Preface

Never, before I had lectured in front of second and third years students of Peking University, did I feel that strongly how much mathematics are universal, a world where human minds think alike. And never before had I enjoyed it so much.

The course I was supposed to give (an advanced undergraduate introduction to Algebra) had no specific syllabus. I decided to let it go, pushed or pulled by the student's reactions and my own feelings. It turned out to become an amazing and delightful encounter between, on one hand, ideas and discoveries of German mathematicians from the end of the XIXth and the beginning of the XXth centuries[1] revisited and taught by some French mathematicians from the XXth century[2], and, on the other hand, young brilliant Chinese students of the XXIst century.

The pleasure of these students while discovering these concepts, results, examples, has been obvious all along the course, and even sometimes expressed loudly. Moreover, the speed of their understanding and handling notions which were mostly new to them was amazing. A couple of times, at the intermission, one of them came and politely told me that he thought he had found a more elegant proof than the one I had just given—and each time he was indeed right, his proof was better, more elegant, more natural.

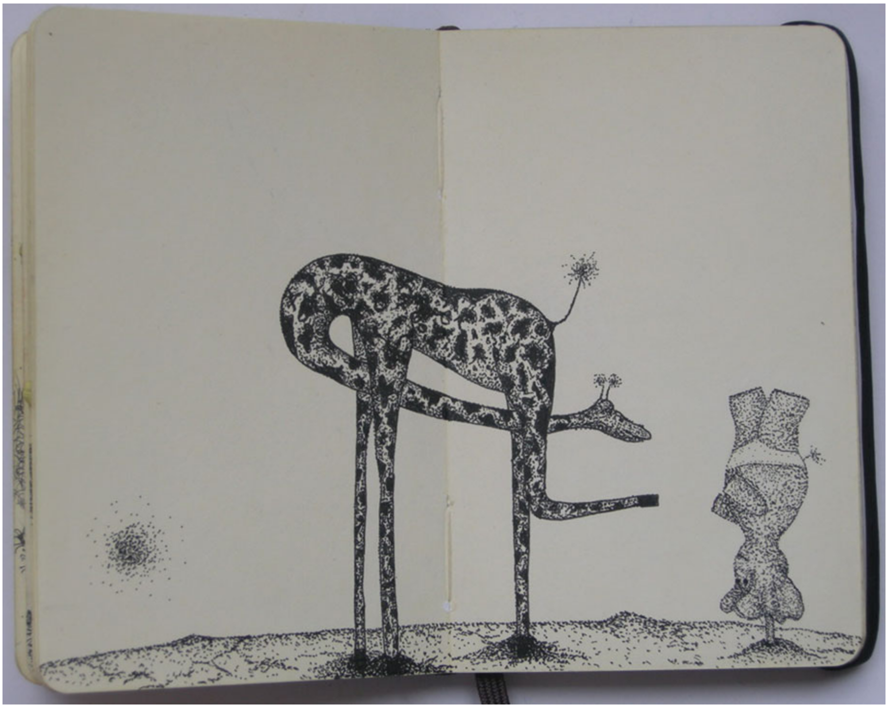
Elegant, efficient, natural, pertinent, beautiful, clever, exciting: these are words sometimes heard when a mathematician discovers a new approach, a new proof, or even a new version of an old result. Whatever country, origin, culture that mathematician may be from: what is beautiful and pertinent for a German Herr Professor of the XIXth Century is also beautiful and pertinent for a young Chinese student of 2013. Of course, universality is not the peculiarity of mathematics, it is certainly shared by most of the arts, and partly by philosophy. But the essence of the universality of mathematics is not directly connected with feelings and events of any human life, pain or joy, love or disaster, war, freedom, death or future. Besides, the

[1] Ideals were first defined by Richard Dedekind in 1876 in the third edition of his book *"Vorlesungen über Zahlentheorie"* (Lectures on Number Theory), after Ernst Kummer had introduced the concept of "ideal numbers". The notion was later expanded by David Hilbert and Emmy Noether.

[2] Like Nicolas Bourbaki.

universality of mathematics is a rule, almost a theorem: what is considered by all as good is indeed good. I do think this is one of the wonders of the world we live in.

The more elegant proofs of the students are integrated in this book, without quotation to their authors since I did not know their names. This is one of the reasons why the book is dedicated to the students of PKU.



CONVERSATION BETWEEN MATHEMATICIANS—©Anouk Grinberg

Abstract

During the Springs of 2011 and 2012, I was invited by the Beijing International Center for Mathematics Research to give an advanced undergraduate algebra course (once a week over two months each year). This is part of the Everest project of Chinese Education Ministry on first class students training.

This book has been written during and for that course. By no way does it pretend to any type of exhaustivity. It is a quick and contingent introduction to Algebra in front of an extremely pleasant and passionate audience, heterogeneous but persistent. It certainly reflects some of my own tastes, and mainly the constraints of such a short period of teaching.

A remark about the last two sections: following a well established tradition, we had planned to conclude by lecturing on the structure of finitely generated modules over principal ideal domains. But during the process of the course, after explaining that the notion of projective module is somehow more natural than the notion of free module, it became rather inevitable to replace principal ideal domains by Dedekind rings; this is less traditional in the literature—but not really more difficult.

Prerequisites

This book requires a certain familiarity with the notions of groups, rings, fields, and specially with the undergraduate knowledge of linear algebra. More specifically, let k be a commutative field. We assume the reader knows

- the definition of the ring of polynomial $k[X_1, \dots, X_n]$ in n indeterminates,
- the Euclidean division in $\mathbb{Z}$ and in $k[X]$, as well as some of the consequences, like: both these rings are principal ideal domains, hence for p a prime number and $P(X)$ an irreducible polynomial, both quotients $\mathbb{Z}/p\mathbb{Z}$ and $k[X]/(P(X))$ are fields;
- the main results of an undergraduate course on k-linear algebra;
- matrices and their determinants.

  The following identity will not be proved: let M be an $n \times n$ matrix with entries in k, let ${}^t\mathrm{Com}(M)$ denote the transpose of its matrix of cofactors, let 1_n be the identity $n \times n$ matrix; then

$$^t\mathrm{Com}(M).M = \det(M).1_n.$$

We take for granted that the reader is familiar with the standard notation $\mathbb{N}$ (for "numbers")—note that by convention $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}$ for "Zahlen"), $\mathbb{Q}$ (for "quotients"), $\mathbb{R}$ (for "reals"), $\mathbb{C}$ (for "complexes"), as well as $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (for "finite")—see [7], p. 3.

By convention, a *field* is a *commutative* ring where all nonzero elements are invertible. A noncommutative ring where all nonzero elements are invertible is called a *division ring*.

We shall also use the following notation.

- For Ω any finite set, $|\Omega|$ will denote the number of its elements.
- For any sequence $\xi_1, \dots, \xi_n$, or any product $\xi_1 \cdots \xi_n$, and for all $j = 1, \dots, n$, we set (with obvious ad hoc convention for ξ_0 and ξ_{n+1}):

$$(\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_n) := (\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_n), \quad \text{and}$$
$$\xi_1 \cdots \widehat{\xi_j} \cdots \xi_n := \xi_1 \cdots \xi_{j-1}\xi_{j+1} \cdots \xi_n.$$

A subset (subgroup, subring, submodule, ...) Ω' of a set (group, ring, module, ...) Ω is said to be *proper* if $\Omega' \neq \Omega$.

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