Preface

Let us start by considering a finite set of operators a_x, called annihilation operators, and a_x^+, called creation operators, indexed by x in the finite set X. They have the commutation relations, for $x, y \in X$,

$$\left[a_x, a_y^+\right] = \delta_{x,y}$$
$$[a_x, a_y] = \left[a_x^+, a_y^+\right] = 0.$$

First we realize these operators in a purely algebraic way. We define them as generators of a complex associative algebra with the above commutation relations as defining relations. We denote this algebra by $\mathfrak{W}(X)$. It is a special form of a *Weyl algebra*. A normal ordered monomial of the $a_x, a_x^+, x \in X$ is what we call a monomial of the form

$$a_{x_1}^+ \cdots a_{x_m}^+ a_{y_1} \cdots a_{y_n}.$$

The normal ordered monomials form a basis of $\mathfrak{W}(X)$. This means any element of $\mathfrak{W}(X)$ can be represented in a unique way according to the formula

$$\sum K(x_1, \ldots, x_m; y_1, \ldots, y_n) a_{x_1}^+ \cdots a_{x_m}^+ a_{y_1} \cdots a_{y_n},$$

where K is a function symmetric both in the x_i and in the y_i.

We can then move on to consider a continuous set of annihilation and creation operators, e.g., $a_x, a_x^+,\ x \in \mathbb{R}$, with the commutation relations

$$\left[a_x, a_y^+\right] = \delta(x - y)$$
$$[a_x, a_y] = \left[a_x^+, a_y^+\right] = 0$$

where $\delta(x-y)$ is Dirac's δ-function. These operators are harder to define rigorously. One possibility is to use the integrals

$$a(\varphi) = \int \mathrm{d}x\ \overline{\varphi}(x) a_x$$

$$a^+(\psi) = \int \mathrm{d}x\ \psi(x) a_x^+,$$

where the arguments φ and ψ are square-integrable functions. Then the non-vanishing commutation relations read

$$\big[a(\varphi), a^+(\psi)\big] = \int \mathrm{d}x\ \overline{\varphi}(x)\psi(x).$$

Everything in this context can be well defined using what is called Fock space.

Another way to approach the problem was chosen by Obata [35]. He uses an infinite system of nested Hilbert spaces, first defines a_x, and then the adjoint a_x^+ in the dual system.

In quantum field theory, one uses for operators the representation developed by Berezin [8]

$$\sum_{m,n} \int \cdots \int \mathrm{d}x_1 \cdots \mathrm{d}x_m \mathrm{d}y_1 \cdots \mathrm{d}y_n K_{m,n}(x_1, \ldots, x_m; y_1, \ldots, y_n)$$
$$\times a_{x_1}^+ \cdots a_{x_m}^+ a_{y_1} \cdots a_{y_n}, \qquad (*)$$

where $K_{m,n}$ might be quite irregular generalized functions. The multiplication of these operators can be performed by using the commutation relations. Berezin provides for that purpose an attractive functional integral.

Another way to perform the multiplication of these operators is to define a convolution for the coefficients K, using the commutation relations formally, and then to forget about the a_x and a_x^+ and work only with the convolution. This can be done in a rigorous way. This is the theory of *kernels* introduced by Hans Maassen [31] and continued by Paul-André Meyer [34]. These kernels are therefore called *Maassen-Meyer-kernels*. The theory works for Lebesgue measurable kernels [41].

We now mention the usual way of defining $a(\varphi)$ and $a^+(\varphi)$. Denote by

$$\mathfrak{R} = \{\emptyset\} + \mathbb{R} + \mathbb{R}^2 + \cdots$$

the space of all finite sequences of real numbers, where we use the $+$ sign for union of disjoint sets. Equip it with the measure

$$\hat{\mathrm{e}}(\lambda)(f) = f(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \mathrm{d}x_1 \cdots \mathrm{d}x_n f(x_1, \ldots, x_n),$$

where the function $f(x_1, \ldots, x_n)$ is supposed to be symmetric in the x_i. The notation $\hat{\mathrm{e}}(\lambda)$ is used because this is essentially the exponential of the Lebesgue measure λ. Then Fock space is defined to be

$$L_{\mathrm{s}}^2\big(\mathfrak{R}, \hat{\mathrm{e}}(\lambda)\big),$$

where the letter s stands for symmetric. If $L^2_{\mathrm{s}}(\mathbb{R}^n) = L(n)$ is the space of symmetric Lebesgue square-integrable functions on $\mathbb{R}^n$, then

$$a(\varphi) : L(n+1) \to L(n),$$
$$\big(a(\varphi) f\big)(x_1, \dots, x_n) = \int \mathrm{d}x_0 \overline{\varphi}(x_0) f(x_0, x_1, \dots, x_n)$$

and

$$\begin{aligned} &a^+(\varphi) : L(n) \to L(n+1), \\ &\big(a^+(\varphi) f\big)(x_0, x_1, \dots, x_n) \\ &\quad = \varphi(x_0) f(x_1, \dots, x_n) + \varphi(x_1) f(x_0, x_2, \dots, x_n) + \cdots \\ &\qquad + \varphi(x_n) f(x_0, x_1, \dots, x_{n-1}). \end{aligned}$$

Thus $a(\varphi)$ and $a^+(\varphi)$ can be defined on the pre-Hilbert space

$$\bigoplus_{n=0,\mathrm{f}}^{\infty} L(n) \subset L^2_{\mathrm{s}}\big(\mathfrak{R}, \hat{\mathrm{e}}(\lambda)\big),$$

where the suffix f means, that any element $f = (f_0, f_1, \dots, f_n, \dots)$ has components $f_n = 0$ for sufficiently large n.

This approach is based on the duality of the Hilbert space $L^2_{\mathrm{s}}(\mathfrak{R}, \mathrm{e}(\lambda))$ with itself. We use Bourbaki's measure theory [10] and employ the duality between measures and functions. The space $\mathfrak{R}$ is locally compact when provided with the obvious topology. Use the notation $\mathscr{M}_{\mathrm{s}}(\mathfrak{R})$ for the space of symmetric measures and $\mathscr{K}_{\mathrm{s}}(\mathfrak{R})$ for the space of symmetric continuous functions of compact support. We can now define, for a measure ν on $\mathbb{R}$ and a symmetric function $f \in \mathscr{K}_{\mathrm{s}}(\mathfrak{R})$,

$$a(\nu) : \mathscr{K}_{\mathrm{s}}(\mathfrak{R}) \to \mathscr{K}_{\mathrm{s}}(\mathfrak{R}),$$
$$\big(a(\nu) f\big)(x_1, \dots, x_n) = \int \overline{\nu}(\mathrm{d}x_0) f(x_0, x_1, \dots, x_n)$$

and for a continuous function φ with compact support in $\mathbb{R}$

$$\begin{aligned} &a^+(\varphi) : \mathscr{K}_{\mathrm{s}}(\mathfrak{R}) \to \mathscr{K}_{\mathrm{s}}(\mathfrak{R}), \\ &\big(a^+(\varphi) f\big)(x_0, x_1, \dots, x_n) \\ &\quad = \varphi(x_0) f(x_1, \dots, x_n) + \varphi(x_1) f(x_0, x_2, \dots, x_n) + \cdots \\ &\qquad + \varphi(x_n) f(x_0, x_1, \dots, x_{n-1}) \end{aligned}$$

which is essentially the same formula as above.

By making use of the δ-function we have raised both a conceptual and a semantic problem. Denote the point measure at the point x by ε_x, with

$$\int \varepsilon_x(\mathrm{d}y)\varphi(y) = \varphi(x).$$

In the physical literature, the δ-function can have three different meanings corresponding to the different differentials with which it is combined:

$$\begin{aligned}\delta(x-y)\mathrm{d}y &= \varepsilon_x(\mathrm{d}y)\\ \delta(x-y)\mathrm{d}x &= \varepsilon_y(\mathrm{d}x)\\ \delta(x-y)\mathrm{d}x\mathrm{d}y &= \Lambda(\mathrm{d}x,\mathrm{d}y),\end{aligned}$$

where Λ is the measure on $\mathbb{R}^2$ concentrated on the diagonal and given by

$$\int \Lambda(\mathrm{d}x,\mathrm{d}y)\varphi(x,y) = \int \mathrm{d}x\varphi(x,x).$$

We will use both types of notation: one is mathematically clearer, the other one is often more convenient for calculations. In mathematics one very often uses δ_x for the point measure ε_x. We tend to avoid this notation.

Now we can define easily

$$\begin{aligned}a(x) &= a(\varepsilon_x) : \mathscr{K}_{\mathrm{s}}(\mathfrak{R}) \to \mathscr{K}_{\mathrm{s}}(\mathfrak{R}),\\ \big(a(x)f\big)(x_1,\ldots,x_n) &= f(x,x_1,\ldots,x_n).\end{aligned}$$

The definition of the creation operator is more difficult. Consider the measure-valued function

$$x \to \varepsilon_x$$

and define

$$\begin{aligned}a^+(\mathrm{d}x) &= a^+\big(\varepsilon(\mathrm{d}x)\big) : \mathscr{K}_{\mathrm{s}}(\mathfrak{R}) \to \mathscr{M}(\mathbb{R}),\\ &\big(a^+(\,\mathrm{d}x)f\big)(x_0,x_1,\ldots,x_n)\\ &\quad = \varepsilon_{x_0}(\mathrm{d}x)f(x_1,\ldots,x_n) + \varepsilon_{x_1}(\mathrm{d}x)f(x_0,x_2,\ldots,x_n) + \cdots\\ &\qquad + \varepsilon_{x_n}(\mathrm{d}x)f(x_0,x_1,\ldots,x_{n-1}),\end{aligned}$$

where the result is a sum of point measures on $\mathbb{R}$. With the help of these operators it is possible to establish a *quantum white noise* calculus.

We have the commutation relation

$$\big[a(x), a^+(\mathrm{d}y)\big] = \varepsilon_x(\mathrm{d}y).$$

There is an important operator called the number operator informally given as

$$N = \int_{\mathbb{R}} \mathrm{d}x \; a^+(x)a(x).$$

The differential of the number operator can be defined rigorously by

$$\mathfrak{n}(\mathrm{d}x) = a^+(\mathrm{d}x)a(x),$$
$$\big(\mathfrak{n}(\mathrm{d}x)f\big)(x_1,\ldots,x_n) = \sum_{i=1}^{n} \varepsilon_{x_i}(\mathrm{d}x) f(x_1,\ldots,x_n).$$

The normal ordered monomials have the form

$$\begin{aligned} M_{lmn} &= M(s_1,\ldots,s_l;t_1,\ldots,t_m;u_1,\ldots,u_n) \\ &= a^+(\mathrm{d}s_1)\cdots a^+(\mathrm{d}s_l)a^+(\mathrm{d}t_1)\cdots a^+(\mathrm{d}t_m)a(t_1)\cdots a(t_m)a(u_1)\cdots \\ &\quad \times a(u_m)\mathrm{d}u_1\cdots\mathrm{d}u_n. \end{aligned}$$

We define a measure on $\mathfrak{R}^5$ by

$$\begin{aligned} \mathfrak{m}_{plmnq} &= \mathfrak{m}(x_1,\ldots,x_p;s_1,\ldots,s_l;t_1,\ldots,t_m;u_1,\ldots,u_n;y_1,\ldots,y_q) \\ &= \langle\emptyset|a(x_1)\cdots a(x_p)\mathrm{d}x_1\cdots\mathrm{d}x_p M_{lmn}(s_1,\ldots,s_l;t_1,\ldots,t_m;u_1,\ldots,u_n) \\ &\quad a^+(\mathrm{d}y_1)\cdots a^+(\mathrm{d}y_q)|\emptyset\rangle. \end{aligned}$$

Fix a Hilbert space $\mathfrak{k}$, and denote by $B(\mathfrak{k})$ the space of bounded operators on it. Consider a Lebesgue locally integrable function

$$F = (F_{lmn})_{lmn\in\mathbb{N}^3} : \mathfrak{R}^3 \to B(\mathfrak{k})$$
$$F_{lmn} = F_{lmn}(s_1,\ldots,s_l;t_1,\ldots,t_m;u_1,\ldots,u_n)$$

which is symmetric in the variables s_i, t_i and u_i, and two functions $f, g \in \mathscr{K}_{\mathrm{s}}(\mathfrak{R},\mathfrak{k})$,

$$f = f_p(x_1,\ldots,x_p)$$
$$g = g_q(y_1,\ldots,y_q).$$

We associate with F the sesquilinear form $\mathscr{B}(F)$ given by

$$\langle f|\mathscr{B}(F)|g\rangle = \sum \frac{1}{p!l!m!n!q!} \int \mathfrak{m}_{plmnq} f_p^+ F_{lmn} g_q$$

where f^+ denotes the adjoint vector to f. This formula may look terrifying, but it becomes more manageable by using multi-indices. It gives to Berezin's formula $(*)$ above a rigorous mathematical meaning, and it has the big advantage that it is a classical integral, so that we have all the tools of classical measure theory available.

These considerations can easily be generalized from $\mathbb{R}$ to any locally compact space X, and to an arbitrary measure λ on X instead of the Lebesgue measure. We will need that in Example 2 below.

The δ-function, or equivalently the point measure ε_0, can be approximated by measures continuous with respect to the Lebesgue measure. If $\varphi \geq 0$ is a continuous function of compact support on $\mathbb{R}$, with $\int \mathrm{d}x\ \varphi(x) = 1$, put

$$\varphi_\zeta(x) = \frac{1}{\zeta}\varphi\left(\frac{x}{\zeta}\right)$$

and

$$\varphi_\zeta^x(y) = \varphi_\zeta(x-y).$$

Then for $\zeta \downarrow 0$

$$\varphi_\zeta^x(y)\mathrm{d}x = \varphi_\zeta(x-y)\mathrm{d}x \to \varepsilon_y(\mathrm{d}x) = \delta(x-y)\mathrm{d}x$$

and

$$\varphi_\zeta^x(y)\mathrm{d}y = \varphi_\zeta(x-y)\mathrm{d}y \to \varepsilon_x(\mathrm{d}y) = \delta(x-y)\mathrm{d}y.$$

Recall

$$a^+(\varphi) = \int \varphi(x)a^+(\mathrm{d}x), \qquad a(\varphi) = \int \mathrm{d}x\ \varphi(x)a_x.$$

These were the operators defined above. We have

$$a^+\big(\varphi_\zeta^x\big)\mathrm{d}x \to a^+(\mathrm{d}x), \qquad a\big(\varphi_\zeta^x\big) \to a_x$$

since

$$\begin{aligned}&\big(a^+\big(\varphi_\zeta^x\big)\mathrm{d}x f\big)(x_0, x_1, \ldots, x_n)\\ &\quad = \big(\varphi_\zeta(x-x_0)f(x_1,\ldots,x_n) + \cdots + \varphi_\zeta(x-x_n)f(x_0,x_1,\ldots,x_{n-1})\big)\mathrm{d}x\\ &\quad \to \varepsilon_{x_0}(\mathrm{d}x)f(x_1,\ldots,x_n) + \cdots + \varepsilon_{x_n}(\mathrm{d}x)f(x_0,x_1,\ldots,x_{n-1}),\end{aligned}$$

and

$$\big(a\big(\varphi_\zeta^x\big)f\big)(x_1,\ldots,x_n) = \int \mathrm{d}x_0\varphi_\zeta^x(x_0)f(x_0,x_1,\ldots,x_n) \to f(x,x_1,\ldots,x_n).$$

In this context the operators $a^+(\varphi_\zeta^x)$ and $a(\varphi_\zeta^x)$ are called *coloured noise* operators, and the transition $\zeta \downarrow 0$ is called, for historical reasons, the *singular coupling limit*.

Without introducing any heavy apparatus we can treat four examples, where we restrict ourselves to the zero-particle case and to the one-particle case, i.e. just to the vacuum $|\emptyset\rangle$ and $L(1) = L^1(\mathbb{R}, \mathfrak{k})$, and do not need the whole Fock space.

1. *A two-level atom coupled to a heat bath of oscillators, or equivalently the damped oscillator*

   We restrict to the one-excitation case: We have either all oscillators in the ground state and the atom in the upper level, or one oscillator is in the first state and the atom is in the lower state. In the rotating wave approximation the Hamiltonian can be reduced to

$$H = \int \omega a^+(\mathrm{d}\omega) a(\omega) + E_{10} a^+(\varphi) + E_{01} a(\varphi),$$

   where

$$E_{01} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{10} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

   and φ is a continuous function ≥ 0, with compact support in $\mathbb{R}$, and $\int \mathrm{d}t \varphi(t) = 1$. We consider $a^+(\varphi)$ and $a^+(\varphi)$ as coloured noise operators, replace φ by φ_ζ, calculate the resolvent and perform the singular coupling limit. This means, in frequency space, that φ approaches 1 and not δ. Then the resolvent converges to the resolvent of a one-parameter strongly continuous unitary group on the space

$$\mathfrak{H} = \left(\mathbb{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathbb{C}|\emptyset\rangle \right) \oplus \left(\mathbb{C} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes L(1) \right).$$

   The one-parameter group can be calculated explicitly, then we obtain the Hamiltonian as a singular operator, and calculate the spectral decomposition of the Hamiltonian explicitly.

   After establishing a more general theory on the entire Fock space we recognize the interaction representation $V(t)$ of the time-development operator in the formal time representation as the restriction of U_0^t to $\mathfrak{H}$, where U_s^t is the solution of the quantum stochastic differential equation (QSDE)

$$\mathrm{d}_t U_s^t = -\mathrm{i}\sqrt{2\pi} E_{01} a^+(\mathrm{d}t) U_s^t - \mathrm{i}\sqrt{2\pi} E_{10} U_s^t a(t)\mathrm{d}t - \pi E_{11} \mathrm{d}t$$

   with $U_s^s = 1$; so U_s^t is an operator on

$$L^2\big(\mathfrak{R}, \mathbb{C}^2\big) \supset \mathfrak{H}.$$

2. *A two-level atom interacting with polarized radiation*

   This is very similar to the first example, but we have to consider not only the frequency but also the direction and the polarization of the photons. So for the photons we are concerned with the space

$$X = L^2\big(\mathbb{R} \times \mathbb{S}^2 \times \{1, 2, 3\}\big),$$

   where the first factor stands for the formal time (replacing the frequency via Fourier transform), the second one for the direction and the third one for the

polarization. We have added a fictional longitudinal polarization in order to make the calculations easier. We provide X with the measure

$$\langle \lambda | f \rangle = \iint \mathrm{d}t\omega_0^2 \mathrm{d}\mathbf{n} \sum_{i=1,2,3} f(t, \mathbf{n}, i),$$

where $\mathrm{d}\mathbf{n}$ is the surface element on the unit sphere such that

$$\int_{\mathbb{S}^2} \mathrm{d}\mathbf{n} = 4\pi$$

and ω_0 is the transition frequency. Define

$$\mathfrak{X} = \{\emptyset\} + X + X^2 + \cdots$$

and consider

$$\Gamma = L^2\big(\mathfrak{X}, \mathbb{C}^2\big).$$

Denote by $\Pi(\mathbf{n})$ the projector on the plane perpendicular to $\mathbf{n}$,

$$\Pi(\mathbf{n})_{ij} = \delta_{ij} - \mathbf{n}_i \mathbf{n}_j.$$

After some approximations we obtain the Hamiltonian

$$H = \int \mathrm{d}\mathbf{n}\omega_0^2 \omega \sum_{i,l} \Pi(\mathbf{n})_{i,l} a^+(\mathrm{d}\omega, \mathbf{n}, i) a(\omega, \mathbf{n}, l)$$
$$+ \int \mathrm{d}\mathbf{n}\omega_0^2 \varphi(\omega) \sum_{i,l} \Pi(\mathbf{k})_{i,l} \big(E_{10} q_i a(\omega, \mathbf{n}, l)\mathrm{d}\omega + E_{01} \overline{q}_i a^+(\mathrm{d}\omega, \mathbf{n}, l)\big)$$

where (q_1, q_2, q_3) is a vector proportional to the dipole moment. We perform the singular coupling limit via the resolvent, and arrive at a strongly continuous unitary one-parameter group on

$$\mathfrak{H} = \left(\mathbb{C}\begin{pmatrix}1\\0\end{pmatrix} \otimes \mathbb{C}|\emptyset\rangle\right) \oplus \left(\mathbb{C}\begin{pmatrix}0\\1\end{pmatrix} \otimes L^2(X, \lambda)\right).$$

We calculate the time evolution explicitly, calculate the Hamiltonian as a singular operator and give its spectral decomposition. If $V(t)$ is the interaction representation of the time evolution in a formal time representation, then $V(t)$ turns out to be the restriction of U_s^t to $\mathfrak{H}$. Here U_s^t is the solution of the differential equation

$$\mathrm{d}_t U_s^t = -\mathrm{i}\sqrt{2\pi} \int_{\mathbb{S}^2} \sum_{il} \Pi(\mathbf{n})_{il} \big(E_{01} \overline{q}_i a^+\big(\mathrm{d}(t, \mathbf{n}), l\big) U_s^t$$
$$+ E_{10} U_s^t q_i a(t, \mathbf{n}, l) \omega_0^2 \mathrm{d}\mathbf{n}\mathrm{d}t\big) - \pi\gamma E_{11} U_s^t \mathrm{d}t$$

with

$$\gamma = \frac{8\pi}{3}|\mathbf{q}|^2.$$

This is a new type of QSDE and should be investigated further.

3. *The Heisenberg equation of the amplified oscillator*

In the coloured noise approximation the Hamiltonian reads

$$H = \int \omega a^+(\mathrm{d}\omega)a(\omega) + \int b^+ a^+(\varphi) + \int ba(\varphi)$$

where b and b^+ are the usual oscillator operators with the non-vanishing commutator $[b, b^+] = 1$. Whereas the evolution corresponding to H is difficult and will be treated in Chap. 9, the Heisenberg evolution is very easy. Define

$$\mathfrak{H} = \mathbb{C}b^+ \oplus \big\{a(\psi) : \psi \in L^2(\mathbb{R})\big\},$$

then $\mathfrak{H}$ stays invariant under the mapping

$$A \mapsto \mathrm{e}^{\mathrm{i}Ht} A \mathrm{e}^{-\mathrm{i}Ht}.$$

Hence we obtain a one-parameter group on the space $\mathfrak{H}$. We perform the weak coupling limit via the resolvent and obtain, similarly to the first example, that evolution forms a strongly continuous one-parameter group on $\mathfrak{H}$. We identify $\mathfrak{H}$ with the $\mathfrak{H}$ of Example 1 and define E_{ij} accordingly. Then the interaction representation $V(t)$ of the evolution is the restriction to $\mathfrak{H}$ of the solution U_s^t to the QSDE

$$\mathrm{d}_t U_s^t = \mathrm{i}\sqrt{2\pi}\, a^+(\mathrm{d}t) E_{01} U_s^t - \mathrm{i}\sqrt{2\pi}\, E_{10} U_s^t a(t)\mathrm{d}t + \pi E_{11} U_s^t \mathrm{d}t.$$

We calculate the evolution on $\mathfrak{H}$ explicitly, determine the Hamiltonian and its spectral decomposition. Whereas this example looks algebraically very similar to the first one, it is analytically very different. The evolution is not unitary, but it does leave invariant the hermitian form

$$(c, f) \mapsto |c|^2 - \|f\|^2.$$

The spectrum of the Hamiltonian consists of the real line and the points $\pm\mathrm{i}\pi$.

4. *The pure number process*

We consider the coloured noise Hamiltonian

$$H = \int \omega a^+(\mathrm{d}\omega)a(\omega) + a^+(\varphi)a(\varphi).$$

The one-particle space $L(1) = L^2(\mathbb{R})$ stays invariant. We calculate on this subspace the resolvent, and determine the weak coupling limit. We again compute

the unitary one-parameter group, the Hamiltonian and its spectral decomposition. The interaction representation is the restriction of the solution of the QSDE

$$\mathrm{d}_t U_s^t = \frac{-\mathrm{i}2\pi}{1+\mathrm{i}\pi} a^+(\mathrm{d}t) U_s^t a(t).$$

After using coloured noise we establish a *white noise theory*. Then we attack *the general Hudson-Parthasarathy differential equation*, i.e., the QSDE

$$\mathrm{d}U_s^t = A_1 a^+(\mathrm{d}t) U_s^t + A_0 a^+(\mathrm{d}t) U_s^t a(t) + A_{-1} U_s^t a(t)\mathrm{d}t + B\mathrm{d}t$$

with $U_s^s = 1$. The solution can be given as an infinite power series in normal ordered monomials. The coefficients A_i, B are in $B(\mathfrak{k})$ for some Hilbert space $\mathfrak{k}$. If the coefficients satisfy some well-known conditions, the evolution is unitary. We give an explicit formula for the Hamiltonian. In Chap. 10 we show how this differential equation can be approximated by coloured noise.

In order to treat *the amplified oscillator* we investigate the QSDE

$$\mathrm{d}_t U_s^t = -\mathrm{i}a^+(\mathrm{d}t) b^+ U_s^t - \mathrm{i}b U_s^t a(t)\mathrm{d}t - \frac{1}{2} bb^+.$$

This is an example of a QSDE with unbounded coefficients. For this we need the white noise theory, and establish an infinite power series in normal ordered polynomials. Using an algebraic theorem due to Wick, we sum the series and obtain an a priori estimate. We prove unitarity, strong continuity and the Heisenberg evolution of Example 3. With the help of the Heisenberg evolution we get estimates which allow the calculation of the Hamiltonian.

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