Chapter 2
Continuous Sets of Creation and Annihilation Operators

Abstract We define first the operators $a(\varphi)$ and $a^+(\varphi)$ on the usual Fock space. Then we exhibit a generalization of the sum-integral lemma to measures. We introduce creation and annihilation operators on locally compact spaces, and use these notions to define creation and annihilation operators localized at points.

2.1 Creation and Annihilation Operators on Fock Space

There are many ways to generalize function spaces on finite sets to function spaces on infinite sets. The usual way to generalize creation and annihilation operators employs Hilbert and Fock spaces. Assume we have a measurable space X and a measure λ on X. We consider the Hilbert space $L^2(X, \lambda)$ and a sequence of Hilbert spaces, for $n = 1, 2, \ldots,$

$$L(n) = L^2_{\mathrm{s}}\big(X^n, \lambda^{\otimes n}\big)$$

of *symmetric* square-integrable functions on X^n, with $L(0) = \mathbb{C}$. The Fock space for X is defined as

$$\Gamma(X, \lambda) = \bigoplus_{n=0}^{\infty} L(n).$$

It is provided with the scalar product

$$\langle f|g\rangle_\lambda = \overline{f}_0 g_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_n) \overline{f}_n(x_1, \ldots, x_n) g_n(x_1, \ldots, x_n)$$

and the norm

$$\|f\|^2_\Gamma = |f_0|^2 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_n) \big|f_n(x_1, \ldots, x_n)\big|^2$$

for $f = f_0 \oplus f_1 \oplus f_2 \oplus \cdots$ with $f_n \in L(n)$, and g accordingly. So f is in Γ, if only and if $\|f\|_\Gamma < \infty$. We define the subspace $\Gamma_{\mathrm{fin}} \subset \Gamma$ of those f such that $f_n = 0$ for n sufficiently large.

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Recall the definition of

$$\mathfrak{X} = \{\emptyset\} + X + X^2 + \cdots$$

and provide $\mathfrak{X}$ with the measure

$$\hat{\mathrm{e}}(\lambda)(f) = f(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \lambda(\mathrm{d}x_1)\cdots\lambda(\mathrm{d}x_n) f_n(x_1, \ldots, x_n).$$

We can make the identification

$$L_{\mathrm{s}}^2\big(\mathfrak{X}, \hat{\mathrm{e}}(\lambda)\big) = \Gamma(X, \lambda).$$

As the values of a function at a given point are generally not defined, we cannot define a_x and a_x^+ for a given $x \in X$. But the definitions at the end of Sect. 1.7 can be generalized. Define for $f \in L(n+1)$ and $g \in L(1)$

$$\big(a(g)f\big)(x_1, \ldots x_n) = \int \lambda(\mathrm{d}x_0)\overline{g}(x_0) f(x_0, x_1, \ldots, x_n)$$

and for $f \in L(n-1)$

$$\big(a^+(g)f\big)(x_1, \ldots, x_n) = \sum_{c \in [1,n]} g(x_c) f(x_{[1,n]\setminus\{c\}}).$$

One obtains in the usual way

$$\begin{aligned}\big\|a(g)\big\|_{\Gamma} &\leq \sqrt{n+1}\|g\|_{\Gamma}\|f\|_{\Gamma},\\ \big\|a^+(g)\big\|_{\Gamma} &\leq \sqrt{n}\|g\|_{\Gamma}\|f\|_{\Gamma}\end{aligned}$$

with, of course,

$$\|g\|_{\Gamma}^2 = \int \lambda(\mathrm{d}x)\big|g(x)\big|^2.$$

The mappings $a(g)$ and $a^+(g)$ can be extended to operators $\Gamma_{\mathrm{fin}}(X, \lambda) \to \Gamma_{\mathrm{fin}}(X, \lambda)$, and one has

$$\big\langle f|a(g)h\big\rangle = \big\langle a^+(g)f|h\big\rangle$$

and the commutator

$$\big[a(f), a^+(g)\big] = \int \lambda(\mathrm{d}x)\overline{f}(x)g(x).$$

2.2 The Sum-Integral Lemma for Measures

In this work we will mainly use another way of generalizing the creation and annihilation operators on finite sets. Instead of $L^2(X, \lambda)$ we will deal with the pairs of spaces of measures and spaces of continuous functions on X. Contrary to the situation described in the last section, we can easily define white noise operators. We have at our disposal the powerful tools of classical measure theory, and may use the positivity of the commutation relations.

This paper is related to the theory of kernels, first used in quantum probability by Maassen [31] and Meyer [34]. The theory of kernels, however, is well known in quantum field theory. Quantum stochastic processes form, to some extent, a quantum field theory in one space coordinate and one time coordinate. Our approach is dual to that of Maassen and Meyer. We introduce the field operators directly and work with them.

The sum-integral lemma is the basic tool of our analysis. It has been well known for diffuse measures for a long time, i.e., for measures where the points have measure 0 [33]. Our lemma is much more general; it holds for all measures.

We shall employ Bourbaki's measure theory. It is a theory of measures on locally compact spaces. If S is a locally compact space, denote by $\mathscr{K}(S)$ the space of complex-valued continuous functions on S with compact support, and by $\mathscr{M}(S)$ the space of complex measures on S. A complex measure is a linear functional $\mu : \mathscr{K}(S) \to \mathbb{C}$, such that for any compact $K \subset S$, there exists a constant C_K such that $|\mu(f)| \leq C_K \max_{x\in S} |f(x)|$ for all $f \in \mathscr{K}(S)$ with support in K. As in other measure theories the set of integrable functions can be extended from functions in $\mathscr{K}(S)$ to much more general functions. All the usual theorems, like the theorem of Lebesgue, are valid. We shall use the *vague* convergence of measures, which is the weak convergence over $\mathscr{K}(S)$, i.e. $\mu_\iota \to \mu$ if $\mu_\iota(f) \to \mu(f)$ for all $f \in \mathscr{K}(S)$.

In order to avoid unnecessary complications, we shall only consider locally compact spaces which are countable at infinity, i.e., which are a union of countably many compact subsets. Assume now that X is a locally compact space, provide X^n with the product topology, and the set

$$\mathfrak{X} = \{\emptyset\} + X + X^2 + \cdots$$

with that topology where the X^n are both open and closed, and where the restrictions to X^n coincide with the natural topology of X^n. Then $\mathfrak{X}$ is locally compact as well, any compact set is contained in a finite union of the X^n, and its intersections with the X^n are compact.

In our case, the space X mostly will be $\mathbb{R}$. But we shall encounter $\mathbb{R} \times \mathbb{S}^2$ and generalizations of $\mathbb{R}$.

If μ is a complex measure on $\mathfrak{X}$, we write

$$\mu = \mu_0 + \mu_1 + \mu_2 + \cdots$$

where μ_n is the restriction of μ to X^n. We denote by Ψ the measure given by

$$\Psi(f) = f(\emptyset).$$

Then μ_0 is a multiple of Ψ. If $A = (A(1), \ldots, A(n))$ is a totally ordered set, we use the notation

$$\mu(\mathrm{d}x_A) = \mu_n(\mathrm{d}x_{A(1)}, \ldots, \mathrm{d}x_{A(n)}).$$

A function f on $\mathfrak{X}$ is called symmetric, if $f(w) = f(\sigma w)$ for all permutations of w. If α is a set without prescribed order and f is symmetric, then $f(x_\alpha)$ is well defined. A measure on X^n is *symmetric*, if for all $f \in \mathscr{K}(X^n)$ and all permutations σ of $[1, n]$, one has $\mu(f) = \mu(\sigma f)$ with $(\sigma f)(w) = f(\sigma w)$ for all $w \in X^n$. A measure on $\mathfrak{X}$ is symmetric if all its restrictions to X^n are symmetric. We then use the notation $\mu(\mathrm{d}x_\alpha)$.

Like a function, a measure μ has an absolute value $|\mu|$. A measure μ is bounded, if the measure of the total space with respect to $|\mu|$ is finite.

If $w \in \mathfrak{X}$, $w = (x_1, \dots, x_n)$, then we set

$$\Delta w = \frac{1}{\#w!} = \frac{1}{n!}.$$

Theorem 2.2.1 (Sum-integral lemma for measures) *Let there be given a measure*

$$\mu(\mathrm{d}w_1, \dots, \mathrm{d}w_k)$$

on

$$\mathfrak{X}^k = \sum_{n_1,\dots,n_k} X^{n_1} \times \cdots X^{n_k},$$

symmetric in each of the variables w_i. Then

$$\mu = \sum_{n_1,\dots,n_k} \mu_{n_1,\dots,n_k}$$

where $\mu_{n_1,\dots,n_k}$ is the restriction of μ to $X^{n_1} \times \cdots \times X^{n_k}$. Assume that

$$\Delta w_1 \cdots \Delta w_k \, \mu(\mathrm{d}w_1, \dots, \mathrm{d}w_k) = \sum \frac{1}{n_1! \cdots n_k!} \, \mu_{n_1,\dots,n_k}(\mathrm{d}w_1, \dots, \mathrm{d}w_k)$$

is a bounded measure on $\mathfrak{X}^k$. Then

$$\int \cdots \int_{\mathfrak{X}^k} \Delta w_1 \cdots \Delta w_k \, \mu(\mathrm{d}w_1, \dots, \mathrm{d}w_k) = \int_{\mathfrak{X}} \Delta w \, \nu(\mathrm{d}w)$$

where ν is a measure on $\mathfrak{X}$, and $\sum(1/n!)\nu_n$ is a bounded measure, in which ν_n is the restriction of ν to X^n and

$$\nu_n(\mathrm{d}x_1, \dots, \mathrm{d}x_n) = \sum_{\beta_1+\cdots+\beta_k=[1,n]} \mu_{\#\beta_1,\dots,\#\beta_k}(\mathrm{d}x_{\beta_1}, \dots, \mathrm{d}x_{\beta_k}),$$

where $\beta_1, \dots, \beta_k$ are disjoint sets.

Proof

$$\int \cdots \int_{\mathfrak{X}^k} \Delta w_1 \cdots \Delta w_k \, \mu(\mathrm{d}w_1, \dots, \mathrm{d}w_k)$$
$$= \sum_{n_1,\dots,n_k} \int_{X^{n_1}} \cdots \int_{X^{n_k}} \frac{1}{n_1! \cdots n_k!} \, \mu_{n_1,\dots,n_k}(\mathrm{d}x_{\alpha_1}, \dots, \mathrm{d}x_{\alpha_n})$$

where the α_i are the intervals

$$\alpha_1 = [1, n_1],$$
$$\alpha_2 = [n_1 + 1, n_1 + n_2], \quad \ldots, \quad \alpha_k = [n_1 + \cdots n_{k-1} + 1, n_1 + \cdots + n_k].$$

Fix $n_1, \ldots, n_k$ and put $n = n_1 + \cdots n_k$. Then for the summand in the above formula we have

$$\int_{X^{n_1}} \cdots \int_{X^{n_k}} \mu_{n_1,\ldots,n_k}(\mathrm{d}x_{\alpha_1}, \ldots, \mathrm{d}x_{\alpha_k})$$
$$= \frac{1}{n!} \sum_{\sigma} \int_{X^{n_1}} \cdots \int_{X^{n_k}} \mu_{n_1,\ldots,n_k}(\mathrm{d}x_{\sigma(\alpha_1)}, \ldots, \mathrm{d}x_{\sigma(\alpha_k)})$$

where the sum runs over all permutations of n elements. The subsets $\sigma(\alpha_i) = \beta_i$ have the property

$$\beta_1 + \cdots + \beta_k = [1, n], \quad \#\beta_i = n_i. \tag{$*$}$$

Fix $\beta_1, \ldots, \beta_k$ with property $(*)$. There are exactly $n_1! \cdots n_k!$ permutations σ such that

$$\sigma(\alpha_i) = \beta_i \quad \text{for } i = 1, \ldots, k.$$

Hence the last integral expression equals

$$\frac{n_1! \cdots n_k!}{n!} \sum_{\beta_1,\ldots,\beta_k} \int \cdots \int \mu_{n_1,\ldots,n_k}(\mathrm{d}x_{\beta_1}, \ldots, \mathrm{d}x_{\beta_k}),$$

for the β_i with $(*)$. Hence

$$\sum_{n_1,\ldots,n_k} \int_{X^{n_1}} \cdots \int_{X^{n_k}} \frac{1}{n_1! \cdots n_k!} \mu_{n_1,\ldots,n_k}(\mathrm{d}x_{\alpha_1}, \ldots, \mathrm{d}x_{\alpha_n})$$
$$= \sum_n \frac{1}{n!} \sum_{\beta_1,\ldots,\beta_k} \int \cdots \int \mu_{n_1,\ldots,n_k}(\mathrm{d}x_{\beta_1}, \ldots, \mathrm{d}x_{\beta_k}).$$

□

Remark 2.2.1 The proof is purely combinatorial. So analogous assertions hold in similar situations.

We want to use the notation of Sect. 1.7. If $\alpha = \{a_1, \ldots, a_n\}$ is a set without a prescribed order and μ is a symmetric measure then

$$\mu(\mathrm{d}x_\alpha) = \mu(\mathrm{d}x_{a_1}, \ldots, \mathrm{d}x_{a_n})$$

is well defined. We have

$$\int_{X^n} \mu(\mathrm{d}w) \Delta w = \int_{X^\alpha} \mu(\mathrm{d}x_\alpha) \Delta\alpha = \frac{1}{n!} \int_{X^\alpha} \mu(\mathrm{d}x_\alpha).$$

For a sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$, with $\#\alpha_n = n$, of sets without prescribed ordering we define for a symmetric measure μ on $\mathfrak{X}$

$$\int_{\mathfrak{X}} \Delta w \mu(\mathrm{d}w) = \sum_n \frac{1}{n!} \int_{X^{\alpha_n}} \mu(\mathrm{d}x_{\alpha_n})$$

and write it, for short, as

$$\int_{\mathfrak{X}} \Delta w \mu(\mathrm{d}w) = \int_{X^\alpha} \mu(\mathrm{d}x_\alpha) \Delta\alpha = \int_\alpha \mu(\mathrm{d}x_\alpha) \Delta\alpha.$$

With this notation we want to reformulate the sum-integral lemma.

Theorem 2.2.2 (Variant of sum-integral lemma) *Let $\alpha_i = (\alpha_{i,0}, \alpha_{i,1}, \ldots)$ be sequences of finite sets, with $\#\alpha_{i,n} = n$ and $\alpha_{n,i} \cap \alpha_{n',j} = \emptyset$ for $i \neq j$, and $\beta = (\beta_0, \beta_1, \ldots)$, with $\#\beta_n = n$ and the β_j disjoint from the α_i, then define*

$$\mu(\mathrm{d}x_{\alpha_1}, \ldots, \mathrm{d}x_{\alpha_k}) = \mu_{\#\alpha_1, \ldots, \#\alpha_k}(\mathrm{d}x_{\alpha_1}, \ldots, \mathrm{d}x_{\alpha_k}).$$

We have

$$\int_{\alpha_1} \cdots \int_{\alpha_k} \Delta\alpha_1 \cdots \Delta\alpha_k \mu(\mathrm{d}x_{\alpha_1}, \ldots, \mathrm{d}x_{\alpha_k}) = \int_\beta \Delta\beta \nu(\mathrm{d}x_\beta)$$

with

$$\nu(\mathrm{d}x_\beta) = \sum_{\beta_1 + \cdots + \beta_k = \beta} \mu(\mathrm{d}x_{\beta_1}, \ldots, \mathrm{d}x_{\beta_k}),$$

$$\mu(\mathrm{d}x_{\beta_1}, \ldots, \mathrm{d}x_{\beta_k}) = \mu_{\#\beta_1, \ldots, \#\beta_k}(\mathrm{d}x_{\beta_1}, \ldots, \mathrm{d}x_{\beta_k}).$$

Remark 2.2.2 We introduced the notation

$$\int_{\mathfrak{X}} \Delta w \mu(\mathrm{d}w) = \int_\alpha \Delta(\alpha) \mu(\mathrm{d}x_\alpha).$$

Later we will often skip the $\Delta\alpha$ completely and write for the last expression simply

$$\int_\alpha \mu(\mathrm{d}x_\alpha)$$

and skipping the $\mathrm{d}x$ as well only

$$\int_\alpha \mu(\alpha).$$

With this simplified notation the sum-integral lemma reads

$$\int_{\alpha_1} \cdots \int_{\alpha_k} \mu(\mathrm{d}x_{\alpha_1}, \ldots, \mathrm{d}x_{\alpha_k}) = \int_\alpha \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} \mu(\mathrm{d}x_{\alpha_1}, \ldots, \mathrm{d}x_{\alpha_k})$$

or by neglecting the dx

$$\int_{\alpha_1} \cdots \int_{\alpha_k} \mu(\alpha_1, \dots, \alpha_k) = \int_{\alpha} \sum_{\alpha_1 + \cdots + \alpha_k = \alpha} \mu(\alpha_1, \dots, \alpha_k).$$

If $X = \mathbb{R}$ and

$$\mathfrak{R} = \{\emptyset\} + \mathbb{R} + \mathbb{R}^2 + \cdots$$

and if λ is the Lebesgue measure

$$\int_{\alpha} \mathrm{e}(\lambda)(\mathrm{d}x_\alpha) f(x_\alpha) \Delta\alpha = \sum_n \int_{x_1 < \cdots < x_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \ f(x_1, \dots, x_n).$$

In the theory of Maassen kernels [34] one defines

$$\int \mathrm{d}\omega f(\omega) = \sum_n \int_{x_1 < \cdots < x_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \ f(x_1, \dots, x_n),$$

where ω runs through all finite subsets of $\mathbb{R}$. The mapping

$$(\omega_1, \dots, \omega_n) \mapsto \omega_1 + \cdots + \omega_n$$

is defined where the ω_i are pairwise disjoint, i.e. Lebesgue almost everywhere. The usual sum-integral lemma is

$$\int \cdots \int \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_k \ f(\omega_1, \dots, \omega_k) = \int \mathrm{d}\omega \sum_{\omega_1 + \cdots + \omega_k = \omega} f(\omega).$$

It can be easily derived from the sum-integral lemma for measures, as multisets with multiple points have Lebesgue measure 0.

2.3 Creation and Annihilation Operators on Locally Compact Spaces

We use the duality between measures and continuous functions of compact support. We define creation and annihilation operators for symmetric functions and measures on $\mathfrak{X}$. Assume given a function $\varphi \in \mathscr{K}(X)$, a function $f \in \mathscr{K}_{\mathrm{s}}(\mathfrak{X})$, the space of symmetric continuous functions on $\mathfrak{X}$ of compact support, a measure $\nu \in \mathscr{M}(X)$, and a measure $\mu \in \mathscr{M}_{\mathrm{s}}(\mathfrak{X})$, the space of symmetric measures on $\mathfrak{X}$. We define

$$\big(a(\nu) f\big)(x_1, \dots, x_n) = \int \overline{\nu}(\mathrm{d}x_0) f(x_0, x_1, \dots, x_n)$$

or in another notation, where $\alpha + c = \alpha + \{c\}$ means that the point c is added to the set α, and similarly using $\alpha \setminus c = \alpha \setminus \{c\}$, we can continue with

$$\big(a(\nu)f\big)(x_\alpha) = \int \overline{\nu}(\mathrm{d}x_c) f(x_{\alpha+c}),$$

$$\big(a^+(\varphi)f\big)(x_\alpha) = \sum_{c\in\alpha} \varphi(x_c) f(x_{\alpha\setminus c}),$$

$$\big(a^+(\nu)\mu\big)(\mathrm{d}x_\alpha) = \sum_{c\in\alpha} \nu(\mathrm{d}x_c)\mu(\mathrm{d}x_{\alpha\setminus c}),$$

$$\big(a(\varphi)\mu\big)(\mathrm{d}x_\alpha) = \int \overline{\varphi(x_c)}\mu(\mathrm{d}x_{\alpha+c}).$$

If Φ is the function defined by

$$\Phi(\emptyset) = 1; \qquad \Phi(x_\alpha) = 0 \quad \text{for } \alpha \neq \emptyset$$

then

$$a(\nu)\Phi = 0.$$

Similarly if Ψ is the measure defined by

$$\Psi(f) = \langle \Psi | f \rangle = f(\emptyset),$$

then

$$a(\varphi)\Psi = 0.$$

We have therefore

$$\langle \Psi | \Phi \rangle = 1.$$

We define the mapping

$$\mu \in \mathscr{M}(\mathfrak{X}) \mapsto \mu(\Phi)$$

and use the notation for it

$$\mu(\Phi) = \Phi(\mu) = \langle \Phi | \mu \rangle.$$

One obtains

$$\big\langle \Psi | a(\nu) a^+(\varphi) \Phi \big\rangle = \int_X \overline{\nu}(\mathrm{d}x)\varphi(x) = \langle \nu | \varphi \rangle$$

$$\big\langle \Phi | a(\varphi) a^+(\nu) \Psi \big\rangle = \int_X \nu(\mathrm{d}x)\overline{\varphi(x)} = \langle \varphi | \nu \rangle$$

and the commutation relations

$$\big[a(\nu), a^+(\varphi)\big] = \int \overline{\nu}(\mathrm{d}x)\varphi(x) = \langle \nu | \varphi \rangle,$$

$$\left[a(\varphi), a^+(\nu)\right] = \int \nu(\mathrm{d}x)\overline{\varphi}(x) = \langle \varphi|\nu\rangle.$$

We define

$$\langle \mu|f\rangle = \int_{\mathfrak{X}} \Delta w\, \overline{\mu}(\mathrm{d}w) f(w) = \int_{\alpha} \Delta\alpha\, \overline{\mu}(\mathrm{d}x_\alpha) f(x_\alpha),$$
$$\langle f|\mu\rangle = \overline{\langle \mu|f\rangle}.$$

Proposition 2.3.1 *We have*

$$\left\langle a^+(\nu)\mu|f\right\rangle = \left\langle \mu|a(\nu)f\right\rangle,$$
$$\left\langle a(\varphi)\mu|f\right\rangle = \left\langle \mu|a(\varphi)^+ f\right\rangle$$

or

$$\int \Delta w\, \overline{\left(a^+(\nu)\mu\right)}(\mathrm{d}w) f(w) = \int \Delta w\, \overline{\mu}(\mathrm{d}w)\left(a(\nu)f\right)(w),$$
$$\int \Delta w\, \overline{\left(a(\varphi)\mu\right)}(\mathrm{d}w) f(w) = \int \Delta w\, \overline{\mu}(\mathrm{d}w)\left(a^+(\varphi)f\right)(w).$$

Proof We prove only one of the equations by using the sum-integral lemma

$$\int_\beta \Delta\beta \overline{\left(a^+(\nu)\mu\right)}(\mathrm{d}x_\beta) f(x_\beta) = \int_\beta \Delta\beta \sum_{c\in\beta} \overline{\nu(\mathrm{d}x_c)\mu(\mathrm{d}x_{\beta\setminus c})} f(x_\beta).$$

Introduce the sequence consisting of $\{c\}$ alone, and the sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$, by putting $\alpha_{n-1} = \beta_n \setminus c$. In this way the integral becomes

$$\int_\alpha \int_c \Delta\alpha\, \overline{\nu(\mathrm{d}x_c)\mu(\mathrm{d}x_\alpha)} f(x_{\alpha+c}) = \left\langle \mu|a(\nu)f\right\rangle. \qquad \square$$

We define the exponential measures and functions

$$\mathrm{e}(\varphi) = \Phi + \varphi + \varphi^{\otimes 2} + \cdots = e^{a^+(\varphi)}\Phi,$$
$$\mathrm{e}(\nu) = \Psi + \nu + \nu^{\otimes 2} + \cdots = e^{a^+(\nu)}\Psi.$$

So, for $\alpha = \{a_1, \ldots, a_n\}$,

$$\mathrm{e}(\varphi)(x_\alpha) = \varphi(x_{a_1}) \cdots \varphi(x_{a_n}),$$
$$\mathrm{e}(\nu)(\mathrm{d}x_\alpha) = \nu(\mathrm{d}x_{a_1}) \cdots \nu(\mathrm{d}x_{a_n}).$$

2.4 Introduction of Point Measures

We consider the function

$$\varepsilon : x \in X \mapsto \varepsilon_x \in \mathscr{M}(X), \qquad \int \varepsilon_x(\mathrm{d}y)\varphi(y) = \varphi(x).$$

So ε_x is the *point measure* at the point $x \in X$.

Lemma 2.4.1 *If μ is a measure on X^n, then*

$$\int_{x_1} \varepsilon_{x_1}(\mathrm{d}y)\mu(\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n) = \mu(\mathrm{d}y, dx_2, \ldots, \mathrm{d}x_n),$$

where the subscript variable x_1 on the integral indicates integration over the range X of that variable.

Proof If $\varphi \in \mathscr{K}(X)$ then

$$\begin{aligned}\int_y \int_{x_1} \varphi(y)\varepsilon_{x_1}(\mathrm{d}y)\mu(\mathrm{d}x_1, dx_2, \ldots, \mathrm{d}x_n) &= \int_{x_1} \varphi(x_1)\mu(\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n)\\ &= \int_y \varphi(y)\mu(\mathrm{d}y, \mathrm{d}x_2, \ldots, \mathrm{d}x_n).\end{aligned}$$

□

We can easily define the mapping

$$a(x) = a(\varepsilon_x) : \mathscr{K}_{\mathrm{s}}(\mathfrak{X}) \to \mathscr{K}_{\mathrm{s}}(\mathfrak{X}),$$

$$\big(a(x)f\big)(x_1, \ldots, x_n) = \int_{x_0} \varepsilon_x(\mathrm{d}x_0) f(x_0, x_1, \ldots, x_n) = f(x, x_1, \ldots, x_n).$$

If $\mu \in \mathscr{M}_{\mathrm{s}}(\mathfrak{X})$ then

$$a^+(\varepsilon_x)\mu(\mathrm{d}x_\alpha) = \sum_{c \in \alpha} \varepsilon_x(\mathrm{d}x_c)\mu(\mathrm{d}x_{\alpha \setminus c}).$$

If ν is a measure on X, then

$$a(\nu) = \int \overline{\nu}(\mathrm{d}x)a(x).$$

We will mostly use the symbol $a^+(\mathrm{d}x)$ for a mapping from $\mathscr{K}_{\mathrm{s}}(\mathfrak{X})$ into the measures on X, which we will now introduce and explain.

If S is a locally compact space, μ a measure on S, and f a Borel function, we define the product $f\mu$ by the formula

$$\int (f\mu)(ds)\varphi(s) = \int \mu(ds)f(s)\varphi(s)$$

for $\varphi \in \mathscr{K}(S)$, and write

$$(f\mu)(ds) = (\mu f)(ds) = f(s)\mu(ds).$$

Let S and T be locally compact spaces. We consider a function $f : S \to \mathscr{M}(T)$, with target the space of measures on T. It can be considered as a function

$$f : S \times \mathscr{K}(T) \to \mathbb{C}$$

and we write it

$$f = f(s, \mathrm{d}t).$$

We extend the notion of the creation operator to functions $f = f(x, \mathrm{d}y) : X \to \mathscr{M}(X)$, where using x indicates the variable and the $\mathrm{d}y$ reminds us that the value is a measure, and define for $g \in \mathscr{K}_{\mathrm{s}}(\mathfrak{X})$

$$\big(a^+(f)g\big)(x_\alpha, \mathrm{d}y) = \sum_{c\in\alpha} f(x_c, \mathrm{d}y)g(x_{\alpha\setminus c}).$$

We apply this notion to the function $\varepsilon : x \mapsto \varepsilon_x$ and write

$$\big(a^+(\mathrm{d}y)g\big)(x_\alpha) = \big(a^+\big(\varepsilon(\mathrm{d}y)\big)g\big)(x_\alpha) = \sum_{c\in\alpha} \varepsilon_{x_c}(\mathrm{d}y)g(x_{\alpha\setminus c}).$$

We may consider $a^+(\varepsilon)$ as an operator-valued measure and write

$$a^+(\varepsilon) = a^+(\varepsilon)(\mathrm{d}y).$$

If $\varphi \in \mathscr{K}(X)$, i.e., φ has one variable, then

$$a^+(\varphi)f = \int a^+(\mathrm{d}x)\varphi(x).$$

We obtain the commutation relations

$$\begin{aligned}
\big[a(\varepsilon_x), a(\varepsilon_y)\big] &= 0,\\
\big[a^+(\varepsilon)(\mathrm{d}x), a^+(\varepsilon)(\mathrm{d}y)\big] &= 0,\\
\big[a(\varepsilon_x), a^+(\varepsilon)(\mathrm{d}y)\big] &= \varepsilon_x(\mathrm{d}y).
\end{aligned}$$

We extend this notion to any Borel function $g : y \in X \mapsto g_y \in \mathscr{K}_{\mathrm{s}}(\mathfrak{X})$ and write

$$\big(a^+(\varepsilon)g_y\big)(x_\alpha, \mathrm{d}y) = \sum_{c\in\alpha} \varepsilon_{x_c}(\mathrm{d}y)g_y(x_{\alpha\setminus c}).$$

In this equation the product of the measure $\varepsilon_{x_c}(\mathrm{d}y)$ with the function g_y appears.

A special case arises if $g_y = a(\varepsilon_y)f$.

Proposition 2.4.1

$$\big(a^+(\varepsilon)a(\varepsilon_y)f\big)(x_\alpha,\mathrm{d}y)=\sum_{c\in\alpha}\varepsilon_{x_c}(\mathrm{d}y)f(x_\alpha).$$

Proof

$$\begin{aligned}\big(a^+(\varepsilon)a(\varepsilon_y)f\big)(x_\alpha,\mathrm{d}y)&=\sum_{c\in\alpha}\varepsilon_{x_c}(\mathrm{d}y)\big(a(\varepsilon_y)f\big)(x_{\alpha\setminus c})\\&=\sum_{c\in\alpha}\varepsilon_{x_c}(\mathrm{d}y)f\big(x_{\alpha\setminus c}+\{y\}\big)=\sum_{c\in\alpha}\varepsilon_{x_c}(\mathrm{d}y)f\big(x_{\alpha\setminus c}+\{x_c\}\big)\\&=\sum_{c\in\alpha}\varepsilon_{x_c}(\mathrm{d}y)f(x_\alpha)\end{aligned}$$

as

$$\varepsilon(x,\mathrm{d}y)g(y)=\varepsilon(x,\mathrm{d}y)g(x).$$ □

So

$$\mathfrak{n}(\mathrm{d}y)=a^+(\varepsilon)(\mathrm{d}y)a(\varepsilon_y)$$

is the operator analogous to the number operator $a_x^+a_x$ in the case of finitely many x considered in Sect. 1.7.

We single out a positive measure λ on X, and introduce in $\mathscr{K}_s(\mathfrak{X})$ the positive sesquilinear form considered already in Sect. 1.7,

$$\langle f|g\rangle_\lambda=\int_\alpha \varDelta\alpha\;\mathrm{e}(\lambda)(\mathrm{d}x_\alpha)\overline{f}(x_\alpha)g(x_\alpha)=\big\langle f\;\mathrm{e}(\lambda)|g\big\rangle=\big\langle f|g\;\mathrm{e}(\lambda)\big\rangle,$$

using the product of a function with the measure

$$\mathrm{e}(\lambda)=\varPsi+\lambda+\lambda^{\otimes 2}+\cdots.$$

More generally, if ν is a measure on X, we have

$$\big\langle f|a(\nu)g\big\rangle_\lambda=\big\langle a^+(\nu)\mathrm{e}(\lambda)f|g\big\rangle.$$

We introduced in Sect. 2.1 the operator $a^+(\varphi)$. One obtains now

$$\big\langle a^+(\varphi)f|g\big\rangle_\lambda=\big\langle f|a(\varphi\lambda)g\big\rangle_\lambda.$$

So $a(\varphi\lambda)$ corresponds to the operator $a(\varphi)$ introduced in Sect. 2.1.

If μ is a symmetric measure on $\mathfrak{X}$, one has

$$\big(a(\varepsilon)(\mathrm{d}x_c)\mu\big)(\mathrm{d}x_\alpha)=\mu(\mathrm{d}x_{\alpha+c})$$

as

$$\big(a(\varepsilon)(\mathrm{d}y)\mu\big)(\mathrm{d}x_\alpha)=\int_{x_c}\varepsilon_{x_c}(\mathrm{d}y)\mu(\mathrm{d}x_{\alpha+c})=\mu(\mathrm{d}x_\alpha,\mathrm{d}y).$$

We can calculate

$$\langle \mu | a^+\big(\varepsilon(\mathrm{d}y)\big) f\rangle = \langle a\big(\varepsilon(\mathrm{d}y)\big)\mu | f\rangle.$$

Proposition 2.4.2 *For $f, g \in \mathscr{K}_{\mathrm{s}}(\mathfrak{X})$*

$$\langle f | a^+\big(\varepsilon(\mathrm{d}y)\big) g\rangle_\lambda = \lambda(\mathrm{d}y)\langle a\big(\varepsilon(y)\big) f | g\rangle_\lambda,$$

$$\int_y \langle f | \mathfrak{n}(dy) g\rangle_\lambda = \langle f | N g\rangle_\lambda$$

where N is the operator on the space of functions on $\mathfrak{X}$ given by

$$(Nf)(x_1, \ldots, x_n) = n f(x_1, \ldots, x_n).$$

Proof

$$\begin{aligned}\langle f | a^+\big(\varepsilon(\mathrm{d}x_c)\big) g\rangle_\lambda &= \langle a\big(\varepsilon(\mathrm{d}x_c)\big) f \mathrm{e}(\lambda) | g\rangle = \int \big(\mathrm{e}(\lambda)\overline{f}\big)(\mathrm{d}x_{\alpha+c}) g(x_\alpha)\Delta(\alpha)\\ &= \lambda(\mathrm{d}x_c)\int \overline{f}(x_{\alpha+c})\big(\mathrm{e}(\lambda)\big)(\mathrm{d}x_\alpha) g(x_\alpha)\Delta(\alpha)\\ &= \lambda(\mathrm{d}x_c)\langle a(x_c) f | g\rangle_\lambda.\end{aligned}$$

Hence

$$\langle f | a^+\big(\varepsilon(\mathrm{d}y)\big) g\rangle_\lambda = \lambda(\mathrm{d}y)\langle a\big(\varepsilon(y)\big) f | g\rangle_\lambda.$$

One obtains, from the definition of $\mathfrak{n}$

$$\langle f | \mathfrak{n}(\mathrm{d}y) g\rangle_\lambda = \langle f | a^+\big(\varepsilon(\mathrm{d}y)\big) a(\varepsilon_y) g\rangle_\lambda = \lambda(\mathrm{d}y)\langle a(\varepsilon_y) f | a(\varepsilon_y) g\rangle_\lambda$$

and

$$\begin{aligned}&\int_y \lambda(\mathrm{d}y)\langle a(\varepsilon_y) f | a(\varepsilon_y) g\rangle_\lambda\\ &= \sum_{n=0}^{\infty}(1/n!)\int \lambda(\mathrm{d}y)\int \lambda(\mathrm{d}x_1)\cdots\lambda(\mathrm{d}x_n) f(y, x_1, \ldots, x_n) g(y, x_1, \ldots, x_n)\\ &= \langle f | N g\rangle_\lambda.\end{aligned}$$

□

If V is a complex vector space with the scalar product $\langle\cdot|\cdot\rangle$, we may write $|f\rangle$ for $f \in V$, and $\langle f|$ for the semilinear functional $g = |g\rangle \mapsto \langle f|g\rangle$. If $c \in \mathbb{C}$, then $\langle cf| = \overline{c}\langle f|$. Given an operator $A : V \to V$, we define the operator $A^\dagger$ operating on $\langle f|$ to the left by

$$\langle f | A^\dagger = \langle Af|.$$

There might be, or there might not be, an operator A^+ acting on $|g\rangle$ to the right with $A^\dagger = A^+$ or $\langle Af|g\rangle = \langle f|A^\dagger g\rangle = \langle f|A^+ g\rangle$.

We apply this definition to $\mathscr{K}_s(\mathfrak{X})$ provided with the scalar product $\langle\cdot\,|\,\cdot\rangle_\lambda$ and, as a corollary of Proposition 2.4.2, we have

$$a^+\big(\varepsilon(\mathrm{d}y)\big) = a^\dagger(\varepsilon_y)\lambda(\mathrm{d}y).$$

We use Bourbaki's terminology in denoting by ε_x the point measure at the point $x \in X$. We compare it to the δ-function on $\mathbb{R}$, as used in physical literature. The δ-function has three different meanings, depending on the differentials with which it is multiplied:

$$\delta(x-y)\mathrm{d}y = \varepsilon_x(\mathrm{d}y),$$

$$\delta(x-y)\mathrm{d}x = \varepsilon_y(\mathrm{d}x),$$

$$\delta(x-y)\mathrm{d}x\,\mathrm{d}y = \Lambda(\mathrm{d}x,\mathrm{d}y),$$

where

$$\int \Lambda(\mathrm{d}x,\mathrm{d}y) f(x,y) = \int \mathrm{d}x f(x,x).$$

Recall

$$\mathfrak{R} = \{\emptyset\} + \mathbb{R} + \mathbb{R}^2 + \cdots,$$

use for λ the Lebesgue measure, treat the δ-function formally as an ordinary function, and put $\delta_x(y) = \delta(x-y)$; then

$$\big(a^+(\delta_x)f\big)(x_\alpha) = \sum_{c\in\alpha}\delta(x-x_c)f(x_{\alpha\setminus c}),$$

$$\big(a(\delta_{x_c})f\big)(x_\alpha) = \int \mathrm{d}x_b\,\delta(x_c-x_b)f(x_{\alpha+b}) = f(x_{\alpha+c}).$$

We have, with this notation, the nice duality relation

$$\big\langle f|a^+(\delta_x)g\big\rangle_\lambda = \big\langle a(\delta_x)f|g\big\rangle_\lambda.$$

For many calculations it is advantageous to work with the δ-function. In doing so there is no difference between a^+ and $a^\dagger$. But the author hopes that the mathematics has become clearer through the use of the ε-measures.

In some calculations we use the terminology of Laurent Schwartz and write

$$\varepsilon_0(\mathrm{d}x) = \delta(x)\mathrm{d}x.$$

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