. \times (1 - 0.2358)^{0.26} \times (1 - 0.6)^{0.26} \times (1 - 0.7579)^{0.24} \right\rangle \\
&= \langle 0.8920 \times 0.8403 \times 0.7776 \times 0.6425, 1 - 0.9126 \times 0.9325 \times 0.7880 \times 0.7115 \rangle \\
&= \langle 0.3745, 0.5229 \rangle,
\end{aligned}$$

$$\begin{aligned}
f_{\omega,w}^{\text{GH}}(A_2) &= f_{\omega,w}^{\text{GH}}(F_{12}, F_{22}, F_{32}, F_{42}) = \prod_{k=1}^4 \widehat{B}_{k2}^{w_k} = \left\langle \prod_{k=1}^4 \widehat{\mu}_{k2}^{w_k}, 1 - \prod_{k=1}^4 (1 - \widehat{\nu}_{k2})^{w_k} \right\rangle \\
&= \left\langle 0.5780^{0.24} \times 0.4146^{0.26} \times 0.42^{0.26} \times 0.3822^{0.24}, 1 - (1 - 0.3789)^{0.24} \right. \\
&\quad \left. \times (1 - 0.4353)^{0.26} \times (1 - 0.45)^{0.26} \times (1 - 0.5253)^{0.24} \right\rangle \\
&= \langle 0.8767 \times 0.7954 \times 0.7981 \times 0.7939, 1 - 0.8920 \times 0.8619 \times 0.8560 \times 0.8363 \rangle \\
&= \langle 0.4418, 0.4496 \rangle,
\end{aligned}$$

$$\begin{aligned}
f_{\omega,w}^{\text{GH}}(A_3) &= f_{\omega,w}^{\text{GH}}(F_{13}, F_{23}, F_{33}, F_{43}) = \prod_{k=1}^4 \widehat{B}_{k3}^{w_k} = \left\langle \prod_{k=1}^4 \widehat{\mu}_{k3}^{w_k}, 1 - \prod_{k=1}^4 (1 - \widehat{\nu}_{k3})^{w_k} \right\rangle \\
&= \left\langle 0.7227^{0.24} \times 0.6266^{0.26} \times 0.5^{0.26} \times 0.1958^{0.24}, 1 - (1 - 0.2208)^{0.24} \right. \\
&\quad \left. \times (1 - 0.3330)^{0.26} \times (1 - 0.42)^{0.26} \times (1 - 0.7816)^{0.24} \right\rangle \\
&= \langle 0.9250 \times 0.8856 \times 0.8351 \times 0.6761, 1 - 0.9419 \times 0.9001 \times 0.8679 \times 0.6941 \rangle \\
&= \langle 0.4625, 0.4893 \rangle
\end{aligned}$$

and

$$\begin{aligned}
f_{\omega,w}^{\text{GH}}(A_4) &= f_{\omega,w}^{\text{GH}}(F_{14}, F_{24}, F_{34}, F_{44}) = \prod_{k=1}^4 \widehat{B}_{k4}^{w_k} = \left\langle \prod_{k=1}^4 \widehat{\mu}_{k4}^{w_k}, 1 - \prod_{k=1}^4 (1 - \widehat{\nu}_{k4})^{w_k} \right\rangle \\
&= \left\langle 0.7829^{0.24} \times 0.6832^{0.26} \times 0.6^{0.26} \times 0.2421^{0.24}, 1 - (1 - 0.1277)^{0.24} \right. \\
&\quad \left. \times (1 - 0.1853)^{0.26} \times (1 - 0.3)^{0.26} \times (1 - 0.6571)^{0.24} \right\rangle \\
&= \langle 0.9430 \times 0.9057 \times 0.8756 \times 0.7115, 1 - 0.9677 \times 0.9481 \times 0.9114 \times 0.7735 \rangle \\
&= \langle 0.5321, 0.3532 \rangle.
\end{aligned}$$

According to Eq. (2.2), the scores of $f_{\omega,w}^{\text{GH}}(A_j)$ ($j = 1, 2, 3, 4$) are obtained as follows:

$$\begin{aligned}
M(f_{\omega,w}^{\text{GH}}(A_1)) &= 0.3745 - 0.5229 = -0.1484, \\
M(f_{\omega,w}^{\text{GH}}(A_2)) &= 0.4418 - 0.4496 = -0.0078, \\
M(f_{\omega,w}^{\text{GH}}(A_3)) &= 0.4625 - 0.4893 = -0.0268
\end{aligned}$$

and

$$M(f_{\omega,w}^{\text{GH}}(A_4)) = 0.5321 - 0.3532 = 0.1789,$$

respectively. Thus, it is obvious that

$$M(f_{\omega,w}^{\text{GH}}(A_4)) > M(f_{\omega,w}^{\text{GH}}(A_2)) > M(f_{\omega,w}^{\text{GH}}(A_3)) > M(f_{\omega,w}^{\text{GH}}(A_1)).$$

Hence, according to the scoring function ranking method of intuitionistic fuzzy sets, the ranking order of the suppliers x_j ($j = 1, 2, 3, 4$) is generated as follows: $x_4 \succ x_2 \succ x_3 \succ x_1$. The best supplier for the enterprise is x_4 .

When $q \rightarrow +\infty$, it is easy to follow from Corollary 2.6 that

$$\begin{aligned} f_{\omega, w}^{\text{GH}}(\mathbf{A}_1) &= f_{\omega, w}^{\text{GH}}(F_{11}, F_{21}, F_{31}, F_{41}) = \widehat{B}_{11} = \widehat{F}_{011} = \langle 0.6211, 0.3168 \rangle, \\ f_{\omega, w}^{\text{GH}}(\mathbf{A}_2) &= f_{\omega, w}^{\text{GH}}(F_{12}, F_{22}, F_{32}, F_{42}) = \widehat{B}_{12} = \widehat{F}_{012} = \langle 0.5780, 0.3789 \rangle, \\ f_{\omega, w}^{\text{GH}}(\mathbf{A}_3) &= f_{\omega, w}^{\text{GH}}(F_{13}, F_{23}, F_{33}, F_{43}) = \widehat{B}_{13} = \widehat{F}_{013} = \langle 0.7227, 0.2208 \rangle \end{aligned}$$

and

$$f_{\omega, w}^{\text{GH}}(\mathbf{A}_4) = f_{\omega, w}^{\text{GH}}(F_{14}, F_{24}, F_{34}, F_{44}) = \widehat{B}_{14} = \widehat{F}_{024} = \langle 0.7829, 0.1277 \rangle.$$

Using Eq. (2.2), we can obtain the scores of $f_{\omega, w}^{\text{GH}}(\mathbf{A}_j)$ ($j = 1, 2, 3, 4$) as follows:

$$\begin{aligned} M(f_{\omega, w}^{\text{GH}}(\mathbf{A}_1)) &= 0.6211 - 0.3168 = 0.3043, \\ M(f_{\omega, w}^{\text{GH}}(\mathbf{A}_2)) &= 0.5780 - 0.3789 = 0.1991, \\ M(f_{\omega, w}^{\text{GH}}(\mathbf{A}_3)) &= 0.7227 - 0.2208 = 0.5019 \end{aligned}$$

and

$$M(f_{\omega, w}^{\text{GH}}(\mathbf{A}_4)) = 0.7829 - 0.1277 = 0.6552,$$

respectively. Evidently, we have

$$M(f_{\omega, w}^{\text{GH}}(\mathbf{A}_4)) > M(f_{\omega, w}^{\text{GH}}(\mathbf{A}_3)) > M(f_{\omega, w}^{\text{GH}}(\mathbf{A}_1)) > M(f_{\omega, w}^{\text{GH}}(\mathbf{A}_2)).$$

Thus, according to the scoring function ranking method of intuitionistic fuzzy sets, the ranking order of the suppliers x_j ($j = 1, 2, 3, 4$) is generated as follows: $x_4 \succ x_3 \succ x_1 \succ x_2$. The best supplier for the enterprise is x_4 .

From the computational results, obviously, the ranking orders of the suppliers x_j ($j = 1, 2, 3, 4$) are completely identical when the parameter $q = 1, 2$, and $q \rightarrow +\infty$, i.e., $x_4 \succ x_3 \succ x_1 \succ x_2$. Moreover, in these cases, the best suppliers are the same as x_4 . However, when $q \rightarrow 0$, the ranking order of the suppliers x_j ($j = 1, 2, 3, 4$) is different from the above those ranking orders, i.e., in this case, the ranking order of the suppliers is $x_4 \succ x_2 \succ x_3 \succ x_1$ whereas the best supplier is still x_4 . In general, different values of the parameter q may affect the ranking orders of the suppliers. Therefore, specific and adequate values of the parameter q should be chosen according to characteristics and needs in real management situations.

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2014, XXI, 444 p. 23 illus., Hardcover

ISBN: 978-3-642-40711-6