Is There a World Behind Shannon? Entropies for Complex Systems

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Abstract In their seminal works, Shannon and Khinchin showed that assuming four information theoretic axioms the entropy must be of Boltzmann-Gibbs type, $S = -\sum_i p_i \log p_i$. In many physical systems one of these axioms may be violated. For non-ergodic systems the so called separation axiom (Shannon-Khinchin axiom 4) is not valid. We show that whenever this axiom is violated the entropy takes a more general form, $S_{c,d} \propto \sum_i^W \Gamma(d+1, 1 - c \log p_i)$, where c and d are scaling exponents and $\Gamma(a, b)$ is the incomplete gamma function. These exponents (c, d) define equivalence classes for *all*!, interacting and non interacting, systems and unambiguously characterize any statistical system in its thermodynamic limit. The proof is possible because of two newly discovered scaling laws which any entropic form has to fulfill, if the first three Shannon-Khinchin axioms hold [1]. (c, d) can be used to define equivalence classes of statistical systems. A series of known entropies can be classified in terms of these equivalence classes. We show that the corresponding distribution functions are special forms of Lambert-$\mathcal{W}$ exponentials containing—as special cases—Boltzmann, stretched exponential, and Tsallis distributions (power-laws). We go on by showing how the dependence of phase space volume $W(N)$ of a classical system on its size N, uniquely determines its extensive entropy, and in particular that the requirement of extensivity fixes the exponents (c, d), [2]. We

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PACS 05.20.-y—Classical statistical mechanics
PACS 02.50.Cw—Probability theory
PACS 05.90.+m—Other topics in statistical physics, thermodynamics, and nonlin. dyn. systems.

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A. Sanayei et al. (eds.), *ISCS 2013: Interdisciplinary Symposium on Complex Systems*,
Emergence, Complexity and Computation 8, DOI: 10.1007/978-3-642-45438-7_2,

give a concise criterion when this entropy is not of Boltzmann-Gibbs type but has to assume a *generalized* (non-additive) form. We showed that generalized entropies can only exist when the dynamically (statistically) relevant fraction of degrees of freedom in the system vanishes in the thermodynamic limit [2]. These are systems where the bulk of the degrees of freedom is frozen and is practically statistically inactive. Systems governed by generalized entropies are therefore systems whose phase space volume effectively collapses to a lower-dimensional 'surface'. We explicitly illustrated the situation for binomial processes and argue that generalized entropies could be relevant for self organized critical systems such as sand piles, for spin systems which form meta-structures such as vortices, domains, instantons, etc., and for problems associated with anomalous diffusion [2]. In this contribution we largely follow the lines of thought presented in [1–3].

Keywords Scaling of entropy · (c, d) entropy · Axiomatic derivation of entropy · Extensivity · Non-ergodic systems · Entropy for non-Markovian processes

1 Introduction

In their seminal works, Shannon and Khinchin showed that assuming four information theoretic axioms the entropy must be of Boltzmann-Gibbs type, $S = -\sum_i p_i \log p_i$. In many physical systems one of these axioms may be violated. For non-ergodic systems the so called separation axiom (Shannon-Khinchin axiom 4) is not valid. We show that whenever this axiom is violated the entropy takes a more general form, $S_{c,d} \propto \sum_i^W \Gamma(d+1, 1-c\log p_i)$, where c and d are scaling exponents and $\Gamma(a, b)$ is the incomplete gamma function. These exponents (c, d) define equivalence classes for *all*!, interacting and non interacting, systems and unambiguously characterize any statistical system in its thermodynamic limit. The proof is possible because of two newly discovered scaling laws which any entropic form has to fulfill, if the first three Shannon-Khinchin axioms hold [1]. (c, d) can be used to define equivalence classes of statistical systems. A series of known entropies can be classified in terms of these equivalence classes. We show that the corresponding distribution functions are special forms of Lambert-$\mathcal{W}$ exponentials containing—as special cases—Boltzmann, stretched exponential, and Tsallis distributions (power-laws). We go on by showing how the dependence of phase space volume $W(N)$ of a classical system on its size N, uniquely determines its extensive entropy, and in particular that the requirement of extensivity fixes the exponents (c, d), [2]. We give a concise criterion when this entropy is not of Boltzmann-Gibbs type but has to assume a *generalized* (non-additive) form. We showed that generalized entropies can only exist when the dynamically (statistically) relevant fraction of degrees of freedom in the system vanishes in the thermodynamic limit [2]. These are systems where the bulk of the degrees of freedom is frozen and is practically statistically inactive. Systems governed by generalized entropies are therefore systems whose phase space volume effectively collapses to a lower-dimensional 'surface'. We explicitly illustrated the situation for binomial processes and argue that generalized entropies could be relevant for self organized critical systems such as sand piles, for spin systems which form meta-structures such as vortices, domains, instantons, etc., and for problems associated with anomalous diffusion [2]. In this contribution we largely follow the lines of thought presented in [1–3].

Theorem number 2 in the seminal 1948 paper, *The Mathematical Theory of Communication* [4], by Claude Shannon, proves the existence of the one and only form of entropy, given that three fundamental requirements hold. A few years later A.I. Khinchin remarked in his *Mathematical Foundations of Information Theory* [5]: "However, Shannon's treatment is not always sufficiently complete and mathematically correct so that, besides having to free the theory from practical details, in many instances I have amplified and changed both the statement of definitions and the statement of proofs of theorems." Khinchin adds a fourth axiom. The three fundamental requirements of Shannon, in the 'amplified' version of Khinchin, are known as the Shannon-Khinchin (SK) axioms. These axioms list the requirements needed for an entropy to be a reasonable measure of the 'uncertainty' about a finite probabilistic system. Khinchin further suggests to also use entropy as a measure of the information *gained* about a system when making an 'experiment', i.e. by observing a realization of the probabilistic system.

Khinchin's first axiom states that for a system with W potential outcomes (states) each of which is given by a probability $p_i \geq 0$, with $\sum_{i=1}^{W} p_i = 1$, the entropy $S(p_1, \ldots, p_W)$ as a measure of uncertainty about the system must take its maximum for the equi-distribution $p_i = 1/W$, for all i.

Khinchin's second axiom (missing in [4]) states that any entropy should remain invariant under adding zero-probability states to the system, i.e. $S(p_1, \ldots, p_W) = S(p_1, \ldots, p_W, 0)$.

Khinchin's third axiom (separability axiom) finally makes a statement of the composition of two finite probabilistic systems A and B. If the systems are independent of each other, entropy should be additive, meaning that the entropy of the combined system $A+B$ should be the sum of the individual systems, $S(A+B) = S(A)+S(B)$. If the two systems are dependent on each other, the entropy of the combined system, i.e. the information given by the realization of the two finite schemes A and B, $S(A+B)$, is equal to the information gained by a realization of system A, $S(A)$, plus the mathematical expectation of information gained by a realization of system B, after the realization of system A, $S(A+B) = S(A) + S|_A(B)$.

Khinchin's fourth axiom is the requirement that entropy is a continuous function of all its arguments p_i and does not depend on anything else.

Given these axioms, the *Uniqueness theorem* [5] states that the one and only possible entropy is

$$S(p_1, \ldots, p_W) = -k \sum_{i=1}^{W} p_i \log p_i , \tag{1}$$

where k is an arbitrary positive constant. The result is of course the same as Shannon's. We call the combination of 4 axioms the Shannon-Khinchin (SK) axioms.

From information theory now to physics, where systems may exist that violate the separability axiom. This might especially be the case for non-ergodic, complex systems exhibiting long-range and strong interactions. Such complex systems may show extremely rich behavior in contrast to simple ones, such as gases. There exists some hope that it should be possible to understand such systems also on a thermodynamical basis, meaning that a few measurable quantities would be sufficient to understand their macroscopic phenomena. If this would be possible, through an equivalent to the second law of thermodynamics, some appropriate entropy would enter as a fundamental concept relating the number of microstates in the system to its macroscopic properties. Guided by this hope, a series of so called generalized entropies have been suggested over the past decades, see [6–11] and Table 1. These entropies have been designed for different purposes and have not been related to a fundamental origin. Here we ask how generalized entropies can look like if they fulfill some of the Shannon-Khinchin axioms, but explicitly violate the separability axiom. We do this axiomatically as first presented in [1]. By doing so we can relate a large class of generalized entropies to a single fundamental origin.

The reason why this axiom is violated in some physical, biological or social systems is *broken ergodicity*, i.e. that not all regions in phase space are visited and many micro states are effectively 'forbidden'. Entropy relates the number of micro states of a system to an *extensive* quantity, which plays the fundamental role in the systems thermodynamical description. Extensive means that if two initially isolated, i.e. sufficiently separated systems, A and B, with W_A and W_B the respective numbers of states, are brought together, the entropy of the combined system $A + B$ is $S(W_{A+B}) = S(W_A) + S(W_B)$. W_{A+B} is the number of states in the combined system $A + B$. This is not to be confused with *additivity* which is the property that $S(W_A W_B) = S(W_A) + S(W_B)$. Both, extensivity and additivity coincide if number of states in the combined system is $W_{A+B} = W_A W_B$. Clearly, for a non-interacting system Boltzmann-Gibbs-Shannon entropy, $S_{\mathrm{BG}}[p] = -\sum_i^W p_i \ln p_i$, is extensive *and* additive. By 'non-interacting' (short-range, ergodic, sufficiently mixing, Markovian, ...) systems we mean $W_{A+B} = W_A W_B$. For interacting statistical systems the latter is in general not true; phase space is only partly visited and $W_{A+B} < W_A W_B$. In this case, an additive entropy such as Boltzmann-Gibbs-Shannon can no longer be extensive and vice versa. To ensure extensivity of entropy, an entropic form should be found for the particular interacting statistical systems at hand. These entropic forms are called *generalized entropies* and usually assume trace form [6–11]

$$S_g[p] = \sum_{i=1}^{W} g(p_i), \tag{2}$$

W being the number of states. Obviously not all generalized entropic forms are of this type. Rényi entropy e.g. is of the form $G(\sum_i^W g(p_i))$, with G a monotonic function. We use trace forms Eq. (2) for simplicity. Rényi forms can be studied in exactly the same way as will be shown, however at more technical cost.

Table 1 Order in the zoo of recently introduced entropies for which SK1-SK3 hold. All of them are special cases of the entropy given in Eq. (3) and their asymptotic behavior is uniquely determined by c and d. It can be seen immediately that $S_{q>1}$, S_b and S_E are asymptotically identical; so are $S_{q<1}$ and S_κ, as well as S_η and S_γ

Entropy		c	d	Reference
$S_{c,d} = er\sum_i \Gamma(d+1, 1-c\ln p_i) - cr$	$(r = (1-c+cd)^{-1})$	c	d	
$S_{BG} = \sum_i p_i \ln(1/p_i)$		1	1	[5]
$S_{q<1}(p) = \frac{1-\sum p_i^q}{q-1}$	$(q<1)$	$c=q<1$	0	[6]
$S_\kappa(p) = -\sum_i p_i \frac{p_i^\kappa - p_i^{-\kappa}}{2\kappa}$	$(0<\kappa\leq 1)$	$c=1-\kappa$	0	[8]
$S_{q>1}(p) = \frac{1-\sum p_i^q}{q-1}$	$(q>1)$	1	0	[6]
$S_b(p) = \sum_i (1-e^{-bp_i}) + e^-b - 1$	$(b>0)$	1	0	[9]
$S_E(p) = \sum_i p_i(1-e^{\frac{p_i-1}{p_i}})$		1	0	[10]
$S_\eta(p) = \sum_i \Gamma(\frac{\eta+1}{\eta}, -\ln p_i) - p_i\Gamma(\frac{\eta+1}{\eta})$	$(\eta>0)$	1	$d=\frac{1}{\eta}$	[7]
$S_\gamma(p) = \sum_i p_i \ln^{1/\gamma}(1/p_i)$		1	$d=1/\gamma$	[14], footnote 11, page 60
$S_\beta(p) = \sum_i p_i^\beta \ln(1/p_i)$		$c=\beta$	1	[15]

Let us revisit the Shannon-Khinchin axioms in the light of generalized entropies of trace form Eq. (2). Specifically axioms SK1-SK3 (now re-ordered) have implications on the functional form of g

- SK1: The requirement that S depends continuously on p implies that g is a continuous function.
- SK2: The requirement that the entropy is maximal for the equi-distribution $p_i = 1/W$ (for all i) implies that g is a concave function.
- SK3: The requirement that adding a zero-probability state to a system, $W+1$ with $p_{W+1} = 0$, does not change the entropy, implies that $g(0) = 0$.
- SK4 (separability axiom): The entropy of a system—composed of sub-systems A and B—equals the entropy of A plus the expectation value of the entropy of B, conditional on A. Note that this also corresponds exactly to Markovian processes.

As mentioned, if SK1 to SK4 hold, the only possible entropy is the Boltzmann-Gibbs-Shannon entropy. We are now going to derive the extensive entropy when the separability axiom SK4 is violated. Obviously this entropy will be more general and should contain BG entropy as a special case.

We now assume that axioms SK1, SK2, SK3 hold, i.e. we restrict ourselves to trace form entropies with g continuous, concave and $g(0) = 0$. These systems we call *admissible* systems. Admissible systems when combined with a maximum entropy principle show remarkably simple mathematical properties [12, 13].

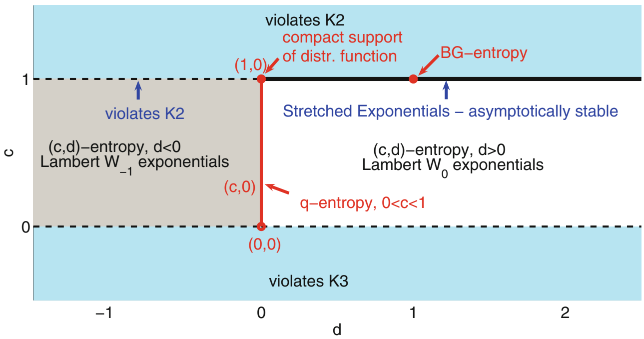

Fig. 1 Entropies parametrized in the (c, d)-plane, with their associated distribution functions. BG entropy corresponds to $(1, 1)$, Tsallis entropy to $(c, 0)$, and entropies for stretched exponentials to $(1, d > 0)$. Entropies leading to distribution functions with compact support, belong to equivalence class $(1, 0)$. Figure from [3]

This generalized entropy for (large) admissible statistical systems (SK1-SK3 hold) is derived from two hitherto unexplored fundamental scaling laws of extensive entropies [1]. Both scaling laws are characterized by exponents c and d, respectively, which allow to uniquely define equivalence classes of entropies, meaning that two entropies are equivalent in the thermodynamic limit if their exponents (c, d) coincide. Each admissible system belongs to one of these equivalence classes (c, d), [1].

In terms of the exponents (c, d) we showed in [1] that all generalized entropies have the form

$$S_{c,d} \propto \sum_i^W \Gamma(d + 1, 1 - c \log p_i) \tag{3}$$

with $\Gamma(a, b) = \int_b^\infty dt\, t^{a-1} \exp(-t)$ the incomplete Gamma-function.

1.1 Special Cases of Entropic Equivalence Classes

Let us look at some specific equivalence classes (c, d)

- Boltzmann-Gibbs entropy belongs to the $(c, d) = (1, 1)$ class. One gets from Eq. (3)

$$S_{1,1}[p] = \sum_i g_{1,1}(p_i) = -\sum_i p_i \ln p_i + 1 \quad . \tag{4}$$

- Tsallis entropy belongs to the $(c, d) = (c, 0)$ class. From Eq. (3) and the choice $r = 1/(1 - c)$ (see below) we get

$$S_{c,0}[p] = \sum_i g_{c,0}(p_i) = \frac{1-\sum_i p_i^c}{c-1} + 1\,. \tag{5}$$

Note, that although the *pointwise* limit $c \to 1$ of Tsallis entropy yields BG entropy, the asymptotic properties $(c, 0)$ do *not* change continuously to $(1, 1)$ in this limit! In other words the thermodynamic limit and the limit $c \to 1$ do not commute.

- The entropy related to stretched exponentials [7] belongs to the $(c, d) = (1, d)$ classes, see Table 1. As a specific example we compute the $(c, d) = (1, 2)$ case,

$$S_{1,2}[p] = 2\left(1 - \sum_i p_i \ln p_i\right) + \frac{1}{2}\sum_i p_i\,(\ln p_i)^2\,, \tag{6}$$

leading to a superposition of two entropy terms, the asymptotic behavior being dominated by the second.

Other entropies which are special cases of our scheme are found in Table 1.

Inversely, for any given entropy we are now in the remarkable position to characterize *all* large SK1-SK3 systems by a pair of two exponents (c, d), see Fig. 1. For example, for $g_{\mathrm{BG}}(x) = -x\ln(x)$ we have $c = 1$, and $d = 1$. S_{BG} therefore belongs to the universality class $(c, d) = (1, 1)$. For $g_q(x) = (x - x^q)/(1-q)$ (Tsallis entropy) and $0 < q < 1$ one finds $c = q$ and $d = 0$, and Tsallis entropy, S_q, belongs to the universality class $(c, d) = (q, 0)$. Other examples are listed in Table 1.

The universality classes (c, d) are equivalence classes with the equivalence relation given by: $g_\alpha \equiv g_\beta \Leftrightarrow c_\alpha = c_\beta$ and $d_\alpha = d_\beta$. This relation partitions the space of all admissible g into equivalence classes completely specified by the pair (c, d).

2 Distribution Functions

Distribution functions associated with our Γ-entropy, Eq. (3), can be derived from so-called generalized logarithms of the entropy. Under the maximum entropy principle (given ordinary constraints) the inverse functions of these logarithms, $\mathcal{E} = \Lambda^{-1}$, are the distribution functions, $p(\epsilon) = \mathcal{E}_{c,d,r}(-\epsilon)$, where for example r can be chosen $r = (1 - c + cd)^{-1}$. One finds [1]

$$\mathcal{E}_{c,d,r}(x) = e^{-\frac{d}{1-c}\left[\mathcal{W}_k\left(B(1-x/r)^{\frac{1}{d}}\right) - \mathcal{W}_k(B)\right]}\,, \tag{7}$$

with the constant $B \equiv \frac{(1-c)r}{1-(1-c)r}\exp\left(\frac{(1-c)r}{1-(1-c)r}\right)$. The function $\mathcal{W}_k$ is the k'th branch of the Lambert-$\mathcal{W}$ function which—as a solution to the equation $x = \mathcal{W}(x)\exp(\mathcal{W}(x))$—has only two real solutions W_k, the branch $k = 0$ and branch $k = -1$. Branch $k = 0$ covers the classes for $d \geq 0$, branch $k = -1$ those for $d < 0$.

2.1 Special Cases of Distribution Functions

It is easy to verify that the class $(c, d) = (1, 1)$ leads to Boltzmann distributions, and the class $(c, d) = (c, 0)$ yields power-laws, or more precisely, Tsallis distributions i.e. q-exponentials. All classes associated with $(c, d) = (1, d)$, for $d > 0$ are associated with stretched exponential distributions. Expanding the $k = 0$ branch of the Lambert-$\mathcal{W}$ function $W_0(x) \sim x - x^2 + \cdots$ for $1 \gg |x|$, the limit $c \to 1$ is shown to be a stretched exponential. It was shown that r does not effect its asymptotic properties (tail of the distributions), but can be used to incorporate finite size properties of the distribution function for small x.

3 How to Determine the Exponents *c* and *d*?

In [2] we have shown that the requirement of extensivity determines uniquely both exponents c and d. What does extensivity mean? Consider a system with N elements. The number of system configurations (microstates) as a function of N are denoted by $W(N)$. Starting with SK2, $p_i = 1/W$ (for all i), we have $S_g = \sum_{i=1}^{W} g(p_i) = W g(1/W)$. As mentioned above extensivity for two subsystems A and B means that

$$W_{A+B}\, g\, (1/W_{A+B}) = W_A\, g\, (1/W_A) + W_B\, g\, (1/W_B). \tag{8}$$

Using this equation one can straight forwardly derive the formulas (for details see [2])

$$\frac{1}{1-c} = \lim_{N\to\infty} N \frac{W'(N)}{W(N)}. \tag{9}$$

$$d = \lim_{N\to\infty} \log W \left(\frac{1}{N} \frac{W}{W'} + c - 1 \right). \tag{10}$$

Here W' means the derivative with respect to N.

3.1 A Note on Rényi-Type Entropies

Rényi entropy is obtained by relaxing SK4 to the unconditional additivity condition. Following the same scaling idea for Rényi-type entropies, $S = G(\sum_{i=1}^{W} g(p_i))$, with G and g some functions, one gets

$$\lim_{W\to\infty} \frac{S(\lambda W)}{S(W)} = \lim_{s\to\infty} \frac{G\left(\lambda f_g(\lambda^{-1}) s\right)}{G(s)}, \tag{11}$$

where $f_g(z) = \lim_{x\to 0} g(zx)/g(x)$. The expression $f_G(s) \equiv \lim_s G(sy)/G(s)$, provides the starting point for deeper analysis which now gets more involved. In particular, for Rényi entropy with $G(x) \equiv \ln(x)/(1-\alpha)$ and $g(x) \equiv x^\alpha$, the asymptotic properties yield the class $(c,d) = (1,1)$, (BG entropy) meaning that Rényi entropy is additive. However, in contrast to the trace form entropies used above, Rényi entropy can be shown to be *not* Lesche stable, as was observed before [16–20]. All of the $S = \sum_i^W g(p_i)$ entropies can be shown to be Lesche stable, see [3].

4 Discussion

We discuss recently discovered scaling laws for trace form entropies for systems that fulfill the first three Shannon-Khinchin axioms. In analogy to critical exponents these laws are characterized by two scaling exponents (c,d), which define generalized entropies. We showed that a particular entropic form—parametrized by these two exponents—covers all *admissible* systems (Shannon-Khinchin axioms 1–3 hold, 4 is violated). In other words every statistical system has its pair of unique exponents in the large size limit, its entropy is then given by $S_{c,d} \sim \sum_i^W \Gamma\,(1+d\,,\,1-c\ln p_i)$. The requirement of extensivity uniquely determines the scaling exponents c and d in terms of the growth of phase space as a function of system size.

The exponents for BG systems are $(c,d) = (1,1)$, systems characterized by stretched exponentials belong to the class $(c,d) = (1,d)$, and Tsallis systems have $(c,d) = (q,0)$. In the context of a maximum entropy principle, the associated distribution functions of *all* systems (c,d) are shown to belong to a class of exponentials involving Lambert-$\mathcal{W}$ functions, given in Eq. (7). There are no other options for tails in distribution functions other than these.

The equivalence classes characterized by the exponents (c,d) form *basins of asymptotic equivalence*. In general these basins characterize interacting statistical (non-additive) systems. There exists an analogy between these basins of asymptotic *equivalence* and the *basin of attraction* of weakly interacting, uncorrelated systems subject to the law of large numbers, i.e. the central limit theorem. Any system within a given equivalence class may show individual characteristics as long as it is small. Systems belonging to the same class will start behaving similarly as they become larger, and in the thermodynamic limit they become identical. Distribution functions converge to those uniquely determined by (c,d). A further interesting feature of all admissible systems is that they all are *Lesche stable*. The proof is found in [3].

Finally, the classification scheme for generalized entropies of type $S = \sum_i g(p_i)$ can be extended to entropies of e.g. Rényi type, i.e. $S = G(\sum_i g(p_i))$, see [3].

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http://www.springer.com/978-3-642-45437-0

ISCS 2013: Interdisciplinary Symposium on Complex Systems
Sanayei, A.; Zelinka, I.; Rössler, O.E. (Eds.)
2014, X, 400 p. 158 illus., 108 illus. in color., Hardcover
ISBN: 978-3-642-45437-0