

Chapter 1

Introduction

1.1 Superconducting Phenomena

The superconductivity is a phenomenon that was discovered first for mercury by Kamerlingh-Onnes in 1911 and has been found for various elements, alloys and compounds. One of the features of superconductivity is that the electric resistance of a material suddenly drops to zero as the temperature decreases through a transition point; such a material is called superconductor. Advantage is taken of this property in the application of the superconducting phenomenon to technology. Later it was found that the origin of the zero electric resistivity is not the perfect conductivity as such but the perfect diamagnetism, i.e., the ability of the superconductor to completely exclude a weak applied magnetic field. A related phenomenon is the complete expulsion of a weak applied magnetic field as the temperature decreases through the transition point. These diamagnetic phenomena are called the Meissner effect. As will be mentioned later, the perfect diamagnetism is broken at sufficiently high magnetic fields. There are two alternative ways in which this break down can take place depending on whether the superconductor is “type-1” or “type-2.” In type-2 materials the superconductivity can be maintained up to very high fields even after the break down. Such superconductors are therefore suitable for use in high-field devices such as magnets, motors, and generators.

Another characteristic of superconductivity is the existence of a gap, just below the Fermi energy, of the energy of the conduction electrons. It turns out that the electron energy in the superconducting state is lower than that in the ground state of the normal state; the difference in the energy per electron between the two states is the energy gap. The size of the energy gap in the superconductor can be measured using the absorption of microwave radiation in the far infrared range, or the tunneling effect of a junction composed of a superconductor and a normal metal separated by a thin insulating layer. In the case of a sufficiently small excitation, the energy gap provides a barrier against the transition of electrons from the superconducting state to the normal state. That is, even when the electrons are scattered by lattice defects, impurities, or thermally oscillating ions, the energy may not be dissipated and hence, electric resistance may not appear. It was theoretically proved by Bardeen,

Cooper and Schrieffer in 1957 that the electrons in the vicinity of the Fermi level exist in so-called Cooper pairs whose condensation yields the superconducting state. This is essentially the BCS theory of superconductivity.

Another essential property of type-2 superconductors is embodied in the so-called Josephson effect. In a junction composed of two superconductors separated by a thin insulating layer, the local property of type-2 superconductor can be directly observed without being averaged. The DC Josephson effect that predicts the superconducting tunneling current is not the tunneling of normal electrons but the tunneling of the Cooper pairs described by a macroscopic wave function. The effect demonstrates that the superconducting state is a coherent state in which the phase of a macroscopic wave function, which is introduced later as the order parameter of the Ginzburg-Landau (G-L) theory, is uniform in the superconductor. In this state the quantum mechanical property is maintained up to a macroscopic scale and the gauge-invariant relation is kept between the macroscopic wave function and the vector potential. This leads to the macroscopic quantization of magnetic flux through the quantization of the total angular momentum, and this phenomenon can be directly seen from the interference of the superconducting tunneling current due to the magnetic field. Another important result, the AC Josephson effect, describes the relation between the time variation rate of the phase of macroscopic wave function and the voltage which in this case appears across the junction. This voltage comes from the motion of the quantized magnetic flux and is identical to the voltage observed in a type-2 superconductor in the flux flow state as will be shown later.

The superconducting state transforms into the normal state at temperatures above the critical temperature and magnetic fields above the critical field. Transitions from the superconducting state to the normal state and vice versa are phase transitions comparable, for example, to the transition between ferromagnetism and paramagnetism. From a microscopic viewpoint, the Cooper-pair condensation of electrons (which can be compared to a Bose-Einstein condensation of Bose particles) results in the superconducting state and the electron energy gap that exists between the superconducting and normal states. From a macroscopic viewpoint, on the other hand, the superconducting state is a thermodynamic phase and thermodynamics is useful in the description of the phenomenon. Finally, since the electron state is coherent in the superconducting state, the G-L theory, in which the order parameter defined as a superposed wave function of coherent superconducting electrons is used. Among its many applications it is suitable for describing the magnetic properties of type-2 superconductors.

In 1986 a La-based copper-oxide superconductor with a higher critical temperature than metallic superconductors was discovered by Bednortz and Müller. Taking advantage of this break through, numerous so-called “high-temperature” superconductors with even higher critical temperatures but containing Y, Bi, Tl and Hg instead of La were discovered. The exact mechanism of superconductivity in these materials is not yet understood. We have yet to wait for a suitable microscopic explanation. However, the macroscopic electromagnetic properties of high-temperature superconductors have been found to be describable phenomenologically in a manner

comparable to those previously applied to metallic superconductors. In this description, the characteristic features of high-temperature superconductors are a large two-dimensional anisotropy originating in the crystal structure and a strong fluctuation effect. The latter feature results from a short coherence length in associating with the high critical temperature, the quasi-two-dimensionality itself, and the condition of high temperature. It was shown theoretically that, as a result of the fluctuation effect, the phase boundary between the superconducting and normal states derived using G-L theory within a mean field approximation is not clear. It follows that a G-L description would be correct only in the region far from the phase boundary. However, because these materials have such high upper critical fields G-L theory is still valid over a wide practical range of temperature and magnetic field.

This book is based on the G-L theory that describes the superconductivity phenomenologically and the Maxwell theory that is the foundation of the electromagnetism. The SI units and the \mathbf{E} - \mathbf{B} analogy are used.

1.2 Kinds of Superconductors

There are two kinds of superconductor—type-1 and type-2. These are classified with respect to their magnetic properties.

The magnetization of a type-1 superconductor is shown in Fig. 1.1(a). When the external magnetic field H_e is lower than some critical field H_c , the magnetization is given by

$$M = -H_e \quad (1.1)$$

and the superconductor shows a perfect diamagnetism ($B = 0$). It is in the Meissner state. The transition from the superconducting state to the normal state occurs at $H_e = H_c$ with a discontinuous variation in the magnetization to $M = 0$ (i.e. $B = \mu_0 H_e$ with μ_0 denoting the permeability of vacuum). For a type-2 superconductor, on the other hand, the perfect diamagnetism given by (1.1) is maintained only up to the lower critical field, H_{c1} , and then the magnetization varies continuously with

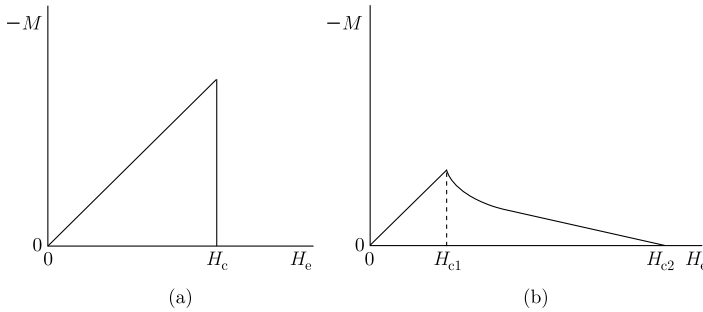


Fig. 1.1 Magnetic field dependence of magnetization for (a) type-1 superconductor and (b) type-2 superconductor

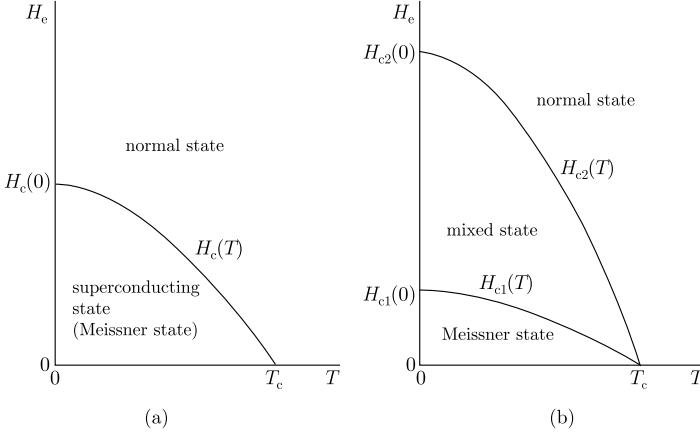


Fig. 1.2 Phase diagram on the magnetic field vs. temperature plane for (a) type-1 superconductor and (b) type-2 superconductor

the penetration of magnetic flux as shown in Fig. 1.1(b) until the diamagnetism disappears at the upper critical field, H_{c2} , where the normal state starts. The partially diamagnetic state between H_{c1} and H_{c2} is called the mixed state. Since the magnetic flux in the superconductor is quantized in the form of “vortices” in this state, it is also called the vortex state.

It is empirically known that the critical field of type-1 superconductors varies with temperature according to

$$H_c(T) = H_c(0) \left[1 - \left(\frac{T}{T_c} \right)^2 \right]. \quad (1.2)$$

The lower and upper critical fields of type-2 superconductors show similar temperature dependences. Obviously they reduce to zero at the critical temperature, T_c . Strictly speaking for type-2 superconductors, whereas the thermodynamic critical field, H_c , shows the temperature dependence of (1.2), that of H_{c2} deviates from this relationship for some superconductors. Especially in high-temperature superconductors and MgB_2 these critical fields have almost linear temperature dependences even in the low temperature region. The phase diagrams of type-1 and type-2 superconductors on the temperature-magnetic field plane are shown in Fig. 1.2(a) and 1.2(b). The superconducting parameters of various superconductors are listed in Table 1.1. Here H_c in type 2 superconductors is the thermodynamic critical field. Since H_{c1} and H_{c2} in high-temperature superconductors and MgB_2 are significantly different depending on the direction of magnetic field with respect to the crystal axes, the doping state of carriers and the electron mean free path, only the value of H_c in the optimally doped state is given in the table. The details of the anisotropy and the dependence on such factors for the critical fields in these superconductors will be described in Sects. 8.1 and 9.1.

Table 1.1 Critical parameters of various superconductors

Superconductor		T_c (K)	$\mu_0 H_c(0)$ (mT)	$\mu_0 H_{c1}(0)$ (mT)	$\mu_0 H_{c2}(0)$ (T)
type-1	Hg(α)	4.15	41	—	—
	In	3.41	28	—	—
	Pb	7.20	80	—	—
	Ta	4.47	83	—	—
type-2	Nb	9.25	199	174	0.404
	Nb ₃₇ Ti ₆₃	9.08	253		15
	Nb ₃ Sn	18.3	530		29
	Nb ₃ Al	18.6			33
	Nb ₃ Ge	23.2			38
	V ₃ Ga	16.5	630		27
	V ₃ Si	16.9	610		25
	PbMo ₆ S ₈	15.3			60
	MgB ₂	39	660		
	YBa ₂ Cu ₃ O ₇	93	1270		
	(Bi,Pb) ₂ Sr ₂ Ca ₂ Cu ₃ O _x	110			
	Tl ₂ Ba ₂ Ca ₂ Cu ₃ O _x	127			
	HgBa ₂ CaCu ₂ O _x	128	700		
	HgBa ₂ Ca ₂ Cu ₃ O _x	138	820		

The practical superconducting materials, Nb-Ti and Nb₃Sn, belong to the type-2 class. Their upper critical fields are very high and hence, their superconducting state can be maintained up to high fields. In high-temperature superconductors the upper critical fields are extremely high and it is considered that a clear phase transition to the normal state does not occur due to the effect of significant fluctuation at the phase boundary, $H_{c2}(T)$, derived from the G-L theory.

1.3 London Theory

The fundamental electromagnetic properties of superconductors, such as the Meissner effect, can be described by a phenomenological theory first propounded by the London brothers in 1935, even before the discovery of type-2 superconductors. Fortunately this theory turned out to be a good approximation for type-2 superconductors with high upper critical fields, or with large values of G-L parameter; several important characteristics of such superconductors can be derived from this theory. So, we shall here briefly introduce the classic London theory.

A steady persistent current can flow through superconductors. Hence, the classical equation of motion of superconducting electrons should be one that can describe

this state. In other words, the deviation from the steady motion, i.e., the acceleration of superconducting electrons is done only by the force due to the electric field. Hence, the equation of motion is given by

$$m^* \frac{d\mathbf{v}_s}{dt} = -e^* \mathbf{e}, \quad (1.3)$$

where m^* , \mathbf{v}_s and $-e^*$ are the mass, the velocity and the electric charge ($e^* > 0$) of the superconducting electron, and \mathbf{e} is the electric field. If the number density of superconducting electrons is represented by n_s , the superconducting current density is written as

$$\mathbf{j} = -n_s e^* \mathbf{v}_s. \quad (1.4)$$

Substitution of this into (1.3) leads to

$$\mathbf{e} = \frac{m^*}{n_s e^{*2}} \cdot \frac{d\mathbf{j}}{dt}. \quad (1.5)$$

If the magnetic field and the magnetic flux density are denoted by \mathbf{h} and \mathbf{b} , respectively, the Maxwell equations are

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t} \quad (1.6)$$

and

$$\nabla \times \mathbf{h} = \mathbf{j}, \quad (1.7)$$

where the displacement current is neglected in (1.7). From these equations and with

$$\mathbf{b} = \mu_0 \mathbf{h}, \quad (1.8)$$

the rotation (curl) of (1.5) is written as

$$\frac{\partial}{\partial t} \left(\mathbf{b} + \frac{m^*}{\mu_0 n_s e^{*2}} \nabla \times \nabla \times \mathbf{b} \right) = 0. \quad (1.9)$$

Thus, the quantity in the parenthesis on the left hand side of (1.9) is a constant. The London brothers showed that, when this constant is zero, the Meissner effect can be explained. That is,

$$\mathbf{b} + \frac{m^*}{\mu_0 n_s e^{*2}} \nabla \times \nabla \times \mathbf{b} = 0. \quad (1.10)$$

Equations (1.5) and (1.10) are called the London equations. Replacing $\nabla \times \nabla \times \mathbf{b}$ by $-\nabla^2 \mathbf{b}$ (since $\nabla \cdot \mathbf{b} = 0$), (1.10) may be written

$$\nabla^2 \mathbf{b} - \frac{1}{\lambda^2} \mathbf{b} = 0, \quad (1.11)$$

where λ is a quantity with the dimension of length defined by

$$\lambda = \left(\frac{m^*}{\mu_0 n_s e^{*2}} \right)^{1/2}. \quad (1.12)$$

Let us assume a semi-infinite superconductor of thickness $x \geq 0$. When an external magnetic field H_e is applied along the z -axis parallel to the surface ($x = 0$), it is reasonable to assume that the magnetic flux density has only a z -component which varies only along the x -axis. Then, (1.11) reduces to

$$\frac{d^2 b}{dx^2} - \frac{b}{\lambda^2} = 0. \quad (1.13)$$

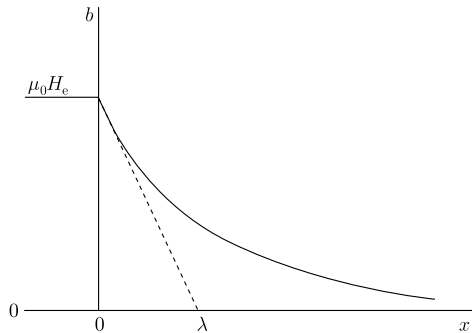
This can be easily solved; and under the conditions that $b = \mu_0 H_e$ at $x = 0$ and is finite at infinity, we have

$$b(x) = \mu_0 H_e \exp\left(-\frac{x}{\lambda}\right) \quad (1.14)$$

This result shows that the magnetic flux penetrates the superconductor only a distance of the order of λ from the surface (see Fig. 1.3). The characteristic distance λ is called the penetration depth. Since the “superconducting electron” is by now well known to be an electron pair, we assign a double electronic charge to e^* , i.e., $e^* = 2e = 3.2 \times 10^{-19}$ C. As for the mass of the superconducting electron, m^* , we assume also a double electron mass in spite of an ambiguity in the mass. Thus $m^* = 2m = 1.8 \times 10^{-30}$ kg. If we substitute a typical free-electron number density for n_s viz. 10^{28} m^{-3} , then $\lambda \simeq 37 \text{ nm}$ from (1.12) and the above quantities. Observed values of λ are of the same order of magnitude as this estimation. Thus, the magnetic flux does not penetrate much below the surface of the superconductor, thereby explaining the Meissner effect. From (1.7), (1.8) and (1.14) it is found that the current is also localized and flows along the y -axis according to

$$j(x) = \frac{H_e}{\lambda} \exp\left(-\frac{x}{\lambda}\right). \quad (1.15)$$

Fig. 1.3 Magnetic flux distribution near the surface of superconductor in the Meissner state



This so-called Meissner current shields the external magnetic field thereby supporting the Meissner effect.

We note that the London equations, (1.5) and (1.10), may be derived just from

$$\mathbf{j} = -\frac{n_s e^{*2}}{m^*} \mathbf{A}, \quad (1.16)$$

where \mathbf{A} is the magnetic vector potential. That is, (1.5) and (1.10) respectively can be obtained by differentiation with respect to time and taking the rotation of (1.16). Equation (1.16) means that the current density of an arbitrary point is determined by the local vector potential at that point. On the other hand, superconductivity is a nonlocal phenomenon and the wave function of electrons is spatially extended. The electrons that contribute to the superconductivity are those within an energy range of the order of $k_B T$ at the Fermi level with k_B denoting the Boltzmann constant. Hence, the uncertainty of the momentum of electrons is $\Delta p \sim k_B T / v_F$ where v_F is the Fermi velocity. Hence, the spatial extent of the wave function of electrons is estimated from the uncertainty principle as

$$\xi_0 \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar v_F}{k_B T_c}, \quad (1.17)$$

where $\hbar = h_P / 2\pi$ with h_P denoting Planck's constant. The characteristic length ξ_0 is called the coherence length.

The London theory predicts that physical quantities such as the magnetic flux density and the current density vary within a characteristic distance λ . Hence, λ is required to be sufficiently long with respect to ξ_0 that the local approximation remains valid. That is, the London theory is a good approximation for superconductors in which $\lambda \gg \xi_0$. Such superconductors are typical type-2 superconductors. In this book the London theory will be used to discuss the structure of quantized magnetic flux in type-2 superconductors (Sect. 1.5) and to derive the induced electric field due to the motion of quantized magnetic flux (Sect. 2.2).

1.4 Ginzburg-Landau Theory

Although the London theory explains the Meissner effect, it is unable to deal with the coexistence of magnetic field and superconductivity such as in the intermediate state of type-1 superconductors or the mixed state of type-2. The theory of Ginzburg and Landau (G-L theory) [1] was proposed for the purpose of treating the intermediate state. This theory is based on the deep insight of Ginzburg and Landau on the essence of superconductivity, namely that the superconducting state is such that the phase of the electrons is coherent on a macroscopic scale. The order parameter defined in the theory is, originally a thermodynamic quantity, which now has the property of a mean wave function describing the coherent motion of the center of a group of electrons. This wave function is comparable to the electron wave function of quantum mechanics.

We define the order parameter, Ψ , as a complex number and assume that the square of its magnitude $|\Psi|^2$ gives the number density of superconducting electrons. The free energy of a superconductor depends on this density of superconducting electrons, and hence, is a function of $|\Psi|^2$. In the vicinity of the transition point $|\Psi^2|$ is expected to be sufficiently small and it is expected that the free energy can be expanded as a power series of $|\Psi^2|$:

$$\text{const.} + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \dots \quad (1.18)$$

For the purpose of describing the phase transition between the superconducting and normal states, the expansion up to the $|\Psi|^4$ term is sufficient, as will be shown later.

It is speculated that the order parameter varies spatially due to existence of the magnetic field. By analogy with quantum mechanics, this should lead to a kinetic energy. The expected value of the kinetic energy density is written in terms of the momentum operator known in the quantum mechanics as

$$\frac{1}{2m^*} \Psi^* (-i\hbar\nabla + 2e\mathbf{A})^2 \Psi, \quad (1.19)$$

where Ψ^* is the complex conjugate of Ψ and m^* is the mass of the superconducting electron, the Cooper pair, and we used the fact that the electric charge of the Cooper pair is $-2e$. The operator of the momentum takes the well known form containing the vector potential, \mathbf{A} , so that the Lorentz force on a moving charge is automatically derived. From the Hermitian property of the operator the kinetic energy density in (1.19) is rewritten as

$$\frac{1}{2m^*} |(-i\hbar\nabla + 2e\mathbf{A})\Psi|^2. \quad (1.20)$$

Thus, the free energy density in the superconducting state including the energy of magnetic field is given by

$$\begin{aligned} F_s = F_n(0) + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 \\ + \frac{1}{2m^*} |(-i\hbar\nabla + 2e\mathbf{A})\Psi|^2, \end{aligned} \quad (1.21)$$

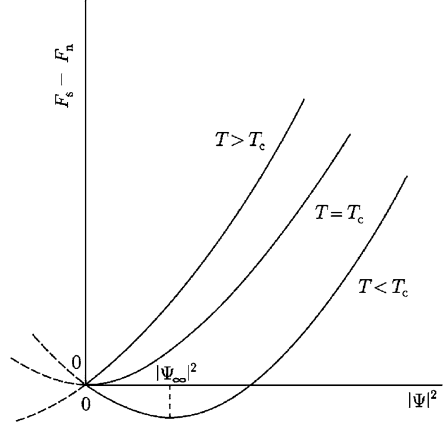
where $F_n(0)$ is the free energy density in the normal state in the absence of the magnetic field.

For simplicity we will first treat the case where the magnetic field is not applied. We may put $\mathbf{A} = 0$ without losing generality. Then, since the order parameter does not vary spatially, (1.21) reduces to

$$F_s(0) = F_n(0) + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4. \quad (1.22)$$

It is necessary for a nonzero equilibrium value of $|\Psi^2|$ to be obtained when the temperature T is lower than the critical value, T_c . This leads to $\alpha < 0$ and $\beta > 0$.

Fig. 1.4 Variation in the free energy density vs. $|\Psi|^2$ at various temperatures



From the condition that the derivative of $F_s(0)$ with respect to $|\Psi|^2$ is zero, we find as the equilibrium value of $|\Psi|^2$

$$|\Psi|^2 = -\frac{\alpha}{\beta} \equiv |\Psi_\infty|^2. \quad (1.23)$$

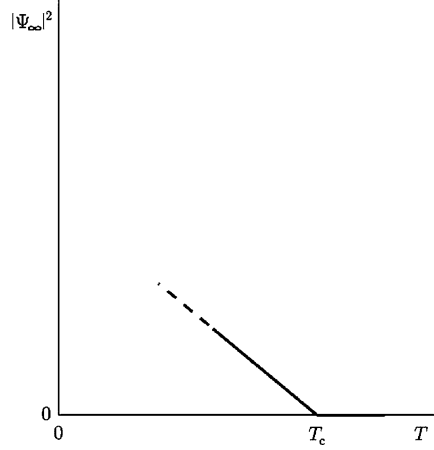
Substitution of this into (1.22) leads to the free energy density in the equilibrium state:

$$F_s(0) = F_n(0) - \frac{\alpha^2}{2\beta}. \quad (1.24)$$

At $T = T_c$ the transition from the superconducting state to the normal one takes place and $|\Psi_\infty|^2$ becomes zero. Thus α is zero at that temperature. The variation of α with temperature in the vicinity of T_c is assumed to be proportional to $(T - T_c)$. α takes a positive value at $T > T_c$ and the free energy density given by (1.22) is minimum at $|\Psi|^2 = 0$. Such variations in the free energy density and the equilibrium value of $|\Psi|^2$ near T_c are shown in Figs. 1.4 and 1.5, respectively. As shown in the above the phase transition at the critical temperature can be explained by the expansion of the free energy density up to the term of the order of $|\Psi|^4$.

Now the phase transition in a magnetic field is treated. We assume that the superconductor is type-1 of a sufficient size. Hence, the superconductor shows the Meissner effect, and a magnetic field does not exist inside it except in a region of about λ from the surface when it is in the superconducting state. Such a surface region can be neglected in a large superconductor, and the spatial variation in the order parameter can be disregarded. The equilibrium state of the superconductor in the magnetic field is determined by minimizing the Gibbs free energy density. If the external magnetic field and the magnetic flux density inside the superconductor are denoted by H_e and B , respectively, the Gibbs free energy density is given by $G_s(H_e) = F_s - BH_e$. If we note that $B = 0$ and $|\Psi|^2$ is given by (1.23) in the

Fig. 1.5 Variation in the equilibrium value of the order parameter, $|\Psi_\infty|^2$, with temperature



superconducting state, we have

$$G_s(H_e) = F_n(0) - \frac{\alpha^2}{2\beta}. \quad (1.25)$$

On the other hand, in the normal state, $|\Psi|^2 = 0$ and $B = \mu_0 H_e$ lead to

$$G_n(H_e) = F_n(0) + \frac{B^2}{2\mu_0} - BH_e = F_n(0) - \frac{1}{2}\mu_0 H_e^2. \quad (1.26)$$

Since G_s and G_n are the same at the transition point, $H_e = H_c$, we have

$$\frac{\alpha^2}{\beta} = \mu_0 H_c^2. \quad (1.27)$$

In the vicinity of T_c , β does not appreciably change with temperature and α is approximately proportional to H_c . That is, we have $\alpha \simeq 2(\mu_0\beta)^{1/2} H_c(0)(T - T_c)/T_c$. Thus, it is found that the above assumption on the temperature dependence of α is satisfied. From (1.25)–(1.27) we obtain

$$G_s(H_e) = G_n(H_e) - \frac{1}{2}\mu_0(H_c^2 - H_e^2). \quad (1.28)$$

This result shows that $G_s(H_e) < G_n(H_e)$ and the superconducting state occurs for $H_e < H_c$ and the normal state occurs for $H_e > H_c$. That is, the transition in the magnetic field is explained by this equation. Especially when $H_e = 0$, the above equation leads to

$$G_s(0) = G_n(0) - \frac{1}{2}\mu_0 H_c^2. \quad (1.29)$$

The maximum difference of the free energy density between the superconducting and normal states, $(1/2)\mu_0 H_c^2$, is called the condensation energy density.

When the superconductor coexists with the magnetic field, $\Psi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ are determined so that the free energy, $\int F_s dV$, is minimized. Hence, the variations of $\int F_s dV$ with respect to $\Psi^*(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ are required to be zero and the following two equations are derived:

$$\frac{1}{2m^*}(-i\hbar\nabla + 2e\mathbf{A})^2\Psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0, \quad (1.30)$$

$$\mathbf{j} = \frac{i\hbar e}{m^*}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{4e^2}{m^*}|\Psi|^2\mathbf{A}, \quad (1.31)$$

with

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \nabla \times \mathbf{A}. \quad (1.32)$$

The above (1.30) and (1.31) are called the Ginzburg-Landau equations, or the G-L equations. In the derivation, the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, and the condition,

$$\mathbf{n} \cdot (-i\hbar\nabla + 2e\mathbf{A})\Psi = 0, \quad (1.33)$$

on the surface were used. In the above, \mathbf{n} is a unit vector normal to the surface and the condition of (1.33) implies that current does not flow across the surface. This is fulfilled for the case where the superconductor is facing a vacuum or an insulating material. On the other hand, if the superconductor is facing a metal, the right hand side in (1.33) is replaced by $ia\Psi$ with a being a real number [2].

The electromagnetic properties in the superconductor are determined by two characteristic lengths, i.e. λ , the penetration depth of magnetic field and ξ , the coherence length. These are related to the spatial variations in the magnetic flux density B and the order parameter Ψ . Here we shall derive these quantities from the G-L theory.

We assume that a weak magnetic field is applied to the superconductor. In this case the variation in the order parameter is expected to be small, and hence, the approximation, $\Psi = \Psi_\infty$, may be allowed. Then, (1.31) reduces to

$$\mathbf{j} = -\frac{4e^2}{m^*}|\Psi_\infty|^2\mathbf{A}. \quad (1.34)$$

This is similar to (1.16) of the London theory. If we recognize that $e^* = 2e$, it follows that $|\Psi_\infty|^2$ corresponds to n_s . Thus, the G-L theory is a more general theory that reduces to the London theory when the order parameter does not vary in space. Hence, the Meissner effect can be derived in the same manner as in Sect. 1.3 and the penetration depth is given by

$$\lambda = \left(\frac{m^*}{4\mu_0 e^2 |\Psi_\infty|^2} \right)^{1/2}. \quad (1.35)$$

Near T_c , $|\Psi_\infty|^2$ is proportional to $(T_c - T)$ and λ varies proportionally to $(T_c - T)^{-1/2}$ and diverges at $T = T_c$. In terms of λ , the coefficients of α and β

are expressed as

$$\alpha = -\frac{(2e\mu_0 H_c \lambda)^2}{m^*}, \quad (1.36)$$

$$\beta = \frac{16e^4 \mu_0^3 H_c^2 \lambda^4}{m^{*2}}. \quad (1.37)$$

Next we shall discuss the spatial variation in the order parameter, Ψ . We treat the case where the magnetic field is not applied and hence that $\mathbf{A} = 0$. For simplicity we assume that Ψ varies only along the x -axis. If we normalize the order parameter according to

$$\psi = \frac{\Psi}{|\Psi_\infty|}, \quad (1.38)$$

equation (1.30) reduces to

$$\xi^2 \frac{d^2 \psi}{dx^2} + \psi - |\psi|^2 \psi = 0, \quad (1.39)$$

where ξ is a characteristic length called the coherence length and is given by

$$\xi = \frac{\hbar}{(2m^*|\alpha|)^{1/2}}. \quad (1.40)$$

We can choose a real function for ψ in (1.39). Suppose that the order parameter varies slightly from its equilibrium value such that $\psi = 1 - f$, where $f \ll 1$. Within this range, (1.39) reduces to

$$\xi^2 \frac{d^2 f}{dx^2} - 2f = 0 \quad (1.41)$$

and hence

$$f \sim \exp\left(-\frac{\sqrt{2}|x|}{\xi}\right). \quad (1.42)$$

This shows that the order parameter varies in space within a distance comparable to ξ . From (1.36) and (1.40) the coherence length can also be expressed as

$$\xi = \frac{\hbar}{2\sqrt{2}e\mu_0 H_c \lambda}. \quad (1.43)$$

It turns out from (1.40) or (1.43) that ξ also increases in proportion to $(T_c - T)^{-1/2}$ in the vicinity of T_c . On the other hand, the coherence length in the BCS theory [3] is given by

$$\xi_0 = \frac{\hbar v_F}{\pi \Delta(0)} = 0.18 \frac{\hbar v_F}{k_B T_c} \quad (1.44)$$

and does not depend on temperature. $\Delta(0)$ is the energy gap at $T = 0$. In spite of such a difference, the two coherence lengths are related to each other. Since the superconductivity is nonlocal, this relation changes with the electron mean free path, l . In the vicinity of T_c , the coherence length in the G-L theory becomes¹

$$\xi(T) = 0.74 \frac{\xi_0}{(1-t)^{1/2}}; \quad l \gg \xi_0, \quad (1.45a)$$

$$= 0.85 \frac{(\xi_0 l)^{1/2}}{(1-t)^{1/2}}; \quad l \ll \xi_0, \quad (1.45b)$$

where $t = T/T_c$. Equations (1.45a) and (1.45b) correspond to the cases of “clean” and “dirty” superconductors, respectively. It is seen that $\xi(T)$ is comparable to ξ_0 in a clean superconductor and is much smaller than ξ_0 in a dirty one.

The penetration depth is also influenced by the electron mean free path l owing to the nonlocal nature of the superconductivity. We call the penetration depth given by (1.35) the London penetration depth and denote it by λ_L . If λ_L is sufficiently longer than ξ_0 and l , we have $\lambda = \lambda_L$ in a clean superconductor ($l \gg \xi_0$) and $\lambda \simeq \lambda_L (\xi_0/l)^{1/2}$ in a dirty superconductor ($l \ll \xi_0$). In a superconductor where $\xi_0 \gg \lambda_L$, i.e. in a “Pippard superconductor,” we have $\lambda \simeq 0.85(\lambda_L^2 \xi_0)^{1/3}$.

The ratio of the two characteristic lengths in G-L theory defined by

$$\kappa = \frac{\lambda}{\xi} \quad (1.46)$$

is called the G-L parameter. According to the G-L theory, λ and ξ have the same temperature dependences, and hence κ is independent of temperature. As a matter of fact, κ decreases slightly with increasing temperature. The G-L parameter is important in describing the magnetic properties of superconductors. In particular the classification into type-1 and type-2 superconductors is determined by the value of this parameter. Also the upper critical field of the type-2 superconductor depends on this parameter.

We next go on to discuss the occurrence of superconductivity in a bulk superconductor in a magnetic field sufficiently high that the higher order term, $\beta|\Psi|^2\Psi$, in (1.30) can be neglected. We assume that the external magnetic field H_e is applied along the z -axis. The magnetic flux density in the superconductor is taken to be uniform in space, $b \simeq \mu_0 H_e$, and hence the vector potential is given by

$$\mathbf{A} = \mu_0 H_e x \mathbf{i}_y, \quad (1.47)$$

where \mathbf{i}_y is a unit vector directed along the y -axis. In the above the choice of x -axis direction is not important in a bulk superconductor and hence generality is still maintained even under (1.47). Since \mathbf{A} depends only on x , it is reasonable to assume that

¹For example, see [4].

Ψ also depends only on x . Hence, (1.30) reduces to

$$-\frac{\hbar^2}{2m^*} \cdot \frac{d^2\Psi}{dx^2} + \frac{2e^2\mu_0^2}{m^*}(H_e^2x^2 - 2H_c^2\lambda^2)\Psi = 0. \quad (1.48)$$

This equation has the same form as the well-known Schrödinger equation for a one-dimensional harmonic oscillator. It has solutions only when the condition

$$\left(n + \frac{1}{2}\right)\hbar H_e = 2e\mu_0 H_c^2 \lambda^2 \quad (1.49)$$

is satisfied with n being nonnegative integers. The maximum value of H_e is obtained for $n = 0$, corresponding to the maximum field within which the superconductivity can exist, i.e., the upper critical field, H_{c2} . Thus, we have

$$H_{c2} = \frac{4e\mu_0 H_c^2 \lambda^2}{\hbar}. \quad (1.50)$$

Using (1.43) and (1.46), the upper critical field may also be written as

$$H_{c2} = \sqrt{2}\kappa H_c. \quad (1.51)$$

Hence, the superconducting state may exist in a magnetic field higher than the critical field H_c for a superconductor with κ larger than $1/\sqrt{2}$. Such is the type-2 superconductor. In this case, the superconductor is in the mixed state and no special characteristic phenomena take place at $H_e = H_c$. That is, H_c cannot be directly measured experimentally. Since it is related to the condensation energy density, H_c is called the thermodynamic critical field in type-2 superconductors. If we now introduce the flux quantum, $\phi_0 = h_P/2e$, to be considered in the next section, the upper critical field may be rewritten in the form

$$H_{c2} = \frac{\phi_0}{2\pi\mu_0\xi^2}. \quad (1.52)$$

This relationship is used in estimating the coherence length from an observed value of H_{c2} .

1.5 Magnetic Properties

A characteristic feature of type-2 superconductors in a magnetic field is that the magnetic flux is quantized on a macroscopic scale. In this book we refer to the quantized magnetic flux as a flux line. The flux lines are isolated from each other at sufficiently low magnetic fields. On the other hand, at high magnetic fields these overlap and interact to form a flux line lattice. In this section such quantization of magnetic flux is discussed in terms of the G-L theory. The superconductor's magnetic properties are discussed in terms of the internal structure of the flux line at low fields and the structure of the flux line lattice at high fields.

1.5.1 Quantization of Magnetic Flux

We suppose a superconductor in a sufficiently weak magnetic field. For simplicity we assume the magnetic flux to be localized at an certain region inside the superconductor. This assumption pre-supposes the quantization of the magnetic flux. It will be shown later that the assumption is actually fulfilled and hence the treatment is self-consistent. Consider a closed loop, C , enclosing the region in which the magnetic flux is localized. The distance between the localized magnetic flux and C is assumed to be sufficiently long to enable the magnetic flux density and the current density to be zero on C . If we write

$$\Psi = |\Psi| \exp(i\phi) \quad (1.53)$$

with ϕ denoting the phase of the order parameter, (1.31) reduces to

$$\mathbf{j} = -\frac{2\hbar e}{m^*} |\Psi|^2 \nabla \phi - \frac{4e^2}{m^*} |\Psi|^2 \mathbf{A}. \quad (1.54)$$

In the above the first term represents the current caused by the gradient of the phase of the order parameter, i.e., the Josephson current. On the loop C , $\mathbf{j} = 0$ and hence

$$\mathbf{A} = -\frac{\hbar}{2e} \nabla \phi. \quad (1.55)$$

Integration of this over C leads to

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int \mathbf{b} \cdot d\mathbf{S} = \Phi, \quad (1.56)$$

where Φ is the magnetic flux that interlinks with C . If we substitute the right hand side of (1.55) for \mathbf{A} in (1.56) the latter becomes

$$\Phi = -\frac{\hbar}{2e} \int_C \nabla \phi \cdot d\mathbf{s} = -\frac{\hbar}{2e} \Delta \phi, \quad (1.57)$$

where $\Delta \phi$ is a variation in the phase after one circulation on C . From the mathematical requirement that the order parameter should be a single-valued function, $\Delta \phi$ must be integral multiple of 2π . That is,

$$\Phi = n\phi_0, \quad (1.58)$$

where n is an integer and

$$\phi_0 = \frac{h_p}{2e} = 2.0678 \times 10^{-15} \text{ Wb}, \quad (1.59)$$

where ϕ_0 is the unit of the magnetic flux and called the flux quantum. Thus we have shown that the magnetic flux in superconductors is quantized. In the above the

curvilinear integral of $\nabla\phi$ on the closed loop C is not zero, since $\nabla\phi$ has a singular point at the center of the flux line. This will be discussed in Sect. 1.5.2.

In the beginning of the above proof we assumed that the magnetic flux was localized in a certain region of a superconductor. This condition is fulfilled at low fields wherein the magnetic flux density decreases as $\exp(-r/\lambda)$ with increasing distance r from the center of the isolated flux line, as will be shown later in (1.62b). At high fields, on the other hand, the flux lines are not localized and there exists a pronounced overlap of the magnetic flux. Under this condition a flux line lattice is formed. But even in this case the magnetic flux is quantized in each unit cell. The proof of this quantization is Exercise 1.3.

1.5.2 Vicinity of Lower Critical Field

Near the lower critical field, the density of magnetic flux penetrating the superconductor is low and the spacing between the flux lines is large. In this subsection we shall discuss the structure of isolated flux line for the case of typical type-2 superconductor with the large G-L parameter, κ . In this case the London theory can be used. It should be noted that (1.10) holds correct only in the region greater than a distance ξ from the center of the flux line in which $|\Psi|$ is approximately constant. As will be shown later, $|\Psi|$ is zero at the center and varies spatially within a region of radius ξ , known as the core. Equation (1.10) cannot be used in the core region. In fact, if we assume that this equation is valid within the entire region, an incorrect result is obtained. This can be seen by integrating (1.10) within a sufficiently wide area including the isolated flux line. From Stokes' theorem the surface integral of the second term in (1.10) is transformed into the integral of the current on the closed loop that surrounds the area. This integral is zero, since the current density is zero at the position sufficiently far from the flux line. This implies that the total magnetic flux in this area is zero. Hence, some modifications are necessary to enable the contribution from the core to the magnetic flux to be equal to ϕ_0 . In the case of superconductor with $\kappa \gg 1$, the area of the core is very narrow in comparison with the total area of the flux line. Hence, we assume most simply that the magnetic structure is described by

$$\mathbf{b} + \lambda^2 \nabla \times \nabla \times \mathbf{b} = \mathbf{i}_z \phi_0 \delta(\mathbf{r}) \quad (1.60)$$

in the region outside the core that occupies most of the area. In the above it is assumed that the magnetic field is applied along the z -axis and \mathbf{i}_z is a unit vector in that direction. \mathbf{r} is a vector in the x - y plane and the center of the flux line exists at $\mathbf{r} = 0$. $\delta(\mathbf{r})$ is a two-dimensional delta function. The coefficient, ϕ_0 , on the right hand side comes from the requirement that the total amount of the magnetic flux of one flux line is ϕ_0 . Equation (1.60) is called the modified London equation.

The solution of this equation is given by

$$b(r) = \frac{\phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right), \quad (1.61)$$

where K_0 is the modified Bessel function of the zeroth order. This function diverges at $r \rightarrow 0$. Since the magnetic flux density should have a finite value, the modified London equation still does not hold correct in the region of $r < \xi$. Outside the core, (1.61) is approximated by

$$b(r) \simeq \frac{\phi_0}{2\pi\lambda^2} \left(\log \frac{\lambda}{r} + 0.116 \right); \quad \xi \ll r \ll \lambda \quad (1.62a)$$

$$\simeq \frac{\phi_0}{2\pi\lambda^2} \left(\frac{\pi\lambda}{2r} \right)^{1/2} \exp\left(-\frac{r}{\lambda}\right); \quad r \gg \lambda, \quad (1.62b)$$

in terms of elementary functions. The current density flowing around the flux line has only the azimuthal component:

$$j(r) = -\frac{1}{\mu_0} \cdot \frac{\partial b}{\partial r} = \frac{\phi_0}{2\pi\mu_0\lambda^3} K_1\left(\frac{r}{\lambda}\right), \quad (1.63)$$

where K_1 is the modified Bessel function of the first order. In particular, the above equation reduces to

$$j(r) = \frac{\phi_0}{2\pi\mu_0\lambda^2 r} \quad (1.64)$$

in the region of $\xi \ll r \ll \lambda$.

In the region of $r < \xi$, the order parameter varies in space. We shall discuss the structures of the order parameter and the magnetic flux density in this region by solving the G-L equations. From symmetry it is reasonable to assume that $|\Psi|$ is a function only of r , the distance from the center of the flux line. Hence we write $\Psi/|\Psi_\infty| = f(r) \exp(i\phi)$ such that when r becomes sufficiently large, $f(r)$ approaches 1. It can be shown according to the argument of Sect. 1.5.1 that the variation of the phase when circulating once around a circle with radius of r should be 2π (recognizing that the number of flux lines inside the circle is 1). Hence, ϕ is a function of the azimuthal angle, θ ; the simplest function satisfying this condition is

$$\phi = -\theta. \quad (1.65)$$

In this case, we easily obtain

$$\nabla\phi = -\frac{1}{r}\mathbf{i}_\theta. \quad (1.66)$$

This shows that the center of the flux line is a singular point at which this function is not differentiable. The relation of $\nabla \times \nabla\phi = 0$ is satisfied except at the singular point, and it can be expressed as

$$\nabla \times \nabla\phi = -2\pi\mathbf{i}_z\delta(\mathbf{r}) \quad (1.67)$$

over all space.

From (1.65) we have

$$\frac{\Psi}{|\Psi_\infty|} = f(r) \exp(-i\theta). \quad (1.68)$$

It is assumed that the vector potential \mathbf{A} is also a function only of r . Then, it turns out that \mathbf{A} has only the θ -component, A_θ . That is, the relation of $b(r) = (1/r)(\partial/\partial r)(rA_\theta)$ results in

$$A_\theta = \frac{1}{r} \int_0^r r' b(r') dr'. \quad (1.69)$$

In the case of high- κ superconductors, since b cannot vary in space in the region of $r < \xi$, we have

$$A_\theta \simeq \frac{b(0)}{2} r. \quad (1.70)$$

Substitution of (1.68) and (1.70) into (1.30) leads to

$$f - f^3 - \xi^2 \left[\left(\frac{1}{r} - \frac{\pi b(0)r}{\phi_0} \right)^2 f - \frac{1}{r} \cdot \frac{d}{dr} \left(r \frac{df}{dr} \right) \right] = 0. \quad (1.71)$$

In the normal core f is considered to be sufficiently small. In fact, it is seen that a nearly constant solution for f does not exist. Hence, we shall assume that $f = cr^n$ with $n > 0$. The dominant terms of the lowest order are those of the order of r^{n-2} . If we take notice of these terms, (1.71) leads to

$$r^{n-2}(1 - n^2) = 0. \quad (1.72)$$

We obtain $n = 1$ from this equation. In the next place we assume as $f = cr(1 + dr^m)$. If the next dominant terms are picked up, we have

$$1 + \frac{b(0)}{\mu_0 H_{c2}} - d\xi^2[1 - (m+1)^2]r^{m-2} = 0, \quad (1.73)$$

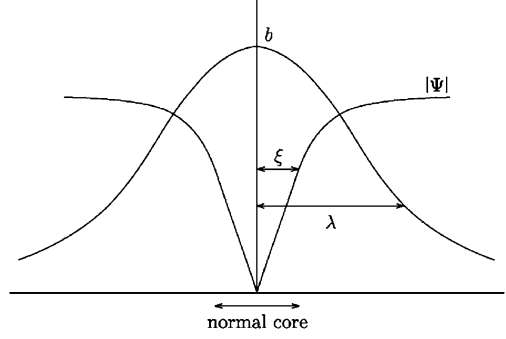
where (1.52) is used. From this equation we obtain $m = 2$ and the value of d . Finally we obtain [5]

$$f \simeq cr \left[1 - \frac{r^2}{8\xi^2} \left(1 + \frac{b(0)}{\mu_0 H_{c2}} \right) \right]. \quad (1.74)$$

It is seen that the order parameter is zero at the center of the core. This is the important feature which proves the current density at the flux line center does not diverge (see (1.54) and (1.66)). Hence, the region of $r \lesssim \xi$ is sometimes called the normal core. At low fields $b(0)/\mu_0 H_{c2}$ is small and may be disregarded, in which case f takes on a maximum value at $r = (8/3)^{1/2}\xi = a_0$. This maximum value should be comparable to 1 at a position sufficiently far from the center, hence $c \sim 1/\xi$. If we approximate as

$$f \simeq \tanh\left(\frac{r}{r_n}\right) \quad (1.75)$$

Fig. 1.6 Spatial variations of the order parameter and the magnetic flux density in an isolated flux line



with $c \simeq 1/r_n$, numerical calculation [6] allows the length to be derived:

$$r_n = \frac{4.16\xi}{\kappa^{-1} + 2.25}, \quad (1.76)$$

which reduces to 1.8ξ in high- κ superconductors. Therefore, in a strict sense the solutions of the London equation, (1.61) and (1.63), are correct only for $r \gtrsim 4\xi$. The structures of the magnetic flux density and the order parameter in the flux line are schematically shown in Fig. 1.6. Since the magnetic flux density in the central part of the core cannot vary steeply in space, its value is approximately given by $(\phi_0/2\pi\lambda^2) \log \kappa$. It will be shown later that this is close to $2\mu_0 H_{c1}$.

We go on to calculate the energy per unit length of the isolated flux line in a bulk high- κ superconductor. From (1.74) we write approximately $f \simeq (3r/2a_0) - (r^3/2a_0^3)$ at low fields. This implies that the core occupies the region $r < a_0$. Outside the core, the important terms in the G-L free energy in (1.21) are the magnetic field energy and the kinetic energy. The kinetic energy density is found to be written as the energy density of the current, $(\mu_0/2)\lambda^2 j^2$, with the replacement of Ψ by Ψ_∞ and the use of (1.34), the London theory. Hence, the contribution from the outside of the core to the energy of a unit length of the flux line is given by

$$\epsilon' = \int \left(\frac{b^2}{2\mu_0} + \frac{\mu_0}{2} \lambda^2 j^2 \right) dV' = \frac{1}{2\mu_0} \int [b^2 + \lambda^2 (\nabla \times \mathbf{b})^2] dV'. \quad (1.77)$$

In the above $\int dV'$ is a volume integral per unit length of the flux line except the area $|\mathbf{r}| \leq a_0$. From the condition that the variation of the kernel of the integral of (1.77) with respect to \mathbf{b} is zero, the London equation is derived. Integrating the second term partially, (1.77) becomes

$$\epsilon' = \frac{1}{2\mu_0} \int (\mathbf{b} + \lambda^2 \nabla \times \nabla \times \mathbf{b}) \cdot \mathbf{b} dV' + \frac{\lambda^2}{2\mu_0} \int [\mathbf{b} \times (\nabla \times \mathbf{b})] \cdot d\mathbf{S}. \quad (1.78)$$

It is found from (1.60) that the first integral is zero. The second integral is carried out on the surfaces of $|\mathbf{r}| = a_0$ and $|\mathbf{r}| = R (R \rightarrow \infty)$. It is easily shown that the latter surface integral at infinity is zero. The former integral on the core surface can

be approximately calculated using (1.62a) and (1.64). As a result we have

$$\begin{aligned}\epsilon' &\simeq \frac{\lambda^2}{2\mu_0} \cdot \frac{\phi_0}{2\pi\lambda^2} \left(\log \frac{\lambda}{a_0} + 0.116 \right) \frac{\phi_0}{2\pi\lambda^2 a_0} \cdot 2\pi a_0 \\ &= \frac{\phi_0^2}{4\pi\mu_0\lambda^2} (\log \kappa - 0.374) = 2\pi\mu_0\xi^2 H_c^2 (\log \kappa - 0.374).\end{aligned}\quad (1.79)$$

The contributions from inside the core to the energy are: $0.995\pi\mu_0 H_c^2 \xi^2$ from the variation in the order parameter and $(8/3)\pi\mu_0 H_c^2 \xi^2 (\log \kappa / \kappa)^2$ from the magnetic field. These are about $(2 \log \kappa)^{-1}$ and $4 \log \kappa / 3\kappa^2$ times as large as the energy given by (1.79). Hence, the second term is found to be very small especially in high- κ superconductors. If this term is disregarded, the energy of a unit length of the flux line becomes

$$\epsilon = 2\pi\mu_0 H_c^2 \xi^2 (\log \kappa + 0.124). \quad (1.80)$$

According to the rigorous calculation of Abrikosov [7] the number in the second term in the above equation is 0.081.

We shall estimate the lower critical field, H_{c1} , from the above result. The Gibbs free energy is continuous during the transition at $H_e = H_{c1}$. The volume of the superconductor is denoted by V . The Gibbs free energy before and after the penetration of a flux line is given by

$$VG_s = VF_s \quad (1.81)$$

and

$$VG_s = VF_s + \epsilon L - H_{c1} \int b dV = VF_s + \epsilon L - H_{c1} \phi_0 L, \quad (1.82)$$

respectively. In the above F_s is the Helmholtz free energy density before the penetration of the flux line and L is the length of the flux line in the superconductor. The second term in (1.82) is a variation in the energy due to the formation of the flux line and the third term is for the Legendre transformation. Comparing (1.81) and (1.82), we have

$$H_{c1} = \frac{\epsilon}{\phi_0} = \frac{H_c}{\sqrt{2}\kappa} (\log \kappa + 0.081), \quad (1.83)$$

where the correct expression by Abrikosov was used for ϵ . This equation can be used for superconductors with high κ values to which the London theory is applicable.

Here we shall calculate the magnetization in the vicinity of H_{c1} . In this case the spacing between the flux lines is so large that the magnetic flux density $b(\mathbf{r})$ is approximately given by a superposition of the magnetic flux density of the isolated flux lines, $b_i(\mathbf{r})$:

$$b(\mathbf{r}) = \sum_n b_i(\mathbf{r} - \mathbf{r}_n), \quad (1.84)$$

where \mathbf{r}_n denotes the position of n -th flux line. The free energy density in this state is again given by (1.78) and we have

$$\begin{aligned} F &= \frac{\phi_0}{2\mu_0} \sum_{m \neq n} \sum b_i(\mathbf{r}_m - \mathbf{r}_n) + \frac{B}{\phi_0} \epsilon \\ &= \frac{B}{2\mu_0} \sum_{n \neq 0} b_i(\mathbf{r}_0 - \mathbf{r}_n) + \frac{B}{\phi_0} \epsilon, \end{aligned} \quad (1.85)$$

where the summation with respect to m is taken within a unit area and the summation with respect to n is taken in the entire region of the superconductor. The first term in (1.85) is the interaction energy among the flux lines and the second term is the self energy of the flux lines, and B is the mean magnetic flux density. In the surface integral around the n -th core in the derivation of the self energy, the contributions from other flux lines are neglected, since these are sufficiently small in the vicinity of H_{c1} . Substituting for b_i using (1.61) results in

$$F = \frac{\phi_0 B}{4\pi\mu_0\lambda^2} \sum_{n \neq 0} K_0\left(\frac{|\mathbf{r}_0 - \mathbf{r}_n|}{\lambda}\right) + B H_{c1}. \quad (1.86)$$

We treat the case of triangular flux line lattice and assume the spacing of flux lines given by

$$a_f = \left(\frac{2\phi_0}{\sqrt{3}B}\right)^{1/2} \quad (1.87)$$

to be sufficiently large. If we take account only the interactions from the six nearest neighbors, the Gibbs free energy density is given by

$$G = F - B H_e = \frac{3\phi_0 B}{2\pi\mu_0\lambda^2} \left(\frac{\pi\lambda}{2a_f}\right)^{1/2} \exp\left(-\frac{a_f}{\lambda}\right) - B(H_e - H_{c1}), \quad (1.88)$$

where H_e is the external magnetic field. The magnetic flux density B at which G is minimum is obtained from the relation:

$$\begin{aligned} B^{-1/4} \left[1 + \frac{5}{2} \left(\frac{\sqrt{3}\lambda^2}{2\phi_0} \right)^{1/2} B^{1/2} \right] \exp \left[- \left(\frac{2\phi_0}{\sqrt{3}\lambda^2 B} \right)^{1/2} \right] \\ = 3.2\mu_0(H_e - H_{c1}) \left(\frac{\lambda^2}{\phi_0} \right)^{5/4}. \end{aligned} \quad (1.89)$$

The exact solution of this equation can be obtained only by numerical calculation. However, if we notice that the variation in B is mostly within the exponential function, the B in the prefactor can be approximately replaced by ϕ_0/λ^2 , and we have

$$B \sim \frac{2\phi_0}{\sqrt{3}\lambda^2} \left\{ \log \left[\frac{\phi_0}{\mu_0(H_e - H_{c1})\lambda^2} \right] \right\}^{-2}. \quad (1.90)$$

It is seen from this equation that B increases rapidly from zero at $H_e = H_{c1}$.

1.5.3 Vicinity of Upper Critical Field

An overlap of the magnetic flux is pronounced and the spacing between the cores is small near the upper critical field. Hence, the London theory cannot be used and an analysis using the G-L theory is necessary. In such a high field the order parameter Ψ is sufficiently small and the higher order term $\beta|\Psi|^2\Psi$ in (1.30) can be neglected. Due to the pronounced flux overlap the magnetic flux density can be regarded as approximately uniform in the superconductor. We assume that the magnetic field is directed along the z -axis. Then to a first approximation the vector potential can be expressed as in (1.47). If we write

$$\Psi(x, y) = e^{-iky}\Psi'(x), \quad (1.91)$$

Ψ' obeys (1.48) with x replaced by $x - x_0$ where

$$x_0 = \frac{\hbar k}{2\mu_0 e H_c}. \quad (1.92)$$

The maximum field at which this equation has a solution is H_{c2} . In this case (1.92) reduces to $x_0 = k\xi^2$. We are interested in the phenomenon at the external magnetic field slightly smaller than H_{c2} , and then, we approximate as $h = H_{c2}$ in the beginning. The equation for Ψ' reduces to

$$-\xi^2 \frac{d^2\Psi'}{dx^2} + \left[\left(\frac{x}{\xi} - k\xi \right)^2 - 1 \right] \Psi' = 0. \quad (1.93)$$

It can be shown easily that Ψ' has a solution of the form:

$$\Psi' \sim \exp \left[-\frac{1}{2} \left(\frac{x}{\xi} - k\xi \right)^2 \right]. \quad (1.94)$$

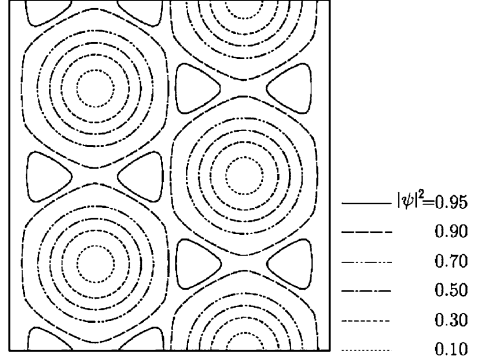
Since the number k is arbitrary, Ψ becomes

$$\Psi = \sum_n C_n e^{-inky} \exp \left[-\frac{1}{2} \left(\frac{x}{\xi} - nk\xi \right)^2 \right]. \quad (1.95)$$

This corresponds to the assumption of a periodic order parameter, i.e., a periodic arrangement of flux lines. This is because such a periodic structure is expected to be favorable with respect to the energy. One of the lattices with the high periodicity is the triangular lattice. This lattice is obtained by putting $C_{2m} = C_0$ and $C_{2m+1} = iC_0$. It is rather difficult to see that (1.95) represents a triangular lattice. Let us make the transformation

$$x = \frac{\sqrt{3}}{2} X, \quad y = \frac{X}{2} + Y \quad (1.96)$$

Fig. 1.7 Contour diagram of normalized $|\Psi|^2$ of the triangular flux line lattice (from Kleiner et al. [8])



and expand $|\Psi|^2$ into a double Fourier series. After a calculation we obtain

$$|\Psi|^2 = |C_0|^2 3^{-1/4} \sum_{m,n} (-1)^{mn} \exp\left[-\frac{\pi}{\sqrt{3}}(m^2 - mn + n^2)\right] \times \exp\left[\frac{2\pi i}{a_f}(mX + nY)\right]. \quad (1.97)$$

In the above we used $k = 2\pi/a_f$ and $a_f^2 = 4\pi\xi^2/\sqrt{3}$, where the latter relation is correct at $h = H_{c2}$. The derivation of (1.97) is Exercise 1.5. The structure of $|\Psi|^2$ for the triangular lattice was derived by Kleiner et al. [8]. Their result is shown in Fig. 1.7. If we pick up only the main terms satisfying $m^2 - mn + n^2 \leq 1$ and rewrite in terms of the original coordinates, (1.97) reduces to

$$|\Psi|^2 = |C_0|^2 3^{-1/4} \left\{ 1 + 2 \exp\left(-\frac{\pi}{\sqrt{3}}\right) \left[\cos \frac{2\pi}{a_f} \left(\frac{2}{\sqrt{3}}x \right) + \cos \frac{2\pi}{a_f} \left(\frac{x}{\sqrt{3}} - y \right) - \cos \frac{2\pi}{a_f} \left(\frac{x}{\sqrt{3}} + y \right) \right] \right\}. \quad (1.98)$$

If we replace the factor, $2 \exp(-\pi/\sqrt{3}) \simeq 0.326$, in the above equation by $1/3$, it is found that $|\Psi|^2$ is zero at $(x, y) = (\sqrt{3}(p \pm 1/4)a_f, (q \mp 1/4)a_f)$ with p and q denoting integers.

The set of the order parameter given by (1.95), which is denoted by Ψ_0 , and $\mathbf{A} = \mu_0 H_{c2} x \mathbf{i}_y = \mathbf{A}_0$ satisfy the linearized G-L equation at $H_e = H_{c2}$. The corrections to these quantities are written as

$$\Psi_1 = \Psi - \Psi_0, \quad \mathbf{A}_1 = \mathbf{A} - \mathbf{A}_0. \quad (1.99)$$

Here we shall estimate the deviation of the magnetic flux density from the uniform distribution, $b = \mu_0 H_{c2}$, that was assumed at the beginning. Substituting (1.95) into (1.31), we find

$$\frac{\partial^2 A_y}{\partial x \partial y} = -\frac{\mu_0 \hbar e}{m^*} \cdot \frac{\partial}{\partial y} |\Psi_0|^2 \quad (1.100)$$

Fig. 1.8 $|\Psi|^2$ and magnetic flux density in the flux line lattice. a_f represents the flux line lattice spacing

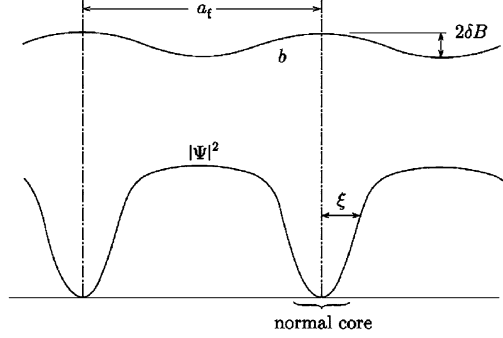
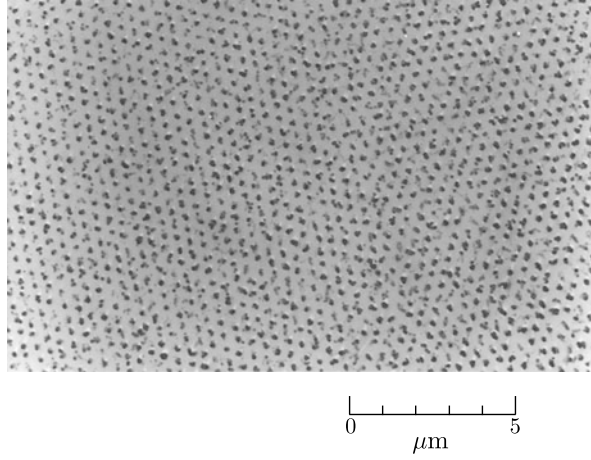


Fig. 1.9 Flux line lattice in Pb-1.6wt%Tl observed by decoration technique after field cooled to 1.2 K at 35 mT and then field is removed. A grain boundary of flux line lattice can be seen. (Courtesy of Dr. B. Obst in Research Center in Karlsruhe)



for the x -component of the current density. Hence, we have

$$b = \mu_0 H_0 - \frac{\mu_0 H_{c2} |\Psi_0|^2}{2\kappa^2 |\Psi_\infty|^2} \quad (1.101)$$

and

$$\mathbf{A} = \left(\mu_0 H_0 x - \frac{\mu_0 H_{c2}}{2\kappa^2 |\Psi_\infty|^2} \int |\Psi_0|^2 dx \right) \mathbf{i}_y, \quad (1.102)$$

where H_0 is an integral constant. It will be shown later that H_0 is equal to the external magnetic field, H_e . Equation (1.101) shows that the local magnetic flux density also varies periodically in the superconductor and becomes maximum where Ψ is zero. Such spatial structures of the magnetic flux density and the density of superconducting electrons, $|\Psi|^2$, are represented in Fig. 1.8. Figure 1.9 is a photograph of the flux line lattice in a superconducting Pb-Tl specimen obtained by the decoration technique.

Since \mathbf{A}_1 is already obtained from (1.102) and $\mathbf{A}_0 = \mu_0 H_{c2} x \mathbf{i}_y$, we shall derive the equation for Ψ_1 , a small quantity. The term, $|\Psi|^2 \Psi$, is also a small quantity.

Substituting (1.99) into (1.30), we have

$$\begin{aligned} & \frac{1}{2m^*}(-i\hbar\nabla + 2e\mathbf{A}_0)^2\psi_1 + \alpha\psi_1 \\ &= \frac{i\hbar e}{m^*}[\nabla \cdot (\mathbf{A}_1\psi_0) + \mathbf{A}_1 \cdot \nabla\psi_0] - \frac{4e^2}{m^*}\mathbf{A}_0 \cdot \mathbf{A}_1\psi_0 - \beta|\psi_0|^2\psi_0. \end{aligned} \quad (1.103)$$

This inhomogeneous equation for ψ_1 has a solution only if the inhomogeneous term on the right hand side is orthogonal to the solution of the corresponding homogeneous equation, i.e., ψ_0 . It means that the integral of the product of the right hand side and ψ_0^* is zero. This leads to

$$\langle \mathbf{A}_1 \cdot \mathbf{j} \rangle - \beta \langle |\psi_0|^4 \rangle = 0 \quad (1.104)$$

where $\langle \rangle$ denotes a spatial average. In the derivation of the above equation a partial integral was carried out and the surface integral of less importance was neglected. \mathbf{j} in (1.104) is the current density that we obtain when ψ_0 and \mathbf{A}_0 are substituted into (1.31). From (1.101) it is given by

$$\mathbf{j} = -\frac{H_{c2}}{2\kappa^2|\psi_\infty|^2}\nabla \times (|\psi_0|^2\mathbf{i}_z). \quad (1.105)$$

A partial integration of (1.104) leads to

$$\frac{H_{c2}}{2\kappa^2|\psi_\infty|^2}\langle |\psi_0|^2(\nabla \times \mathbf{A}_1)_z \rangle + \beta \langle |\psi_0|^4 \rangle = 0. \quad (1.106)$$

From (1.99) and (1.102) we have

$$(\nabla \times \mathbf{A}_1)_z = -\mu_0(H_{c2} - H_0) - \frac{\mu_0 H_{c2}|\psi_0|^2}{2\kappa^2|\psi_\infty|^2}. \quad (1.107)$$

Hence, (1.106) reduces to

$$\left(1 - \frac{H_0}{H_{c2}}\right)|\psi_\infty|^2\langle |\psi_0|^2 \rangle - \left(1 - \frac{1}{2\kappa^2}\right)\langle |\psi_0|^4 \rangle = 0. \quad (1.108)$$

Using this relation, the mean magnetic flux density is obtained from (1.101) in the form

$$B = \langle b \rangle = \mu_0 H_0 - \frac{\mu_0(H_{c2} - H_0)}{(2\kappa^2 - 1)\beta_A}, \quad (1.109)$$

where

$$\beta_A = \frac{\langle |\psi_0|^4 \rangle}{\langle |\psi_0|^2 \rangle^2} \quad (1.110)$$

is a quantity independent of H_0 .

Now we shall calculate the free energy density. If we take zero for $F_n(0)$, the mean value of the free energy density given by (1.21) is calculated as

$$\langle F_s \rangle = \left\langle \frac{b^2}{2\mu_0} - \frac{\mu_0 H_c^2 |\Psi|^4}{2|\Psi_\infty|^4} \right\rangle, \quad (1.111)$$

where (1.30) was used. If we approximately substitute Ψ_0 into Ψ and eliminate H_0 by the use of (1.101), (1.108) and (1.109), (1.111) becomes

$$\langle F_s \rangle = \frac{B^2}{2\mu_0} - \frac{(\mu_0 H_{c2} - B)^2}{2\mu_0[(2\kappa^2 - 1)\beta_A + 1]}. \quad (1.112)$$

It is found from this equation that β_A should take on a minimum value in order to minimize the free energy. Initially Abrikosov [7] thought that the square lattice was most stable and obtained $\beta_A = 1.18$ for this. Later Kleiner et al. [8] showed that the triangular lattice was most stable with $\beta_A = 1.16$. However, the difference between the two lattices is small.

When (1.112) is differentiated with respect to B , we have

$$\frac{\partial \langle F_s \rangle}{\partial B} = \frac{(2\kappa^2 - 1)\beta_A B + \mu_0 H_{c2}}{\mu_0[(2\kappa^2 - 1)\beta_A + 1]} = H_0, \quad (1.113)$$

where (1.109) is used. Since the derivative of the free energy with respect to the internal variable B gives the corresponding external variable, i.e., the external magnetic field H_e , it follows that H_0 is the external magnetic field as earlier stated. The magnetization then becomes

$$M = \frac{B}{\mu_0} - H_e = -\frac{H_{c2} - H_e}{(2\kappa^2 - 1)\beta_A}. \quad (1.114)$$

This result suggests that the diamagnetism decreases linearly with increasing magnetic field and reduces to zero at $H_e = H_{c2}$ with the transition to the normal state. The magnetic susceptibility, dM/dH_e , is of the order of $1/2\kappa^2\beta_A$ and takes a very small value for a type-2 superconductor with a high κ value. According to (1.101), the deviation of the local magnetic flux density from its mean value is given by

$$\delta B = \frac{\mu_0 H_{c2} \langle |\Psi_0|^2 \rangle}{2\kappa^2 |\Psi_\infty|^2} = -\mu_0 M \quad (1.115)$$

(see Fig. 1.8: note that $b = \mu_0 H_e$ at the point where $|\Psi_0|^2 = 0$ and that b is minimum at the point where $|\Psi_0|^2$ in (1.98) takes on a maximum value, $2\langle |\Psi_0|^2 \rangle$). Hence, the magnetic flux density is almost uniform and the spatial variation is very small in a high- κ superconductor. For example, the relative fluctuation of the magnetic flux density at $H_e = H_{c2}/2$ is $\delta B/B \sim 1/2\kappa^2\beta_A$ and takes a value as small as 10^{-4} in Nb-Ti with $\kappa \simeq 70$.

Here we shall argue the transition at H_{c2} from another viewpoint. Since the transition in a magnetic field is treated, the Gibbs free energy density, $G_s = F_s - H_e B$

is suitable. The local magnetic flux density b is given by (1.101) and a part of the energy reduces to

$$\frac{1}{2\mu_0}\langle b^2 \rangle - H_e B = -\frac{1}{2}\mu_0 H_e^2 \quad (1.116)$$

where the equation, $H_e = H_0$, was used and the small term proportional to $(b - \mu_0 H_e)^2$ was neglected. Hence, using the expression on the kinetic energy density shown in Exercise 1.1, the Gibbs free energy density is rewritten as

$$G_s = \alpha |\Psi|^2 + \frac{\hbar^2}{2m^*} (\nabla |\Psi|)^2 + \frac{\mu_0}{2} \lambda^2 \left(\frac{|\Psi_\infty|}{|\Psi|} \right)^2 j^2 - \frac{1}{2} \mu_0 H_e^2 \quad (1.117)$$

in the vicinity of the transition point. The first term is the condensation energy density and has a constant negative value. Thus, it can be understood that the transition to the normal state at H_{c2} occurs, since the kinetic energy given by the second and third terms consumes the gain of condensation energy. We shall ascertain that this speculation is correct. For this purpose the approximate solution of $|\Psi|^2$ of (1.98) in the vicinity of H_{c2} is used: the quantity in $\{\dots\}$ is represented by g , for simplicity. Hence, we have $|\Psi|^2/|\Psi_\infty|^2 = g \langle |\psi|^2 \rangle$ with $\Psi/|\Psi_\infty| = \psi$. Since the error around the zero points of Ψ in this expression is large, the factor of $2 \exp(-\pi/3)$ in front of $[\dots]$ is replaced by $1/3$ so that the zero points are reproduced. Rewriting as $(\nabla |\Psi|)^2 = (\nabla |\Psi|^2)^2/4|\Psi|^2$, the second term of (1.117) leads to

$$\frac{\hbar^2}{2m^*} (\nabla |\Psi|)^2 = \frac{1}{4} \mu_0 H_c^2 \xi^2 \langle |\psi|^2 \rangle \frac{(\nabla g)^2}{g}. \quad (1.118)$$

After a calculation using (1.101), the third term of (1.117) leads to

$$\frac{\mu_0}{2} \lambda^2 \left(\frac{|\Psi_\infty|}{|\Psi|} \right)^2 j^2 = \frac{1}{4} \mu_0 H_c^2 \xi^2 \langle |\psi|^2 \rangle \frac{(\nabla g)^2}{g}. \quad (1.119)$$

Thus, it is found that the second and third terms are the same. Hence, (1.117) can be written as

$$G_s = \mu_0 H_c^2 [-|\psi|^2 + 2\xi^2 (\nabla |\psi|)^2] - \frac{1}{2} \mu_0 H_e^2. \quad (1.120)$$

Since $B = \mu_0 H_e$ in the normal state, the third term of (1.120) is the same as the Gibbs free energy density in the normal state, G_n . Hence, the transition point, H_{c2} , is given by the magnetic field at which the sum of the first and second terms reduces to zero. This condition is given by

$$\langle -|\psi|^2 + 2\xi^2 (\nabla |\psi|)^2 \rangle = \langle |\psi|^2 \rangle \left[-1 + \frac{\xi^2}{2} \left\langle \frac{(\nabla g)^2}{g} \right\rangle \right] = 0. \quad (1.121)$$

A numerical calculation leads to $\langle (\nabla g)^2/g \rangle = 14.84/a_f^2$, and the flux line lattice spacing at H_{c2} is obtained: $a_f^2 = 7.42\xi^2$. Thus, from (1.52) and the relationship of

$a_f = (2\phi_0/\sqrt{3}B)^{1/2}$, we have [9]

$$H_e = \frac{B}{\mu_0} = 0.98 H_{c2}. \quad (1.122)$$

Thus, it is found that the transition point can also be obtained fairly correctly even by such a simple approximation.

In the above the magnetic properties of type-2 superconductors are described using the G-L theory. Especially the fundamental properties are determined by the two physical quantities, H_c and κ . That is, the critical fields, H_{c1} (1.83) and H_{c2} (1.51), and the magnetization in their vicinities given by (1.90) and (1.114) are described only by the two quantities (note that $\phi_0/\lambda^2 = 2\sqrt{2}\pi\mu_0 H_c/\kappa$ in (1.90)). In addition, from the argument on thermodynamics we have the general relation

$$-\int_0^{H_{c2}} \mu_0 M(H_e) dH_e = \frac{1}{2} \mu_0 H_c^2. \quad (1.123)$$

In the above we assumed that κ is a general parameter decreasing slightly with increasing temperature. Strictly speaking, the κ values defined by (1.51) (κ_1), (1.114) (κ_2) and (1.83) (κ_3) are slightly different.

1.6 Surface Superconductivity

In the previous section the magnetic properties and the related superconducting order parameter in a bulk superconductor were investigated. In practice, the superconductor has a finite size and the surface. A special surface property different from that of the bulk is expected. Here we assume a semi-infinite type-2 superconductor occupying $x \geq 0$ with the magnetic field applied parallel to the surface along the z -axis for simplicity. On the surface where the superconductor is facing vacuum or an insulating material, the boundary condition on the order parameter is given by (1.33). Under this condition the vector potential A can be chosen so that it contains only the y -component. Hence, the above boundary condition may be written

$$\left. \frac{\partial \Psi}{\partial x} \right|_{x=0} = 0. \quad (1.124)$$

We shall solve again the linearized G-L equation (by ignoring the small term to the third power of Ψ). We assume the order parameter of the form [10]

$$\Psi = e^{-iky} e^{-ax^2} \quad (1.125)$$

referring to (1.91) and (1.94). This order parameter satisfies the condition (1.124). In the following we shall obtain approximate values of the parameters, k and a , by the variation method. Under the present condition in which the external variable is given, the quantity to be minimized is the Gibbs free energy density; this is given

by the free energy density in (1.21) minus BH_e . If the small term proportional to the fourth power of Ψ is neglected, the Gibbs free energy per unit length in the directions of the y - and z -axes measured from the value in the normal state is given by

$$G = \frac{1}{2m^*} \int_0^\infty \left[|(-i\hbar\nabla + 2e\mathbf{A})\Psi|^2 - \frac{\hbar^2}{\xi^2} |\Psi|^2 \right] dx \quad (1.126)$$

under the approximation $A_y = \mu_0 H_e x$. After substitution of (1.125) into this equation and a simple calculation we have

$$G = \frac{\hbar^2}{4m^*} \left[\left(\frac{\pi}{2a} \right)^{1/2} \left(k^2 - \frac{1}{\xi^2} \right) - \frac{2e\mu_0 H_e k}{\hbar a} + \left(\frac{\pi}{2a^3} \right)^{1/2} \left(a^2 + \frac{e^2 \mu_0^2 H_e^2}{\hbar^2} \right) \right]. \quad (1.127)$$

When minimizing this with respect to k , we obtain

$$k = \left(\frac{2}{\pi a} \right)^{1/2} \frac{e\mu_0 H_e}{\hbar}, \quad (1.128)$$

after which, G becomes

$$G_e = \frac{\hbar^2}{4m^*} \left(\frac{\pi}{2} \right)^{1/2} \left[a^{1/2} - \frac{1}{\xi^2} a^{-1/2} + \frac{e^2 \mu_0^2 H_e^2}{\hbar^2} \left(1 - \frac{2}{\pi} \right) a^{-3/2} \right]. \quad (1.129)$$

From the requirements that G_e is minimum with respect to a and that $G_e = 0$ at the transition point, we obtain a and the critical value of H_e denoted by H_{c3} as [10]

$$a = \frac{1}{2\xi^2}, \quad (1.130)$$

$$H_{c3} = \frac{\hbar}{2\xi^2 e \mu_0} \left(1 - \frac{2}{\pi} \right)^{-1/2} \simeq 1.66 H_{c2}. \quad (1.131)$$

The exact calculation was carried out by Saint-James and de Gennes [11] who obtained

$$H_{c3} = 1.695 H_{c2}. \quad (1.132)$$

Thus, the superconductivity appears in the surface region even at the magnetic field above H_{c2} .

The surface critical field H_{c3} depends on the angle between the surface and the magnetic field. H_{c3} decreases from the value given by (1.132) with increasing angle and reduces to the bulk upper critical field H_{c2} at the angle normal to the surface.

1.7 Josephson Effect

It was predicted by Josephson [12] that a DC superconducting tunneling current can flow between superconductors separated by a thin insulating layer. This is the DC Josephson effect. The intuitive picture of this effect was given by (1.54), based on phenomenological theory. That is, it was expected from the first term in this equation that, if a phase difference occurs between the order parameters of superconductors separated by an insulating layer, a superconducting tunneling current proportional to that phase difference flows across the insulating barrier. Here we suppose a Josephson junction as schematically shown in Fig. 1.10 and assume that the physical quantities vary only along the x -axis along which the current flows. If we assume that the order parameter is constant and that the gradient of the phase is uniform in the insulating region, (1.54) leads to

$$j = j_c \theta, \quad (1.133)$$

where j_c is given by

$$j_c = \frac{2\hbar e}{m^* d} |\Psi|^2 \quad (1.134)$$

with d denoting the thickness of the insulating layer. In (1.133) θ , which is the difference of the gauge-invariant phase of the two superconductors, is given by

$$\theta = \phi_1 - \phi_2 - \frac{2\pi}{\phi_0} \int_1^2 A_x dx \quad (1.135)$$

with ϕ_1 and ϕ_2 denoting the phases of superconductors 1 and 2, respectively. Equation (1.133) is correct when the phase difference θ is small. When θ becomes large, the relationship between the current density and θ starts to deviate from this equation. This can be understood from the physical requirement that the current should vary periodically with θ the period of the variation being 2π . Hence, a relationship of the form

$$j = j_c \sin \theta \quad (1.136)$$

Fig. 1.10 Structure of Josephson junction

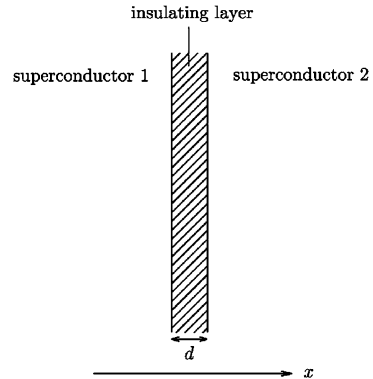
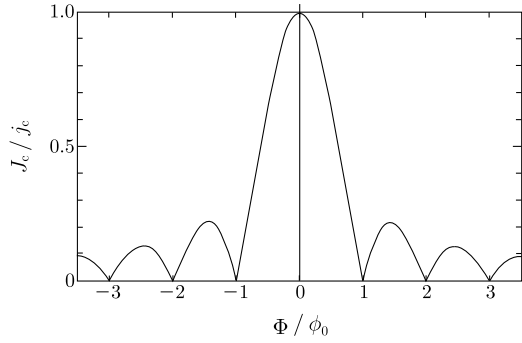


Fig. 1.11 Relation between the critical current density in a Josephson junction and the magnetic flux inside the junction



is expected instead of (1.133). In fact, this relationship was derived by Josephson using the BCS theory. Equation (1.136) can also be derived using the G-L theory, if (1.30) and (1.54) are solved simultaneously [13].

Since the phase difference θ contains the effect of the magnetic field in a gauge-invariant form, the critical current density, i.e., the maximum value of (1.136) averaged in the junction, varies with the magnetic field as

$$J_c = j_c \left| \frac{\sin(\pi \Phi / \phi_0)}{\pi \Phi / \phi_0} \right| \quad (1.137)$$

due to interference (see Fig. 1.11), where Φ is the magnetic flux inside the junction. This form is similar to the interference pattern due to the Fraunhofer diffraction by a single slit. For example, when the magnetic flux just equal to one flux quantum penetrates the junction, the critical current density of the junction is zero. In this situation the phase inside the junction varies over 2π and the zero critical current density results from the interference of the positive and negative currents of the same magnitude. This influence of the magnetic field gives a direct proof of the DC Josephson effect. The SQUID (Superconducting Quantum Interference Device) in which a very small magnetic flux density can be measured is a device that relies on this property.

Another effect predicted by Josephson is the AC Josephson effect. In this phenomenon, when a voltage with V is applied to the junction, an AC superconducting current flows with angular frequency, ω , given by

$$\hbar\omega = 2eV. \quad (1.138)$$

In the voltage state the magnetic flux flows through the junction region and the phase of the order parameter varies in time. As will be shown in Sect. 2.2, the angular frequency given by (1.138) is the same as the rate of variation of the phase. When the junction is irradiated by microwave energy of this frequency, resonant absorption occurs and a DC step of the superconducting current, i.e., a “Shapiro step,” appears. The AC Josephson effect was demonstrated by this kind of measurement. The present voltage standard is established by the AC Josephson effect expressed by (1.138) in association with an extremely exact frequency measuring technique.

1.8 Critical Current Density

The maximum superconducting current density that the superconductor can carry is a very important factor from an engineering standpoint. Some aspects of this property are mentioned in this section. According to the G-L theory, the superconducting current density may be transcribed from (1.54) into the form

$$\mathbf{j} = -2e|\Psi|^2 \mathbf{v}_s, \quad (1.139)$$

where

$$\mathbf{v}_s = \frac{1}{m^*} (\hbar \nabla \phi + 2e \mathbf{A}) \quad (1.140)$$

is the velocity of the superconducting electrons. If the size of superconductor is sufficiently small compared to the coherence length ξ , $|\Psi|$ can be probably regarded as approximately constant over the cross section of the superconductor. If we note that $\nabla \Psi \simeq i\Psi \nabla \phi$, the free energy density in (1.21) reduces to

$$F_s = F_n(0) + \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{2} m^* |\Psi|^2 v_s^2 + \frac{B^2}{2\mu_0}. \quad (1.141)$$

Minimizing the free energy density with respect to $|\Psi|$, we have

$$|\Psi|^2 = |\Psi_\infty|^2 \left(1 - \frac{m^* v_s^2}{2|\alpha|} \right). \quad (1.142)$$

From (1.139) the corresponding current density is given by

$$\mathbf{j} = 2e |\Psi_\infty|^2 \left(1 - \frac{m^* v_s^2}{2|\alpha|} \right) \mathbf{v}_s. \quad (1.143)$$

This becomes maximum when $m^* v_s^2 = (2/3)|\alpha|$, the maximum value being

$$j_c = \left(\frac{2}{3} \right)^{3/2} \frac{H_c}{\lambda}. \quad (1.144)$$

Under the condition that j is maximum, $|\Psi|$ takes a finite value, $(2/3)^{1/2} |\Psi_\infty|$, and the depairing of the superconducting electron pairs has not yet occurred. In fact, the velocity at which the depairing takes place resulting in zero $|\Psi|$ is $\sqrt{3}$ times as large as the velocity corresponding to j_c . However, according to the BCS theory the current density almost attains its maximum value when v_s is such that the energy gap diminishes to zero in the limit of $T = 0$. Thus there is a clear relationship between the depairing velocity and the maximum current density. For this reason the current density given by (1.144) is sometimes called the depairing current density.

The Meissner current is another current associated with the superconducting phenomena. This current, which is localized near the surface according to (1.15), brings

about the perfect diamagnetism. In type-2 superconductors its maximum value is

$$j_{c1} = \frac{H_{c1}}{\lambda}. \quad (1.145)$$

Here we shall investigate the above two critical current densities quantitatively. Take the practical superconducting material Nb_3Sn for example. From $\mu_0 H_c \simeq 0.5$ T, $\mu_0 H_{c1} \simeq 20$ mT and $\lambda \simeq 0.2$ μm , we have $j_c \simeq 1.1 \times 10^{12}$ A m^{-2} and $j_{c1} \simeq 8.0 \times 10^{10}$ A m^{-2} at 4.2 K. It is seen that these values are very high. However, the size of superconductor should be smaller than ξ to attain the depairing current density j_c over its entire cross section. Since ξ in Nb_3Sn is approximately 3.9 nm, the fabrication of superconducting wires sufficiently thinner than ξ is difficult. Furthermore, suppose that multifilamentary subdivision is adopted for keeping the current capacity at a sufficient level; i.e., suppose that a large number of fine superconducting filaments are embedded in a normal metal. In this case we have to confront an essential problem; viz. the proximity effect in which the superconducting electrons in the superconducting region soak into the surrounding normal metal matrix. Two consequences follow: (1) the superconducting property in the superconducting region becomes degraded. (2) Since superconductivity is induced in the normal metal, the superconducting filaments become coupled and the whole wire behaves as a single superconductor. This is contradictory to the premise that the size of superconductor is sufficiently smaller than the coherence length. Hence, it is necessary to embed the superconducting filaments in an insulating material to avoid the proximity effect. However, such a wire is hopelessly unstable. Application of the Meissner current j_{c1} is strongly restricted by the condition that the surface field should be lower than H_{c1} . In Nb_3Sn $\mu_0 H_{c1}$ is as low as 20 mT. Hence j_{c1} cannot be practically used except some special uses.

Since the magnetic energy density is proportional to the second power of the magnetic field, superconducting materials are sometimes used as high-field magnets to store large amounts of energy. Therefore, the superconductivity is required to persist up to high magnetic fields. For such applications a type-2 superconductor with the short coherence length is required; the superconductor is then in the mixed state and is penetrated by flux lines. If the superconductor carries a transport current under this condition (suppose a superconducting wire composing a superconducting magnet under an operating condition), the relative direction of the magnetic field and the current is like the one shown in Fig. 1.12 and the flux lines in the superconductor experience a Lorentz force. The driving force on the flux lines will be described in more detail in Sect. 2.1. If the flux lines are driven by this Lorentz force with velocity \mathbf{v} , the electromotive force induced is:

$$\mathbf{E} = \mathbf{B} \times \mathbf{v}, \quad (1.146)$$

where \mathbf{B} is the macroscopic magnetic flux density. When this state is maintained steadily, an energy dissipation, and hence an electric resistance, should appear as in a normal metal. Microscopically, the central region of each flux line is almost in the normal state as shown in Fig. 1.6, and the normal electrons in this region are

Fig. 1.12 Situation of a current-carrying superconductor in the magnetic field. The Lorentz force acts on the flux lines in the direction shown by the arrow

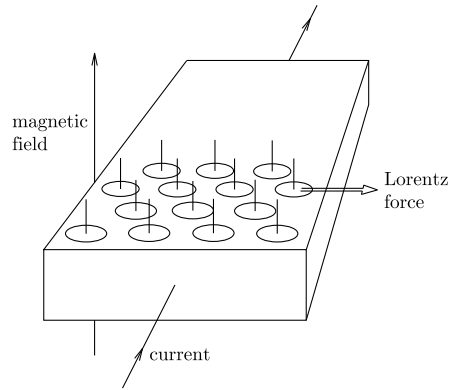
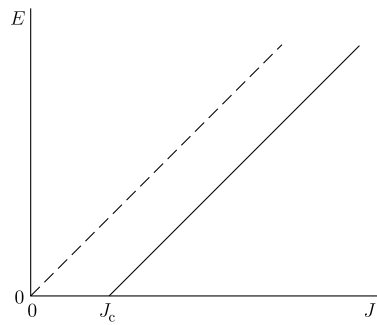


Fig. 1.13 Current-voltage characteristics in the presence of flux pinning interactions. The *broken line* shows the characteristics in the absence of pinning interactions



driven by the electromotive force, resulting in an ohmic loss. This phenomenon is inevitable as long as an electromotive force exists. Hence, it is necessary to stop the motion of flux lines ($v = 0$) in order to prevent the electromotive force. This so-called flux pinning is provided by inhomogeneities and various defects such as dislocations, normal precipitates, voids and grain boundaries. These inhomogeneities and defects are therefore called pinning centers. Flux pinning is like a frictional force in that it prevents the motion of flux lines until the Lorentz force exceeds some critical value. In this state only the superconducting electrons are able to flow and energy dissipation does not occur. For the Lorentz force larger than the critical value the motion of flux lines sets in and the electromotive force reappears, resulting in the current-voltage characteristics shown in Fig. 1.13. The total pinning force that all the elementary pinning centers in a unit volume can exert on the flux lines is called the pinning force density; it is denoted by F_p . At the critical current density J_c , at which the electromotive force starts to appear, the Lorentz force of $J_c B$ acting on the flux lines in a unit volume is balanced by the pinning force density. Hence, we have the relation:

$$J_c = \frac{F_p}{B}. \quad (1.147)$$

The practical critical current density in commercial superconducting materials is

determined by this flux pinning mechanism. This implies that this J_c is not an intrinsic property like the two critical current densities previously mentioned but is an acquired property determined by the macroscopic structure of introduced defects. That is, the critical current density depends on the density, type of, and arrangement of pinning centers. It is necessary to increase the flux pinning strength in order to increase the critical current density. In the above-mentioned Nb_3Sn , a critical current density of the order of $J_c \simeq 1 \times 10^{10} \text{ A m}^{-2}$ is obtained at $B = 5 \text{ T}$.

As a matter of fact, the current-voltage characteristics are not the ideal ones shown in Fig. 1.13 and the electric field is not completely zero for $J \leq J_c$. This comes from the motion of flux lines that have been depinned due to the thermal agitation. This phenomenon called the flux creep will be considered in detail in Sect. 3.8. However, in most cases at sufficiently low temperatures the critical current density J_c can be defined as in Fig. 1.13. Henceforth we will assume that the E - J relation depicted in Fig. 1.13 is approximately correct and J_c can be well defined in most cases. Some practical definitions of J_c are considered in Sect. 5.1.

1.9 Flux Pinning Effect

The practical critical current density in superconductors originates from the flux pinning interactions between the flux lines and defects. The flux line has spatially varying structures of order parameter Ψ and magnetic flux density b as shown in Fig. 1.6. The materials parameters such as T_c , H_c , ξ , etc., in the pinning center are different from those in the surrounding region. Hence, when the flux line is virtually displaced near the pinning center, the free energy given by (1.21) varies due to the interference between the spatial variation in Ψ or b and that of α or β . The rate of variation in the free energy, i.e., the gradient of the free energy gives the interaction force.

Each such individual pinning interaction is vectored in various directions depending on the relative location of the flux line and the pinning center. On the other hand, the resultant macroscopic pinning interaction force density is a force directed opposite to the motion of flux lines in the manner of a macroscopic frictional force. While the individual pinning force comes from the potential and is reversible, the macroscopic pinning force is irreversible. Furthermore, the macroscopic pinning force density is not generally equal to the sum all the elementary pinning forces, the maximum forces of individual interactions, in a unit volume; and the relationship between the macroscopic pinning force density and the elementary pinning force is not simple. The so-called pinning force summation problem will be considered in Chap. 7.

At first glance it might seem that the superconductor can carry some current of the density smaller than J_c without energy dissipation. However, this is correct only in the case of steady direct current. For an AC current or a varying current, loss occurs even when the current is smaller than the critical value. The loss is caused by the electromotive force given by (1.146) due to the motion of flux lines in the superconductor under the AC or varying condition. That is, the mechanism of the loss

is the same as that of ohmic loss in normal metals. Hence, the resultant loss seems to be of the nature that the loss energy per cycle is proportional to the frequency, similarly to the eddy current loss in copper. However, it is the hysteresis loss independent of the frequency. What is the origin for such an apparent contradiction? This originates also from the fact that the flux pinning interaction comes from the potential. This will be mentioned in Chap. 2.

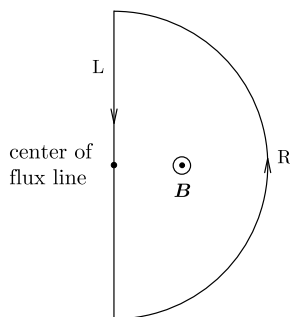
1.10 Exercises

- 1.1. Compare the energy treated in the London theory and that in the G-L theory.
- 1.2. With the use of the G-L equation (1.30), prove that the free energy density given by (1.21) is written as

$$F_s = F_n(0) + \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 - \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{4m^*}\nabla^2|\psi|^2.$$

- 1.3. Prove that the magnetic flux is quantized in a unit cell of the flux line lattice.
- 1.4. Calculate the contributions from the following matter to the energy of the flux line in the low field region:
 - (1) the spatial variation in the order parameter inside the core and
 - (2) the magnetic field inside the core.
 Use (1.74).
- 1.5. Derive (1.97). We can write as $C_n = C_0 \exp(i\pi n^2/2)$ so as to satisfy $C_{2m} = C_0$ and $C_{2m+1} = iC_0$.
- 1.6. It was shown by the approximate solution of (1.98) that $(x, y) = ((\sqrt{3}/4)a_f, -a_f/4)$ is one of the zero points of Ψ . Prove that Ψ given by (1.95) is exactly zero at this point.
- 1.7. Derive (1.111).
- 1.8. We calculate the magnetic flux of one flux line in the area shown in Fig. 1.14. The surface integral is given by the curvilinear integral of the vector potential \mathbf{A} . Since the current density \mathbf{j} is perpendicular to the straight line L, the curvilinear integral of \mathbf{A} is equal to the curvilinear integral of $-(\hbar/2e)\nabla\phi$ on

Fig. 1.14 Closed loop consisting of the *straight line* L passing through the center of the quantized magnetic flux and the *half circle* R at sufficiently long distance



L with ϕ denoting the phase of the order parameter. Equation (1.55) is valid also on the half circle R at sufficiently long distance. As a result the magnetic flux in the region shown in the figure should be an integral multiple of the flux quantum ϕ_0 . This is clearly incorrect. Examine the reason why such an incorrect result was derived.

- 1.9. Discuss the reason why the center of quantized flux line is in the normal state.

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