Chapter 2
The GSH Distribution Family and Skew Versions

Abstract The generalized secant hyperbolic (GSH) distribution denotes a popular symmetric subclass of Perk's family which was already introduced in 1932. It allows for any kurtosis higher than 1.8 and, hence, admits both thin and fat tail behavior. Under a slightly different parameterization, the GSH family was re-examined by [1] who also derived additional properties. Based on the GSH family, there are three different proposals in the literature—related to Fischer and Vaughan [2], Fischer [3], and Vaughan [4]—how to additionally introduce skewness which are discussed within this chapter.

Keywords Definition and properties · Perk's distribution · Scale parameter split · Esscher transformation · Vaughan's skew version

2.1 Perk's Distribution Family

Already in 1932, the British actuary Wilfred Perks [5]—being interested in general functions for graduating life-table data—introduced a large class of probability densities of the form

$$f(x) = \frac{a_0 + a_1 e^{-x} + a_2 e^{-2x} + \cdots + a_m e^{-mx}}{b_0 + b_1 e^{-x} + b_2 e^{-2x} + \cdots + b_n e^{-nx}} \tag{2.1}$$

with parameters $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n$ such that f is actually a probability density. Setting $m = 1, a_0 = 0, a_1 = 1$ and $n = 2, b_0 = 1, b_1 = 0, b_2 = 1$, Eq. (2.1) reduces to hyperbolic secant distribution:

$$f(x) = \frac{2}{\pi} \cdot \frac{e^{-x}}{1 + e^{-2x}} = \frac{1}{\pi} \cdot \frac{1}{\cosh(x)}, \quad x \in \mathbb{R}.$$

M. J. Fischer, *Generalized Hyperbolic Secant Distributions*,
SpringerBriefs in Statistics, DOI: 10.1007/978-3-642-45138-6_2,

For $b_0 = b_2 = 2$, the logistic distribution is recovered. Slightly more generally, Talacko [6] discussed specific distribution families with $m = 1$, $a_0 = 0$ and $n = 2$, $b_0 = b_2$, i.e., densities of the form

$$f(x) = \frac{a_1 e^{-x}}{b_0 + b_1 e^{-x} + b_0 e^{-2x}} = \frac{c}{e^x + k + e^{-x}} = \frac{ce^x}{e^{2x} + ke^x + 1}, \qquad x \in \mathbb{R} \tag{2.2}$$

where $c \equiv a_1/b_0$ is a normalizing constant and $k \equiv b_1/b_0 > -2$ makes sure that (2.2) is actually a density. For $-2 < k \leq 2$ but $k \neq 0$ replace k in (2.2) by $2\cos(\lambda)$ with $0 \leq \lambda < \pi$. Talacko [6] calculated the corresponding characteristic functions as follows:

$$\begin{aligned}
\mathscr{C}(t) = \mathbb{E}(e^{itX}) &= c \int_{-\infty}^{\infty} \frac{e^{itx}dx}{e^x + 2\cos(\lambda) + e^{-x}} = c \int_{-\infty}^{\infty} \frac{e^{(it+1)x}dx}{e^{2x} + 2\cos(\lambda)e^x + 1} \\
&= c \int_{-\infty}^{\infty} \frac{e^{(it+1)x}dx}{(e^x + e^{i\lambda})(e^x + e^{-i\lambda})} = c \int_C \frac{e^{(it+1)z}dz}{(e^z + e^{i\lambda})(e^z + e^{-i\lambda})} \\
&= c \cdot \frac{\pi}{\sin(\lambda)} \cdot \frac{\sinh(\lambda t)}{\sinh(\pi t)} = \frac{\pi}{\lambda} \cdot \frac{\sinh(\lambda t)}{\sinh(\pi t)}.
\end{aligned}$$

Note that from $\lim_{t \to 0} \mathscr{C}(t) = 1$ we concluded that $c = \frac{\sin(\lambda)}{\lambda}$. For $k > 2$, replace λ by $i\theta$, i.e. k by $\cos(i\theta) = \cosh(\theta)$ in order to obtain with a similar calculation

$$\mathscr{C}(t) = \frac{\pi}{\theta} \cdot \frac{\sin(\theta t)}{\sinh(\pi t)} \quad \text{and} \quad c = \frac{\sinh(\theta)}{\theta}.$$

It took about 50 years until Talacko's generalized secant hyperbolic (GSH) distribution was re-examined by Vaughan [1] under the slightly different parameterization

$$k = k(\eta) = \begin{cases} \cos(\eta), & -\pi < \eta \leq 0, \\ \cosh(\eta), & \eta \geq 0 \end{cases}$$

and with scaling constant $c_2 = c_2(\eta)$ such that zero mean and unit variance is achieved:

$$f(x; \eta) = c_1(\eta) \cdot \frac{\exp(c_2(\eta)x)}{\exp(2c_2(\eta)x) + 2a(\eta)\exp(c_2(\eta)x) + 1} \tag{2.3}$$

$$= \frac{c_1(\eta)}{2\left(\cosh(c_2(\eta)x) + a(\eta)\right)} \tag{2.4}$$

with

$$\begin{aligned}
a(\eta) &= \cos(\eta), \quad c_2(\eta) = \sqrt{\tfrac{\pi^2 - \eta^2}{3}}, \quad c_1(\eta) = \tfrac{\sin(\eta)}{\eta} \cdot c_2(\eta) \quad \text{for } \eta \in (-\pi, 0], \\
a(\eta) &= \cosh(\eta), \quad c_2(\eta) = \sqrt{\tfrac{\pi^2 + \eta^2}{3}}, \quad c_1(\eta) = \tfrac{\sinh(\eta)}{\eta} \cdot c_2(\eta) \quad \text{for } \eta > 0.
\end{aligned}$$

Vaughan [1] also derived the cumulative distribution function, given by

$$F(x;\eta) = \begin{cases} 1 + \frac{1}{\eta}\operatorname{arccot}\left(-\frac{\exp(c_2(\eta)x)+\cos(\eta)}{\sin(\eta)}\right) & \text{for } \eta \in (-\pi, 0), \\ \frac{\exp(\pi x/\sqrt{3})}{1+\exp(\pi x/\sqrt{3})} & \text{for } \eta = 0, \\ 1 - \frac{1}{\eta}\operatorname{arccoth}\left(\frac{\exp(c_2(\eta)x)+\cosh(\eta)}{\sinh(\eta)}\right) & \text{for } \eta > 0 \end{cases}$$

and the inverse distribution function, given by

$$F^{-1}(u;\eta) = \begin{cases} \frac{1}{c_2(\eta)}\ln\left(\frac{\sin(\eta u)}{\sin(\eta(1-u))}\right) & \text{for } \eta \in (-\pi, 0), \\ \frac{\sqrt{3}}{\pi}\ln\left(\frac{u}{1-u}\right) & \text{for } \eta = 0, \\ \frac{1}{c_2(\eta)}\ln\left(\frac{\sinh(\eta u)}{\sinh(\eta(1-u))}\right) & \text{for } \eta > 0. \end{cases}$$

2.2 Properties of the GSH Family

The density from (2.3) is chosen so that the GSH variable has zero mean and unit variance, the range of the "kurtosis parameter" η is $\in (-\pi, \infty)$. Actually, Fischer and Klein [16] proved that the η is a kurtosis parameter in the sense of van Zwet [7]. The GSH distribution includes the logistic distribution ($\eta = 0$) and the hyperbolic secant distribution ($\eta = -\pi/2$) as special cases and the uniform distribution on $(-\sqrt{3}, \sqrt{3})$ as limiting case for $\eta \to \infty$. Figure 2.1 displays different densities and log-density. All densities are unimodal.

The moment-generating function also depends on η and is given by

$$\mathscr{M}(u;\eta) = \begin{cases} \frac{\pi}{\eta}\sin(u\eta/c_2(\eta))\csc(u\pi/c_2(\eta)) & \text{for } \eta \in (-\pi, 0), \\ \sqrt{3}u\csc(\sqrt{3}u) & \text{for } \eta = 0, \\ \frac{\pi}{\eta}\sinh(u\eta/c_2(\eta))\csc(u\pi/c_2(\eta)) & \text{for } \eta > 0. \end{cases}$$

It also satisfies (see Vaughan [1], p. 222)

$$\mathscr{M}(u;\eta) = \begin{cases} 1 + \frac{1}{2}u^2 + \frac{1}{4!}\frac{21\pi^2-9\eta^2}{5\pi^2-5\eta^2}u^4 + \mathscr{O}(u^5) & \text{for } \eta \in (-\pi, 0], \\ 1 + \frac{1}{2}u^2 + \frac{1}{4!}\frac{21\pi^2+9\eta^2}{5\pi^2+5\eta^2}u^4 + \mathscr{O}(u^5) & \text{for } \eta > 0 \end{cases}$$

establishing that $Var(X) = 1$, so that the kurtosis coefficient $m_4 = \mathbb{E}(X^4)$ is

$$m_4 = \begin{cases} \frac{21\pi^2-9\eta^2}{5\pi^2-5\eta^2} & \text{for } \eta \in (-\pi, 0], \\ \frac{21\pi^2+9\eta^2}{5\pi^2+5\eta^2} & \text{for } \eta > 0. \end{cases}$$

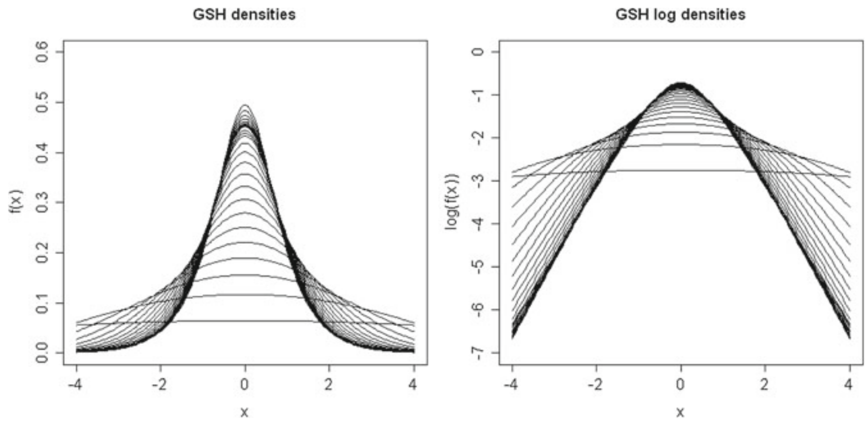

Fig. 2.1 GSH distribution: Log-density, density for different $\eta \in [-\pi, 5]$

It is readily apparent that m_4 decreases as η tends to ∞ and that $m_4 \in (1.8, \infty)$. Vaughan [1] also states that there is a unique member of the GSH family that corresponds to any given kurtosis for regular unimodal distributions:

$$\eta = -\pi\sqrt{\frac{5m_4 - 21}{5m_4 - 9}} \text{ for } m_4 \geq 4.2 \text{ and } \eta = \pi\sqrt{\frac{21 - 5m_4}{5m_4 - 9}} \text{ for } m_4 \leq 4.2.$$

In particular, when $\eta = \pi$ then $m_4 = 3$, the kurtosis of a normal distribution. Note also that if ν denotes the degrees of freedom for a Student-t distribution with a given (finite) kurtosis, then the parameter η in the GSH family with the same first four moments is $-\pi\sqrt{(9-\nu)/(\nu+1)}$ for $4 < \nu < 9$, 0 for $\nu = 9$ and $\pi\sqrt{(\nu-9)/(\nu+1)}$ for $\nu > 9$.

2.3 Introducing Skewness by Splitting the Scale Parameter

The first skew version of Vaughan's GSH distribution was proposed by Fischer and Vaughan [2]. Application to unconditional and conditional financial return models followed up with Fischer [8] and Palmitesta and Provasi [9]. The main idea of this approach is to split the scale parameter of the GSH distribution into two parameters representing the left and the right part across the expectation value. Note that this idea was already used by Fernández et al. [10] and Fernández and Steel [11] in order to design a skew Student-t distribution. Let $\mathbf{I}^+(x)$ denote the indicator function for x on $\mathbb{R}_+$ and $\mathbf{I}^-(x)$ the indicator function for x on $\mathbb{R}_-$. It follows that

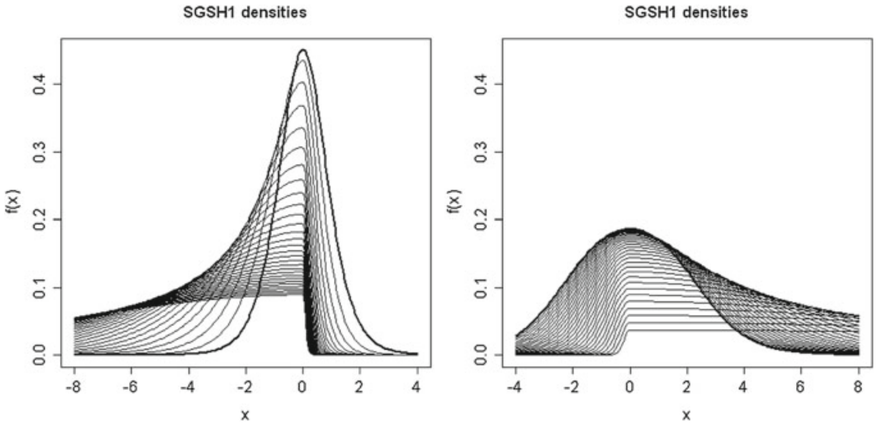

Fig. 2.2 SGSH1 distribution: Effect of γ for fixed $\eta = 0.2$ and $\gamma \in [1, 10]$ (*left panel*) and $\eta = 2.5$ and $\gamma \in [0.1, 1]$ (*right panel*)

$$f(x; \eta, \gamma) = \frac{2\gamma}{\gamma^2 + 1} \left\{ f_{GSH}(x/\gamma; \eta) \cdot \mathbf{I}^-(x) + f_{GSH}(\gamma x; \eta) \cdot \mathbf{I}^+(x) \right\}$$

$$= \frac{2c_1}{\gamma + \frac{1}{\gamma}} \cdot \left(\frac{\exp(c_2 x/\gamma) \cdot \mathbf{I}^-(x)}{\exp(2c_2 x/\gamma) + 2a \exp(c_2 x/\gamma) + 1} + \frac{\exp(c_2 \gamma x) \cdot \mathbf{I}^+(x)}{\exp(2c_2 \gamma x) + 2a \exp(c_2 \gamma x) + 1} \right)$$

is a density function which is symmetric for $\gamma = 1$, skewed to the right for $\gamma > 1$ and skewed to the left for $0 < \gamma < 1$. The corresponding distribution will be termed the *skewed GSH distribution of type I* (SGSH1) in the sequel. The effect of γ on the GSH density is illustrated in Fig. 2.2.

Following Fischer [3], both cumulative distribution function and quantile function admit closed forms, namely

$$F(x; \eta, \gamma) = \frac{2\gamma^2}{\gamma^2 + 1} \cdot \left(F_{GSH}(x/\gamma) \cdot \mathbf{I}^-(x) + \left(\frac{\gamma^2 - 1 + 2F_{GSH}(\gamma x)}{2\gamma^2} \right) \cdot \mathbf{I}^+(x) \right),$$

$$F^{-1}(x; \eta, \gamma) = \gamma F_{GSH}^{-1}\left(x \cdot \frac{\gamma^2 + 1}{2\gamma^2}; \eta \right) \mathbf{I}_A^-(x) + \frac{1}{\gamma} F_{GSH}^{-1}\left(x \cdot \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}; \eta \right) \mathbf{I}_A^+(x).$$

with

$$\mathbf{I}_A^-(x) = \begin{cases} 1, & \text{if } x < \frac{\gamma^2}{1+\gamma}, \\ 0, & \text{if } x \geq \frac{\gamma^2}{1+\gamma}. \end{cases} \quad \text{and} \quad \mathbf{I}_A^+(x) = 1 - \mathbf{I}_A^-(x).$$

Referring to Fernández and Steel [11], the power moments of an SGSH1-variable Z can be derived using the following calculation scheme:

$$\mathbb{E}(Z^r) = \mathbb{E}^+(X^r) \cdot \frac{2\gamma}{\gamma^2+1} \cdot \left(\gamma^{-r-1} + (-1)^r \gamma^{r+1}\right) \text{ with } \mathbb{E}^+(X^r) \equiv \int_0^\infty x^r f_{GSH}(x)dx.$$

Evidently, $\mathbb{E}^+(X^r)$ equals the r-th power moment of the GSH distribution (which can be obtained from the corresponding moment-generating function) divided by two when r is even. For odd r and $t \neq 0$, Palmitesta and Provasi [9] derive the following expression[1]:

$$\mathbb{E}^+(X^r) = \frac{c_1 \Gamma(r+1)}{2c_2^{r+1}\sqrt{a^2-1}} \cdot \mathscr{L}_{r,a}$$

defining

$$\mathscr{L}_{r,a} \equiv \left[\mathscr{L}_{r+1}\left(-\frac{1}{\sqrt{a^2-1}+a}\right) - \mathscr{L}_{r+1}\left(\frac{1}{\sqrt{a^2-1}-a}\right)\right], \text{ where } \mathscr{L}_r(x) \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^r}$$

denotes the *polylogarithmic function* (see Lewin [12]) which is defined for $x \in \mathbb{C}$ and $|x| < 1$. Consequently, the first four power moments derive as

$$\mathbb{E}(Z) = \frac{c_1 \mathscr{L}_{1,a}(1-\gamma^2)}{\gamma c_2^2 \sqrt{a^2-1}}, \quad \mathbb{E}(Z^2) = \frac{\gamma^4 - \gamma^2 + 1}{\gamma^2},$$

$$\mathbb{E}(Z^3) = \frac{6c_1 \mathscr{L}_{2,a}(1-\gamma^6+\gamma^4-\gamma^2)}{\gamma^3 c_2^4 \sqrt{a^2-1}}$$

and

$$\mathbb{E}(Z^4) = \frac{(21\pi^2 + \text{sgn}(t)9t^2)(1+\gamma^8-\gamma^2-\gamma^6+\gamma^4)}{\gamma^4(5\pi^2+\text{sgn}(t)5\eta^2)}.$$

Using the relationship between the centered and uncentered moments (e.g., Stuart and Ord [13]), the derivation of the third and fourth standardized moments is tedious but straightforward. As expected, both η and γ determine the skewness and the kurtosis. It should also be noted that all other higher moments exist.

2.4 Introducing Skewness by Means of the Esscher Transformation

A second way to introduce skewness into Vaughan's GSH distribution was recently discussed by Fischer [3], who used the existence of the moment-generating function to construct asymmetric densities by means of the so-called *Esscher transformation*.

[1] We exclude the case $\eta = 0$, which corresponds to the logistic distribution and refers to Palmitesta and Provasi [9], instead.

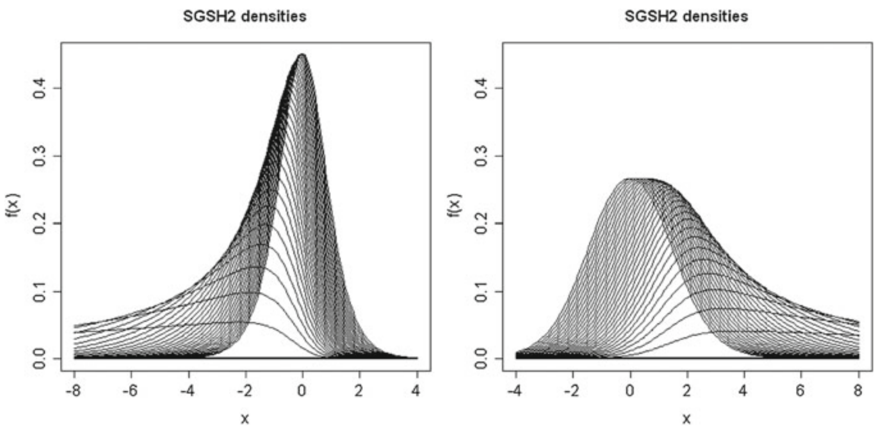

Fig. 2.3 SGSH2 distribution: h for fixed $\eta = 0.2$ and $h \in [-1, 0]$ (*left panel*), and $\eta = 2$ and $h \in [0, 1]$ (*right panel*)

Esscher transformation: Originally, this concept was a tool in actuarial science suggested by Esscher [14], which was popularized by Gerber and Shiu [15] who applied this concept to value derivative securities. Given a random variable X with moment-generating function $\mathscr{M}_X(t)$ and density $f_X(x)$, the Esscher-transformed density with parameter h is defined by

$$f(x; h) \equiv \exp(hx) f(x)/\mathscr{M}(h). \tag{2.5}$$

Note that if X is Gaussian, the resulting Esscher-transformed variable is again Gaussian (and thus symmetric) but with different scale and location. In contrast, Esscher-transformations of symmetric non-Gaussian densities frequently produce asymmetric distributions, where $h \neq 0$ governs the amount of skewness, and symmetry is obtained for $h = 0$.

Plugging (2.3) into (2.5), the Esscher-transformed GSH density for $h \neq 0$ derives as

$$f(x; \eta, h) \equiv \begin{cases} \frac{\sin(h\pi)\sin(\eta)}{\pi\sin(ht)} \cdot \frac{\exp((h+1)x)}{\exp(2x)+2\cos(\eta)\exp(x)+1} & \text{for } -\pi < \eta < 0, \\ \frac{\sin(h\pi)}{h\pi} \cdot \frac{\exp((h+1)x)}{\exp(2x)+2\exp(x)+1} & \text{for } \eta = 0, \\ \frac{\sin(h\pi)\sinh(\eta)}{\pi\sinh(h\eta)} \cdot \frac{\exp((h+1)x)}{\exp(2x)+2\cosh(\eta)\exp(x)+1} & \text{for } \eta > 0 \end{cases} \tag{2.6}$$

and will be termed as *skew GSH densities* of type II, or briefly SGSH2 densities in the sequel. Examples of SGSH2 densities are plotted in Fig. 2.3.

Note that for $-\pi < \eta < 0$ and $h \neq 0$ we can derive the corresponding *moment-generating function* of a SGSH2 variable X,

$$
\begin{aligned}
\mathbb{E}(e^{uX}) &= \int_{-\infty}^{\infty} \frac{\sin(h\pi)\sin(\eta)}{\pi\sin(ht)} \frac{\exp((h+u+1)x)}{\exp(2x)+2\cos(\eta)\exp(x)+1} dx \\
&= \frac{\sin((h+u)\eta)}{\sin((h+u)\pi)} \frac{\sin(h\pi)}{\sin(h\eta)} \int_{-\infty}^{\infty} \frac{\sin((h+u)\pi)\sin(\eta)\exp(((h+u)+1)x)dx}{\pi\sin((h+u)\eta)(\exp(2x)+2\cos(\eta)\exp(x)+1)}. \\
&= \frac{\sin((h+u)\eta)}{\sin((h+u)\pi)} \frac{\sin(h\pi)}{\sin(h\eta)}.
\end{aligned}
$$

Similar reformulations hold for $\eta \geq 0$ and we finally arrive at

$$
\mathcal{M}(u) = \begin{cases} (\sin((h+u)\eta)\sin(h\pi)) \,/\, (\sin((h+u)\pi)\sin(h\eta)) & \text{for } -\pi < \eta < 0, \\ ((h+u)\cdot\sin(h\pi)) \,/\, (h\sin((h+u)\pi)) & \text{for } \eta = 0, \\ (\sinh((h+u)\eta)\sin(h\pi)) \,/\, (\sin((h+u)\pi)\sinh(h\eta)) & \text{for } \eta > 0. \end{cases}
$$

All moments of the SGSH2 distribution exist. In particular, the first four power moments are given by

$$
\mathbb{E}(X) = \begin{cases} \eta\cot(h\eta) - \pi\cot(h\pi) & \text{for } -\pi < \eta < 0, \\ (1 - h\pi\cot(h\pi))/h & \text{for } \eta = 0, \\ \eta\coth(h\eta) - \pi\cot(h\pi) & \text{for } \eta > 0, \end{cases}
$$

$$
\mathbb{E}(X^2) = \begin{cases} \pi^2 - \eta^2 - 2\eta\pi\cot(h\eta)\cot(h\pi) + 2\pi^2\cot^2(h\pi) & \text{for } -\pi < \eta < 0, \\ \pi^2 - 2\pi/h\cdot\cot(h\pi) + 2\pi^2\cot^2(h\pi) & \text{for } \eta = 0, \\ \eta^2 + \pi^2 - 2\eta\pi\coth(h\eta)\cot(h\pi) + 2\pi^2\cot^2(h\pi) & \text{for } \eta > 0, \end{cases}
$$

$$
\mathbb{E}(X^3) = \begin{cases} -\eta^3\cot(h\eta) + 3\eta^2\pi\cot(h\pi) + 6\eta\pi^2\cot(h\eta)\cot^2(h\pi) + 3\eta\pi^2\cot(h\eta) \\ \quad -6\pi^3\cot^3(h\pi) - 5\pi^3\cot(h\pi) \quad \text{for } -\pi < \eta < 0, \\ 6\pi^2/h\cdot\cot^2(h\pi) + 3\pi^2/h - 6\pi^3\cot^3(h\pi) - 5\pi^3\cot(h\pi) \quad \text{for } \eta = 0, \\ \eta^3\coth(h\eta) - 3\eta^2\pi\cot(h\pi) + 6\eta\pi^2\coth(h\eta)\cot^2(h\pi) + 3\eta\pi^2\coth(h\eta) \\ \quad -6\pi^3\cot^3(h\pi) - 5\pi^3\cot(h\pi) \quad \text{for } \eta > 0, \end{cases}
$$

$$
\mathbb{E}(X^4) = \begin{cases} \eta^4 + 5\pi^4 - 4\eta^3\pi\coth(h\eta)\cot(h\pi) + 12\eta^2\pi^2\cot^2(h\pi) + 6\eta^2\pi^2 \\ \quad -24\eta\pi^3\coth(h\eta)\cot^3(h\pi) - 20\eta\pi^3\coth(h\eta)\cot(h\pi) \\ \quad +24\pi^4\cot^4(h\pi) + 28\pi^4\cot^2(h\pi) \quad \text{for } -\pi < \eta < 0. \\ 5\pi^4 - 24\pi^3/h\cdot\cot^3(h\pi) - 20\pi^3/h\cot(h\pi) + 24\pi^4\cot^4(h\pi) \\ \quad +28\pi^4\cot^2(h\pi) \quad \text{for } \eta = 0, \\ \eta^4 + 5\pi^4 + 4\eta^3\pi\cot(h\eta)\cot(h\pi) - 12\eta^2\pi^2\cot^2(h\pi) - 6\eta^2\pi^2 \\ \quad -24\eta\pi^3\cot(h\eta)\cot^3(h\pi) - 20\eta\pi^3\cot(h\eta)\cot(h\pi) + 24\pi^4\cot^4(h\pi) \\ \quad +28\pi^4\cot^2(h\pi) \quad \text{for } \eta > 0, \end{cases}
$$

Consequently, the variance of an SGSH2 variable is given by

$$Var(X) = \begin{cases} \pi^2(1+\cot^2(h\pi)) - \eta^2(1+\cot^2(h\eta)) & \text{for} \quad -\pi < \eta < 0, \\ \pi^2(1+\cot^2(h\pi)) - 1/h^2, & \text{for} \quad \eta = 0, \\ \pi^2(1+\cot^2(h\pi)) + \eta^2(1-\coth^2(h\eta)) & \text{for} \quad \eta > 0. \end{cases}$$

2.5 Vaughan's Skew Extension

Recently, Vaughan [4] advocated skew-extended GSH (S-EGSH) distribution families as a natural generalization of the GSH representative. For constants $c > h \geq 0$, $k, \omega q > 0$ and parameters a and b satisfying either $\omega kb > a > 0$ or $0 > a > \omega kb$ Vaughan [4] discusses, e.g.,

$$f(x) = C_1 \frac{\exp(ax)}{\left[(\exp(bx)+c)^k - h^k\right]^{\omega}} \tag{2.7}$$

with normalizing constant

$$C_1 = bc^{-\lambda+k\omega}\left[\sum_{j=0}^{\infty} \frac{\Gamma(\omega+j)}{\Gamma(\omega)j!}\kappa^{kj} B(\lambda, k(j+\omega)-\lambda)\right]^{-1},$$

where $B(a,b)$ denotes the Beta function, $\lambda = a/b$ and $\kappa = h/c$. The original GSH family in (2.4) is recovered by setting

$$a = b = c_2(\eta),\ \ c = a(\eta),\ \ h = \sqrt{a(\eta)^2 - 1},\ \ k = 2,\ \ w = 1$$

in (2.7). The corresponding cumulative function is given by

$$F(x) = C_1 \frac{\sum_{j=0}^{\infty} \frac{\Gamma(\omega+j)}{j!}\kappa^{kj} B_{\tau(u)}(\lambda, k(j+\omega)-\lambda)}{\sum_{j=0}^{\infty} \frac{\Gamma(\omega+j)}{j!}\kappa^{kj} B(\lambda, k(j+\omega)-\lambda)} \text{ for } \tau(u) \equiv \frac{c}{\exp(bu)+c}$$

and where $B_u(a,b)$ denotes the incomplete Beta function. The above conditions on the parameters ensure the densities are positive, and further that the distribution has well-defined moment-generating functions, and hence all moments finite. They can be expressed in terms of the Gamma function and its derivatives. For all S-EGSH members there is a unique mode (Fig. 2.4).

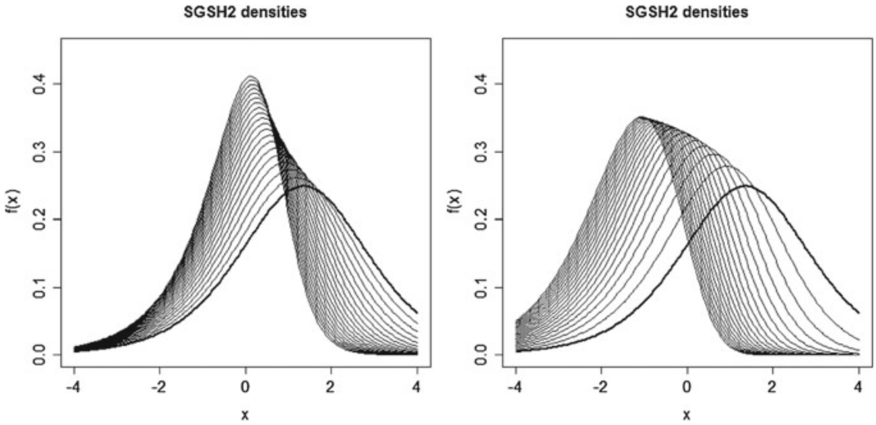

Fig. 2.4 S-EGSH distribution: Densities with $a = 1$, $C = 4$, $h = 1$, $k = 2$, $w = 1$ and $b \in [1, 2]$ (*left panel*), and $a = 1$, $b = 1$, $C = 4$, $h = 1$, $k = 2$ and $w \in [1, 6]$ (*right panel*)

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