Chapter 2
Tree-Level Techniques

In this chapter, we discuss techniques to compute tree-level scattering amplitudes. Feynman rules allow one to write down any tree-level amplitude as a sum of all Feynman diagrams contributing, as we did in the last chapter for the four-gluon amplitude. In practice, this straightforward approach can quickly become cumbersome, as the number of Feynman diagrams typically grows factorially with the number of external legs. To give an example: for the scattering of two gluons into n gluons the number of contributing Feynman diagrams grows from 4 ($n = 2$), to 220 ($n = 4$), to 34,300 ($n = 6$) to an enormous 10,525,900 for $n = 8$ [1]. Moreover, one often finds that the final answer is much simpler than the intermediate steps of the calculation.

Here, we will present techniques that exploit the reasons behind this simplicity. Very powerful insights can be gained from thinking about tree-level amplitudes as algebraic functions of the external momenta. As we will see, their analyticity properties under complex deformations of the momenta can be used to derive simple, yet powerful recursion relations known as Britto-Cachazo-Feng-Witten (BCFW) on-shell recursions. These recursion relations use as input on-shell amplitudes, so that the gauge redundancy, which is partly responsible for the complexity of conventional Feynman graph calculations, is absent.

We will also discuss in some detail the symmetry properties of tree-level scattering amplitudes, that is Poincaré symmetry and in the massless case the extension to conformal symmetry. We then briefly introduce the concept of supersymmetry and present the maximally supersymmetric gauge theory in four dimensions, namely $\mathcal{N} = 4$ super Yang-Mills theory, which plays a distinguished role in the space of all gauge theories. We discuss its scattering amplitudes, a supersymmetric variant of the BCFW recursion and the extended superconformal symmetry, and discuss how results for non-supersymmetric theories can be obtained from the supersymmetric case.

2.1 Britto-Cachazo-Feng-Witten (BCFW) On-shell Recursion

The Britto-Cachazo-Feng-Witten (BCFW) recursion relations are an efficient way to compute higher-point partial amplitudes from lower point ones. As we will see,

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the mere knowledge of three-point amplitudes allows the construction of *all* higher point amplitudes in a recursive fashion.

To begin with let us consider an n-gluon amplitude $A_n(p_1, \dots, p_n)$. The key to the BCFW recursion is to study how the amplitude behaves under a complex deformation of the particle momenta preserving the on-shell conditions. For this we perform the following complex shift of the helicity spinors for two neighboring legs 1 and n,

$$\begin{aligned} \lambda_1 &\to \hat{\lambda}_1(z) = \lambda_1 - z\lambda_n, \\ \tilde{\lambda}_n &\to \hat{\tilde{\lambda}}_n(z) = \tilde{\lambda}_n + z\tilde{\lambda}_1, \end{aligned} \tag{2.1}$$

with $z \in \mathbb{C}$, and $\tilde{\lambda}_1$ and λ_n are left inert. We denote the shifted quantities by a hat. This corresponds to a complexification of momenta

$$\begin{aligned} p_1^{\alpha\dot{\alpha}} &\to \hat{p}_1^{\alpha\dot{\alpha}}(z) = (\lambda_1 - z\lambda_n)^{\alpha}\tilde{\lambda}_1^{\dot{\alpha}}, \\ p_2^{\alpha\dot{\alpha}} &\to \hat{p}_2^{\alpha\dot{\alpha}}(z) = \lambda_n^{\alpha}(\tilde{\lambda}_n + z\tilde{\lambda}_1)^{\dot{\alpha}}. \end{aligned} \tag{2.2}$$

Importantly, the deformation preserves both overall momentum conservation and the on-shell conditions,

$$\hat{p}_1^2(z) = 0, \qquad \hat{p}_n^2(z) = 0, \qquad \hat{p}_1(z) + \hat{p}_n(z) = p_1 + p_2, \tag{2.3}$$

and the z-deformed partial amplitude may be written as

$$\mathscr{A}_n(z) = \delta^{(4)}\left(\sum_{i=1}^{n} p_i\right) A_n(z). \tag{2.4}$$

The BCFW recursion relations rely on an understanding of the behavior of the function $A_n(z)$ in the complex z plane. The derivation proceeds in three steps. First, the locations of the poles of $A_n(z)$ are analyzed. Then, it is shown that the residues of the poles correspond to products of lower-point tree amplitudes. Finally, the large z behavior of $A_n(z)$ is determined.

What are the analytical properties in z of the deformed amplitude $A_n(z)$? In order to answer that question, it is helpful to think of $A_n(z)$ in terms of tree-level Feynman diagrams that contribute to it. The sum of Feynman diagrams is gauge invariant. Therefore we can choose the Feynman gauge for the following discussion, without loss of generality. It is clear that $A_n(z)$ is a rational function of the $\lambda_i, \tilde{\lambda}_i$ and z. Moreover, $A_n(z = 0)$ can only have poles where the denominators of Feynman propagators become zero. Given that the partial amplitudes are color-ordered, the propagators take the form

$$\frac{1}{(p_i + p_{i+1} + \cdots + p_j)^2}, \tag{2.5}$$

i.e. they are given by a sum of adjacent momenta. Cf. Fig. 1.4 for illustration. It follows that the deformed amplitude $A_n(z)$ will only have simple poles in z of the

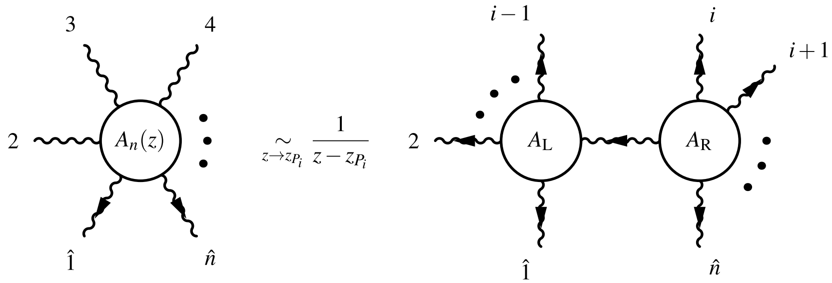


Fig. 2.1 The factorization of the z deformed amplitude on the pole $z = z_{P_i}$

form

$$\frac{1}{\hat{P}_i(z)} := \frac{1}{(\hat{p}_1(z) + p_2 + \cdots + p_{i-1})^2} = \frac{1}{(p_i + p_{i+1} + \cdots + \hat{p}_n(z))^2}$$
$$= \frac{1}{P_i^2 - z\langle n|P_i|1]}, \tag{2.6}$$

with the region momenta $P_i := p_1 + p_2 + \cdots + p_{i-1}$ and $\langle n|P_i|1] = \lambda_{n\alpha} P_i^{\alpha\dot{\alpha}} \tilde{\lambda}_{1\dot{\alpha}}$. Note that any region momentum containing both $\hat{p}_1$ and $\hat{p}_n$ is independent of z, and hence cannot contribute any poles. Therefore the only propagators that can produce poles are the ones considered above.

We deduce that $A_n(z)$ has *simple poles* in z at positions

$$z_{P_i} = \frac{P_i^2}{\langle n|P_i|1]}, \qquad \forall i \in [3, n-1]. \tag{2.7}$$

We also need to know what the residues at the poles are. In fact tree-level amplitudes have universal factorization properties when propagators go on-shell. This is easy to see from Feynman diagrams. The on-shell propagator splits the Feynman diagrams into two parts, or clusters. One can convince oneself that each cluster contains all Feynman diagrams that would be required to compute a scattering amplitude. Put differently the propagator going on-shell can be represented as inserting a complete set of all on-shell states. In conclusion, near the pole z_{P_i} the amplitude $A_n(z)$ factorizes into a product of lower-point amplitudes, which we refer to as "left" and "right" amplitudes A^L and A^R, respectively (see Fig. 2.1),

$$\lim_{z \to z_{P_i}} A_n(z) = \frac{1}{z - z_{P_i}} \frac{-1}{\langle n|P_i|1]} \sum_s A^L\big(\hat{1}(z_{P_i}), 2, \ldots, i-1, -\hat{P}^s(z_{P_i})\big)$$
$$\times A^R\big(\hat{P}^{\bar{s}}(z_{P_i}), i, \ldots, n-1, \hat{n}(z_{P_i})\big). \tag{2.8}$$

The key point is that the internal propagator goes on-shell, so that one has a product of on-shell subamplitudes A^L and A^R. The sum over s in Eq. (2.8) runs over all possible on-shell states propagating between A^L and A^R and $\bar{s} = -s$. In general this will depend on the field content of the theory under consideration. For gluons it becomes a sum over the helicities $s = \{+1, -1\}$.

The emergence of this sum may be justified with the following argument. When an internal gluon propagator goes on-shell, $P^2 = 0$, the tree-level amplitude will factorize as

$$P^2 A_n \xrightarrow{P^2=0} -i M^L_\mu(1, \dots, i-1, -P)\eta^{\mu\nu} M^R_\nu(P, i, \dots, n), \tag{2.9}$$

where $M^{L/R}_\mu$ are lower-point amplitudes with the polarization vector on the $\pm P$ leg stripped off. Without loss of generality let us parametrize the on-shell momentum four-vector as $P^\mu = E(1, 0, 0, 1)$ and the gluon polarization vectors as

$$\varepsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0). \tag{2.10}$$

A straightforward calculation yields

$$\sum_{s=+,-} e^\mu_s M^L_\mu e^\mu_{\bar{s}} M^R_\mu = M^L_1 M^R_1 + M^L_2 M^R_2, \tag{2.11}$$

while the transversality of the amplitude $P^\mu M^{L/R}_\mu = 0$ implies $M^{L/R}_0 - M^{L/R}_3 = 0$. Adding this zero to the above Eq. (2.11) yields

$$\begin{aligned} -i M^L_\mu \eta^{\mu\nu} M^R_\nu &= i\big(M^L_1 M^R_1 + M^L_2 M^R_2 + M^L_3 M^R_3 - M^L_0 M^R_0\big) \\ &= i \sum_{s=+,-} e^\mu_s M^L_\mu e^\mu_{\bar{s}} M^R_\mu \\ &= i \sum_{s=+,-} A^L\big(1, \dots, i-1, -P^s\big) A^R\big(P^{\bar{s}}, i, \dots, n\big), \end{aligned} \tag{2.12}$$

as stated in Eq. (2.8). We note that the same argument would also go through if we were to put an internal fermion propagator on-shell. Then the numerator of the "cut" propagator reads $(|P\rangle[P| + |P]\langle P|)$, which immediately introduces the sum over fermion helicities $\pm 1/2$. For a scalar propagator put on-shell, finally, there is no helicity sum and Eq. (2.8) is obvious.

Returning to our discussion, we can now use complex analysis to construct $A_n(z = 0)$ from the knowledge of the poles of $A_n(z)$. Consider the function $A_n(z)/z$. It behaves as

$$\lim_{z \to z_{P_i}} \frac{A_n(z)}{z} = -\frac{1}{z - z_{P_i}} \sum_s A^s_L(z_{P_i}) \frac{1}{P_i^2} A^{\bar{s}}_R(z_{P_i}), \tag{2.13}$$

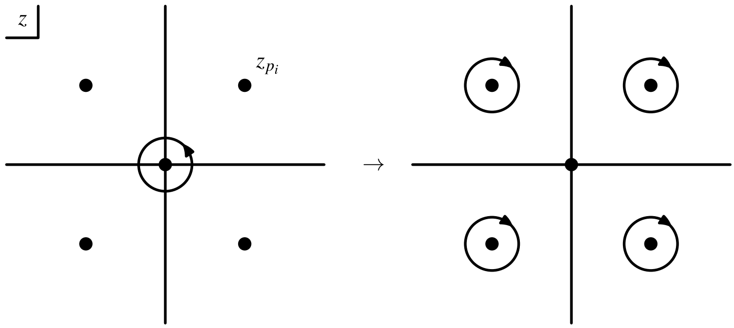


Fig. 2.2 Pulling the initial circle C_0 off to infinity in Eq. (2.16)

with the abbreviations

$$A_L^s(z_{P_i}) = A_i^L\big(\hat{1}(z_{P_i}), 2, \ldots, i-1, -\hat{P}^s(z_{P_i})\big), \tag{2.14}$$

$$A_R^{\bar{s}}(z_{P_i}) = A_i^R\big(\hat{P}^{\bar{s}}(z_{P_i}), i, \ldots, n-1, \hat{n}(z_{P_i})\big). \tag{2.15}$$

Of course we are interested in the original amplitude $A_n = A_n(z=0)$. Using the residue theorem it may be written as

$$\begin{aligned} A_n = A_n(z=0) &= \oint_{C_0} \frac{dz}{2\pi i} \frac{A(z)}{z} \\ &= \sum_{i=2}^{n-1} \sum_s A_L^s(z_{P_i}) \frac{1}{P_i^2} A_R^{\bar{s}}(z_{P_i}) + \mathrm{Res}(z=\infty). \end{aligned} \tag{2.16}$$

Here C_0 is a small circle around the origin at $z=0$ not embracing any of the poles z_{P_i}. To reach the final expression we have pulled this circle off to infinity capturing all the poles z_{P_i} in the complex plane, now encircled in the opposite orientation, see Fig. 2.2. If $A_n(z) \to 0$ as $z \to \infty$ we can drop the residue at $z = \infty$ (also called boundary term). As we show presently, this is the case for gauge theories, under certain conditions. Assuming this for now we arrive at the BCFW recursion relation [2]:

$$A_n = \sum_{i=2}^{n-1} \sum_s A_L^s(z_{P_i}) \frac{1}{P_i^2} A_R^{\bar{s}}(z_{P_i}). \tag{2.17}$$

The following comments are in order. We chose neighboring legs $\hat{1}$ and $\hat{n}$ to perform the complex shifts. One may generalize to non-adjacent legs. In that case, there are typically more BCFW diagrams to consider. In general, different BCFW deformations lead to equivalent representations of the same amplitude. The equivalence may not always be easy to see analytically. Multi-line shifts involving more than two legs have also been considered in the literature [3].

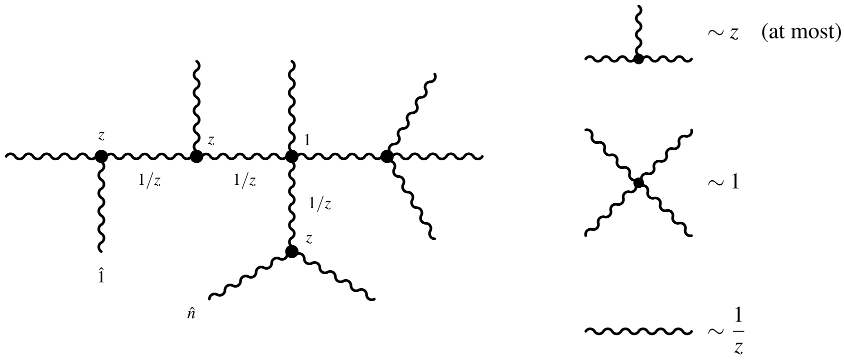


Fig. 2.3 Maximal z scaling of an individual Feynman diagram contributing to $A_n(z)$

In order to complete the derivation of the BCFW recursion (2.17), we need to determine the large z behavior of $A_n(z)$. To have $\oint_\infty \frac{dz}{2\pi i}\frac{A_n(z)}{z} = 0$ we need that $A_n(z)$ vanishes as $z \to \infty$.

In order to estimate the large z behavior of generic tree-level amplitudes we perform an analysis based on Feynman graphs. There are three sources for z dependence: the denominators of the propagators, the interaction vertices, and the polarization vectors.

Consider a generic graph contributing to the tree-level n-gluon amplitude ($\hat{1}$ and $\hat{n}$ are assumed to be neighbors). As one can see from Fig. 2.3 the z dependence occurs only along the path from $\hat{1}$ to $\hat{n}$. Along this path, each three-gluon vertex, being linear in the momenta, contributes a factor of z at most, four-gluon vertices contribute 1, and propagators behave as $1/z$.

We can derive an upper bound by considering the least favorable case. This is when the line $\hat{1}$ to $\hat{n}$ contains only three-valent vertices. In that case it is easy to see that the graph behaves as $\sim z$.

Finally, there is an additional z-dependence arising from polarization vectors at legs 1 and n:

$$\varepsilon_1^{+\alpha\dot{\alpha}} = -\sqrt{2}\frac{\tilde{\lambda}_1^{\dot{\alpha}}\mu_1^{\alpha}}{\langle\hat{\lambda}_1(z)\mu_1\rangle} \sim \frac{1}{z}, \qquad \varepsilon_1^{-\alpha\dot{\alpha}} = \sqrt{2}\frac{\hat{\lambda}_1^{\alpha}(z)\tilde{\mu}_1^{\dot{\alpha}}}{[\lambda_1\mu_1]} \sim z, \tag{2.18}$$

$$\varepsilon_n^{+\alpha\dot{\alpha}} = -\sqrt{2}\frac{\hat{\tilde{\lambda}}_n^{\dot{\alpha}}(z)\mu_n^{\alpha}}{\langle\lambda_n\mu_n\rangle} \sim z, \qquad \varepsilon_n^{-\alpha\dot{\alpha}} = \sqrt{2}\frac{\lambda_n^{\alpha}\tilde{\mu}_n^{\dot{\alpha}}}{[\hat{\lambda}_n(z)\mu_n]} \sim \frac{1}{z}. \tag{2.19}$$

Therefore, taking all sources of z dependence into account, we conclude that individual graphs scale at worst as

$$A\big(\hat{1}^+\hat{n}^-\big) \sim \frac{1}{z}, \qquad A\big(\hat{1}^+\hat{n}^+\big) \sim z, \tag{2.20}$$

$$A\big(\hat{1}^-\hat{n}^-\big) \sim z, \qquad A\big(\hat{1}^-\hat{n}^+\big) \sim z^3. \tag{2.21}$$

This shows that the $(+-)$-shift has the desired falloff properties that allow to drop the boundary term at infinity in the BCFW formula. By cyclicity it is always possible to find a $\{\hat{1}^+, \hat{n}^-\}$ pair. In fact, the above bound is too conservative. One can show [4] that the $(++)$ and $(--)$-shifts also lead to an overall $\frac{1}{z}$ scaling once the sum over all Feynman graphs is performed, as the terms scaling as z or 1 cancel out. Only the $(-+)$-shift gives a non-vanishing $\mathrm{Res}(z=\infty)$ in general.

So far we have discussed the recursion for pure gluon amplitudes. In fact the arguments used to reach Eq. (2.16) go through for an arbitrary quantum field theory. However, the vanishing of the residue at infinity may not. For example one might consider a seemingly simple scalar ϕ^4-theory. Here it is easy to convince oneself that the residue at infinity will not vanish. For two neighboring shifted legs in the ϕ^4 there are always diagrams contributing to the amplitude in which $\hat{1}$ and $\hat{n}$ are connected via a single ϕ^4-vertex. As the external legs have no polarization degrees of freedom such a Feynman diagram will not depend on z. There is no chance of a cancellation either, as all other diagrams in which $\hat{1}$ and $\hat{n}$ are connected by more than one vertex vanish $z \to \infty$. Thus the amplitude $A_n(z)$ for a scalar field theory will not vanish for $z \to \infty$. In a sense the apparent simplicity of ϕ^4 theory at the level of the Feynman rules is misleading, as this theory does not give rise to a simple on-shell recursion relation. The ϕ^4-theory is often used as a testing ground for quantum field theory. Here we see a first hint that, from the on-shell perspective, Yang-Mills theory is actually simpler.

Example As an example of the general result above, consider for concreteness the four-gluon amplitude derived in Eq. (1.130) Under a $(--)$-shift, we have

$$A_4\big(\hat{1}^-, 2^+, 3^+, \hat{4}^-\big) = \frac{\langle\hat{1}\hat{4}\rangle^4}{\langle\hat{1}2\rangle\langle 23\rangle\langle 3\hat{4}\rangle\langle\hat{4}\hat{1}\rangle} = \frac{\langle 14\rangle^4}{\langle\hat{1}2\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \sim \frac{1}{z}, \qquad (2.22)$$

where we have used

$$\langle\hat{4}\hat{1}\rangle \overset{|\hat{4}\rangle=|4\rangle}{=} \langle 4\hat{1}\rangle = \langle 41\rangle - z\langle 44\rangle = \langle 41\rangle. \qquad (2.23)$$

This is in line with the claim that the actual z-scaling of the $(--)$ and $(++)$ shifts is better than estimated in Eq. (2.20). On the other hand, under a $(-+)$ shift, the amplitude behaves as

$$A_4\big(\hat{1}^-, 2^-, 3^+, \hat{4}^+\big) = \frac{\langle\hat{1}2\rangle^4}{\langle\hat{1}2\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \sim z^3. \qquad (2.24)$$

This is consistent with the general bounds derived above.

We will now proceed by explaining an important subtlety concerning three-point amplitudes. This will complete all prerequisites for using the BCFW recursion relations, and we will then derive the n-point MHV amplitudes as a first example.

2.2 The Gluon Three-Point Amplitude

In order to use the BCFW recursion relations derived above we need to understand the starting point of the recursion, in other words the 'smallest' or atomistic amplitudes. It may be slightly surprising that these are certain three-particle amplitudes. As we discuss presently, their definition is somewhat peculiar due to the constraints imposed by the on-shell conditions and momentum conservation.

In fact, for real null momenta there are no three-point amplitudes, since

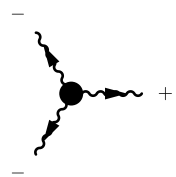
$$p_1^\mu + p_2^\mu + p_3^\mu = 0 \quad \text{implies } p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_1 = 0. \tag{2.25}$$

All Mandelstam invariants vanish and there are no other Lorentz scalars an amplitude could depend on

$$p_i \cdot p_j = 0 \quad \leftrightarrow \quad \langle ij\rangle[ji] = 0. \tag{2.26}$$

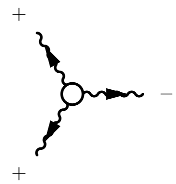
The situation is different for *complex momenta* $p_i \in \mathbb{C}$. In this case the helicity spinors λ_i and $\tilde{\lambda}_i$ are independent, and the conditions $p_i \cdot p_j = 0$ can be solved either by $\langle ij\rangle = 0\,\forall i, j = 1, 2, 3$ or by $[ij] = 0\,\forall i, j = 1, 2, 3$. Hence either $\lambda_1^\alpha \propto \lambda_2^\alpha \propto \lambda_3^\alpha$ (collinear left-handed spinors) or $\tilde{\lambda}_1^{\dot\alpha} \propto \tilde{\lambda}_2^{\dot\alpha} \propto \tilde{\lambda}_3^{\dot\alpha}$ (collinear right-handed spinors) solve the constraints $p_i \cdot p_j = 0$. The two choices correspond to the three-gluon MHV_3 and $\overline{\text{MHV}}_3$ amplitudes, respectively. They are given by

$$A_3^{\text{MHV}}(i^-, j^-) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \qquad [12] = [23] = [31] = 0, \tag{2.27}$$



and

$$A_3^{\overline{\text{MHV}}}(i^+, j^+) = -\frac{[ij]^4}{[12][23][31]}, \qquad \langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0, \tag{2.28}$$



respectively. These three-point amplitudes can of course be straightforwardly read off from the color-ordered Feynman rules. Alternatively, one can argue that these are the only functional forms compatible with the helicity assignments of the external particles and the vanishing of the $[ij]$ and $\langle ij\rangle$, respectively, up to a free constant which is identified with the coupling constant [5]. Let us illustrate this point for the MHV_3 amplitude. Without loss of generality we can take the negative helicity states to be $i = 1, j = 2$. From the kinematical discussion above it is clear that the amplitude can depend only on the variables $\langle 12\rangle$, $\langle 23\rangle$ and $\langle 31\rangle$. The helicity assignment implies

$$h_{1,2}A_3^{\mathrm{MHV}} = -A_3^{\mathrm{MHV}}, \qquad h_3 A_3^{\mathrm{MHV}} = A_3^{\mathrm{MHV}}, \tag{2.29}$$

where we recall that the helicity generator was defined as

$$h_i = -\frac{1}{2}\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \frac{1}{2}\tilde{\lambda}_i^{\dot\alpha}\frac{\partial}{\partial \tilde{\lambda}_i^{\dot\alpha}}. \tag{2.30}$$

These equations fix the answer to be $\langle 12\rangle^3/(\langle 23\rangle\langle 31\rangle)$, up to the overall normalization.

In summary, we have arrived at a remarkable result. Via the BCFW recursion we can produce all n-gluon tree amplitudes from the three-point amplitude. The structure of the latter follows solely from kinematical considerations, i.e. helicity assignments and momentum conservation. In particular, the explicit form of the four-gluon vertex in Yang-Mills theory is not needed!

Exercise 2.1 (Three-Point Amplitudes from Color-Ordered Feynman Rules)
Verify the expressions for the three-point MHV amplitude of Eq. (2.27) and the three-point anti-MHV amplitude of Eq. (2.28) by an explicit calculation using the color-ordered Feynman rules of Chap. 1, given in Table 1.4.

2.3 An Example: MHV Amplitudes

Having established the BCFW recursion relations, we will now give a simple yet non-trivial example, and derive the formula for n-point MHV amplitudes with arbitrary multiplicity n.

For simplicity, but without loss of generality, let us assume that the negative helicity gluons are at positions 1 and n. Then, by the discussion above we will obtain a valid BCFW relation without boundary term if we shift the momenta p_1 and p_n, as in Eq. (2.1), as this corresponds to a $(--)$-shift.

We then have to consider all BCFW diagrams that can contribute to this channel, see Fig. 2.4. There are two possible helicity assignments for the internal propagator, $(-+)$ and $(+-)$. As we will see presently, this imposes further constraints on the BCFW diagrams. In the $(+-)$ case, A_L has only one negative helicity gluon. As we saw in Sect. 1.12, such an amplitude vanishes unless it has three external legs. This leads us to draw the first BCFW diagram shown in Fig. 2.4. Likewise, the $(-+)$ helicity assignment leads to a zero term unless only three legs enter A_R. However, there is an additional subtlety. In fact, with the $\overline{\mathrm{MHV}}_3$ amplitude on the right the three-point condition $\hat{\lambda}_n \propto \lambda_{n-1}$ implies $\langle \hat{n} n-1\rangle = \langle n n-1\rangle = 0$, i.e. the collinearity of p_n and p_{n-1}. For generic external momenta, which is the case we are

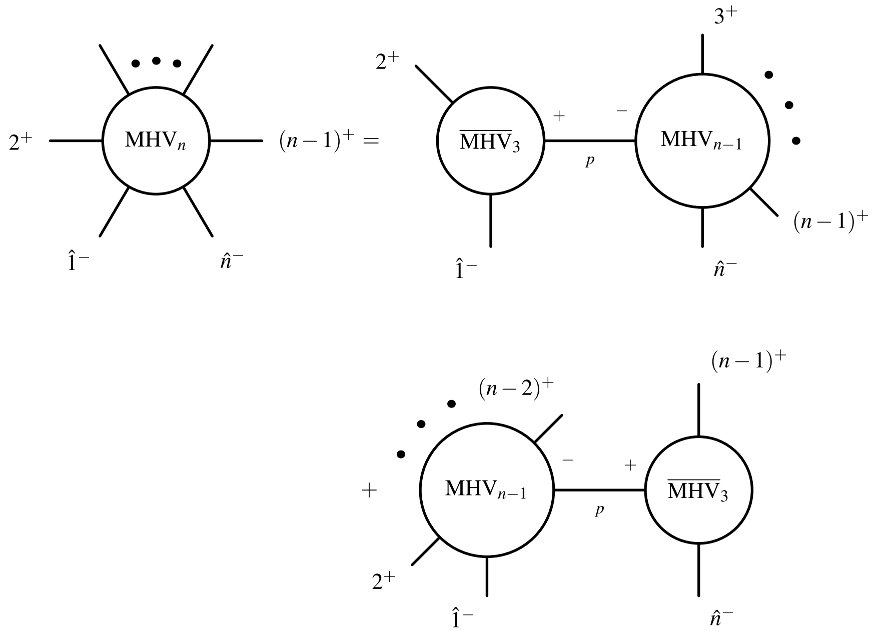


Fig. 2.4 The BCFW recursion for an MHV$_n$ amplitude

interested in, this collinearity is not fulfilled and therefore this contribution vanishes, since the amplitude on the right does. Therefore, we are left with only one BCFW diagram, the first one on the RHS of Fig. 2.4.

Let us now prove the formula for MHV amplitudes by induction. We want to prove that

$$A_n^{\text{MHV}}\left(n^-, 1^-\right) = \frac{\langle n1\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}. \tag{2.31}$$

We already know that this formula is true for $n = 3$, see Eq. (2.27) (and also for $n = 4$, from the Feynman graph calculation of Sect. 1.12). Therefore we only need to prove the inductive step.

We will also need the formula for the three-point $\overline{\text{MHV}}$ amplitude that was given in Eq. (2.28). In the BCFW channel that we are considering, we have

$$z_P = \frac{P^2}{\langle n|P|1]}, \qquad P^\mu = p_1^\mu + p_2^\mu, \tag{2.32}$$

and hence

$$z_P = \frac{(p_1 + p_2)^2}{\langle n|P|1]} = \frac{\langle 12\rangle[21]}{\langle n2\rangle[21]} = \frac{\langle 12\rangle}{\langle n2\rangle}. \tag{2.33}$$

The amplitudes A_L and A_R are given by the induction assumption

$$A_L = A_3^{\overline{\mathrm{MHV}}}\big(\hat{1}^-, 2^+, -\hat{P}^+\big) = -\frac{[2(-\hat{P})]^3}{[12][(-\hat{P})1]}, \tag{2.34}$$

$$A_R = A_{n-1}^{\mathrm{MHV}}\big(\hat{P}^-, 3^+, 4^+, \dots, (n-1)^+, \hat{n}^-\big) = \frac{\langle \hat{n}\hat{P}^-\rangle^3}{\langle \hat{P}3\rangle\langle 34\rangle\cdots\langle (n-1)\hat{n}\rangle}. \tag{2.35}$$

Then, using $|-\tilde{\lambda}_P] = -|\tilde{\lambda}_P]$ and $|-\lambda_P\rangle = +|\lambda_P\rangle$, and

$$[\hat{1}*] = [1*], \qquad \langle\hat{n}*\rangle = \langle n*\rangle, \qquad \langle n\hat{P}\rangle[\hat{P}2] = \langle n\hat{1}\rangle[12] = \langle n1\rangle[12], \tag{2.36}$$

$$(p_1+p_2)^2 = \langle 12\rangle[21], \qquad \langle 3\hat{P}\rangle[\hat{P}1] = \langle 32\rangle[2\hat{1}] = \langle 32\rangle[21], \tag{2.37}$$

we find

$$\hat{A}_L\frac{1}{(p_1+p_2)^2}\hat{A}_R = -\frac{\langle n1\rangle^3[12]^3}{[12][21]\langle 32\rangle[21]\langle 12\rangle\langle 34\rangle\cdots\langle (n-1)n\rangle} = \frac{\langle n1\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}, \tag{2.38}$$

in perfect agreement with Eq. (2.31). This completes the inductive proof of Eq. (2.31).

Exercise 2.2 (The 6-Gluon Split-Helicity NMHV Amplitude)
Determine the first non-trivial next-to-maximally-helicity-violating (NMHV) amplitude

$$A_6^{\mathrm{tree}}\big(1^+, 2^+, 3^+, 4^-, 5^-, 6^-\big)$$

from the BCFW recursion relation and our knowledge of the MHV amplitudes. Consider a shift of the two helicity states 1^+ and 6^- and show that

$$\begin{aligned} A_6^{\mathrm{tree}}\big(1^+, 2^+, 3^+, 4^-, 5^-, 6^-\big) = {} & \frac{\langle 6|p_{12}|3]^3}{\langle 61\rangle\langle 12\rangle[34][45][5|p_{16}|2\rangle}\frac{1}{(p_6+p_1+p_2)^2} \\ & + \frac{\langle 4|p_{56}|1]^3}{\langle 23\rangle\langle 34\rangle[16][65][5|p_{16}|2\rangle}\frac{1}{(p_5+p_6+p_1)^2}, \end{aligned}$$

where $p_{ij} = p_i + p_j$.

2.4 Factorization Properties of Tree-Level Amplitudes

As we have already seen in the derivation of the BCFW recursion relations, the analytic structure of tree-level amplitudes is governed by propagator poles and their residues. The residues are given by lower-point tree-level amplitudes. Here we summarize these properties.

2.4.1 Factorization on Multi-Particle Poles

Partial or color-ordered amplitudes can have poles when region momenta $P_{i,j} := p_i + p_{i+1} + \cdots + p_j$ go on shell. Then the amplitude factorizes according to

$$A_n^{\text{tree}}(1,\ldots,n) \sim \sum_{\lambda} A_L\big(i,\ldots,j,P^{\lambda}\big)\frac{1}{P_{i,j}^2}A_R\big(P^{-\lambda},j+1,\ldots,i-1\big). \tag{2.39}$$

We call this a two-particle pole if the region momentum $P_{i,j}$ is formed by two external momenta, and multi-particle pole otherwise.

2.4.2 Absence of Multi-Particle Poles in MHV Amplitudes

Multi-gluon amplitudes will in general have multi-particle poles. However, MHV amplitudes are special, and in fact, they only have two-particle poles. The reason is the following. Consider the general factorization formula Eq. (2.39). In a factorization of an MHV amplitude there are only three negative helicity legs (corresponding to the two external negative helicities, and one for the internal on-shell propagator) that are distributed over two partial amplitudes. However, we saw in the previous chapter that $A_n(1^{\pm},2^{+},\ldots,n^{+}) = 0$ (for $n > 3$). Therefore, this is always zero unless one partial amplitude is a three-particle amplitude, and this corresponds to a two-particle pole. In fact we have seen this principle at work in the previous Sect. 2.3. Here the BCFW recursion for MHV_n amplitudes reduced to a single term with a $\overline{\text{MHV}}_3$ amplitude on the left-hand-side, as a consequence of the absence of multi-particle poles.

2.4.3 Collinear Limits

A special case of the factorization formula (2.39) occurs for $j = i+1$, when we have a two-particle singularity. In fact, since the factorization involves a three-particle amplitude, such a pole can only occur for collinear external momenta. We already know from Sect. 2.2 that this is very subtle. For real momenta, the limiting configuration is $p_i \sim p_{i+1}$, which we may parameterize by $p_i = zP$ and $p_{i+1} = (1-z)P$ with the total collinear momentum $P = p_i + p_{i+1}$. It is convenient to write the null-momenta p_i and p_{i+1} in terms of spinors $P = \lambda_P\tilde{\lambda}_P$ associated to P. We then have $\lambda_i = \sqrt{z}\lambda_P$, $\tilde{\lambda}_i = \sqrt{z}\tilde{\lambda}_P$, and similarly for p_{i+1} with z replaced by $1-z$. If in addition $z \to 1$ (or $z \to 0$), we have a *soft limit*, where the four-momentum of one of the external momenta goes to zero. Tree (and loop-level) amplitudes possess important factorization properties with universal features in collinear as well as in soft limits. We will discuss the tree-level case here (the same analysis also applies to loop integrands).

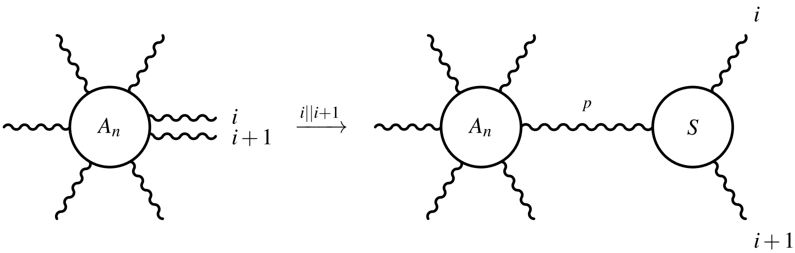


Fig. 2.5 Factorization in the collinear limit of legs i and $i+1$ where S denotes the splitting function $\mathrm{Split}^{\mathrm{tree}}_{-\lambda}(z, i, i+1)$

An analysis based on Feynman diagrams shows that tree-level amplitudes have a universal (singular) behavior in the collinear limit. It is governed by splitting functions,

$$\begin{aligned} &A_n^{\mathrm{tree}}\big(\ldots, i^{\lambda_i}, (i+1)^{\lambda_{i+1}}, \ldots\big) \\ &\xrightarrow{i\|i+1} \sum_{\lambda=\pm} \mathrm{Split}^{\mathrm{tree}}_{-\lambda}(z, i, i+1) A^{\mathrm{tree}}_{n-1}\big(\ldots, P^{\lambda}, \ldots\big), \end{aligned} \tag{2.40}$$

see Fig. 2.5. The splitting amplitude depends on the helicities of the collinear gluons but is independent of the helicities of the other legs not participating in the collinear limit. This is known as the universality of the splitting functions. We have (see [6] and references therein)

$$\mathrm{Split}^{\mathrm{tree}}_{-}\big(z, a^{-}, b^{-}\big) = 0, \tag{2.41}$$

$$\mathrm{Split}^{\mathrm{tree}}_{-}\big(z, a^{+}, b^{+}\big) = \frac{1}{\sqrt{z(1-z)}\langle ab\rangle}, \tag{2.42}$$

$$\mathrm{Split}^{\mathrm{tree}}_{-}\big(z, a^{+}, b^{-}\big) = -\frac{z^2}{\sqrt{z(1-z)}[ab]}, \tag{2.43}$$

$$\mathrm{Split}^{\mathrm{tree}}_{-}\big(z, a^{-}, b^{+}\big) = -\frac{(1-z)^2}{\sqrt{z(1-z)}[ab]}. \tag{2.44}$$

The remaining splitting amplitudes may be obtained from the ones above by parity,

$$\mathrm{Split}^{\mathrm{tree}}_{-(-\lambda)}\big(z, a^{-\lambda_a}, b^{-\lambda_b}\big) = -\mathrm{Split}^{\mathrm{tree}}_{-\lambda}\big(z, a^{\lambda_a}, b^{\lambda_b}\big)\big|_{\langle ab\rangle \leftrightarrow [ab]}. \tag{2.45}$$

We shall now derive these expressions from the MHV amplitudes.

2.4.3.1 Example: Collinear Limits of the Five-Point MHV Amplitude

Let us test the above splitting functions using the example of a five-point MHV amplitude,

$$A_5^{\text{tree}}\big(1^-,2^-,3^+,4^+,5^+\big) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \tag{2.46}$$

$$\xrightarrow{4\|5} \frac{1}{\sqrt{z(1-z)}\langle 45\rangle} \times \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3P\rangle\langle P1\rangle} \tag{2.47}$$

$$= \text{Split}_-^{\text{tree}}\big(z,4^+,5^+\big) \times A_4^{\text{tree}}\big(1^-,2^-,3^+,P^+\big), \tag{2.48}$$

where we parametrized the collinear limit by

$$\lambda_4 = \sqrt{z}\lambda_P, \qquad \lambda_5 = \sqrt{1-z}\lambda_P, \tag{2.49}$$

$$\tilde\lambda_4 = \sqrt{z}\tilde\lambda_P, \qquad \tilde\lambda_5 = \sqrt{1-z}\tilde\lambda_P. \tag{2.50}$$

Indeed agreement with Eq. (2.42) is found. Similarly, we can take the collinear limit in a $(+-)$ channel. We have

$$A_5^{\text{tree}}\big(1^-,2^-,3^+,4^+,5^+\big) \xrightarrow{2\|3} \frac{z^2}{\sqrt{z(1-z)}}\frac{1}{\langle 23\rangle}\frac{\langle 1P\rangle^4}{\langle 1P\rangle\langle P4\rangle\langle 45\rangle\langle 51\rangle} \tag{2.51}$$

$$= \text{Split}_+^{\text{tree}}\big(z,2^-,3^+\big) \times A_4^{\text{tree}}\big(1^-,P^-,4^+,5^+\big) \tag{2.52}$$

from which we deduce

$$\text{Split}_+^{\text{tree}}\big(z,a^-,b^+\big) = \frac{z^2}{\sqrt{z(1-z)}\langle ab\rangle}. \tag{2.53}$$

Converting this to $\text{Split}_-^{\text{tree}}(z,a^+,b^-)$ via the parity transformation Eq. (2.45) recovers Eq. (2.43). In order to check Eq. (2.44) we consider the collinear limit in a $(-+)$-channel

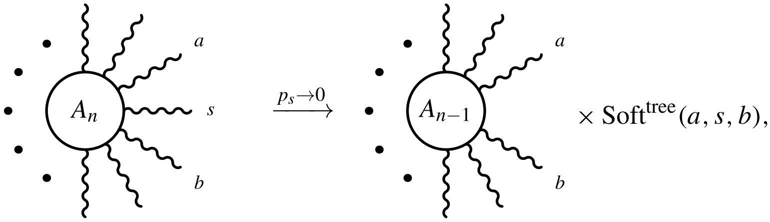
$$A_5^{\text{tree}}\big(1^-,2^-,3^+,4^+,5^+\big) \xrightarrow{5\|1} \underbrace{\frac{(1-z)^2}{\sqrt{z(1-z)}\langle 51\rangle}}_{\text{Split}_+^{\text{tree}}(z,5^+,1^-)} A_4^{\text{tree}}\big(P^-,2^-,3^+,4^+\big) \tag{2.54}$$

yielding Eq. (2.44) via parity. In order to check the vanishing of Eq. (2.41) one has to study the collinear factorization of the 6-point MHV amplitude with the helicity distributions $A_6^{\text{tree}}(1^-,2^-,3^+,4^+,5^+,6^+)$ along the legs 5 and 6.

2.4.4 Soft Limit

One speaks of the soft limit of an external gluon when its energy vanishes. This is equivalent to sending the on-shell four-momentum k_s of the gluon leg s to zero.

This limit also leads to a factorization and has universal features,



$$A_n^{\text{tree}}\big(\dots, a, s^{\pm}, b, \dots\big) \xrightarrow{k_s \to 0} \text{Soft}^{\text{tree}}\big(a, s^{\pm}, b\big) A_{n-1}^{\text{tree}}(\dots, a, b, \dots). \tag{2.55}$$

The factorized function depends on the momenta and helicities of the soft gluon and the momenta of the color-ordered neighbors a and b. It is independent, however, of the helicities and particle types of the neighboring legs. From considering the soft limit of an MHV amplitude one easily establishes

$$\text{Soft}^{\text{tree}}\big(a, s^+, b\big) = \frac{\langle ab\rangle}{\langle as\rangle\langle sb\rangle}. \tag{2.56}$$

Via parity in analogy to Eq. (2.45) we have

$$\text{Soft}^{\text{tree}}\big(a, s^-, b\big) = -\frac{[ab]}{[as][sb]}. \tag{2.57}$$

The factorization properties under collinear and soft limits may be used to test obtained results for their consistency. For example, the 6-gluon split-helicity amplitude $A_6^{\text{tree}}(1^+, 2^+, 3^+, 4^-, 5^-, 6^{+-})$ has to factorize into a soft function and the 5-point MHV amplitude when its legs 4, 5 or 6 are taken soft. It is instructive to work this out in detail for the leg 4.

In Exercise 2.1 the 6-gluon split-helicity amplitude was computed using the BCFW recursion, with the result

$$\begin{aligned} &A_6^{\text{tree}}\big(1^+, 2^+, 3^+, 4^-, 5^-, 6^-\big) \\ &\quad= \frac{\langle 6|p_{12}|3]^3}{\langle 61\rangle\langle 12\rangle[34][45][5|p_{16}|2\rangle} \frac{1}{(p_6 + p_1 + p_2)^2} \\ &\qquad+ \frac{\langle 4|p_{56}|1]^3}{\langle 23\rangle\langle 34\rangle[16][65][5|p_{16}|2\rangle} \frac{1}{(p_5 + p_6 + p_1)^2}, \\ &\quad\text{where } p_{ij} := p_i + p_j. \end{aligned} \tag{2.58}$$

Taking the soft limit on leg 4, i.e. $|4\rangle \to 0$, we see that the second term in the above formula vanishes, while the first term develops the expected pole of Soft$^{\text{tree}}(3, 4^-, 5)$ from (2.57)

$$A_6^{\text{tree}} \xrightarrow{4^- \to 0} -\frac{[35]}{[34][45]} \frac{\langle 6|p_1 + p_2|3]^3}{\langle 61\rangle\langle 12\rangle[53][5|p_1 + p_6|2\rangle} \frac{1}{(p_6 + p_1 + p_2)^2}. \tag{2.59}$$

Using momentum conservation $p_1 + p_2 + p_3 + p_5 + p_6 = 0$ for the limiting kinematics one has

$$\begin{aligned} &\langle 6|p_1 + p_2|3] = -\langle 65\rangle[53], \qquad [5|p_1 + p_6|2\rangle = -[53]\langle 32\rangle, \\ &(p_6 + p_1 + p_2)^2 = \langle 35\rangle[53]. \end{aligned} \tag{2.60}$$

Plugging this into Eq. (2.59), canceling out factors of [53], one arrives at

$$\begin{aligned} A_6^{\text{tree}} &\xrightarrow{4^- \to 0} \text{Soft}^{\text{tree}}\big(3, 4^-, 5\big) \frac{\langle 56\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 35\rangle\langle 61\rangle} \\ &= \text{Soft}^{\text{tree}}\big(3, 4^-, 5\big) A_5^{\text{tree}}\big(1^+, 2^+, 3^+, 5^-, 6^{+-}\big), \end{aligned} \tag{2.61}$$

which is the correct factorized result of Eq. (2.55).

Exercise 2.3 (The Vanishing Splitting Function $\text{Split}_+^{\text{tree}}(z, a^+, b^+)$)
Show by studying the factorization properties of the six-point MHV-amplitude $A_6^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ in the collinear limit $5 \parallel 6$ that

$$\text{Split}_+^{\text{tree}}\big(z, a^+, b^+\big) = 0.$$

Exercise 2.4 (Soft Limit of 6-Gluon Split-Helicity Amplitude)
Check the consistency of the 6-gluon split-helicity amplitude of Eq. (2.58) with the soft limit of leg 5. In contrast to the discussion in Sect. 2.4.4, this tests both terms contributing to the amplitude in Eq. (2.58).

Exercise 2.5 (A $\bar{q}qggg$ Amplitude from Collinear and Soft Limits)
In Chap. 1 we established the following color-ordered $\bar{q}qgg$ amplitudes involving a massless quark and anti-quark using color-ordered Feynman rules:

$$A_4^{\text{tree}}\big(1_{\bar{q}}^-, 2_q^+, 3^+, 4^+\big) = 0, \tag{2.62}$$

$$A_4^{\text{tree}}\big(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+\big) = \frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{2.63}$$

Use these and the discussed splitting and soft factorization properties for gluonic legs to make a sophisticated guess for the five-point single quark-line tree amplitude $A_4^{\text{tree}}(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+, 5^+)$. Check your guess against all known factorization properties.

Can you generalize your guess to the partial amplitudes $A_n^{\text{tree}}(1_{\bar{q}}^-, 2_q^+, 3^+, \ldots, n^+)$ and $A_n^{\text{tree}}(1_{\bar{q}}^-, 2_q^+, 3^-, 4^+, \ldots, n^+)$?

2.5 On-shell Recursion for Amplitudes with Massive Particles

So far our discussion focused on massless amplitudes, mostly involving gluons and massless fermions. Of course amplitudes with massive particles are of great relevance as well and on-shell recursions have also been developed for this case. Let us then briefly discuss tree-level scattering amplitudes of n particles, some of which (but not all) are allowed to be massive

$$A_n(p_1, p_2, \ldots, p_n), \quad p_i^2 = m_i^2. \tag{2.64}$$

Concretely, we consider gauge theory amplitudes involving at least two massless gluons, which we moreover take to be neighboring for simplicity of the discussion, say $p_1^2 = p_n^2 = 0$. Let us now see how the BCFW on-shell recursion derived in Sect. 2.1 may be generalized to gauge theory amplitudes with massive particles. We closely follow reference [7] in our exposition.

As before we consider a complex shift of the null gluon momenta at positions 1 and n by a parameter $z \in \mathbf{C}$

$$\begin{aligned}
\hat{p}_1(z) &= p_1 - z|n\rangle[1|, \\
\hat{p}_n(z) &= p_n + z|n\rangle[1|, \\
\hat{P}_i(z) &= P_i - z|n\rangle[1|,
\end{aligned} \tag{2.65}$$

where $P_i = p_1 + \cdots + p_{i-1}$ with $i \in [3, n-1]$ is the sum of adjacent momenta from 1 to $i-1$ which featured in the BCFW recursion relation of Eq. (2.17). In fact, the arguments leading to the BCFW recursion relation are very general and are applicable to a generic quantum field theory. They were obtained by thinking about the poles of the deformed amplitude $A(z)$ as an analytic function in z which arise whenever an internal propagator associated to the momentum $\hat{P}_i(z)$ goes on-shell. This reasoning does not change in the massive case, i.e. when $\hat{P}_i^2(z) = m^2$. The pole then reads in generalization of Eq. (2.6) as

$$\begin{aligned}
\frac{1}{\hat{P}_i(z) - m_{P_i}^2} &= \frac{1}{(\hat{p}_1(z) + p_2 + \cdots + p_{i-1})^2 - m_{P_i}^2} \\
&= \frac{1}{(p_i + p_{i+1} + \cdots + \hat{p}_n(z))^2 - m_{P_i}^2} \\
&= \frac{1}{P_i^2 - m_{P_i}^2 - z\langle n|P_i|1]},
\end{aligned} \tag{2.66}$$

where m_{P_i} is the mass of the associated particle going on-shell. The location of the pole is then simply shifted to

$$z_{P_i} = \frac{P_i^2 - m_{P_i}^2}{\langle n|P_i|1]}, \quad \forall i \in [3, n-1], \tag{2.67}$$

generalizing Eq. (2.7) to the massive case. Using again the theorem that the sum of residues of the rational function $A(z)/z$ on the Riemann sphere is zero, one immediately arrives at the on-shell recursion relation for amplitudes including massive particles

$$A_n(1,\dots,n)=\sum_{i=2}^{n-1}\sum_{s} A_L\big(\hat{1}(z_{P_i}),2,\dots,i-1,-\hat{P}^s(z_{P_i})\big)\frac{1}{P_i^2-m_{P_i}^2}$$
$$\times A_R\big(\hat{P}^{\bar{s}}(z_{P_i}),i,\dots,n-1,\hat{n}(z_{P_i})\big)+\mathrm{Res}(z=\infty). \qquad (2.68)$$

where the sum over s is over all contributing states and the legs 1 and n are massless. Of course this formula is only of use if one can show that the residue at infinity vanishes. This turns out to be the case if the gluon helicities of the shifted legs are not of the $(-,+)$ type as before, i.e. the statement is

$$\mathrm{Res}(z=\infty)=0 \quad \text{iff} \quad (h_1,h_n)=(+,-),(+,+),(-,-), \qquad (2.69)$$

see [7] for a derivation. This concludes our discussion of the massive on-shell recursion.

Let us now study as an example four point amplitudes involving gluons and massive scalars.

Example (Amplitudes with Gluons and Massive Scalars) We consider a theory of a massive complex scalar field coupled to gauge theory. Concretely, we want to evaluate the four-point amplitude involving two neighboring gluons of positive helicity

$$A_4\big(1^+,2_\phi,3_{\bar{\phi}},4^+\big). \qquad (2.70)$$

The scalars have mass m^2. In fact this amplitude vanishes in the massless limit $m=0$, similar to the vanishing of the above amplitude when the scalars are replaced by massless fermions, as was shown in chapter one. This is related to a hidden supersymmetry of massless gauge theory amplitudes to be discussed soon. In fact amplitudes of the above type are of interest even in massless theories at the one-loop level. There the need to regulate divergences leads one to consider internal particles propagating in $D=4-2\varepsilon$ dimensions, as we shall discuss in detail in the next chapter. A massless particle in D dimensions may be alternatively viewed as a massive one in four dimension and hence the above amplitude becomes relevant here. This will be used later on in Sect. 3.5.3.

Returning to our concrete example we employ the massive on-shell recursion of Eq. (2.68). Only a single channel contributes

$$A_4\big(1^+,2_\phi,3_{\bar{\phi}},4^+\big)=A_3\big(\hat{1}^+,2_\phi,-\hat{P}_{\bar{\phi}}\big)\frac{1}{P^2-m^2}A_3\big(\hat{P}_\phi,3_{\bar{\phi}},4^+\big). \qquad (2.71)$$

All that is needed are—again—the atomistic $(\phi g\bar{\phi})$-amplitudes. These follow from the color ordered Feynman vertices of two scalars of mass m and momenta l_1,l_2, and a single gluon with momentum p

$$V_3\big(l_1^+,p^\mu,l_2^-\big)=\frac{i}{\sqrt{2}}\big(l_1^\mu-l_2^\mu\big), \qquad (2.72)$$

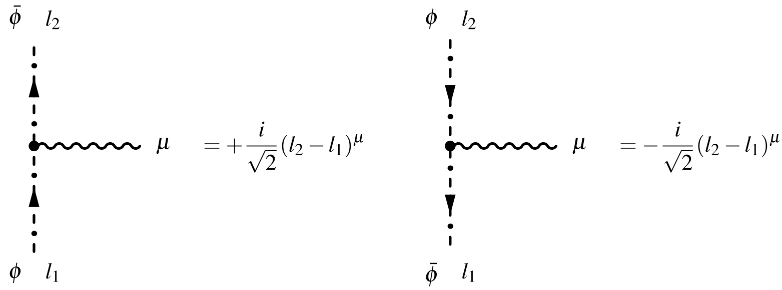


Fig. 2.6 Color ordered Feynman rules for a massive complex scalar field

where the $+$ or $-$ index for the scalars denotes a scalar or anti-scalar state respectively, see Fig. 2.6. Contracting this with the positive helicity gluon polarization of Eq. (1.82) one obtains the on-shell three-point amplitudes

$$A_3\big(l_1^+, p^+, l_2^-\big) = \frac{\langle q|l_1|p]}{\langle qp\rangle} = A_3\big(l_1^-, p^+, l_2^+\big), \tag{2.73}$$

where the last relation follows by reflection. Note that here q is the arbitrary null reference momentum of the gluon leg related to the local gauge invariance of the theory. By similar arguments one establishes the three point amplitudes involving a negative helicity gluon

$$A_3\big(l_1^+, p^-, l_2^-\big) = \frac{\langle p|l_1|q]}{[pq]} = A_3\big(l_1^-, p^-, l_2^+\big). \tag{2.74}$$

An alert reader might object at this point that there appears to be a serious problem with these amplitudes, as they explicitly depend on the reference momentum q. How should one be able to build all scattering amplitudes in this theory on such an arbitrariness? In fact, the amplitudes of Eqs. (2.73) and (2.74) are independent of the choice of q. This is seen as follows. Taking $|q\rangle$ and $|p\rangle$ as a basis in Weyl spinor space we may parametrize an arbitrary reference spinor different from $|q\rangle$ as $|q'\rangle = \alpha|q\rangle + \beta|p\rangle$. Clearly Eq. (2.73) is invariant under rescalings of the reference spinor, thus without loss of generality we may parameterize $|q'\rangle = |q\rangle + \gamma|p\rangle$ or infinitesimally $\delta_q|q\rangle \propto |p\rangle$. This entails that the amplitude Eq. (2.73) changes under a variation of the reference spinor as

$$\delta_q A_3\big(l_1^+, p^+, l_2^-\big) \propto \frac{\langle p|l_1|p]}{\langle qp\rangle} = 0, \tag{2.75}$$

where the vanishing follows from $\langle p|l_1|p] = 2p\cdot l_1 = 0$, which is a consequence of the three point kinematics

$$(l_1 + p)^2 = l_2^2 \quad \rightarrow \quad l_1\cdot p = 0 \quad \text{as } l_1^2 = l_2^2 = m^2,\ p^2 = 0. \tag{2.76}$$

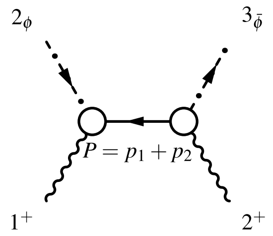


Fig. 2.7 On-shell recursion for the massive $A_4(1^+, 2_\phi, 3_{\bar\phi}, 4^+)$ amplitude. All external momenta are outgoing, P runs from right to left

Coming back to the recursive construction of the amplitude Eq. (2.70) we then have (cf. Fig. 2.7)

$$\begin{aligned} A_4\big(1^+, 2_\phi, 3_{\bar\phi}, 4^+\big) &= A_3\big(-\hat P_{\bar\phi}, \hat 1^+, 2_\phi\big)\frac{1}{P^2 - m^2} A_3\big(3_{\bar\phi}, \hat 4^+, \hat P_\phi\big) \\ &= -\frac{\langle q_1|\hat P|\hat 1]}{\langle q_1 \hat 1\rangle}\frac{1}{P^2 - m^2}\frac{\langle q_4|p_3|\hat 4]}{\langle q_4 \hat 4\rangle}. \end{aligned} \tag{2.77}$$

Here $q_{1/4}$ denote the reference momenta of the gluon legs 1 and 4. Things are simplified considerably with the gauge choice

$$q_1 = \hat p_4, \qquad q_4 = \hat p_1. \tag{2.78}$$

Noting that $|\hat 1] = |1]$ and $|\hat 4\rangle = |4\rangle$ we then have

$$\begin{aligned} &\langle q_1|\hat P|\hat 1] = \langle 4|\hat P|1] = \langle 4|P|1] = -\langle 4|p_3|1], \qquad \langle q_1 \hat 1\rangle = \langle 4\hat 1\rangle = \langle 41\rangle, \\ &\langle q_4|p_3|\hat 4] = \langle \hat 1|p_3|\hat 4], \qquad \langle q_4 \hat 4\rangle = \langle \hat 1 4\rangle = \langle 14\rangle. \end{aligned} \tag{2.79}$$

Plugging these into the above we find

$$A_4 = \frac{\langle 4|p_3|1]\langle \hat 1|p_3|\hat 4]}{\langle 14\rangle^2[(p_1 + p_2)^2 - m^2]}. \tag{2.80}$$

The numerator may be simplified with a trace identity to

$$\begin{aligned} \langle 4|p_3|1]\langle \hat 1|p_3|\hat 4] &= \frac{1}{2}\,\mathrm{Tr}(\hat{\not p}_4 \not p_3 \hat{\not p}_1 \not p_3) \\ &= 2\big(2(p_3 \cdot \hat p_4)(p_3 \cdot \hat p_1) - p_3^2(\hat p_1 \cdot \hat p_4)\big). \end{aligned} \tag{2.81}$$

In fact $p_3 \cdot \hat p_4 = 0$, which follows from momentum conservation

$$\hat P^2 = (p_3 + \hat p_4)^2 \quad \Rightarrow \quad m^2 = 2p_3 \cdot \hat p_4 + p_3^2 \quad \Rightarrow \quad p_3 \cdot \hat p_4 = 0. \tag{2.82}$$

Putting everything together we arrive at the compact expression for the four-point amplitude

$$A_4\big(1^+, 2_\phi, 3_{\bar\phi}, 4^+\big) = \frac{m^2[14]}{\langle 14\rangle[(p_1 + p_2)^2 - m^2]}, \tag{2.83}$$

which indeed vanishes in the massless limit as promised.

Exercise 2.6 (Four Point Scalar-Gluon Scattering)
Find the four-point massive-scalar-gluon amplitude

$$A_4\big(1^+, 2_\phi, 3_{\bar{\phi}}, 4^-\big),$$

with one positive and one negative gluon using the above recursive techniques.

2.6 Poincaré and Conformal Symmetry

We now turn to a more conceptual yet important subject: the question how the global symmetries of a gauge field theory manifest themselves at the level of scattering amplitudes. This has proven to be a very rich subject in particular at tree-level. The symmetries of the scattering amplitudes may be grouped into obvious and less obvious symmetries.

The obvious symmetries are the Poincaré transformations under which scattering amplitudes should be invariant. As we are working in the spinor helicity formulation of momentum space the momentum generator $p^{\alpha\dot{\alpha}}$ is represented by a multiplicative operator

$$p^{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \tag{2.84}$$

and the amplitude $\mathscr{A}_n$ should obey

$$p^{\alpha\dot{\alpha}} \mathscr{A}_n(\lambda_i, \tilde{\lambda}_i) = 0. \tag{2.85}$$

This is in fact true in the distributional sense of

$$p\delta(p) = 0, \tag{2.86}$$

thanks to the momentum conserving delta-function present in each amplitude

$$\mathscr{A}_n(\lambda_i, \tilde{\lambda}_i) = \delta^{(4)}\bigg(\sum_i p_i\bigg) A_n(\lambda_i, \tilde{\lambda}_i). \tag{2.87}$$

The Lorentz generators in the helicity spinor basis come in two pairs of symmetric rank-two tensors $m_{\alpha\beta}$ and $\overline{m}_{\dot{\alpha}\dot{\beta}}$ originating from the projections $M^{\mu\nu}(\sigma_{\mu\nu})_{\alpha\beta} = m_{\alpha\beta}$ and $M^{\mu\nu}(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} = \overline{m}_{\dot{\alpha}\dot{\beta}}$, see Appendix B for conventions. They are first order differential operators in helicity spinor space,

$$m_{\alpha\beta} = \sum_{i=1}^{n} \lambda_{i(\alpha} \partial_{i\beta)}, \qquad \overline{m}_{\dot{\alpha}\dot{\beta}} = \sum_{i=1}^{n} \tilde{\lambda}_{i(\dot{\alpha}} \partial_{i\dot{\beta})}, \tag{2.88}$$

where $\partial_{i\alpha} := \frac{\partial}{\partial\lambda_i^\alpha}$, $\partial_{i\dot\alpha} := \frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}$ and $r_{(\alpha\beta)} := \frac{1}{2}(r_{\alpha\beta} + r_{\beta\alpha})$ denotes symmetrization with unit weight. The invariance of $\mathscr{A}_n(\lambda_i, \tilde\lambda_i)$ under Lorentz-transformations

$$m_{\alpha\beta}\mathscr{A}_n(\lambda_i, \tilde\lambda_i) = 0 = \overline{m}_{\dot\alpha\dot\beta}\mathscr{A}_n(\lambda_i, \tilde\lambda_i) \tag{2.89}$$

is manifest, as it is an immediate consequence of the proper contraction of all Weyl indices within $\mathscr{A}_n$, i.e. the fact that the spinor brackets $\langle ij\rangle$ and $[ij]$ are invariant under $m_{\alpha\beta}$ and $\overline{m}_{\dot\alpha\dot\beta}$. For example,

$$m_{\alpha\beta}\langle jk\rangle = \sum_{i=1}^{n} \lambda_{i(\alpha}\partial_{i\beta)}\lambda_j^\gamma\lambda_{k\gamma} = \lambda_{j\alpha}\lambda_{k\beta} - \lambda_{j\beta}\lambda_{k\alpha} + (\alpha \leftrightarrow \beta) = 0. \tag{2.90}$$

Let us now discuss the less obvious symmetries of $\mathscr{A}_n(\lambda_i, \tilde\lambda_i)$. Classical Yang-Mills theory is invariant under a larger symmetry group than the four-dimensional Poincaré group: Due to the absence of dimensionful parameters in the theory (the coupling g is dimensionless) pure Yang-Mills theory or massless QCD is invariant under a scale transformation

$$x^\mu \to \kappa^{-1}x^\mu, \quad \text{respectively} \quad p^\mu \to \kappa p^\mu. \tag{2.91}$$

The scale transformations of the momenta are generated by the dilatation operator d, whose representation in spinor helicity variables acting on amplitudes is

$$d = \sum_{i=1}^{n}\left(\frac{1}{2}\lambda_i^\alpha\partial_{i\alpha} + \frac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + d_0\right), \quad d_0 \in \mathbb{R}, \tag{2.92}$$

reflecting the dilatation weight $1/2$ of the λ_i and $\tilde\lambda_i$, i.e. $[d, \lambda_i] = \frac{1}{2}\lambda_i$ and $[d, \tilde\lambda_i] = \frac{1}{2}\tilde\lambda_i$. The constant d_0 is undetermined at this point. It may be fixed by requiring invariance of the MHV amplitudes

$$\mathscr{A}_n^{\mathrm{MHV}} = \delta^{(4)}\left(\sum_i p_i\right)\frac{\langle\lambda_s\lambda_t\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}. \tag{2.93}$$

The dilatation operator d of Eq. (2.92) simply measures the weight in units of mass of the amplitude it acts on plus nd_0

$$d\mathscr{A}_n = \big([\mathscr{A}_n] + nd_0\big)\mathscr{A}_n. \tag{2.94}$$

We note the weights $[\delta^{(4)}(p)] = -4$, $[\langle\lambda_s\lambda_t\rangle^4] = 4$ and $[\frac{1}{\langle 12\rangle\cdots\langle n1\rangle}] = -n$, hence

$$d\mathscr{A}_n^{\mathrm{MHV}} = (-4 + 4 - n + nd_0)\mathscr{A}_n^{\mathrm{MHV}}, \tag{2.95}$$

which vanishes for the choice $d_0 = 1$. One easily checks the invariance under dilatations of the $q\bar q gg$-amplitude of Eq. (1.132) and of the $\overline{\mathrm{MHV}}_n$ amplitudes as well.

There is a further symmetry of scale invariant theories, namely the special conformal transformations $k_{\alpha\dot{\alpha}}$. This is a less obvious symmetry generator realized in terms of a second order differential operator in the spinor variables,

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \partial_{i\alpha} \partial_{i\dot{\alpha}}. \tag{2.96}$$

Together with the Poincaré and dilatation generators the set $\{p_{\alpha\dot{\alpha}}, k_{\alpha\dot{\alpha}}, m_{\alpha\beta}, \overline{m}_{\dot{\alpha}\dot{\beta}}, d\}$ generate the conformal group in four dimensions, $SO(2,4)$.

Let us now prove the invariance of the MHV amplitudes under special conformal transformations. As the only dependence of $\mathscr{A}_n^{\mathrm{MHV}}$ on the conjugate spinors $\tilde{\lambda}_i$ resides in the momentum conserving delta-function we have

$$\begin{aligned} k_{\alpha\dot{\alpha}} \mathscr{A}_n^{\mathrm{MHV}} &= \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \big(\delta^{(4)}(p) A_n^{\mathrm{MHV}} \big) \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^{\alpha}} \left(\frac{\partial p^{\beta\dot{\beta}}}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \left(\frac{\partial}{\partial p^{\beta\dot{\beta}}} \delta^{(4)}(p) \right) A_n^{\mathrm{MHV}} \right) \\ &= \left[\left(n \frac{\partial}{\partial p^{\alpha\dot{\alpha}}} + p^{\beta\dot{\beta}} \frac{\partial}{\partial p^{\beta\dot{\alpha}}} \frac{\partial}{\partial p^{\alpha\dot{\beta}}} \right) \delta^{(4)}(p) \right] A_n^{\mathrm{MHV}} \\ &\quad + \left(\frac{\partial \delta^{(4)}(p)}{\partial p^{\beta\dot{\alpha}}} \right) \sum_{i=1}^{n} \lambda_i^{\beta} \frac{\partial}{\partial \lambda_i^{\alpha}} A_n^{\mathrm{MHV}}. \end{aligned} \tag{2.97}$$

The last term may be rewritten as follows. First, we note the relation

$$\sum_{i=1}^{n} \lambda_{i\alpha} \partial_{i\beta} = \sum_{i=1}^{n} \lambda_{i(\alpha} \partial_{i\beta)} + \frac{1}{2} \varepsilon_{\alpha\beta} \sum_{i} \lambda_i^{\gamma} \partial_{i\gamma}, \tag{2.98}$$

which follows from decomposing the l.h.s. in a symmetric and anti-symmetric piece. The first term on the right-hand-side is the Lorentz generator $m_{\alpha\beta}$ which we already know annihilates A_n^{MHV}. The remaining term yields

$$\sum_{i=1}^{n} \lambda_i^{\beta} \frac{\partial}{\partial \lambda_i^{\alpha}} A_n^{\mathrm{MHV}} = \frac{1}{2} \delta_{\alpha}^{\beta} \sum_{i} \lambda_i^{\delta} \partial_{i\delta} A_n^{\mathrm{MHV}} = (4-n) A_n^{\mathrm{MHV}}. \tag{2.99}$$

Hence Eq. (2.97) turns into

$$k_{\alpha\dot{\alpha}} \mathscr{A}_n^{\mathrm{MHV}} = \left[\left(4 \frac{\partial}{\partial p^{\alpha\dot{\alpha}}} + p^{\beta\dot{\beta}} \frac{\partial}{\partial p^{\beta\dot{\alpha}}} \frac{\partial}{\partial p^{\alpha\dot{\beta}}} \right) \delta^{(4)}(p) \right] A_n^{\mathrm{MHV}}. \tag{2.100}$$

Indeed in a distributional sense we have

$$p^{\beta\dot{\beta}} \frac{\partial}{\partial p^{\beta\dot{\alpha}}} \frac{\partial}{\partial p^{\alpha\dot{\beta}}} \delta^{(4)}(p) = -4 \frac{\partial}{\partial p^{\alpha\dot{\alpha}}} \delta^{(4)}(p), \tag{2.101}$$

which one sees by integrating the second derivative expression against a test function $F(p)$,

$$\begin{aligned}
&\int d^4 p F(p) p^{\beta\dot{\beta}} \frac{\partial}{\partial p^{\beta\dot{\alpha}}} \frac{\partial}{\partial p^{\alpha\dot{\beta}}} \delta^{(4)}(p) \\
&= \int d^4 p \left(\left[\frac{\partial}{\partial p^{\beta\dot{\alpha}}} F(p) \right] 2\delta^{\beta}_{\alpha} + \left[\frac{\partial}{\partial p^{\alpha\dot{\beta}}} F(p) \right] 2\delta^{\dot{\beta}}_{\dot{\alpha}} \right) \\
&= 4 \int d^4 p \left[\frac{\partial}{\partial p^{\alpha\dot{\alpha}}} F(p) \right] \delta^{(4)}(p).
\end{aligned} \tag{2.102}$$

This proves the vanishing of $k_{\alpha\dot{\alpha}} \mathscr{A}_n^{\mathrm{MHV}}$, as claimed.

Summarizing, we have constructed a representation of the conformal group whose generators are represented by differential operators of degree one ($m_{\alpha\beta}$, $\overline{m}_{\dot{\alpha}\dot{\beta}}$, d), of degree two ($k_{\alpha\dot{\alpha}}$) and as a multiplicative operator ($p_{\alpha\dot{\alpha}}$) in an n-particle helicity spinor space. This representation is natural as amplitudes are functions in this space. All the generators leave the scattering amplitudes invariant. We have verified this explicitly for the MHV amplitudes. The representation obeys the commutation relations of the conformal algebra $\mathfrak{so}(2,4)$

$$\begin{aligned}
&\left[d, p^{\alpha\dot{\alpha}}\right] = p^{\alpha\dot{\alpha}}, \qquad [d, k_{\alpha\dot{\alpha}}] = -k_{\alpha\dot{\alpha}}, \qquad [d, m_{\alpha\beta}] = 0 = [d, \overline{m}_{\dot{\alpha}\dot{\beta}}], \\
&\left[k_{\alpha\dot{\alpha}}, p^{\beta\dot{\beta}}\right] = \delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}} d + m_{\alpha}{}^{\beta}\delta^{\dot{\beta}}_{\dot{\alpha}} + \overline{m}_{\dot{\alpha}}{}^{\dot{\beta}}\delta^{\beta}_{\alpha},
\end{aligned} \tag{2.103}$$

plus the Poincaré commutators discussed in Sect. 1.1.

The origin of this helicity spinor space representation becomes clear if one looks at the more familiar representation of the conformal group in configuration space x^μ which reads ($\partial_\mu := \frac{\partial}{\partial x^\mu}$)

$$\begin{aligned}
M_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu), \qquad P_\mu = -i\partial_\mu, \\
D &= -ix^\mu \partial_\mu, \qquad K_\mu = i\left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu\right).
\end{aligned} \tag{2.104}$$

A Fourier transform $\int d^4 x e^{ip\cdot x} \mathscr{O}(x, \partial_x)$ brings this representation into momentum space, which in turn can be mapped to the helicity spinor representation discussed in Sect. 1.6. From this point of view it is clear why $p^{\alpha\dot{\alpha}}$ becomes a multiplication operator and $k_{\alpha\dot{\alpha}}$ a second order derivative operator in momentum space.

Summary: Conformal Generators Here we collect the generators of the conformal algebra. For simplicity of notation, we write their single-particle action.

$$\begin{aligned}
p^{\alpha\dot{\alpha}} &= \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, \qquad k_{\alpha\dot{\alpha}} = \partial_\alpha \partial_{\dot{\alpha}}, \\
m_{\alpha\beta} &= \lambda_{(\alpha}\partial_{\beta)} := \frac{1}{2}(\lambda_\alpha \partial_\beta + \lambda_\beta \partial_\alpha), \qquad \overline{m}_{\dot{\alpha}\dot{\beta}} = \tilde{\lambda}_{(\dot{\alpha}}\partial_{\dot{\beta})}, \\
d &= \frac{1}{2}\lambda^\alpha \partial_\alpha + \frac{1}{2}\tilde{\lambda}^{\dot{\alpha}}\partial_{\dot{\alpha}} + 1.
\end{aligned} \tag{2.105}$$

The helicity generator is given by $h=-\frac{1}{2}\lambda^{\alpha}\partial_{\alpha}+\frac{1}{2}\tilde{\lambda}^{\dot{\alpha}}\partial_{\dot{\alpha}}$. It commutes with all generators of the conformal algebra.

Exercise 2.7 (Conformal Algebra)

Show that the representation constructed in the above Eq. (2.105) indeed obeys the commutation relations of the conformal algebra given in Eq. (2.103).

Exercise 2.8 (Inversion and Special Conformal Transformations)

The generator K_{μ} of Eq. (2.104) generates infinitesimal special conformal transformations. A finite special conformal transformation is given by

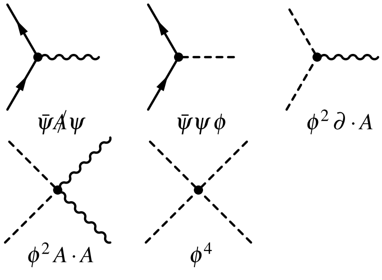
$$K^{\mu}\colon \quad x^{\mu}\to x'^{\mu}=\frac{x^{\mu}-a^{\mu}x^{2}}{1-2a\cdot x+a^{2}x^{2}}, \qquad a^{\mu}: \text{transformation parameter.} \tag{2.106}$$

An intuition on the character of these transformations may be found by noting that the action of K^{μ} may be also written as $K^{\mu}=IP^{\mu}I$, i.e. an inversion $Ix^{\mu}=\frac{x^{\mu}}{x^{2}}$ followed by a translation by a^{μ} followed by another inversion. Show that $K^{\mu}=IP^{\mu}I$ is equivalent to Eq. (2.106).

2.7 $\mathscr{N}=4$ Super Yang-Mills Theory

So far we have mostly discussed pure Yang-Mills theory or massless QCD. Our external states were either gluons ($h=\pm1$) or quarks ($h=\pm1/2$). However, a renormalizable quantum field theory in four dimensions could also contain scalar fields with helicity $h=0$. Of course the Higgs is an example for an elementary scalar particle realized in Nature. In particular if we continue to insist on conformal symmetry at tree-level, i.e. the absence of any dimensionful quantity in the bare Lagrangian of the theory, then the only allowed terms for a massless scalar in the Lagrangian are quartic.

Interestingly, irrespective of the details of the content and couplings of the fermionic and scalar matter fields in a gauge theory, the n-gluon *tree* level amplitudes are identical to the pure Yang-Mills theory ones. This is the case because scalars and fermions interact with each other and the gauge field via vertices of the type



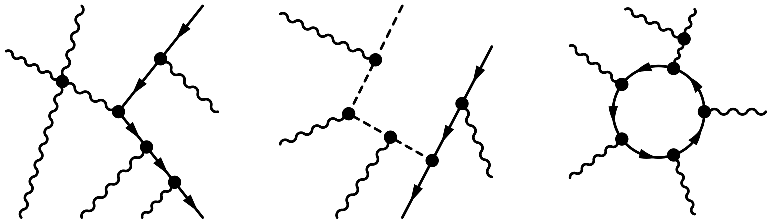


Fig. 2.8 Tree-level and one-loop examples for a gauge field theory coupled to scalars and fermions: at tree-level a scalar of fermion line always has to exit the diagram

Hence in a tree-level diagram with only external gluon legs, scalars or fermions cannot appear: they are always produced in pairs from gluon lines and thus have to exit the diagram at tree-level, see Fig. 2.8. Thus as long as one is interested in pure gluon amplitudes at tree-level[1] one may assemble the matter content of the gauge theory according to one's preference. It then proves useful to maximize the amount of symmetries of the theory in this choice. There is a distinguished and very special gauge theory which surpasses all others in its remarkable properties: it is the *maximally supersymmetric* Yang-Mills theory or $\mathcal{N}=4$ super Yang-Mills theory, which we now introduce. It may be thought of as a supersymmetric version of QCD.

Supersymmetry is a very attractive concept in elementary particle physics albeit not yet discovered in Nature. It proposes a symmetry between fermionic and bosonic fields generated by Grassmann odd supersymmetry generators leading to a graded (supersymmetric) extension of the Poincaré algebra. In its simplest ($\mathcal{N}=1$) gauge theoretical version there is one super-partner to every gluon A^a_μ: the spin $1/2$ gluino ψ^a_α. Indeed, the supersymmetry of gauge field theories can be extended to a maximal degree of four ($\mathcal{N}=4$ supersymmetry):[2] one then has *four* gluinos $\psi^a_{\alpha A}$ ($A=1,2,3,4$) as the super-partners to the gluon field A^a_μ. Closure of the supersymmetry algebra requires the additional presence of six real scalar fields $\phi^{aAB}=-\phi^{aBA}$. In total the field content of the $\mathcal{N}=4$ super Yang-Mills theory (we consider again the $SU(N)$ gauge group) is

$$\begin{array}{ll} A^a_\mu: & \text{gluon } a=1,\dots,N^2-1, \\ \psi^a_{\alpha A},\bar\psi^{\dot\alpha aA}: & 4 \text{ gluinos } \alpha,\dot\alpha=1,2;\ A=1,2,3,4, \\ \phi^{aAB}: & 6 \text{ scalars antisymmetric in } AB. \end{array}$$

For the scalars and gluinos we have the complex conjugation properties

$$\left(\phi^{AB}\right)^* = \phi_{AB} = \frac{1}{2}\varepsilon_{ABCD}\phi^{CD}, \qquad \left(\psi^a_{\alpha A}\right)^* = \bar\psi^{aA}_{\dot\alpha}. \tag{2.107}$$

[1] One can also show that QCD tree-level amplitudes with quarks and gluons can be deduced from their supersymmetric cousins, at least for up to four quark-antiquark pairs.

[2] In fact one may extend beyond $\mathcal{N}=4$ supersymmetries at the prize of leaving the realm of renormalizable quantum field theories. This leads to supergravity with a graviton and gravitinos in the spectrum. The maximally extended supersymmetric model is then $\mathcal{N}=8$ supergravity.

The extended supersymmetry also forces all fields to transform in the adjoint representation, distinguishing the gluinos from the quarks, which transform in the fundamental representation. Nevertheless, this difference is not visible at the level of color-ordered amplitudes where all gauge group dependences have been stripped off. The $\mathcal{N} = 4$ super Yang-Mills action reads

$$S = \frac{1}{g_{\mathrm{YM}}^2} \int d^4x \,\mathrm{Tr}\Big(-\frac{1}{4} F_{\mu\nu}^2 - (D_\mu \phi_{AB}) D^\mu \phi^{AB} - \frac{1}{2} [\phi_{AB}, \phi_{CD}] \big[\phi^{AB}, \phi^{CD}\big] \\ + i \bar{\psi}^A_{\dot{\alpha}} \sigma^{\dot{\alpha}\alpha}_\mu D^\mu \psi_{\alpha A} - \frac{i}{2} \psi^\alpha_A \big[\phi^{AB}, \psi_{\alpha B}\big] - \frac{i}{2} \bar{\psi}^A_{\dot{\alpha}} \big[\phi_{AB}, \bar{\psi}^{\dot{\alpha} B}\big]\Big). \tag{2.108}$$

Its form is uniquely fixed by the $\mathcal{N} = 4$ supersymmetry. For a review and more details, see Ref. [8]. There are only two tunable parameters: the gauge coupling g_{YM} and the rank N of the gauge group $SU(N)$. Notably in $\mathcal{N} = 4$ super Yang-Mills the gauge coupling g_{YM} is not renormalized at the quantum level, i.e. the theory is ultraviolet finite. In other words, the conformal symmetry at tree-level survives the quantization process without anomalies, and we have an interacting four dimensional quantum conformal field theory.

In fact this does not imply that there are no divergences arising in scattering amplitudes and correlation functions in this theory: radiative corrections to scattering amplitudes suffer from infrared (IR) divergences, just as in QCD, but are free of ultraviolet (UV) divergences reflecting the non-renormalization of the coupling constant.[3] Also composite gauge invariant operators, such as $\mathrm{Tr}(\phi_{AB}\phi^{AB})$, are renormalized in order to cure their inherent short-distance UV divergences. This induces anomalous scaling dimensions which are non-trivial functions of g_{YM}.

It may be said that $\mathcal{N} = 4$ super Yang-Mills is the interacting four dimensional quantum field theory with the largest amount of symmetry: maximally extended supersymmetry, quantum conformal symmetry and local $SU(N)$ gauge invariance.

We shall be interested in tree-level and loop-level color-ordered amplitudes in this highly symmetric quantum field theory, which we shall analyze in the following.

2.7.1 On-shell Superspace and Superfields

In a supersymmetric theory the on-shell degrees of freedom are balanced between bosons and fermions. In the $\mathcal{N} = 4$ super Yang-Mills (SYM) model we have 8 bosonic and 8 fermionic on-shell degrees of freedom, which may be grouped in Table 2.1.

[3] Depending on the formalism used, the elementary fields may need to be renormalized.

Table 2.1 The $\mathscr{N}=4$ super Yang-Mills on-shell field content

Field	Bosons			Fermions	
	g_+	g_-	S_{AB}	$\tilde{g}_A$	$\bar{\tilde{g}}^A$
Name	gluon		scalar	gluino	anti-gluino
Helicity	$+1$	-1	0	$+1/2$	$-1/2$
Degrees of freedom	1	1	6	4	4
$SU(4)_R$ representation	singlet		anti-symmetric (**6**)	fundamental (**4**)	anti-fund ($\bar{\mathbf{4}}$)

This $\mathscr{N}=4$ SYM on-shell multiplet may be assembled into one on-shell superfield Φ upon introducing the Grassmann odd parameter η^A with $A=1,2,3,4$

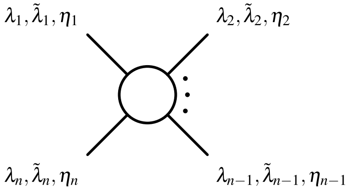
$$\Phi(p,\eta)=g_+(p)+\eta^A\tilde{g}_A(p)+\frac{1}{2!}\eta^A\eta^B S_{AB}(p)$$
$$+\frac{1}{3!}\eta^A\eta^B\eta^C\varepsilon_{ABCD}\bar{\tilde{g}}^D(p)+\frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\varepsilon_{ABCD}g_-(p). \quad (2.109)$$

If we assign the helicity $h=1/2$ to the Grassmann variable η^A then the on-shell superfield $\Phi(\eta)$ carries uniform helicity $h=1$. This extends our definition Eq. (1.75) for the helicity operator h to the supersymmetric case

$$h=\frac{1}{2}\big[-\lambda^\alpha\partial_\alpha+\tilde{\lambda}^{\dot{\alpha}}\partial_{\dot{\alpha}}+\eta^A\partial_A\big], \qquad \partial_A:=\frac{\partial}{\partial\eta^A}, \quad (2.110)$$

with $h\Phi(\eta)=\Phi(\eta)$. The introduced superspace $\{\lambda^\alpha,\tilde{\lambda}^{\dot{\alpha}},\eta^A\}$ is chiral in the following sense: the complex conjugate of η^A is not part of the superspace: $\overline{(\eta^A)}=\bar{\eta}_A$.

It is natural to consider color ordered superamplitudes in $\mathscr{N}=4$ SYM whose external legs are parametrized by a point in super-momentum space $\Lambda_i:=\{\lambda_i,\tilde{\lambda}_i,\eta_i\}$ associated to an on-shell superfield $\Phi(\Lambda_i)$, i.e.



$$\mathbb{A}_n\big(\{\lambda_i,\tilde{\lambda}_i,\eta_i\}\big)=\big\langle\Phi(\lambda_1,\tilde{\lambda}_1,\eta_1)\cdots\Phi(\lambda_n,\tilde{\lambda}_n,\eta_n)\big\rangle. \quad (2.111)$$

This prescription packages all possible component field amplitudes involving gluons, gluinos and scalars as external states into a single object. The component level amplitudes may then be extracted from a known $\mathbb{A}_n(\{\lambda_i,\tilde{\lambda}_i,\eta_i\})$ upon expanding it in the Grassmann odd η_i^A variables.

For example, the expansion of the η_i^A-polynomial of $\mathbb{A}_n(\Lambda_i)$ will contain terms such as

$$\begin{aligned}
&(\eta_1)^4(\eta_2)^4\mathscr{A}_n(-,-,+,\dots,+) \quad \text{with } \eta_i^4 := \frac{1}{4!}\varepsilon_{ABCD}\eta_i^A\eta_i^B\eta_i^C\eta_i^D, \\
&\big(\eta_1^4\big)\varepsilon_{ACDE}\eta_2^C\eta_2^D\eta_2^E\eta_3^B\mathscr{A}_n\big(-,\tilde{g}^A,\bar{\tilde{g}}_B,+,\dots,+\big).
\end{aligned} \tag{2.112}$$

Here $\mathscr{A}_n$ denotes the resulting component field amplitudes, in this example two MHV amplitudes, namely a pure gluon and a gluon-$\tilde{g}\bar{\tilde{g}}$ amplitude. We also stress the property

$$\begin{aligned}
&h_i\,\mathbb{A}_n(1,\dots,n) = \mathbb{A}_n(1,\dots,n), \\
&h_i = \frac{1}{2}\big[-\lambda_i^\alpha\partial_{i\alpha} + \tilde{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} + \eta_i^A\partial_{iA}\big], \quad \forall i = 1,\dots,n,
\end{aligned} \tag{2.113}$$

which holds 'locally' for each individual leg. This is a consequence of the uniform helicity 1 of any on-shell superfield $\Phi(\eta)$.

2.7.2 *Superconformal Symmetry*

The $\mathcal{N}=4$ supersymmetry transformations are generated by the operators $q^{\alpha A}$ and $\bar{q}_A^{\dot{\alpha}}$. By definition their anti-commutator yields the translation operator

$$\big\{q^{\alpha A},\bar{q}_B^{\dot{\alpha}}\big\} = \delta_B^A p^{\alpha\dot{\alpha}}, \tag{2.114}$$

which is the key commutation relation of supersymmetry: The supersymmetry transformation may be thought of as the 'square-root' of the translation. As $p^{\alpha\dot{\alpha}} = \lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$ the natural representation of the supersymmetry generators in our on-shell superspace then is

$$q^{\alpha A} = \lambda^\alpha\eta^A, \qquad \bar{q}_A^{\dot{\alpha}} = \tilde{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\eta^A}, \tag{2.115}$$

manifestly obeying Eq. (2.114). This representation enables us to read off the supersymmetry q-transformations of the on-shell components of the superfield $\Phi(p,\eta)$

$$\delta_q\Phi(p,\eta) = \xi_{\alpha A}\big(q^{\alpha A}\Phi(p,\eta)\big), \tag{2.116}$$

with the Grassmann odd transformation parameter $\xi_{\alpha A}$. The respective left- and right-hand sides of this equation then take the form

$$\begin{aligned}
&\delta_q \Phi(p,\eta) \\
&\quad = \delta_q g_+ + \eta^A \delta_q \tilde{g}_A(p) + \frac{1}{2!}\eta^A\eta^B \delta_q S_{AB}(p) \\
&\qquad + \frac{1}{3!}\eta^A\eta^B\eta^C \varepsilon_{ABCD}\delta_q \bar{\tilde{g}}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \varepsilon_{ABCD}\delta_q g_-(p), \\
&\xi_{\alpha A}\big(q^{\alpha A}\Phi(p,\eta)\big) \\
&\quad = \xi_{\alpha A}\lambda^\alpha\bigg(\eta^A g_+ + \eta^A\eta^B \tilde{g}_B + \frac{1}{2!}\eta^A\eta^B\eta^C S_{BC} \\
&\qquad + \frac{1}{3!}\eta^A\eta^B\eta^C\eta^D \varepsilon_{BCDE}\bar{\tilde{g}}^E\bigg).
\end{aligned} \tag{2.117}$$

This result implies the q-variations of the component on-shell fields by comparing the left-hand and right-hand side of Eq. (2.116)

$$\begin{aligned}
&\delta_q g_+ = 0, \qquad \delta_q \tilde{g}_A = -\langle \xi_A \lambda\rangle g_+, \qquad \delta_q S_{AB} = -\langle \xi_A\lambda\rangle \tilde{g}_B + \langle \xi_B \lambda\rangle \tilde{g}_A, \\
&\delta_q \bar{\tilde{g}}_A = \varepsilon^{ABCD}\langle \xi_B\lambda\rangle S_{CD}, \qquad \delta_q g_- = -\langle \xi_A \lambda\rangle \bar{\tilde{g}}^A.
\end{aligned} \tag{2.118}$$

With the same method one establishes the $\bar{q}$-variations

$$\begin{aligned}
\delta_{\bar{q}} g_+ &= \big[\tilde{\lambda}\tilde{\xi}^A\big]\tilde{g}_A, \qquad \delta_{\bar{q}} \tilde{g}_A = \big[\tilde{\lambda}\tilde{\xi}^B\big] S_{BA}, \\
\delta_{\bar{q}} S_{AB} &= \varepsilon_{ABCD}\big[\tilde{\lambda}\tilde{\xi}^C\big]\bar{\tilde{g}}^D, \\
\delta_{\bar{q}} \bar{\tilde{g}}^A &= -\big[\tilde{\lambda}\tilde{\xi}^A\big] g_-, \qquad \delta_{\bar{q}} g_- = 0,
\end{aligned} \tag{2.119}$$

with the Grassmann odd parameter $\tilde{\xi}^A_{\dot{\alpha}}$.

Having established the supersymmetry generators, let us now discuss the remaining generators of the superconformal symmetry algebra. In addition to the known Lorentz symmetry generators $m_{\alpha\beta}$ and $\overline{m}_{\dot{\alpha}\dot{\beta}}$, discussed in Sect. 2.6, one now has an additional global $SU(4)$ R-symmetry generated by $r^A{}_B$ which acts as an internal rotation in the η^A space

$$\begin{aligned}
r^A{}_B &= \eta^A \partial_B - \frac{1}{4}\delta^A_B \eta^C \partial_C, \qquad \partial_A := \frac{\partial}{\partial \eta^A}, \\
m_{\alpha\beta} &= \lambda_{(\alpha}\partial_{\beta)}, \qquad \overline{m}_{\dot{\alpha}\dot{\beta}} = \tilde{\lambda}_{(\dot{\alpha}}\tilde{\partial}_{\dot{\beta})}.
\end{aligned} \tag{2.120}$$

In the conformal sector the generator of special conformal transformations of Sect. 2.6

$$k_{\alpha\dot{\alpha}} = \partial_\alpha \tilde{\partial}_{\dot{\alpha}} \tag{2.121}$$

is augmented by two superconformal symmetry generators $s_{\alpha A}$ and $\bar{s}^{A}_{\dot\alpha}$ which arise from the commutators

$$\begin{aligned} \left[k_{\alpha\dot\alpha}, q^{\beta A}\right] &= \delta^{\beta}_{\alpha}\bar{s}^{A}_{\dot\alpha}, \qquad & \bar{s}^{A}_{\dot\alpha} &= \eta^{A}\tilde{\partial}_{\dot\alpha}, \\ \left[k_{\alpha\dot\alpha}, \bar{q}^{\dot\beta}_{A}\right] &= \delta^{\dot\beta}_{\dot\alpha} s_{\alpha A}, \qquad & s_{\alpha A} &= \partial_{\alpha}\partial_{A}. \end{aligned} \tag{2.122}$$

The complete super-conformal symmetry algebra reads

$$\begin{aligned} &\left\{q^{\alpha A}, \bar{q}^{\dot\alpha}_{B}\right\} = \delta^{A}_{B} p^{\alpha\dot\alpha}, \qquad \left\{s_{\alpha A}, \bar{s}^{B}_{\dot\alpha}\right\} = \delta^{A}_{B} k_{\alpha\dot\alpha}, \\ &\left\{q^{\alpha A}, s_{\beta B}\right\} = m^{\alpha}{}_{\beta}\delta^{A}_{B} + \delta^{\alpha}_{\beta} r^{A}{}_{B} + \frac{1}{2}\delta^{\alpha}_{\beta}\delta^{A}_{B}(d+c), \\ &\left\{\bar{q}^{\dot\alpha}_{A}, \bar{s}^{B}_{\dot\beta}\right\} = \overline{m}^{\dot\alpha}{}_{\dot\beta}\delta^{A}_{B} - \delta^{\dot\alpha}_{\dot\beta} r^{B}{}_{A} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\delta^{B}_{A}(d-c), \\ &\left[p^{\alpha\dot\alpha}, s_{\beta A}\right] = \delta^{\alpha}_{\beta}\bar{q}^{\dot\alpha}_{A}, \qquad \left[p^{\alpha\dot\alpha}, \bar{s}^{A}_{\dot\beta}\right] = \delta^{\dot\alpha}_{\dot\beta} q^{\alpha A}, \end{aligned} \tag{2.123}$$

with the dilatation generator and central charge

$$d = \frac{1}{2}\left[\lambda^{\alpha}\partial_{\alpha} + \tilde\lambda^{\dot\alpha}\partial_{\dot\alpha} + 1\right], \qquad c = 1 + \frac{1}{2}\left(\lambda^{\alpha}\partial_{\dot\alpha} - \tilde\lambda^{\dot\alpha}\partial_{\dot\alpha} - \eta^{A}\partial_{A}\right) = 1 - h. \tag{2.124}$$

While c commutes with all generators of the above algebra, d measures their weight in momentum units. Also note that the central charge vanishes on super-amplitudes $c_i\mathbb{A}_n(\Lambda_i) = 0$ locally for every leg as a consequence of Eq. (2.113). Together with the conformal algebra

$$\begin{aligned} &\left[k_{\alpha\dot\alpha}, p^{\beta\dot\beta}\right] = \delta^{\beta}_{\alpha}\delta^{\dot\beta}_{\dot\alpha} d + m_{\alpha}{}^{\beta}\partial^{\dot\beta}_{\dot\alpha} + \overline{m}_{\dot\alpha}{}^{\dot\beta}\delta^{\beta}_{\alpha}, \\ &\left[d, p^{\alpha\dot\alpha}\right] = p^{\alpha\dot\alpha}, \qquad [d, k_{\alpha\dot\alpha}] = -k_{\alpha\dot\alpha}, \qquad [c, *] = 0, \end{aligned} \tag{2.125}$$

and the Ponicaré algebra this constitutes the super-conformal $\mathfrak{psu}(2,2|4)$ symmetry algebra. Super-amplitudes are invariant under all these generators.

2.7.3 *Super-amplitudes, and Extraction of Components*

The super-amplitudes are invariant under the superconformal symmetry algebra $\mathfrak{psu}(2,2|4)$ discussed above. As before the symmetry generators acting on n-leg super-amplitudes are given by the sum of the single-particle representations

$$p^{\alpha\dot\alpha} = \sum_{i=1}^{n}\lambda^{\alpha}_{i}\tilde\lambda^{\dot\alpha}_{i}, \qquad q^{\alpha A} = \sum_{i=1}^{n}\lambda^{\alpha}_{i}\eta^{A}_{i}, \qquad \bar{q}^{\dot\alpha}_{A} = \sum_{i=1}^{n}\tilde\lambda^{\dot\alpha}_{i}\partial_{iA}, \qquad \text{etc.}, \tag{2.126}$$

and we have the conservation of total charges in the sense of

$$\left\{p^{\alpha\dot\alpha}, d, k_{\alpha\dot\alpha}, m_{\alpha\beta}, \overline{m}_{\alpha\beta}, r^{A}{}_{B}; q^{\alpha A}, \bar{q}^{\dot\alpha}_{A}, s_{\alpha A}, \bar{s}^{B}_{\dot\alpha}; h_i\right\} \circ \mathbb{A}_n = 0. \tag{2.127}$$

Note that only $p^{\alpha\dot{\alpha}}$ and $q^{\alpha A}$ act multiplicatively, whereas the set of generators $\{d, m_{\alpha\beta}, \overline{m}_{\dot{\alpha}\dot{\beta}}, r^A{}_B; \bar{q}^{\dot{\alpha}}_A, \bar{s}^B_{\dot{\alpha}}; h_i\}$ are first order differential operators, while $k_{\alpha\dot{\alpha}}$ and $s_{\alpha A}$ are second order differential operator in the variables $\{\lambda^\alpha_i, \tilde{\lambda}^{\dot{\alpha}}_i, \eta^A_i\}$.

The invariance under the multiplicatively represented operators $p^{\alpha\dot{\alpha}}$ and $q^{\alpha A}$ then requires the general form of the super-amplitudes to be (for $n > 3$)

$$\mathbb{A}_n(\lambda_i, \tilde{\lambda}_i, \eta_i) = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} P_n(\lambda_i, \tilde{\lambda}_i, \eta_i), \tag{2.128}$$

enforcing the p and q conservation through delta functions. In the above the analytic function P_n is arbitrary and we took out the MHV-like numerator factor $\langle 12\rangle\cdots\langle n1\rangle$ from the definition of P_n. This is a pure convention.

Due to the $\mathfrak{su}(4)$ R-symmetry $P_n(\lambda_i, \tilde{\lambda}_i, \eta_i)$ has an η-expansion of the form

$$\begin{array}{cccccccc} P_n(\lambda_i, \tilde{\lambda}_i, \eta_i) = & P_n^{(0)} & + & P_n^{(4)} & + & P_n^{(8)} & + \cdots + & P_n^{(4n-16)}, \\ & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ & \text{MHV} & & \text{NMHV} & & \text{N}^2\text{MHV} & & \overline{\text{MHV}} \end{array} \tag{2.129}$$

where $P_n^{(l)} \sim \mathscr{O}(\eta^l)$, which corresponds directly to the indicated helicity classification. We shall see shortly that $P_n^{(0)} = 1$.

As we encounter them for the first time, let us briefly discuss the formalism of Grassmann odd delta-functions. For a single real Grassmann odd variable θ we have the integration rules

$$\int d\theta \cdot \theta = 1, \qquad \int d\theta \cdot 1 = 0, \quad \text{thus} \int d\theta \,\hat{=}\, \frac{\partial}{\partial\theta}. \tag{2.130}$$

Indeed $\delta(\theta) = \theta$ as one sees upon integrating against a test-function $F(\theta) = F_0 + \theta F_1$

$$\int d\theta\delta(\theta - \theta_0)F(\theta) = \int d\theta(\theta - \theta_0)(F_0 + \theta F_1) = \int d\theta(-\theta_0 F_0 + \theta(F_0 + \theta_0 F_1)$$
$$= F_0 + \theta_0 F_1 = F(\theta_0). \tag{2.131}$$

We also note the integral representation

$$\delta(\theta - \theta_0) = \int d\bar{\theta} e^{\bar{\theta}(\theta-\theta_0)}. \tag{2.132}$$

The eight-dimensional delta function in Eq. (2.128) refers to the explicit expression

$$\delta^{(8)}(q) = \delta^{(8)}\left(\sum_{i=1}^n \lambda^\alpha_i \eta^A_i\right) := \prod_{\alpha=1}^2 \prod_{A=1}^4 \left(\sum_{i=1}^n \lambda^\alpha_i \eta^A_i\right) \sim \mathscr{O}(\eta^8). \tag{2.133}$$

Hence, $\mathbb{A}_n$ has an η-expansion starting at order η^8, which in turn implies that large classes of component-field amplitudes vanish. In particular we see that

$$\begin{aligned}
\mathbb{A}_n|_{\eta^0}=0 \quad &\Rightarrow \quad \mathscr{A}_n^{\text{gluon}}\big(1^+,2^+,\dots,n^+\big)=0 \\
&\qquad\quad \text{from 1 term}, &(2.134)\\
\mathbb{A}_n|_{\eta^4}=0 \quad &\Rightarrow \quad \mathscr{A}_n^{\text{gluon}}\big(1^-,2^+,\dots,n^+\big)=0 \\
&\qquad\quad \text{from } \varepsilon_{ABCD}\eta_1^A\eta_1^B\eta_1^C\eta_1^D \text{ term}, &(2.135)\\
&\Rightarrow \quad \mathscr{A}_n^{\phi\phi g^{n-2}}\big(1_\phi,2_\phi,3^+,\dots,n^+\big)=0 \\
&\qquad\quad \text{from } \varepsilon_{ABCD}\eta_1^A\eta_1^B\eta_2^C\eta_2^D \text{ term}, &(2.136)\\
&\Rightarrow \quad \mathscr{A}_n^{\bar{\tilde{g}}\tilde{g}g^{n-2}}\big(1^+_{\tilde{g}_A},2^-_{\bar{\tilde{g}}^A},3^+,\dots,n^+\big)=0 \\
&\qquad\quad \text{from } \varepsilon_{ABCD}\eta_1^A\eta_2^B\eta_2^C\eta_2^D \text{ term}. &(2.137)
\end{aligned}$$

Moreover note that due to the $\mathfrak{su}(4)$ R-symmetry of the super-amplitudes the η-expansion of $\mathbb{A}_n$ is an expansion in powers of η^4. These results carry over to the following vanishing tree-amplitudes in massless QCD

$$\mathscr{A}_{g^n}^{\text{QCD}}\big(1^\pm,2^+,3^+,\dots,n^+\big)=0, \tag{2.138}$$

$$\mathscr{A}_{\bar{q}qg^{n-2}}^{\text{QCD}}\big(1_q^+,2^+,\dots,(i-1)^+,i_{\bar{q}}^-,(i+1)^+,\dots,n^+\big)=0, \tag{2.139}$$

reproducing the results of Sect. 1.11 based on an explicit evaluation of the color-ordered Feynman graphs. As promised we now understand this vanishing as following from a hidden supersymmetry for QCD tree-amplitudes with external gluon legs and up to one quark-anti-quark line. We also see the vanishing of the scalar-gluon amplitudes when all gluons have positive helicity, as shown for $n=4$ in Sect. 2.5.

The alert reader may object that there could be internal scalar contributions to the $\mathcal{N}=4$ tree-amplitudes which would invalidate the equivalence to massless QCD-trees. However, scalars may only be exchanged between gluino lines at tree-level due to the absence of a ϕAA interaction in the theory. Hence, as long as there is only one gluino line running through the tree-diagram with otherwise external gluon legs, there can be no intermediate scalar exchange. In fact, this property generalizes to an arbitrary number of gluino lines, as long as they are all of the same flavor: the $\mathcal{N}=4$ Yukawa vertices of Eq. (2.108) are of the type $\psi_A\psi_B\phi^{AB}$ or $\bar{\psi}^A\bar{\psi}^{\dot{\alpha}B}\phi_{AB}$ respectively. These clearly vanish for $A=B$. In this sense massless QCD at tree-level is effectively $\mathcal{N}=1$ supersymmetric.

The component amplitudes can be extracted in an elegant way by carrying out integrations over the Grassmann parameters. The latter are always localized thanks to corresponding Grassmann delta functions in the amplitudes, such that in general extracting component amplitudes from superamplitudes amounts to linear algebra. We will see examples of this in exercises at the end of this chapter.

2.7.4 Super BCFW-Recursion

We now want to construct a super-amplitude formulation of the BCFW-recursion relations generalizing the results of Sect. 2.1. Such a recursion should exist as the super-amplitudes merely represent a packaging of component field amplitudes with the help of the Grassmann odd η_i's and the component field amplitudes do enjoy a BCFW-recursion relation. For the derivation of the BCFW-recursion in Sect. 2.1 we considered the complex shifts of the helicity spinors at neighboring legs 1 and n

$$\begin{aligned} \lambda_1 &\to \lambda_1 - z\lambda_n = \hat{\lambda}_1, \\ \tilde{\lambda}_n &\to \tilde{\lambda}_n + z\tilde{\lambda}_1 = \hat{\tilde{\lambda}}_n. \end{aligned} \tag{2.140}$$

Recall that this shift preserves the total momentum

$$\begin{aligned} &p_1 \to \hat{p}_1 = \lambda_1\tilde{\lambda}_1 - z\lambda_n\tilde{\lambda}_1, \quad p_n \to \hat{p}_n = \lambda_n\tilde{\lambda}_n + z\lambda_n\tilde{\lambda}_1, \\ &\Rightarrow \quad \hat{p}_1 + \hat{p}_n = p_1 + p_n. \end{aligned} \tag{2.141}$$

For the super-amplitude it is natural to consider an additional z-dependent shift in the η_i^A variables at leg positions 1 and n. The question is which one to take. Again the guideline is the conservation of 'super-momentum' $q^{\alpha A}$ of the superamplitude $q^{\alpha A} = \sum_{i=1}^{n} \lambda_i^\alpha \eta_i^A$ whose leg 1 and n components transform under the shifts Eq. (2.140) as

$$\begin{aligned} q_1^{\alpha A} &\to \hat{q}_1^{\alpha A} = (\lambda_1 - z\lambda_n)\hat{\eta}_1, \\ q_n^{\alpha A} &\to \hat{q}_n^{\alpha A} = \lambda_n\hat{\eta}_n. \end{aligned} \tag{2.142}$$

with so far unspecified shifted $\hat{\eta}_1$ and $\hat{\eta}_n$. The unique choice to preserve total $q^{\alpha A}$ is

$$\begin{aligned} &\hat{\eta}_1 = \eta_1, \quad \hat{\eta}_n = \eta_n + z\eta_1, \\ &\Rightarrow \quad \hat{q}_1^{\alpha A} + \hat{q}_n^{\alpha A} = q_1^{\alpha A} + q_n^{\alpha A}. \end{aligned} \tag{2.143}$$

The derivation of the super-BCFW recursion then follows the same steps as in Sect. 2.1: the unshifted super-amplitude $\mathbb{A}_n(z=0)$ results from the knowledge of the poles of $\mathbb{A}_n(z)$ in the complex plane where the amplitude factorizes into a left $\mathbb{A}^L$ and right $\mathbb{A}^R$ part. The super-recursion relation reads

$$\begin{aligned} \mathbb{A}_n(1,\ldots,n) = \sum_{i=3}^{n-1} \int d^4\eta_{\hat{P}_i}\, &\mathbb{A}_i^L\big(\hat{1}(z_{P_i}), 2, \ldots, i-1, -\hat{P}(z_{P_i})\big) \\ &\times \frac{1}{P_i^2}\mathbb{A}_{n-i+2}^R\big(\hat{P}(z_{P_i}), i, \ldots, n-1, \hat{n}(z_{P_i})\big) + \mathrm{Res}(z=\infty), \end{aligned} \tag{2.144}$$

where the $\mathbb{A}_n$ entering this formula are the super-amplitudes with stripped-off *bosonic* delta-functions $\delta^{(4)}(p)$. The fermionic delta-functions of Eq. (2.128) remain included. Moreover we have

$$\begin{aligned}
\hat{1}(z_{P_i}) &= \{\hat{\lambda}_1(z_{P_i}), \tilde{\lambda}_1, \eta_1\}, \qquad \hat{n}(z_{P_i}) = \{\lambda_1, \hat{\tilde{\lambda}}_n, \hat{\eta}_n\}, \\
\hat{P}_i(z_{P_i}) &= \{\lambda_{\hat{P}_i}, \tilde{\lambda}_{\hat{P}_i}, \eta_{\hat{P}_i}\}, \qquad -\hat{P}_i(z_{P_i}) = \{-\lambda_{\hat{P}_i}, \tilde{\lambda}_{\hat{P}_i}, \eta_{\hat{P}_i}\}, \\
z_{P_i} &= \frac{P_i^2}{\langle n|P_i|1]}, \qquad P_i = p_1 + p_2 + \cdots + p_{i-1}, \\
\lambda_{\hat{P}_i}\tilde{\lambda}_{\hat{P}_i} &= \hat{\lambda}_1(z)\tilde{\lambda}_1 + \sum_{j=2}^{i-1}\lambda_j\tilde{\lambda}_j = -\sum_{j=i}^{n-1}\lambda_j\tilde{\lambda}_j - \lambda_n\hat{\tilde{\lambda}}_n(z).
\end{aligned} \tag{2.145}$$

Note that in the super BCFW-recursion the sum over intermediate states present in the ordinary BCFW-recursion of Eq. (2.16) is now elegantly replaced by an integral over η_{P_i}. What remains to be shown is the vanishing of the residue at $z=\infty$.

The key point here is the observation quoted in Sect. 2.1 based on [4] that a shift with a $(1^+, n^+)$ gluon helicity configurations has a falloff as $1/z$ for $z\to\infty$. How can we relate this to the super-amplitude case? The fact that the super-amplitude is also invariant under the supersymmetry $\bar{q}$, i.e. $\bar{q}^{\dot{\alpha}}_A \mathbb{A}_n = 0$ with $\bar{q}^{\dot{\alpha}}_A = \tilde{\lambda}^{\dot{\alpha}}\partial_{\eta^A}$, may be used to set two arbitrary η_i's to zero. This is seen as follows: a *finite* $\bar{q}$-transformation acts as a translation in η_i-space

$$\eta_i^A \to \eta_i^A + [\xi^A \tilde{\lambda}_i], \tag{2.146}$$

where $\xi^A_{\dot{\alpha}}$ is the associated Grassmann odd transformation parameter. The special choice

$$\xi^A_{\dot{\alpha}} = \frac{\tilde{\lambda}_{1\dot{\alpha}}\eta_n^A - \tilde{\lambda}_{n\dot{\alpha}}\eta_1^A}{[n1]} \tag{2.147}$$

sets η_1 and η_n to zero, as $[\xi^A\tilde{\lambda}_{1|n}] = -\eta^A_{1|n}$. In fact the shifts of Eqs. (2.140), (2.143) leave the parameter $\xi^A_{\dot{\alpha}}$ invariant

$$\xi^A_{\dot{\alpha}}(z) = \frac{\tilde{\lambda}_{1\dot{\alpha}}\hat{\eta}_n^A - \hat{\tilde{\lambda}}_{n\dot{\alpha}}\eta_1^A}{[\hat{n}1]} = \frac{\tilde{\lambda}_{1\dot{\alpha}}\eta_n^A - \tilde{\lambda}_{n\dot{\alpha}}\eta_1^A}{[n1]} = \xi^A_{\dot{\alpha}}. \tag{2.148}$$

Thus by using this finite $\bar{q}$-supersymmetry transformation we can relate the z-dependence of the full superamplitude to that of its component with a $(1^+, n^+)$ positive gluon helicity configuration. The later is known to vanish as $1/z$ for z going to infinity. This then proves the $1/z$ falloff for the full super-amplitude and the vanishing of the residuum at infinity of Eq. (2.144) for any two neighboring legs in $\mathbb{A}_n(1,\ldots,n)$.

2.7.5 Three-Point Super-amplitudes

As before the seed for solving the recursion are the 3-point super-amplitudes. As in the pure Yang-Mills case we have a MHV_3 and $\overline{\mathrm{MHV}}_3$ super-amplitude characterized by either $[ij] = 0$ or $\langle ij\rangle = 0$ for $i, j = 1, 2, 3$, respectively.

Up to a proportionality constant the 3-point MHV super-amplitude is uniquely determined. It must have the form $\mathbb{A}_3^{\mathrm{MHV}} \sim \delta^{(4)}(p)\delta^{(8)}(q)$. The 3-point kinematics $[ij] = 0$ but $\langle ij\rangle \neq 0\ \forall i, j \in \{1, 2, 3\}$ along with the local helicity requirement $h_i \circ \mathbb{A}_3 = \mathbb{A}_3$, $\forall i$, uniquely leads via Eq. (2.124) to the result

$$\mathbb{A}_3^{\mathrm{MHV}} = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \tag{2.149}$$

Its cousin, the $\overline{\mathrm{MHV}}_3$ super-amplitude follows from parity and a Fourier transformation in $\bar{\eta}$-space. This is seen as follows: conjugating the MHV amplitude Eq. (2.149) we have

$$\overline{\mathbb{A}_3^{\mathrm{MHV}}} = -\frac{\delta^{(4)}(p)}{[12][23][31]}\delta^{(8)}\left(\sum_{i=1}^{3}\tilde{\lambda}_i\bar{\eta}_i\right). \tag{2.150}$$

As we are dealing with a chiral on-shell superspace initially $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$ we need to Fourier transform this result back to η-space, i.e. we have the relation

$$\mathbb{A}_3^{\overline{\mathrm{MHV}}} = \int\left(\prod_{i=1}^{3} d^4\bar{\eta}_i\right)e^{i\sum_{i=1}^{3}\eta_i^A\bar{\eta}_{iA}}\,\overline{\mathbb{A}_3^{\mathrm{MHV}}}. \tag{2.151}$$

It is instructive to perform this integral.

For this we need the following identity (to be shown in Exercise 2.9)

$$\delta^{(2)}\big(\lambda^\alpha a + \mu^\alpha b\big) = \begin{cases} \frac{\delta(a)\delta(b)}{|\langle\lambda\mu\rangle|} & \text{for } a, b \text{ Grassmann even,} \\ \delta(a)\delta(b)\langle\lambda\mu\rangle & \text{for } a, b \text{ Grassmann odd} \end{cases} \tag{2.152}$$

which is a consequence of the linear independence of the two-vectors λ^α and μ^α. Using this one shows that

$$\delta^{(8)}\left(\sum_{i=1}^{3}\tilde{\lambda}_i^{\dot{\alpha}}\bar{\eta}_{iA}\right) = [12]^4\delta^{(4)}\left(\bar{\eta}_{1,A} - \frac{[23]}{[12]}\bar{\eta}_{3A}\right)\delta^{(4)}\left(\bar{\eta}_{2,A} - \frac{[31]}{[12]}\bar{\eta}_{3A}\right). \tag{2.153}$$

Inserting this expression in Eq. (2.151) and performing the $\bar{\eta}_1$ and $\bar{\eta}_2$ integrals via the delta functions yields

$$\int \left(\prod_{i=1}^{3} d^4\bar{\eta}_i\right) \delta^{(8)}\left(\sum_{i=1}^{3} \tilde{\lambda}_i^{\dot{\alpha}} \bar{\eta}_{iA}\right) = [12]^4 \int d^4\bar{\eta}_3 e^{i\bar{\eta}_{3A}(\eta_3^A + \frac{[23]}{[12]}\eta_1^A + \frac{[31]}{[12]}\eta_2^A)}$$
$$= [12]^4 \prod_{A=1}^{4} \delta\left(\eta_3^A + \frac{[23]}{[12]}\eta_1^A + \frac{[31]}{[12]}\eta_2^A\right)$$
$$= \delta^{(4)}\big([12]\eta_3^A + [23]\eta_1^A + [31]\eta_2^A\big), \qquad (2.154)$$

where we have also used the integral representation for the Grassmann odd delta function of Eq. (2.132). In summary, we find the final expression for the $\overline{\text{MHV}}_3$ amplitude

$$\mathbb{A}_3^{\overline{\text{MHV}}} = -\frac{\delta^{(4)}(p)\delta^{(4)}([12]\eta_3 + [23]\eta_1 + [31]\eta_2)}{[12][23][31]}. \qquad (2.155)$$

At first sight it is surprising that for this amplitude the invariance under q supersymmetry does not require the $\delta^{(8)}(q)$ factor as claimed in Eq. (2.128) for $n > 3$. The reason lies in the special kinematics of the $\overline{\text{MHV}}_3$ amplitude where $\langle ij\rangle = 0$ requires all λ_i to be parallel. Therefore q factorizes into $q^{\alpha A} = \lambda_F^{\alpha}\eta_F^A$ for suitable λ_F and η_F. Hence q-invariance only demands a factor of $\delta^{(4)}(\eta_F)$ which is what we indeed found in Eq. (2.155).

Exercise 2.9 (Super Delta Function Manipulations)
Prove the following relations for the helicity spinors λ and μ

$$\delta^{(2)}\big(\lambda^\alpha a + \mu^\alpha b\big) = \begin{cases} \frac{\delta(a)\delta(b)}{|\langle\lambda\mu\rangle|} & \text{for } a, b \text{ Grassmann even,} \\ \delta(a)\delta(b)\langle\lambda\mu\rangle & \text{for } a, b \text{ Grassmann odd.} \end{cases}$$

Use this to show that

$$\delta^{(8)}\left(\sum_{i=1}^{3} \lambda_i^{\alpha}\eta_i^A\right) = \langle 12\rangle^4 \delta^{(4)}\left(\eta_1^A - \frac{\langle 23\rangle}{\langle 12\rangle}\eta_3^A\right)\delta^{(4)}\left(\eta_2^A - \frac{\langle 31\rangle}{\langle 12\rangle}\eta_3^A\right),$$

which we used in its complex conjugated version in the derivation of the $\mathbb{A}_3^{\overline{\text{MHV}}}$ amplitude above.

2.7.6 *Solving the Super-BCFW Recursion: MHV Case*

If one decomposes the super-BCFW recursion into contributions of different Grassmann degrees, i.e. decomposes the super-amplitudes into their various N^pMHV

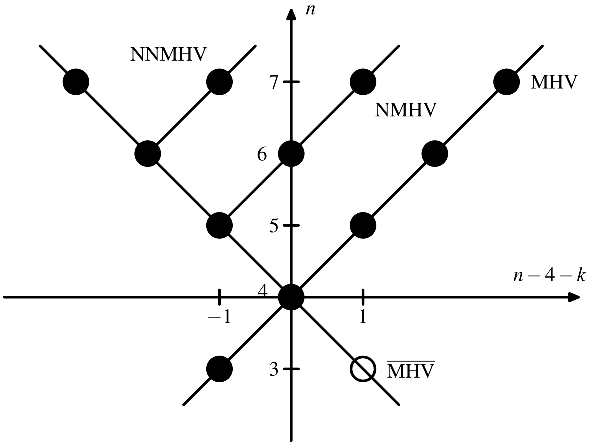


Fig. 2.9 Classification of amplitudes according to helicity violating level k and number of external legs n. *Every black dot* denotes an N^kMHV_n (super-)amplitude. The three-point amplitudes are special. The recursion relations can be solved iteratively in n and k, as explained in the main text

parts as in Eq. (2.129), one obtains a recursion of the form

$$\begin{aligned}\mathbb{A}_n^{\text{N}^p\text{MHV}} &= \int \frac{d^4\eta_P}{P^2}\mathbb{A}_3^{\overline{\text{MHV}}}(z_P)\mathbb{A}_{n-1}^{\text{N}^p\text{MHV}}(z_P)\\ &\quad + \sum_{m=0}^{p-1}\sum_{i=4}^{n-1}\int \frac{d^4\eta_{P_i}}{P_i^2}\mathbb{A}_i^{\text{N}^m\text{MHV}}(z_{P_i})\mathbb{A}_{n-i+2}^{\text{N}^{(p-m-1)}\text{MHV}}(z_{P_i}).\end{aligned} \tag{2.156}$$

The reason for this decomposition lies in the fact that the total η-count on the left-hand-side, η^{4p+8}, has to equal the η-count on the right-hand-side. Due to the Grassmann integral $\int d^4\eta_P$ the combined η-count of the integrand $\mathbb{A}^L \cdot \mathbb{A}^R$ has to be by four larger than the left-hand-side. Note that we have not included a term where the left subamplitude is $\mathbb{A}_3^{\text{MHV}}$. This cannot happen since $\mathbb{A}_3^{\text{MHV}}(1, 2, *)$ implies the vanishing of the square bracket $[12]$. In the left subamplitude $\tilde{\lambda}_1$ is unshifted and hence $[12] = 0$ implies $(p_1 + p_2)^2 = 0$, which is a restriction on the kinematics not generally true. In consequence such a term does not contribute to the recursion relation as there is no solution to the on-shell conditions for this channel for generic external momenta. Similarly, the right sub-amplitude may never be $\mathbb{A}_3^{\overline{\text{MHV}}}$.

This form of the super-BCFW recursion (2.156) leads to the following iterative structure: the MHV_n super-amplitudes follow from the lower-point $\text{MHV}_{k<n}$ and the $\overline{\text{MHV}}_3$ super-amplitudes. The NMHV_n super-amplitudes arise from the NMHV_{n-1}, $\text{MHV}_{k<n}$ and $\overline{\text{MHV}}_3$. An iterative solution of the NNMHV super-amplitudes requires the knowledge of the NMHV, MHV and $\overline{\text{MHV}}_3$ superamplitudes etc. See Fig. 2.9 for illustration.

To start this iteration we look at the MHV sector. We already know the form of the MHV super-amplitude by lifting the result of the MHV amplitude at component level to superspace

$$\mathbb{A}_n^{\text{MHV}}(\lambda_i, \tilde{\lambda}_i, \eta_i) = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{2.157}$$

Let us nevertheless rederive this formula from the super-BCFW recursion for the 4-point case. For $n = 4$ and $p = 0$ Eq. (2.156) only has one term and simply reads

$$\mathbb{A}_4^{\mathrm{MHV}} = \int \frac{d^4\eta_P}{P^2} \mathbb{A}_3^{\overline{\mathrm{MHV}}}(z_P) \mathbb{A}_3^{\mathrm{MHV}}(z_P)$$
$$= -\int \frac{d^4\eta_P}{P^2} \frac{\delta^{(4)}(\eta_1[2,\hat{P}] + \eta_2[\hat{P}1] + \eta_P[12])\delta^{(8)}(\lambda_{\hat{P}}\eta_P + \lambda_3\eta_3 + \lambda_4\hat{\eta}_4)}{[12][2\hat{P}][\hat{P}1]\langle\hat{P}3\rangle\langle 34\rangle\langle 4\hat{P}\rangle}. \tag{2.158}$$

We use the $\delta^{(4)}$-function to localize η_P

$$\eta_P = -\frac{1}{[12]}\big(\eta_1[2\hat{P}] + \eta_2[\hat{P}1]\big). \tag{2.159}$$

Inserting this into the $\delta^{(8)}$-function and using momentum conservation yields

$$\delta^{(8)}\left(-\frac{\lambda_{\hat{P}}}{[12]}\big(\eta_1[2\hat{P}] + \eta_2[\hat{P}1]\big) + \lambda_3\eta_3 + \lambda_4\hat{\eta}_4\right)$$
$$\overset{\lambda_{\hat{P}}\tilde{\lambda}_{\hat{P}} = \hat{\lambda}_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2}{=} \delta^{(8)}\left(-\frac{\hat{\lambda}_1\eta_1[21] + \lambda_2\eta_2[21]}{[12]} + \lambda_3\eta_3 + \lambda_4\hat{\eta}_4\right) = \delta^{(8)}(q), \tag{2.160}$$

recovering the expected q-preserving delta-function. Note that in the last step there is a cancelation of the z_P dependent terms in $\hat{\lambda}_1\eta_1 + \lambda_4\hat{\eta}_4 = \lambda_1\eta_1 + \lambda_4\eta_4$. In the following, we will often perform similar calculations using the Grassmann delta functions. In this way, the intermediate state sums, which in this formalism are given by Grassmann integrals, can be trivially performed, and this considerably streamlines the computations. All that remains are the bosonic factors which we group as follows

$$-\frac{1}{P^2}[12]^4 \frac{1}{[12]\langle 34\rangle} \frac{1}{[2\hat{P}]\langle 4\hat{P}\rangle} \frac{1}{[\hat{P}1]\langle\hat{P}3\rangle}. \tag{2.161}$$

Here the second term arises from pulling the factor [12] out of the $\delta^{(4)}$-fermionic delta function. Using momentum conservation $|\hat{P}]\langle\hat{P}| = |1]\langle\hat{1}| + |2]\langle 2|$ we have

$$[2\hat{P}]\langle 4\hat{P}\rangle = [21]\langle 4\hat{1}\rangle = [21]\langle 41\rangle, \qquad [\hat{P}1]\langle\hat{P}3\rangle = [21]\langle 23\rangle, \qquad P^2 = \langle 12\rangle[21], \tag{2.162}$$

which enables one to bring the expression in (2.161) into the form

$$-\frac{1}{P^2}[12]^4 \frac{1}{[12]\langle 34\rangle} \frac{1}{[2\hat{P}]\langle 4\hat{P}\rangle} \frac{1}{[\hat{P}1]\langle\hat{P}3\rangle} = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{2.163}$$

This proves Eq. (2.157) for $n = 4$. The proof for general n-point MHV superamplitudes works analogously.

2.7.7 Solving the Super-BCFW Recursion: NMHV Case

Finally let us solve the recursion for the NMHV case closely following the approach of [9]. The general recursion formula Eq. (2.156) for $p = 1$ reads

$$\begin{aligned}\mathbb{A}_n^{\text{NMHV}} &= \int \frac{d^4P}{P^2} \int d^4\eta_{\hat{P}} \mathbb{A}_3^{\overline{\text{MHV}}}(z_P)\mathbb{A}_{n-1}^{\text{NMHV}}(z_P) \\ &\quad + \sum_{i=4}^{n-1} \int \frac{d^4P_i}{P_i^2} \int d^4\eta_{P_i} \mathbb{A}_i^{\text{MHV}}(z_{P_i})\mathbb{A}_{n-i+2}^{\text{MHV}}(z_{P_i}) \\ &\equiv A + B,\end{aligned} \tag{2.164}$$

resulting in a homogeneous term A and an inhomogeneous term B.

The inhomogeneous term may be straightforwardly computed from the known MHV amplitudes. Writing the Grassmann delta function coming from the left $\mathbb{A}_i^{\text{MHV}}(z_P)$ in the following way,

$$\begin{aligned}&\delta^{(8)}\left(\hat{\lambda}_1\eta_1 + \sum_{j=2}^{i-1}\lambda_j\eta_j - \lambda_{\hat{P}_i}\eta_{P_i}\right) \\ &= \langle\hat{1}\hat{P}_i\rangle^4\delta^{(4)}\left(\sum_{j=2}^{i-1}\frac{\langle\hat{1}j\rangle}{\langle\hat{1}\hat{P}_i\rangle}\eta_j - \eta_{P_i}\right)\delta^{(4)}\left(\eta_1 + \sum_{j=2}^{i-1}\frac{\langle j\hat{P}_i\rangle}{\langle\hat{1}\hat{P}_i\rangle}\eta_j\right),\end{aligned} \tag{2.165}$$

the integration over η_{P_i} may be performed straightforwardly. In this way, we obtain the following contribution to the n-point NMHV amplitude:

$$B = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\,j+1\rangle}\sum_{i=4}^{n-1} R_{n;2i}. \tag{2.166}$$

Here $R_{r;st}$ (called R-invariant because of its properties under dual superconformal symmetry, cf. Chap. 4) is given by

$$R_{r;st} = \frac{\langle s\,s-1\rangle\langle t\,t-1\rangle\delta^{(4)}(\Xi_{r;st})}{x_{st}^2\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rt}x_{ts}|s\rangle\langle r|x_{rt}x_{ts}|s-1\rangle}, \tag{2.167}$$

where

$$x_{ab} := p_a + p_{a+1} + \cdots + p_{b-1}, \qquad \theta_{ab} := q_a + q_{a+1} + \cdots + q_{b-1}, \tag{2.168}$$

are the dual variables or region momenta which will play a more prominent rôle in chapter four. The Grassmann odd quantity $\Xi_{r;st}$ in the above is given by

$$\Xi_{r;st} = \langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle. \tag{2.169}$$

In the following we will often deal with the quantity $\Xi_{n;st}$ for $1 < s < t < n$. It is instructive to switch from the dual θ_i to the η_i,

$$\Xi_{n;st} = \langle n| \left[x_{ns} x_{st} \sum_{i=t}^{n-1} |i\rangle \eta_i + x_{nt} x_{ts} \sum_{i=s}^{n-1} |i\rangle \eta_i \right], \tag{2.170}$$

to see that $\Xi_{n;st}$ is independent of η_n and η_1. Alternatively, using the $\delta^{(8)}(q)$ present in all physical amplitudes to rewrite the sums we can obtain

$$\delta^{(8)}(q)\Xi_{n;st} = -\delta^{(8)}(q)\langle n| \left[x_{ns} x_{st} \sum_{i=1}^{t-1} |i\rangle \eta_i + x_{nt} x_{ts} \sum_{i=1}^{s-1} |i\rangle \eta_i \right], \tag{2.171}$$

such that the only dependence on η_{n-1} and η_n on the l.h.s. of (2.171) is contained in $\delta^{(8)}(q)$.

Moreover, it is useful to realize that terms like $\langle r|x_{rs} x_{st}|t\rangle$ in (2.167) and similar terms in (2.169) can always be written as

$$\langle r|x_{rs} x_{st}|t\rangle = \langle r|x_{r+1s} x_{st}|t\rangle, \tag{2.172}$$

such that it is clear that they only depend explicitly on λ_r, but not on $\tilde{\lambda}_r$.

Finally note that the superamplitude must have cyclic symmetry. This implies

$$\delta^{(8)}(q)R_{5;24} = \delta^{(8)}(q)R_{1;35} = \delta^{(8)}(q)R_{2;41} = \delta^{(8)}(q)R_{3;52} = \delta^{(8)}(q)R_{4;13}. \tag{2.173}$$

This is just the first example of a general identity for n points

$$\delta^{(8)}(q) \sum_{s,t} R_{r;st} = \delta^{(8)}(q) \sum_{s,t} R_{r';st}, \tag{2.174}$$

where the sum goes over all values of s, t such that r, s, t (or r', s, t) are ordered cyclically with r and s (or r' and s) and s and t separated by at least two.

Let us first analyze the 5-point NMHV amplitude. This is a somewhat trivial case as for five points, $\text{NMHV}_5 = \overline{\text{MHV}}_5$, and therefore we could simply obtain the NMHV_5 amplitude from a Grassmann Fourier transform of the known $\overline{\text{MHV}}_5$ amplitude similar to the way we obtained the $\overline{\text{MHV}}_3$ amplitude above. Nevertheless it is instructive to evaluate the recursion (2.164) in this case.

One immediately concludes that only the second term in (2.164) contributes, because there is no four-point NMHV amplitude. Hence for five points, the complete amplitude is given by the inhomogeneous term (2.166), i.e.

$$\mathbb{A}_5^{\text{NMHV}} = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\prod_{j=1}^{5}\langle j j+1\rangle} R_{5;24}. \tag{2.175}$$

Moving on to the general NMHV_n case it can be seen that there is a general pattern of how the n-point solution is generated from the $(n-1)$-point one. We therefore postulate that the ansatz

$$\mathbb{A}_n^{\mathrm{NMHV}} = \mathbb{A}_n^{\mathrm{MHV}} \mathscr{P}_n^{\mathrm{NMHV}} = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \sum_{2\leq s<t\leq n-1} R_{n;st}, \tag{2.176}$$

solves the supersymmetric BCFW-recursion. In this expression it is assumed that s and t are separated by at least two. We can verify that Eq. (2.176) is correct for $n = 5$ by comparing to Eq. (2.175).

We now prove (2.176) by induction. Let us assume that the form (2.176) is valid for $n-1$ points. Then it follows from the cyclicity of superamplitudes that (2.174) is also true for $n-1$ points. Now, we notice that $\mathbb{A}_{n-1}^{\mathrm{NMHV}}(z_P)$ in the homogeneous term, A on the RHS of (2.164), only involves the quantities $R_{n-1;st}$ where the first subscript is always equal to $n-1$. Cyclic symmetry allows us to insert $\mathbb{A}_{n-1}^{\mathrm{NMHV}}(z_P)$ into (2.164) in our favorite orientation. It is convenient to insert it such that the legs $\{1, 2, 3, \ldots, n-1\}$ of $\mathbb{A}_{n-1}^{\mathrm{NMHV}}(z_P)$ are identified with the legs $\{\hat{P}, 3, 4, \ldots, n\}$ in the recursion relation

$$A = \int \frac{d^4 P}{P^2} \int d^4\eta_{\hat{P}} \mathbb{A}_3^{\overline{\mathrm{MHV}}}(z_P)\mathbb{A}_{n-1}^{\mathrm{MHV}} \mathscr{P}_{n-1}(\hat{P}, 3, \ldots, \bar{n}). \tag{2.177}$$

After carrying out this change of labels in $\mathbb{A}_{n-1}^{\mathrm{NMHV}}(z_P)$ is clear from Eqs. (2.170) and (2.172) that the obtained $R_{n;st}$ does not depend on $\eta_{\hat{P}}$. Indeed the range of η-dependence is only $\{\eta_3, \ldots, \eta_{n-1}\}$. When the lower summation variable attains its minimum value, there is an explicit dependence on the spinor $\langle\hat{P}|$. However, due to the three-point kinematics, this spinor is proportional to $\langle 2|$ and since it appears homogeneously in R with degree zero it can simply be replaced by $\langle 2|$. Thus we find

$$A = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j j+1\rangle} \sum_{3\leq s<t\leq n-1} R_{n;st}. \tag{2.178}$$

We see that (2.166) is just the missing first term (for $s = 2$) to complete (2.178) to the ansatz (2.176) for n points, i.e.

$$A + B = \mathbb{A}_n^{\mathrm{NMHV}} = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j j+1\rangle} \sum_{2\leq s<t\leq n-1} R_{n;st}. \tag{2.179}$$

This completes the inductive proof for the general NMHV super-amplitude.

In fact it is possible to *completely solve* the super-BCFW recursion and write down an exact analytic expression of all tree super-amplitudes in $\mathscr{N} = 4$ super Yang-Mills [9].

Exercise 2.10 (The n-Point MHV Superamplitude and Component Amplitudes)

Use the super-BCFW recursion to prove the MHV super-amplitude formula at n-points. Use this to establish the four point gluino-quark component field amplitudes

$$\begin{aligned} A_4\big(1^-_{\bar{\tilde{g}}}, 2^+_{\tilde{g}}, 3^-, 4^+\big) &= \delta^{(4)}(p)\frac{\langle 31\rangle^3\langle 23\rangle}{\langle 12\rangle\cdots\langle n1\rangle}, \\ A_4\big(1^-_{\bar{\tilde{g}}}, 2^+_{\tilde{g}}, 3^-_{\bar{\tilde{g}}}, 4^+_{\tilde{g}}\big) &= -\delta^{(4)}(p)\frac{\langle 31\rangle^3\langle 24\rangle}{\langle 12\rangle\cdots\langle n1\rangle}. \end{aligned} \tag{2.180}$$

What follows from this result for the 4-point single-flavor massless QCD tree-level amplitudes with one and two quark lines?

Exercise 2.11 (Extraction of Split-Helicity Gluon Amplitudes from NMHV Superamplitude)

Start from the formula for the n-point NMHV superamplitude. As a component in its η expansion, it contains all pure Yang-Mills gluon amplitudes with 3 minus-helicity and $(n-3)$ plus-helicity gluons. Here we focus on the so-called split-helicity gluon amplitudes, where the negative helicity gluons are adjacent. It is found in the superamplitude in the following way,

$$\begin{aligned} \mathscr{A}_n^{\text{NMHV}} &= (\eta_{n-2})^4(\eta_{n-1})^4(\eta_n)^4 \\ &\quad \times A\big(1^+,\ldots,(n-3)^+,(n-2)^-,(n-1)^-,n^-\big) + \cdots, \end{aligned} \tag{2.181}$$

where the dots denote other terms in the η expansion. A convenient way to project out the amplitude that we are interested in is

$$\begin{aligned} &A\big(1^+,\ldots,(n-3)^+,(n-2)^-,(n-1)^-,n^-\big) \\ &\quad = \int d^4\eta_{n-2}\int d^4\eta_{n-1}\int d^4\eta_n \mathscr{A}_n^{\text{NMHV}}. \end{aligned} \tag{2.182}$$

Carry out the Grassmann integrals in (2.182).

- Hint 1: Write the super momentum conserving Grassmann delta function as

$$\begin{aligned} \delta^{(8)}\big(q^A_\alpha\big) &= \langle n-1n\rangle^4\delta^{(4)}\left(\eta^A_{n-1} + \sum_{i=1}^{n-2}\frac{\langle in\rangle}{\langle n-1n\rangle}\eta^A_i\right) \\ &\quad \times \delta^{(4)}\left(\eta^A_n + \sum_{i=1}^{n-2}\frac{\langle n-1i\rangle}{\langle n-1n\rangle}\eta^A_i\right). \end{aligned} \tag{2.183}$$

- Hint 2: Use the cyclic symmetry of the NMHV superamplitude in order to write it in a form where the dependence on the relevant η's is simple. Different choices of the representation of $\mathscr{A}_n^{\text{NMHV}}$ will lead to equivalent answers, but may differ e.g. in the number of terms.

The solution to this exercise can be found in Sect. 8 of Ref. [9].

2.8 References and Further Reading

In Sect. 2.1 we introduced recursion relations enabling the construction of higher-point tree-level amplitudes from lower-point ones using exclusively on-shell data. The described formalism based on a two leg shift is due to Britto, Cachazo, Feng and Witten from 2005 [2, 10]. It is the most efficient technique in a number of recursion relation families.

The factorization properties of scattering amplitudes into sub-amplitudes discussed in Sect. 2.4 are a direct consequence of the color-ordered Feynman rules. The universality of the soft limit was shown by Berends and Giele [11] based on their off-shell recursion relation [12] discussed below. The universality of the factorization on multi-particle poles and collinear limits may also be explained via the point-like limit of string theory amplitudes, see [1].

Our discussion of the conformal invariance of gluon scattering amplitudes of Sect. 2.5 follows the paper of Witten [13].

The $\mathcal{N} = 4$ super Yang-Mills (SYM) theory introduced in Sect. 2.6 was first written down in 1976 [14, 15], its finiteness properties were analyzed in the early 1980s [16–18] and the concept of super-amplitudes in an on-shell chiral superspace goes back to Nair [19] who wrote down the MHV n-point super amplitudes. The supersymmetrized version of the BCFW recursion discussed in Chap. 2.6 was derived in [20] and a related construction was given in [21, 22]. The complete analytic solution of the super BCFW recursion was achieved in 2008 in [9] giving compact analytical formulae for *all* tree-amplitudes in $\mathcal{N} = 4$ SYM. This analytic solution was subsequently used to generate *all* tree-level QCD amplitudes with up to four massless quark-anti-quark pairs and an arbitrary number of gluons [23] including an implementation of the formulas in Mathematica.

In fact, recursive techniques for the construction of tree-level amplitudes existed before the BCFW-recursion discussed in these notes. In 1988 Berends and Giele introduced an off-shell recursion [12] using as building blocks color ordered amplitudes with one off-shell leg. This method is also highly efficient and easily implemented in numerical applications. For a pedagogical introduction to the method see e.g. Chap. 3 of [24].

An alternative and purely on-shell recursion in gauge theories is the Cachazo, Svrcek and Witten MHV vertex expansion from 2004 [25]. This CSW or MHV vertex expansion uses as building blocks exclusively MHV n-point amplitudes. Here all N^pMHV amplitudes may be represented as on-shell diagrams with MHV amplitudes as vertices. Although it was historically not developed in this way, it may be viewed as a particular form of the BCFW recursions, where one shifts several external momenta in a specific way [26]. See also [3] for the supersymmetric case.

Furthermore a series of papers established a Lagrangian formulation of the MHV-vertex expansion [27–31]. Here field redefinitions and suitable gauge choices reformulate the (super) Yang-Mills theory as a model with MHV vertices of arbitrary multiplicity whose Feynman rules yield the CSW expansion. A good overview to the MHV vertex expansion at tree and loop-level is given in [32].

The recursive techniques discussed here can be generalized to scattering amplitudes with massive particles as we briefly discussed. Here the spinor helicity formalism may be extended to massive momenta by representing them as the sum of two null momenta. Concretely one introduces a reference null-momenta q and writes $p_i^\mu = p_{\perp i}^\mu + \frac{m_i^2}{2q\cdot p_i} q^\mu$ with a null $p_{\perp i}^\mu$. This decomposition ensures $p_i^2 = m_i^2$, for a full exposition of this construction see [33]. The analogue of the BCFW recursion relation for amplitudes with massive particles was discussed in [7, 34]. An analogue of the CSW vertex expansion also exists in the massive case [35, 36].

Finally, in these notes we have focused on pure Yang-Mills and on maximally supersymmetric Yang-Mills theory. A detailed account of on-shell superspace techniques for theories with $\mathcal{N} < 4$ supersymmetry is given in [37].

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