Chapter 2
Optimality Concepts and Their Characterization

A variable ordering defined on the real linear space Y induces by the relation $\leq_1$ (given in (1.1)) and by the relation $\leq_2$ (given in (1.2)) an optimality concept which will be denoted as nondominated and as minimal elements, respectively. These two, fundamentally different concepts coincide only in the case of a non-variable ordering, i.e. if $\mathcal{D}(y) = K$ for all $y \in Y$ and some pointed convex cone K. Also weaker and stronger concepts can be derived as weakly or strongly nondominated/minimal elements or properly nondominated/minimal elements. In this chapter we first recall the optimality concepts from partially ordered spaces and then we introduce the various optimality notions for vector optimization problems with variable ordering structures. In addition to that we collect some basic properties of these notions.

We start by discussing optimal elements of a set. Having these concepts available directly allows us to define optimal solutions of a vector optimization problem

$$\min_{x \in S} f(x)$$

for some vector-valued map $f : X \to Y$ with X, Y real linear spaces and $S \subset X$ a nonempty subset: one only needs to determine the optimal elements of the set $f(S)$ and then to find their pre-images.

2.1 Optimality Concepts in Partially Ordered Spaces

In this section we recall the optimality concepts used in vector optimization in a partially ordered space. For that, we assume that Y is a real linear space which is partially ordered by a convex cone $K \subset Y$. Let A be a nonempty subset of Y. In partially ordered spaces we denote optimal elements as efficient elements according to the following definition.

G. Eichfelder, *Variable Ordering Structures in Vector Optimization*, Vector Optimization,
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Definition 2.1. An element $\bar{y} \in A$ is an *efficient element* of the set A if

$$(\{\bar{y}\} - K) \cap A \subset \{\bar{y}\} + K. \tag{2.1}$$

If K is additionally pointed, then (2.1) reduces to

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\}. \tag{2.2}$$

A partial ordering which is not antisymmetric, i.e. which is induced by a non-pointed ordering cone, is difficult to interpret. For that reason in most cases K is assumed to be pointed and thus, later, we also assume for the given variable ordering that the images of the ordering map are pointed convex cones. For handling also non-pointed cones, Borwein [24] replaced K in (2.2) with $\tilde{K} := (K \backslash (K \cap (-K))) \cup \{0_Y\}$. This could also be done for the images $\mathcal{D}(y)$ of an ordering map.

In the following, we always assume K to be a pointed convex cone.

For efficient elements in a partially ordered space, weaker (assuming $\operatorname{cor}(K) \neq \emptyset$) and stronger concepts are known.

Definition 2.2. (a) Assume $\operatorname{cor}(K) \neq \emptyset$. An element $\bar{y} \in A$ is a *weakly efficient* element of the set A, if

$$(\{\bar{y}\} - \operatorname{cor}(K)) \cap A = \emptyset.$$

(b) An element $\bar{y} \in A$ is a *strongly efficient* element of the set A, if

$$A \subset \{\bar{y}\} + K.$$

Weakly efficient elements are useful as they are completely characterized by linear scalarizations, but in applications one is usually not interested in only weakly efficient elements. Any efficient element of A is also a weakly efficient element (provided that K is pointed and $\operatorname{cor}(K) \neq \emptyset$), and any strongly efficient element of A is also an efficient element of A. If there is some strongly efficient element $\bar{y} \in A$, then it is the unique efficient element of the set. By replacing K by $-K$ in the above definitions, we obtain corresponding concepts of *(weakly/strongly) max-efficient* elements of a set A.

There are also other, stronger optimality notions, compared to efficiency, in vector optimization with a partially ordered space, known as properly efficient elements. We assume for all of these definitions for simplicity that $(Y, \|\cdot\|)$ is a real normed space and K is a closed pointed convex cone. Various types of these optimality concepts have been introduced. We give here the definitions in the sense of Henig, Benson and Borwein. For the definition of Borwein properly efficient elements, we need the concept of the contingent cone:

Definition 2.3. Let Ω be a nonempty subset of a real normed space $(Y, \|\cdot\|)$ and some $\bar{y} \in \operatorname{cl}(\Omega)$ be given. The set

$$T(\Omega,\bar{y}) := \{h \in Y \mid \exists (\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{++},\ \exists (y_n)_{n\in\mathbb{N}} \subset \Omega \text{ such that } \lim_{n\to\infty} y_n = \bar{y} \text{ and } h = \lim_{n\to\infty} \lambda_n (y_n - \bar{y})\}$$

is called *contingent cone* (or the *Bouligand tangent cone*) to Ω at $\bar{y}$.

Here, $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.

In the convex case, contingent cones to some set Ω at $\bar{y}$ are related to the closure of the cone generated by the set $\Omega - \{\bar{y}\}$:

Lemma 2.4 ([94, Chap. 3]). *Let Ω be a nonempty subset of a real normed space $(Y, \|\cdot\|)$ and some $\bar{y} \in \Omega$ be given.*

(i) It holds

$$T(\Omega,\bar{y}) \subset cl(cone(\Omega - \{\bar{y}\})).$$

(ii) If Ω is starshaped w.r.t. $\bar{y}$, i.e. if for any $y \in \Omega$,

$$\lambda\, y + (1-\lambda)\,\bar{y} \in \Omega \ \textit{ for all } \ \lambda \in [0,1],$$

then

$$T(\Omega,\bar{y}) = cl(cone(\Omega - \{\bar{y}\})).$$

Definition 2.5. (a) An element $\bar{y} \in A$ is a *properly efficient* element of A (in the sense of Henig [83]) if it is an efficient element of A and if there is a convex cone $C \subset Y$ with $K \setminus \{0_Y\} \subset \mathrm{int} C$ such that $\bar{y}$ is an efficient element of A w.r.t. the partial ordering introduced by C, i.e.

$$\bar{y} \notin \{y\} + C \ \forall\ y \in A \setminus \{\bar{y}\}.$$

(b) An element $\bar{y} \in A$ is a *properly efficient* element of A (in the sense of Benson [16]) if it is an efficient element of A and if $\bar{y}$ is an efficient element of the set

$$\{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(A + K - \{\bar{y}\})). \tag{2.3}$$

(c) An element $\bar{y} \in A$ is a *properly efficient* element of A (in the sense of Borwein [24]) if it is an efficient element of A and if $\bar{y}$ is an efficient element of the set

$$\{\bar{y}\} + T(A + K, \bar{y}). \tag{2.4}$$

The original definitions are slightly different but equivalent to the above ones. For instance, in (b), instead of requiring that $\bar{y}$ is an efficient element of the set defined in (2.3), i.e.

$$(\{\bar{y}\} - K) \cap (\{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(A + K - \{\bar{y}\}))) = \{\bar{y}\},$$

in the original definition it is required that 0_Y is an efficient element of the set $\mathrm{cl}(\mathrm{cone}(A + K - \{\bar{y}\}))$, i.e. that

$$(-K) \cap (\mathrm{cl}(\mathrm{cone}(A + K - \{\bar{y}\}))) = \{0_Y\}.$$

The following example illustrates some of the optimality notions:

Example 2.6. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and

$$A := \{y \in \mathbb{R}^2 \mid \|y\|_2 \leq 1\} \cup \{y \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 \geq -1\}.$$

Then all elements of the set

$$\{y \in \mathbb{R}^2 \mid \|y\|_2 = 1,\ y_1 \leq 0,\ y_2 \leq 0\}$$

are efficient elements of A w.r.t. $\mathcal{D}$. All elements of the set

$$\{y \in \mathbb{R}^2 \mid \|y\|_2 = 1,\ y_1 \leq 0,\ y_2 \leq 0\} \cup \{y \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 = -1\}$$

are weakly efficient elements of A w.r.t. $\mathcal{D}$. There is no strongly efficient element of A. All elements of the set

$$\{y \in \mathbb{R}^2 \mid \|y\|_2 = 1,\ y_1 < 0,\ y_2 < 0\}$$

are properly efficient elements (in the sense of Henig/Benson/Borwein).

By Lemma 2.4, any properly efficient element in the sense of Benson in also properly efficient in the sense of Borwein. Moreover, if the set A is convex, then an element $\bar{y}$ is properly efficient in the sense of Benson if and only if it is properly efficient in the sense of Borwein.

2.2 Optimality Concepts for Variable Ordering Structures

In the following we assume Y to be a real linear space equipped with a variable ordering structure defined by the ordering map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ a pointed convex cone for all $y \in Y$. Additionally, let A be a nonempty subset of Y.

Based on the relation $\leq_1$ defined in (1.1) a candidate element is called a nondominated element if it is not dominated by other reference elements w.r.t. their cone. Instead, in view of the relation $\leq_2$ defined in (1.2), a candidate element is called a minimal element if it is not dominated by any other reference element w.r.t. the cone associated with the candidate element. These two notions are formally defined by:

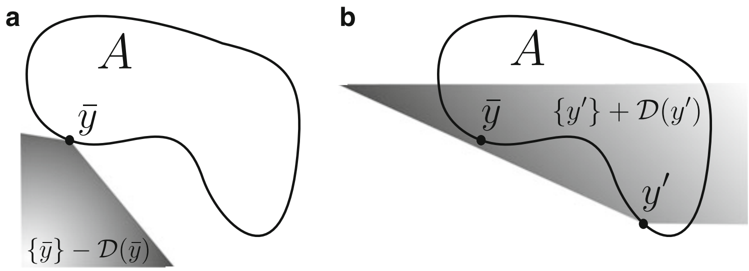


Fig. 2.1 (**a**) $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$. (**b**) $\bar{y}$ is not a nondominated element of A w.r.t. $\mathcal{D}$, cf. [46]

Definition 2.7. (a) An element $\bar{y} \in A$ is a *nondominated element* of the set A w.r.t. the ordering map $\mathcal{D}$, if $y \not\leq_1 \bar{y}$ for all $y \in A \setminus \{\bar{y}\}$, i.e. if no $y \in A$ exists such that

$$\bar{y} \in \{y\} + \mathcal{D}(y) \setminus \{0_Y\}, \tag{2.5}$$

or, equivalently,

$$\bar{y} \notin \bigcup_{y \in A} \{y\} + (\mathcal{D}(y) \setminus \{0_Y\}).$$

(b) An element $\bar{y} \in A$ is a *minimal element* of the set A w.r.t. the ordering map $\mathcal{D}$, if $y \not\leq_2 \bar{y}$ for all $y \in A \setminus \{\bar{y}\}$, i.e. if no $y \in A$ exists such that

$$\bar{y} \in \{y\} + \mathcal{D}(\bar{y}) \setminus \{0_Y\},$$

or, equivalently,

$$(\{\bar{y}\} - \mathcal{D}(\bar{y})) \cap A = \{\bar{y}\}. \tag{2.6}$$

For an illustration of both optimality concepts see Fig. 2.1.

Remark 2.8. For the definitions given in Definition 2.7 we do not require that the sets $\mathcal{D}(y)$ are pointed convex cones. Instead, $\mathcal{D}(y)$ can be an arbitrary set (with $0_Y \in \mathcal{D}(y)$) for any $y \in Y$.

For $\mathcal{D}(y) = K$ for all $y \in Y$ for some pointed convex cone K both definitions, Definition 2.7(a) and (b), coincide. I that case, K defines a partial ordering on Y and the minimal and the nondominated elements are exactly the efficient elements w.r.t. this partial ordering. Otherwise these notions are generally not related to each other in the sense that the one does not imply the other. This is illustrated with the following examples. We will show, however, in Chap. 7, that the notions are connected via duality in the sense that the nondominated elements of a set are the

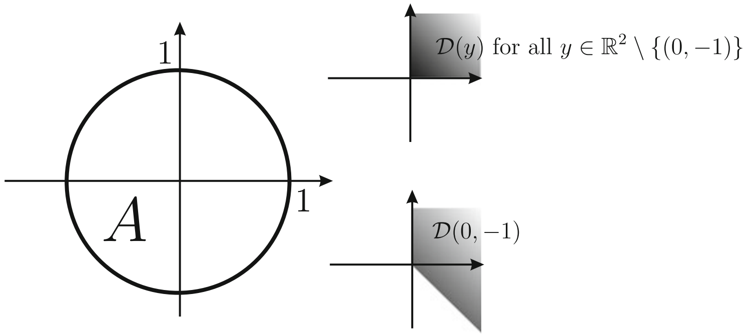


Fig. 2.2 Set A and the cones $\mathcal{D}(y)$ of Example 2.10, cf. [46]

minimal elements of some dual set (w.r.t. the "negative" of the variable ordering structure).

Example 2.9. Let Y be the Euclidean space $\mathbb{R}^2$,

$$A := \{y \in \mathbb{R}^2 \mid \|y\|_2 \leq 1\}$$

and let $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathcal{D}(y) := \begin{cases} \mathbb{R}^2_+ & \forall\, y \in \mathbb{R}^2 \setminus \{(0,-1),(-1,0)\}, \\ \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{for } y = (0,-1), \\ \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 \geq 0,\ z_2 \leq 0\} & \text{for } y = (-1,0). \end{cases}$$

Then all elements of the set $\{(y_1,y_2) \in \mathbb{R}^2 \mid \|y\|_2 = 1,\ y_1 \leq 0,\ y_2 \leq 0\}$ are minimal elements w.r.t. $\mathcal{D}$ but there is no nondominated element of A w.r.t. $\mathcal{D}$.

Example 2.10. Let Y and A be specified as in Example 2.9, i.e.

$$A := \{y \in \mathbb{R}^2 \mid \|y\|_2 \leq 1\},$$

and let $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathcal{D}(y) := \begin{cases} \mathbb{R}^2_+ & \forall\ y \in \mathbb{R}^2 \setminus \{(0,-1)\}, \\ \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 + z_2 \geq 0, z_1 \geq 0\} & \text{for } \ y = (0,-1), \end{cases}$$

see Fig. 2.2. Then $(0,-1)$ is a nondominated element of A w.r.t. $\mathcal{D}$ but not a minimal element of A w.r.t. $\mathcal{D}$. The set of all nondominated elements of A w.r.t. $\mathcal{D}$ is $\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \in [-1,0],\ y_2 = -\sqrt{1-y_1^2}\}$, and the set of all minimal elements of A w.r.t. $\mathcal{D}$ is $\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \in [-1,0),\ y_2 = -\sqrt{1-y_1^2}\}$.

$$\begin{tikzcd} \boxed{\bar{y} \text{ nondominated w.r.t. } \mathcal{D}} \arrow[r, shift left, "\mathcal{D}(\bar{y}) \subset \mathcal{D}(y)\ \forall y \in A"] & \boxed{\bar{y} \text{ minimal w.r.t. } \mathcal{D}} \arrow[l, shift left, "\mathcal{D}(y) \subset \mathcal{D}(\bar{y})\ \forall y \in A"] \end{tikzcd}$$

Fig. 2.3 Diagram illustrating the result of Lemma 2.11

Under strong assumptions on the ordering map $\mathcal{D}$ we obtain the following obvious relations between the ordering concepts, see also Fig. 2.3:

Lemma 2.11. *(i) If $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$ and $\mathcal{D}(y) \subset \mathcal{D}(\bar{y})$ for all $y \in A$, then $\bar{y}$ is also a nondominated element of A w.r.t. $\mathcal{D}$.*
(ii) If $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$ and $\mathcal{D}(\bar{y}) \subset \mathcal{D}(y)$ for all $y \in A$, then $\bar{y}$ is also a minimal element of A w.r.t. $\mathcal{D}$.

For variable ordering structures we obtain analogously to Definition 2.2 the following weaker and stronger notions.

Definition 2.12. (a) Assume $\operatorname{cor}(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. An element $\bar{y} \in A$ is a *weakly nondominated element* of the set A w.r.t. $\mathcal{D}$ if no $y \in A$ exists such that

$$\bar{y} \in \{y\} + \operatorname{cor}(\mathcal{D}(y)).$$

(b) An element $\bar{y} \in A$ is a *strongly nondominated element* of the set A w.r.t. $\mathcal{D}$ if

$$\bar{y} \in \{y\} - \mathcal{D}(y) \quad \text{for all} \quad y \in A.$$

(c) Assume $\operatorname{cor}(\mathcal{D}(\bar{y})) \neq \emptyset$ for some element $\bar{y} \in A$. Then $\bar{y}$ is a *weakly minimal element* of the set A w.r.t. $\mathcal{D}$ if no $y \in A$ exists such that

$$\bar{y} \in \{y\} + \operatorname{cor}(\mathcal{D}(\bar{y})),$$

i.e. if

$$(\{\bar{y}\} - \operatorname{cor}(\mathcal{D}(\bar{y}))) \cap A = \emptyset.$$

(d) An element $\bar{y} \in A$ is a *strongly minimal element* of the set A w.r.t. $\mathcal{D}$ if

$$A \subset \{\bar{y}\} + \mathcal{D}(\bar{y}).$$

The concept of strongly minimal elements is a stronger notion compared to minimal elements w.r.t. $\mathcal{D}$: we do not only demand that $y \notin \{\bar{y}\} - \mathcal{D}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$, but even that $y \in \{\bar{y}\} + \mathcal{D}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$, i.e. there should be no other element which is in some sense "close" to being a preferred element. The same considerations hold for the concept of strongly nondominated elements w.r.t. $\mathcal{D}$. There, we do not only assume that the elements are nondominated elements w.r.t.

$\mathcal{D}$, i.e. that $\bar{y} \notin \{y\} + \mathcal{D}(y)$ for all $y \in A \setminus \{\bar{y}\}$, but even that $\bar{y} \in \{y\} - \mathcal{D}(y)$ for all $y \in A \setminus \{\bar{y}\}$, what can be interpreted in the sense that the elements are "far away" from being dominated by any other element.

We illustrate the weaker and stronger concepts by the following examples.

Example 2.13. Let Y be the Euclidean space $\mathbb{R}^2$,

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 \geq 0,\ y_2 \geq 1 - y_1\}$$

and $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathcal{D}(y_1, y_2) := \begin{cases} \text{cone conv}\,\{(y_1, y_2), (1, 0)\} & \text{if } (y_1, y_2) \in \mathbb{R}^2_+,\ y_2 \neq 0\,, \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

One can check that $\{(y_1, y_2) \in A \mid y_1 + y_2 = 1\}$ is the set of all nondominated elements of A w.r.t. $\mathcal{D}$ and

$$\{(y_1, y_2) \in A \mid y_1 + y_2 = 1\ \vee y_1 = 0\ \vee\ y_2 = 0\}$$

is the set of all weakly nondominated elements of A w.r.t. $\mathcal{D}$. There is no strongly nondominated element of A w.r.t. $\mathcal{D}$.

Example 2.14. Let Y be the Euclidean space $\mathbb{R}^2$,

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq y_2 \leq 2y_1\}$$

and $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathcal{D}(y_1, y_2) := \begin{cases} \mathbb{R}^2_+ & \text{if } y_2 = 0\,, \\ \text{cone conv}\,\{(y_1, |y_2|), (1, 0)\} & \text{otherwise.} \end{cases}$$

One can check that $(0, 0) \in A$ is a strongly minimal and also a strongly nondominated element of A w.r.t. $\mathcal{D}$.

Again, for $\mathcal{D}(y) = K$ for all $y \in Y$ with K a pointed convex cone (and $\text{cor}(K) \neq \emptyset$ when needed) the above weak and strong concepts coincide with those in a partially ordered space. In the following we study the connections between the optimal elements w.r.t. a variable ordering structure and the optimal elements w.r.t. a partial ordering in more detail.

Lemma 2.15. *(i) An element $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$ if and only if it is an efficient element of A in the linear space Y partially ordered by the cone $K := \mathcal{D}(\bar{y})$.*

(ii) Let $\text{cor}(\mathcal{D}(\bar{y})) \neq \emptyset$ for some $\bar{y} \in A$. The element $\bar{y}$ is a weakly minimal element of A w.r.t. $\mathcal{D}$ if and only if it is a weakly efficient element of A in the linear space Y partially ordered by the cone $K := \mathcal{D}(\bar{y})$.

Proof. The assertion follows directly from the definitions. □

For nondominated elements w.r.t. a variable ordering we obtain the following sufficient conditions.

Lemma 2.16. *Let $\mathcal{D}(A)$ be convex and pointed.*

(i) Any efficient element of A in the linear space Y partially ordered by the cone $K := \mathcal{D}(A)$ is also a nondominated element of A w.r.t $\mathcal{D}$.
(ii) Let $cor(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. Any weakly efficient element of A in the linear space Y partially ordered by $K := \mathcal{D}(A)$ is also a weakly nondominated element of A w.r.t $\mathcal{D}$.

Proof. (i) If $\bar{y}$ is efficient, then $(\{\bar{y}\} - \mathcal{D}(A)) \cap A = \{\bar{y}\}$, i.e. for any $y \in A \setminus \{\bar{y}\}$ it holds $y \notin \{\bar{y}\} - \mathcal{D}(A)$ and thus $\bar{y} \notin \{y\} + \mathcal{D}(y)$.
(ii) If $\bar{y}$ is weakly efficient, then $(\{\bar{y}\} - \mathrm{cor}(\mathcal{D}(A))) \cap A = \emptyset$, i.e. for any $y \in A$ it holds $y \notin \{\bar{y}\} - \mathrm{cor}(\mathcal{D}(A))$ and thus $\bar{y} \notin \{y\} + \mathrm{cor}(\mathcal{D}(y))$.

□

Analogously, we can formulate necessary conditions for nondominated elements w.r.t. a variable ordering structure based on efficient elements. A slightly weaker result is given later in Lemma 2.37.

Lemma 2.17. *(i) Any nondominated element of A w.r.t. $\mathcal{D}$ is also an efficient element of A with the linear space Y partially ordered by $K := \bigcap_{y \in A} \mathcal{D}(y)$.*
(ii) Let $cor(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$ and set $K := \bigcap_{y \in A} \mathcal{D}(y)$. If $cor(K) \neq \emptyset$ then any weakly nondominated element of A w.r.t. $\mathcal{D}$ is also a weakly efficient element of A with the linear space Y partially ordered by K.

Proof. (i) $\bar{y}$ nondominated of A w.r.t. $\mathcal{D}$ is equivalent to

$$\bar{y} \notin \{y\} + \mathcal{D}(y) \setminus \{0_Y\} \ \text{ for all } y \in A,$$

and hence $\bar{y} \notin \{y\} + K \setminus \{0_Y\}$ for all $y \in A$ or

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\}.$$

(ii) $\bar{y}$ weakly nondominated of A w.r.t. $\mathcal{D}$ is equivalent to

$$\bar{y} \notin \{y\} + \mathrm{cor}(\mathcal{D}(y)) \ \text{ for all } \ y \in A,$$

and hence $\bar{y} \notin \{y\} + \mathrm{cor}(K)$ for all $y \in A$ or

$$(\{\bar{y}\} - \mathrm{cor}(K)) \cap A = \emptyset.$$

□

We sum up the result of the Lemmas 2.15–2.17 together with some obvious relations in a diagram in Fig. 2.4.

$$\begin{tikzcd}
\boxed{\bar{y} \text{ nondominated w.r.t. } \mathcal{D}} \arrow[r] \arrow[d, shift right=2, "K=\bigcap_{y\in A}\mathcal{D}(y)"'] & \boxed{\bar{y} \text{ weakly nondominated w.r.t. } \mathcal{D}} \arrow[d, shift right=2, "K=\bigcap_{y\in A}\mathcal{D}(y)"'] \\
\boxed{\bar{y} \text{ efficient w.r.t. } K} \arrow[r] \arrow[u, shift right=2, "K=\mathcal{D}(A) \text{ convex pointed}"'] \arrow[d, shift right=2, "K=\mathcal{D}(\bar{y})"'] & \boxed{\bar{y} \text{ weakly efficient w.r.t. } K} \arrow[u, shift right=2, "K=\mathcal{D}(A) \text{ convex pointed}"'] \arrow[d, shift right=2, "K=\mathcal{D}(\bar{y})"'] \\
\boxed{\bar{y} \text{ minimal w.r.t. } \mathcal{D}} \arrow[r] \arrow[u, shift right=2, "K=\mathcal{D}(\bar{y})"'] & \boxed{\bar{y} \text{ weakly minimal w.r.t. } \mathcal{D}} \arrow[u, shift right=2, "K=\mathcal{D}(\bar{y})"']
\end{tikzcd}$$

Fig. 2.4 Diagram illustrating the results of Lemmas 2.15–2.17 under the assumption $\operatorname{cor}(\mathcal{D}(y)) \neq \emptyset$ and $\operatorname{cor}(K) \neq \emptyset$ whenever considered

In the following theorem we extend Lemma 2.17 to a necessary condition which is, under some assumptions, also sufficient. We need the notion of external stability which is also denoted domination property, see [113] and the references therein.

Definition 2.18. Let the linear space Y be partially ordered by some convex cone K and let Ω be a nonempty subset of A. Then Ω is said to be *externally stable* if for all $y \in A \setminus \Omega$ there exists some $\bar{y} \in \Omega$ such that $y \in \{\bar{y}\} + K$.

Thus, external stability is satisfied if for all $y \in A$ there exists some $\bar{y} \in \Omega$ such that $y \in \{\bar{y}\} + K$, i.e. $A \subset \Omega + K$. In [134, Sect. 3.2] and [113] conditions ensuring the external stability of the set of efficient elements are given. Such a sufficient condition for the external stability of the set of efficient elements in a real linear space partially ordered by a convex cone K is given in the following.

Lemma 2.19 ([134, Theorem 3.2.10]). *Let Y be a real topological linear space, $K \subset Y$ a closed pointed convex cone and A a K-compact set, i.e. let the sets*

$$(\{y\} - K) \cap A$$

be compact for all $y \in A$. Let $\mathcal{E}_K$ denote the set of efficient elements of A w.r.t. the partial ordering introduced by K. Then $\mathcal{E}_K$ is externally stable, i.e. $A \subset \mathcal{E}_K + K$.

For instance, if A is compact then it is also K-compact for any closed cone K. In case A is an open set, the set of efficient elements $\mathcal{E}_K$ is empty and thus not an externally stable set of A. Next we give the announced theorem.

Theorem 2.20. *Let $K \subset \bigcap_{y\in A} \mathcal{D}(y)$ be a pointed convex cone and let $\mathcal{E}_K$ denote the set of efficient elements of A w.r.t. the partial ordering introduced by K. Let $\mathcal{E}_K$ be externally stable and let*

$$y^1 \in \{y^2\} + K \text{ imply } \mathcal{D}(y^1) \subset D(y^2) \text{ for all } y^1, y^2 \in A. \tag{2.7}$$

Then $\bar{y} \in A$ is a nondominated element of A w.r.t. $\mathcal{D}$ if and only if $\bar{y} \in \mathcal{E}_K$ and $\bar{y}$ is a nondominated element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$.

Proof. By Lemma 2.17(i) and as $\mathcal{E}_K \subset A$, the condition is necessary.

To show that it is also sufficient, assume $\bar{y}$ is in $\mathcal{E}_K$ and a nondominated element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$ but not of A. Then there exists some $y \in A \setminus \mathcal{E}_K$ with $\bar{y} \in \{y\} + \mathcal{D}(y) \setminus \{0_Y\}$. As $y \in A \setminus \mathcal{E}_K$ and $\mathcal{E}_K$ is externally stable, there exists some $\hat{y} \in \mathcal{E}_K$ with $y \in \{\hat{y}\} + K \setminus \{0_Y\}$. The condition (2.7) implies $\mathcal{D}(y) \subset \mathcal{D}(\hat{y})$ and we obtain

$$\begin{aligned}\bar{y} &\in \{\hat{y}\} + K + \mathcal{D}(y)\\ &\subset \{\hat{y}\} + \mathcal{D}(y)\\ &\subset \{\hat{y}\} + \mathcal{D}(\hat{y})\end{aligned}$$

and $\bar{y} \neq \hat{y} \in \mathcal{E}_K$ in contradiction to $\bar{y}$ a nondominated element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$. □

The following lemma shows that condition (2.7) is satisfied if the binary relation defined by the ordering map is transitive.

Lemma 2.21. *Let $\mathcal{D}(y)$ be algebraically closed for all $y \in Y$ and let $K \subset \bigcap_{y\in A} \mathcal{D}(y)$. If the binary relation $\leq_1$ defined in (1.1) is transitive, then (2.7) is satisfied, i.e. $y^1 \in \{y^2\} + K$ implies $\mathcal{D}(y^1) \subset D(y^2)$ for all $y^1, y^2 \in A$.*

Proof. The relation $\leq_1$ is transitive according to Lemma 1.10(ii) if and only if

$$\mathcal{D}(y + d) \subset \mathcal{D}(y) \text{ for all } y \in Y \text{ and for all } d \in \mathcal{D}(y).$$

For $y^1 \in \{y^2\} + K$ it holds $y^1 = y^2 + d$ with $d \in K \subset \mathcal{D}(y^2)$ and thus $\mathcal{D}(y^1) = \mathcal{D}(y^2 + d) \subset \mathcal{D}(y^2)$. □

We obtain a similar result for minimal elements w.r.t a variable ordering structure. However, note that we need no condition like (2.7) or any other transitivity-related assumption.

Theorem 2.22. *Let $K \subset \bigcap_{y\in A} \mathcal{D}(y)$ be a pointed convex cone and let $\mathcal{E}_K$ denote the set of efficient elements of A w.r.t. the partial ordering introduced by K and let $\mathcal{E}_K$ be externally stable. Then $\bar{y} \in A$ is a minimal element of A w.r.t. $\mathcal{D}$ if and only if $\bar{y} \in \mathcal{E}_K$ and $\bar{y}$ is a minimal element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$.*

Proof. By Lemma 2.15, the definition of an efficient element and the fact that $K \subset \mathcal{D}(y)$ for all $y \in A$ and as $\mathcal{E}_K \subset A$, the condition is necessary.

To show that the condition is also sufficient, assume $\bar{y}$ is in $\mathcal{E}_K$ and a minimal element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$ but not of A. Then there exists some $y \in A \setminus \mathcal{E}_K$ with $\bar{y} \in \{y\} + \mathcal{D}(\bar{y}) \setminus \{0_Y\}$. As $\mathcal{E}_K$ is externally stable, there exists some $\hat{y} \in \mathcal{E}_K$ with $y \in \{\hat{y}\} + K \setminus \{0_Y\}$. Then

$$\begin{aligned}\bar{y} &\in \{\hat{y}\} + K + \mathcal{D}(\bar{y})\\ &\subset \{\hat{y}\} + \mathcal{D}(\bar{y}),\end{aligned}$$

and $\bar{y} \neq \hat{y} \in \mathcal{E}_K$ in contradiction to $\bar{y}$ a minimal element of $\mathcal{E}_K$ w.r.t. $\mathcal{D}$. □

Next, we collect some results on weakly and strongly optimal elements w.r.t. a variable ordering structure.

Lemma 2.23. *(i) Any strongly nondominated element of A w.r.t. $\mathcal{D}$ is also a nondominated element of A w.r.t. $\mathcal{D}$. Any strongly minimal element of A w.r.t. $\mathcal{D}$ is also a minimal element of A w.r.t. $\mathcal{D}$.*

(ii) Let $cor(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. Then any nondominated element of A w.r.t. $\mathcal{D}$ is also a weakly nondominated element of A w.r.t. $\mathcal{D}$. Any minimal element of A w.r.t. $\mathcal{D}$ is also a weakly minimal element of A w.r.t. $\mathcal{D}$.

(iii) If $\bar{y}$ is a strongly minimal element of A w.r.t. $\mathcal{D}$ and if $\mathcal{D}(\bar{y}) \subset \mathcal{D}(y)$ for all $y \in A$, then $\bar{y}$ is also a strongly nondominated element of A.

(iv) If $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$, then the set of minimal elements of A w.r.t. $\mathcal{D}$ is empty or equals $\{\bar{y}\}$. If additionally $\mathcal{D}(A) = \bigcup_{y \in A} \mathcal{D}(y)$ is pointed, then $\bar{y}$ is the unique minimal element of A w.r.t. $\mathcal{D}$.

(v) If $\mathcal{D}(A)$ is pointed, then there is at most one strongly nondominated element of A w.r.t. $\mathcal{D}$.

Proof. (i)–(iii) follow directly from the definitions.

(iv) As $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$,

$$\bar{y} \in (\{y\} - \mathcal{D}(y)) \cap A \quad \text{for all} \quad y \in A,$$

i.e. all elements in $A \backslash \{\bar{y}\}$ are not minimal elements of A w.r.t. $\mathcal{D}$. If $\mathcal{D}(A)$ is pointed, for any $y \in A$ the assumption $y \in \{\bar{y}\} - \mathcal{D}(\bar{y})$ together with $y \in \{\bar{y}\} + \mathcal{D}(y)$ (as $\bar{y}$ is strongly nondominated) implies

$$y \in (\{\bar{y}\} - \mathcal{D}(A)) \cap (\{\bar{y}\} + \mathcal{D}(A)) ,$$

i.e. $y = \bar{y}$ and thus $(\{\bar{y}\} - \mathcal{D}(\bar{y})) \cap A = \{\bar{y}\}$.

(v) Let $\bar{y}$ be a strongly nondominated element of A w.r.t. $\mathcal{D}$. If $\mathcal{D}(A)$ is pointed, then $\bar{y} - y \in -\mathcal{D}(y) \subset -\mathcal{D}(A)$ implies $\bar{y} - y \notin \mathcal{D}(A)$ for all $y \in A \setminus \{\bar{y}\}$, i.e. $y \notin \{\bar{y}\} - \mathcal{D}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$ and thus no other element of A can be strongly nondominated w.r.t. $\mathcal{D}$. $\square$

For a diagram illustrating the relations between the notions of (weakly, strongly) nondominated and minimal elements see Fig. 2.5.

The following example illustrates that the assumption that $\mathcal{D}(A)$ is pointed is necessary for the conclusion in Lemma 2.23(iv).

Example 2.24. Let Y be the Euclidean space $\mathbb{R}^2$, $A := \mathbb{R}^2_+$ and $\mathcal{D}: \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathcal{D}(y_1, y_2) := \begin{cases} \text{cone conv}\,\{(y_1, y_2), (1,0)\} & \text{if } (y_1, y_2) \in \mathbb{R}^2_+ \setminus \{(0,0)\}, \\ \text{cone conv}\,\{(1,0), (-1,-1)\} & \text{if } (y_1, y_2) = (0,0), \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
\boxed{\bar{y} \text{ strongly nondominated w.r.t. } \mathcal{D}} \arrow[d] & \boxed{\bar{y} \text{ strongly minimal w.r.t. } \mathcal{D}} \arrow[d] \\
\boxed{\bar{y} \text{ nondominated w.r.t. } \mathcal{D}} \arrow[d, "\mathrm{cor}\mathcal{D}(y) \neq \emptyset\ \forall y \in A"'] & \boxed{\bar{y} \text{ minimal w.r.t. } \mathcal{D}} \arrow[d, "\mathrm{cor}\mathcal{D}(\bar{y}) \neq \emptyset"] \\
\boxed{\bar{y} \text{ weakly nondominated w.r.t. } \mathcal{D}} & \boxed{\bar{y} \text{ weakly minimal w.r.t. } \mathcal{D}}
\end{tikzcd}$$

Fig. 2.5 Diagram illustrating the relation between the notions of (weakly, strongly) nondominated and minimal elements

Then $(0,0)$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$ but not a minimal element of A w.r.t. $\mathcal{D}$. The cone $\mathcal{D}(A)$ is not pointed.

By replacing $\mathcal{D}$ by $\tilde{\mathcal{D}}$ with $\tilde{\mathcal{D}}(y) := -\mathcal{D}(y)$ for all $y \in Y$ in the Definitions 2.7 and 2.12, we obtain corresponding concepts of *(weakly/strongly) max-nondominated* and *(weakly/strongly) maximal elements* of a set A w.r.t. the ordering map $\mathcal{D}$.:

Definition 2.25. (a) An element $\bar{y} \in A$ is a *max-nondominated element* of the set A w.r.t. the ordering map $\mathcal{D}$, if there is no $y \in A$ such that

$$\bar{y} \in \{y\} - \mathcal{D}(y) \setminus \{0_Y\}.$$

(b) Assume $\mathrm{cor}(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. An element $\bar{y} \in A$ is a *weakly max-nondominated element* of the set A w.r.t. the ordering map $\mathcal{D}$, if there is no $y \in A$ such that

$$\bar{y} \in \{y\} - \mathrm{cor}(\mathcal{D}(y)).$$

Definition 2.26. (a) An element $\bar{y} \in A$ is a *maximal element* of the set A w.r.t. the ordering map $\mathcal{D}$, if

$$(\{\bar{y}\} + \mathcal{D}(\bar{y})) \cap A = \{\bar{y}\}.$$

(b) Assume $\mathrm{cor}(\mathcal{D}(\bar{y})) \neq \emptyset$ for some element $\bar{y} \in A$. Then $\bar{y}$ is a *weakly maximal element* of the set A w.r.t. the ordering map $\mathcal{D}$, if

$$(\{\bar{y}\} + \mathrm{cor}(\mathcal{D}(\bar{y}))) \cap A = \emptyset.$$

Finally, we also give various notions of properly nondominated and properly efficient elements which generalize the definitions introduced in partially ordered real linear spaces to vector optimization with variable ordering structures.

Definition 2.27. Let $(Y, \|\cdot\|)$ be a real normed space and let $\mathcal{D}(y)$ be additionally closed for all $y \in Y$.

(a) An element $\bar{y} \in A$ is a *properly nondominated element* of A w.r.t. the ordering map $\mathcal{D}$ in the sense of Henig if it is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ and if there is a cone-valued map $\mathcal{K}\colon Y \to 2^Y$ with $\mathcal{K}(y)$ a convex cone for all $y \in Y$ and with $\mathcal{D}(y) \setminus \{0_Y\} \subset \operatorname{int}(\mathcal{K}(y))$ for all $y \in Y$ such that $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{K}$, i.e.

$$\bar{y} \notin \{y\} + \mathcal{K}(y) \ \forall\, y \in A \setminus \{\bar{y}\}.$$

(b) An element $\bar{y} \in A$ is a *properly nondominated element* of A w.r.t. the ordering map $\mathcal{D}$ in the sense of Benson if it is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ and if $\bar{y}$ is a nondominated element of the set

$$\{\bar{y}\} + \operatorname{cl}(\operatorname{cone}(\bigcup_{a\in A}(\{a\} + \mathcal{D}(a)) - \{\bar{y}\})).$$

(c) An element $\bar{y} \in A$ is a *properly nondominated element* of A w.r.t. the ordering map $\mathcal{D}$ in the sense of Borwein if it is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ and if $\bar{y}$ is a nondominated element of the set

$$\{\bar{y}\} + T(\bigcup_{a\in A}(\{a\} + \mathcal{D}(a)), \bar{y}).$$

(d) An element $\bar{y} \in A$ is a *properly minimal element* of A w.r.t. the ordering map $\mathcal{D}$ in the sense of Henig if it is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ and if there is a cone-valued map $\mathcal{K}\colon Y \to 2^Y$ with $\mathcal{K}(y)$ a convex cone for all $y \in Y$ and with $\mathcal{D}(y) \setminus \{0_Y\} \subset \operatorname{int}(\mathcal{K}(y))$ for all $y \in Y$ such that $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{K}$, i.e.

$$y \notin \{\bar{y}\} - \mathcal{K}(\bar{y}) \ \forall\, y \in A \setminus \{\bar{y}\}.$$

(e) An element $\bar{y} \in A$ is a *properly minimal element* of A w.r.t. the ordering map $\mathcal{D}$ in the sense of Benson if it is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ and if $\bar{y}$ is a minimal element of the set

$$\{\bar{y}\} + \operatorname{cl}(\operatorname{cone}(A + \mathcal{D}(\bar{y}) - \{\bar{y}\})).$$

(f) An element $\bar{y} \in A$ is a *properly minimal element* of A w.r.t. the ordering map $\mathcal{D}$ in the sense of Borwein if it is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ and if $\bar{y}$ is a minimal element of the set

$$\{\bar{y}\} + T(A + \mathcal{D}(\bar{y}), \bar{y}).$$

If $\mathcal{D}(y) = K$ for all $y \in Y$ then the definitions of a properly nondominated element w.r.t. $\mathcal{D}$ and of a properly minimal element w.r.t. $\mathcal{D}$ coincide with the concepts of a properly efficient element in a partially ordered space ordered by the closed pointed convex cone K, in the sense of Henig/Benson/Borwein. The following example illustrates some of the above notions:

Example 2.28. Let $A := \{y \in \mathbb{R}^2 \mid \|y\|_2 \leq 1\}$ be the unit ball in the Euclidean space $Y = \mathbb{R}^2$ and let an ordering map $\mathcal{D}$ be defined by

$$\mathcal{D}(y) = \begin{cases} \{z \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{if } y \in \{(0,1),(2,0)\}, \\ \mathbb{R}^2_+ & \text{else.} \end{cases}$$

The point $\bar{y} = (-1/\sqrt{2}, -1/\sqrt{2})$ is a properly nondominated and also a properly minimal point in the sense of Borwein.

Lemma 2.29. *Let $(Y, \|\cdot\|)$ be a real normed space and let $\mathcal{D}(y)$ be additionally closed for all $y \in Y$. The element $\bar{y}$ is a properly minimal element of A w.r.t. $\mathcal{D}$ in the sense of Henig if and only if it is a minimal element of A w.r.t. $\mathcal{D}$ and if there is a convex cone K with $\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subset int(K)$ such that $y \notin \{\bar{y}\} - K$ for all $y \in A \setminus \{\bar{y}\}$.*

Proof. The only-if-part is immediate from Definition 2.27(d) by setting $K := \mathcal{K}(\bar{y})$. For the if-part, the set-valued map $\mathcal{K}$ can be defined by

$$\mathcal{K}(y) := \begin{cases} Y & \text{if } y \neq \bar{y}, \\ K & \text{if } y = \bar{y}. \end{cases}$$

Then $\mathcal{D}(y) \setminus \{0_Y\} \subset \text{int}(\mathcal{K}(y))$ for all $y \in Y$ and we are done. □

The next lemma follows directly from the definitions.

Lemma 2.30. *Let $(Y, \|\cdot\|)$ be a real normed space and let $\mathcal{D}(y)$ be additionally closed for all $y \in Y$. $\bar{y}$ is a properly minimal element of A in the sense of Henig/Benson/Borwein w.r.t. $\mathcal{D}$ if and only if it is a properly efficient element in the sense of Henig/Benson/Borwein of A with Y partially ordered by the closed convex cone $K := \mathcal{D}(\bar{y})$.*

Remark 2.31. As a consequence of the above Lemma and [75, Theorem 4.2], if $\bar{y}$ is a properly minimal element in the sense of Henig with $\mathcal{K}(\bar{y})$ pointed, then $\bar{y}$ is also a properly minimal element in the sense of Benson. As a consequence of the above Lemma and [100, Theorem 5.2, Remark 5.3], if $\bar{y}$ is a properly minimal element in the sense of Benson and if $\mathcal{D}(\bar{y})$ has a weakly compact base, then $\bar{y}$ is also a properly minimal element in the sense of Henig w.r.t. $\mathcal{D}$.

We also have the following relations between the proper optimality notions:

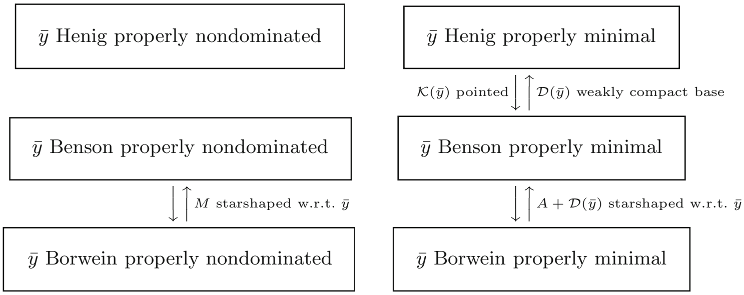


Fig. 2.6 Diagram illustrating the results of Remarks 2.31 and 2.32 with $M := \bigcup_{a\in A}(\{a\} + \mathcal{D}(a))$, cf. [55]

Remark 2.32. (i) By Lemma 2.4, any Benson properly nondominated/ minimal element of A w.r.t. $\mathcal{D}$ is also a Borwein properly nondominated/minimal element of A w.r.t. $\mathcal{D}$.

(ii) If the set $M := \bigcup_{a\in A}(\{a\} + \mathcal{D}(a))$ is starshaped w.r.t. some element $\bar{y} \in A$, then by Lemma 2.4 the element $\bar{y}$ is a Benson properly nondominated element if and only if it is a Borwein properly nondominated element.

(iii) Also by Lemma 2.4, if $A + \mathcal{D}(\bar{y})$ is starshaped w.r.t. $\bar{y} \in A$, then the element $\bar{y}$ is a Benson properly minimal element if and only if it is a Borwein properly minimal element.

(iv) If the cones $\mathcal{D}(y)$ are closed, pointed, nontrivial and convex cones in $\mathbb{R}^m$, then by [134, Theorem 3.1.2], [75, Theorem 4.2], Henig's proper minimality is equivalent to Benson's proper minimality.

We sum up the results of the Remarks 2.32 and 2.31 in a diagram in Fig. 2.6.

The remaining relations between Henig properly nondominated and Benson properly nondominated are not as straightforward and are discussed in [55]: Henig properly nondominated implies in general not Benson properly nondominated. In case $\mathcal{D}(y)$ has a weakly compact base for each $y \in Y$, then there is an implication from Benson to Henig proper nondominatedness.

In addition to considering optimal elements of a set w.r.t. a variable ordering structure, all concepts apply also to an optimization problem with a vector-valued (or set-valued) objective map and with the objective space equipped with a variable ordering structure. For that assume that X and Y are real linear spaces and Y is equipped with a variable ordering structure defined by the cone-valued map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ a pointed convex cone for all $y \in Y$. Let $F : X \to 2^Y$ be a set-valued map and $S \subset X$ a nonempty set. We consider the following optimization problem:

$$\text{Minimize } F(x) \text{ subject to } x \in S. \tag{VP}$$

In case $F = f$ is a single-valued map $f\colon X \to Y$ we denote problem (VP) as a vector optimization problem. Otherwise we speak of a set optimization problem. Further, we denote the image of S under F with $F(S) = \bigcup_{x\in S} F(x)$.

The various notions of nondominated and minimal elements w.r.t. the ordering map $\mathcal{D}$ for sets naturally induce corresponding notions of solutions to the optimization problem (VP) as follows.

Definition 2.33. Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$.

(a) The pair $(\bar{x}, \bar{y})$ is a *globally "N" solution* of the problem (VP) w.r.t. the ordering map $\mathcal{D}$, if $\bar{y}$ is an "N" element of the set $F(S)$. Here, "N" may be (weakly/strongly) nondominated or max-nondominated, (weakly/strongly) minimal or maximal.

(b) The pair $(\bar{x}, \bar{y})$ is a *locally "N" solution* of the problem (VP) w.r.t. the ordering map $\mathcal{D}$, if $\bar{y}$ is an "N" element of the set $F(S \cap U_{\bar{x}})$, where $U_{\bar{x}}$ is some neighborhood of $\bar{x}$. Here, "N" may be (weakly/strongly) nondominated or max-nondominated or (weakly, strongly) minimal or maximal.

Analogously to the above definition one can define (weakly/strongly) efficient solutions of the problem (VP) w.r.t. a partial ordering introduced by a convex cone K. The same for all notions of proper optimality.

When no confusion occurs, we omit "globally" in the above definition and when $F = f$ is a single-valued map $f : X \to Y$, we put $\bar{y} = f(\bar{x})$ in Definition 2.33 and say $\bar{x}$ is a (globally) "N" solution.

In Definition 2.33 we define optimal solutions of a set optimization problem by the so-called vector approach. This approach is widespread, but might not be suitable for some applications. See Sect. 2.5 for a short discussion on this topic.

2.3 Characterization of Nondominated Elements

We examine in this section basic properties of nondominated elements of a set A w.r.t. a variable ordering structure. In the following we again assume Y to be a real linear space equipped with a variable ordering structure defined by the ordering map $\mathcal{D}: Y \to 2^Y$ with $\mathcal{D}(y)$ a pointed convex cone for all $y \in Y$. Additionally, let A be a nonempty subset of Y. Under rather weak assumptions, all weakly nondominated elements are a subset of the algebraic boundary ∂A of the set A.

Lemma 2.34. *(i) Let $cor(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. If*

$$\bigcap_{y \in A} cor(\mathcal{D}(y)) \neq \emptyset$$

and $\bar{y} \in A$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$, then $\bar{y} \in \partial A$.

(ii) If $\bigcap_{y \in A} \mathcal{D}(y) \neq \{0_Y\}$ and $\bar{y} \in A$ is a nondominated element of A w.r.t. $\mathcal{D}$, then $\bar{y} \in \partial A$.

Proof. (i) We assume that $\bar{y} \in \mathrm{cor}(A)$. Let $d \in \bigcap_{y \in A} \mathrm{cor}(\mathcal{D}(y))$. Then $d \neq 0_Y$ and there exists $\lambda > 0$ with $\bar{y} - \lambda d \in A \setminus \{\bar{y}\}$. As

$$-\lambda d \in -\bigcap_{y \in A} \mathrm{cor}(\mathcal{D}(y)) \subset -\mathrm{cor}(\mathcal{D}(\bar{y} - \lambda d))$$

we have $\bar{y} - \lambda d \in A \cap (\{\bar{y}\} - \mathrm{cor}(\mathcal{D}(\bar{y} - \lambda d)))$ or

$$\bar{y} \in \{\bar{y} - \lambda d\} + \mathrm{cor}(\mathcal{D}(\bar{y} - \lambda d)),$$

being a contradiction to $\bar{y}$ weakly nondominated. As also $\bar{y} \in A$ and thus $\bar{y} \notin \mathrm{cor}(Y \setminus A)$ we get $\bar{y} \in \partial A$

(ii) We assume that $\bar{y} \in \mathrm{cor}(A)$. Let $d \in \left(\bigcap_{y \in A} \mathcal{D}(y)\right) \setminus \{0_Y\}$. Then there exists $\lambda > 0$ with $\bar{y} - \lambda d \in A \setminus \{\bar{y}\}$. As

$$-\lambda d \in -\bigcap_{y \in A} \mathcal{D}(y) \subset -\mathcal{D}(\bar{y} - \lambda d)$$

we have $\bar{y} - \lambda d \in A \cap (\{\bar{y}\} - \mathcal{D}(\bar{y} - \lambda d))$ or $\bar{y} \in \{\bar{y} - \lambda d\} + \mathcal{D}(\bar{y} - \lambda d)$, being a contradiction to $\bar{y}$ nondominated. As $\bar{y} \in A$ we get $\bar{y} \in \partial A$.

□

The following example demonstrates that the assumption in Lemma 2.34(i) is necessary.

Example 2.35. Let Y be the Euclidean space $\mathbb{R}^2$, $A = [1,3] \times [1,3]$ and the ordering map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathcal{D}(y_1, y_2) := \begin{cases} \mathbb{R}^2_+ & \text{if } y_1 \geq 2\,, \\ \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{otherwise.} \end{cases}$$

Then $\bar{y} = (2,2)$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$ but $\bar{y} \notin \partial A$.

It is easy to see that $A_1 \subset A_2$ implies that an element $\bar{y} \in A_1$, which is a (weakly) nondominated element of A_2 w.r.t. $\mathcal{D}$ is also a (weakly) nondominated element of A_1 w.r.t. $\mathcal{D}$. The following remark states a similar result for two ordering maps $\mathcal{D}_1$ and $\mathcal{D}_2$ with the one including the other in the sense that $\mathcal{D}_1(y) \subset \mathcal{D}_2(y)$ for all $y \in A$.

Remark 2.36. Let $\mathcal{D}_1, \mathcal{D}_2\colon Y \to 2^Y$ be cone-valued maps with $\mathcal{D}_1(y)$ and $\mathcal{D}_2(y)$ a pointed convex cone for all $y \in Y$. According to the Definitions 2.7 and 2.12, if $\mathcal{D}_1(y) \subset \mathcal{D}_2(y)$ for all $y \in A$, then any nondominated element of the set A w.r.t. the ordering map $\mathcal{D}_2$ is also a nondominated element of the set A w.r.t. the ordering map $\mathcal{D}_1$. If $\mathrm{cor}(\mathcal{D}_1(y)) \neq \emptyset$ for all $y \in A$, then this also holds if we replace nondominated by weakly nondominated.

This leads to the following relation—which also directly follows from Lemmas 2.16 and 2.17—between nondominated elements w.r.t. a variable ordering structure and efficient elements in a partially ordered space. Of course, corresponding statement can be formulated for weakly efficient and nondominated elements.

Lemma 2.37. *(i) Let $\bar{y} \in A$ be a nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$. Then for any convex cone K with $K \subset \mathcal{D}(y)$ for all $y \in A$, $\bar{y}$ is also an efficient element of the set A with the space Y partially ordered by K.*

(ii) Let $\bar{y}$ be an efficient element of the set A with the space Y partially ordered by some pointed convex cone K. If $\mathcal{D}(y) \subset K$ for all $y \in A$, then $\bar{y}$ is also a nondominated element of the set A w.r.t. the ordering map $\mathcal{D}$.

In partially ordered spaces ordered by some pointed convex cone K it is known [158, Lemma 4.1] that an element is an efficient element of a set A if and only if it is an efficient element of the set $A + K$. This can be generalized to variable ordering structures only under the additional assumption of the transitivity of the binary relation on the set A. The advantage of studying $A + K$ instead of A lies for instance in a better applicability of linear scalarization results if $A + K$ is convex while the set A is not. In that case A is called *convex-like*, see [59] or [1, 2].

Lemma 2.38. *Define the set*

$$M := \bigcup_{y \in A} \{y\} + \mathcal{D}(y).$$

(i) If $\bar{y} \in M$ is a nondominated element of M w.r.t. $\mathcal{D}$, then $\bar{y} \in A$ and $\bar{y}$ is also a nondominated element of A w.r.t. $\mathcal{D}$.

(ii) If $\bar{y} \in A$ is a nondominated element of A w.r.t. $\mathcal{D}$, and if

$$\mathcal{D}(y + d) \subset \mathcal{D}(y) \text{ for all } y \in A \text{ and for all } d \in \mathcal{D}(y)\,, \tag{2.8}$$

then $\bar{y}$ is a nondominated element of M w.r.t. $\mathcal{D}$.

Proof. (i) If $\bar{y} \in M \setminus A$ then $\bar{y} \in \{y\} + (\mathcal{D}(y) \setminus \{0_Y\})$ for some $y \in A \subset M$ in contradiction to $\bar{y}$ a nondominated element of M w.r.t. $\mathcal{D}$. Thus $\bar{y} \in A$. Due to $A \subset M$, $\bar{y}$ is then also a nondominated element of A w.r.t. $\mathcal{D}$.

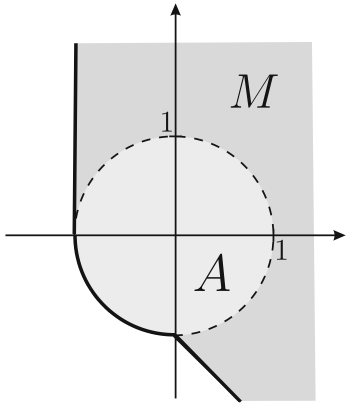
(ii) We assume that $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$ but not of M, i.e. there exist $y \in A$ and $d_y \in \mathcal{D}(y) \setminus \{0_Y\}$ with $\bar{y} \in \{y + d_y\} + (\mathcal{D}(y + d_y) \setminus \{0_Y\})$. As $\mathcal{D}(y)$ is a pointed convex cone this implies

$$\begin{aligned} \bar{y} &\in \{y\} + (\mathcal{D}(y) \setminus \{0_Y\}) + (\mathcal{D}(y + d_y) \setminus \{0_Y\}) \\ &\subset \{y\} + (\mathcal{D}(y) \setminus \{0_Y\}) + (\mathcal{D}(y) \setminus \{0_Y\}) \\ &\subset \{y\} + (\mathcal{D}(y) \setminus \{0_Y\}), \end{aligned}$$

in contradiction to $\bar{y}$ being a nondominated element of A w.r.t. $\mathcal{D}$.

□

Fig. 2.7 Set M of Example 2.39, cf. [46]



If the binary relation $\leq_1$ defined in (1.1) by the ordering map $\mathcal{D}$ is transitive, then condition (2.8) is satisfied, compare Lemma 1.10(ii).

In the following example all assumptions of Lemma 2.38(ii) are fulfilled, but $\mathcal{D}$ is not transitive on Y.

Example 2.39. Let Y, A and $\mathcal{D}$ be defined as in Example 2.10. Then

$$\begin{aligned} M = &\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in [-1, 0], y_2 \geq -\sqrt{1-y_1^2}\} \\ &\cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \geq -1 - y_1\}, \end{aligned}$$

see Fig. 2.7.

The set of nondominated elements of M w.r.t. $\mathcal{D}$ is

$$\left\{ (y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in [-1, 0],\ y_2 = -\sqrt{1-y_1^2} \right\}$$

and thus equals the set of nondominated elements of A w.r.t. $\mathcal{D}$. It is easy to verify that $\mathcal{D}(y+d) \subset \mathcal{D}(y)$ for all $y \in A$ and for all $d \in \mathcal{D}(y)$. However, the relation $\leq_1$ defined by the ordering map $\mathcal{D}$ is not transitive. For instance $(-1,-2) \leq_1 (0,-1)$ and $(0,-1) \leq_1 (2,-3)$ but $(-1,-2) \not\leq_1 (2,-3)$ because of $(2,-3) \notin \{(-1,-2)\} + \mathbb{R}^2_+$.

In general only the cones $\mathcal{D}(y)$ for $y \in A$ are of interest for modeling a decision making problem. Thus we have the freedom of setting $\mathcal{D}(y) := \{0_Y\}$ for all $y \in Y \setminus A$. This allows us to make the assumption (2.8) dispensable for the result in Lemma 2.38(ii):

Lemma 2.40. *Let $\mathcal{D}\colon Y \to 2^Y$ be given with $\mathcal{D}(y) = \{0_Y\}$ for all $y \in Y \setminus A$ and let $M := \bigcup_{y \in A} \{y\} + \mathcal{D}(y)$. Then an element $\bar{y} \in Y$ is a nondominated element of the set A w.r.t. $\mathcal{D}$ if and only if it is a nondominated element of the set M w.r.t. $\mathcal{D}$.*

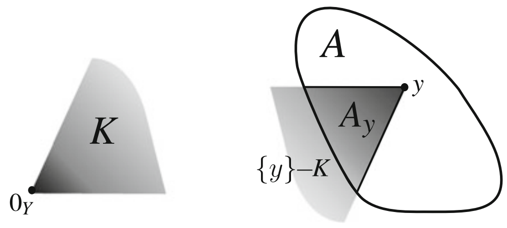


Fig. 2.8 A section $A_y = (\{y\} - K) \cap A$ of some set A

Proof. First assume $\bar{y}$ is a nondominated element of the set A w.r.t. $\mathcal{D}$. If it is not also nondominated of M w.r.t. $\mathcal{D}$ then there exists some $y \in A$ and some $d \in \mathcal{D}(y)$ such that

$$\bar{y} \in \{y + d\} + \mathcal{D}(y + d) \setminus \{0_Y\} \text{ with } y + d \notin A. \tag{2.9}$$

Thus $y + d \in M \setminus A$ and $\mathcal{D}(y + d) = \{0_Y\}$ in contradiction to (2.9). The other implication follows from Lemma 2.38(i). □

For obtaining existence results for optimal elements in a partially ordered space under weak assumptions on the set, so-called sections are considered. If $K \subset Y$ denotes the ordering cone then to any $y \in Y$ with

$$A_y := (\{y\} - K) \cap A \neq \emptyset,$$

the set A_y is denoted a *section*, see Fig. 2.8. It holds that any (weakly) efficient element of a section is also a (weakly) efficient element of the set in the real linear space Y partially ordered by K, cf. [94, Lemma 6.2]. Using Lemma 2.16 we can extend this result to variable ordering structures.

Lemma 2.41. *Let $K := \mathcal{D}(A)$ be pointed and convex and let K thus introduce a partial ordering on Y. Consider for some $y \in Y$ the set $A_y := (\{y\} - K) \cap A$.*

(i) Any efficient element of A_y w.r.t. K is also a nondominated element of A w.r.t. $\mathcal{D}$.
(ii) Let $\operatorname{cor}(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. Then any weakly efficient element of A_y w.r.t. K is also a weakly nondominated element of A w.r.t. $\mathcal{D}$.

Proof. (i) According to [94, Lemma 6.2(a)], any efficient element of A_y is also an efficient element of A and thus by Lemma 2.16(i) also a nondominated element of A w.r.t. $\mathcal{D}$.
(ii) According to [94, Lemma 6.2(b)], any weakly efficient element of A_y is also a weakly efficient element of A and thus by Lemma 2.16(ii) also a weakly nondominated element of A w.r.t. $\mathcal{D}$.

□

The result of Lemma 2.41 is useful for applying existence results for efficient elements as provided in [94, Chap. 6] also to nondominated elements—in case $\mathcal{D}(A)$

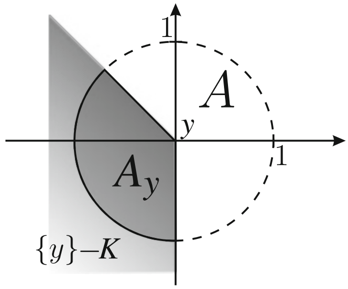


Fig. 2.9 Section A_y of Example 2.43

is pointed and convex. The following lemma directly relates nondominated elements of a section w.r.t. an ordering map $\mathcal{D}$ with those of the set itself.

Lemma 2.42. *Let $K := \mathcal{D}(A)$ be convex and consider for some $y \in Y$ the set $A_y = (\{y\} - K) \cap A$.*

(i) Any nondominated element of A_y w.r.t. $\mathcal{D}$ is also a nondominated element of A w.r.t. $\mathcal{D}$.
(ii) Let $cor(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. Then any weakly nondominated element of A_y w.r.t. $\mathcal{D}$ is also a weakly nondominated element of A w.r.t. $\mathcal{D}$.

Proof. (i) Assume $\bar{y}$ is a nondominated element of A_y w.r.t. $\mathcal{D}$ but there exists some $\hat{y} \in A \setminus A_y$ with $\bar{y} \in \{\hat{y}\} + \mathcal{D}(\hat{y})$. As $\bar{y} \in A_y$,

$$\begin{aligned}\hat{y} &\in \{\bar{y}\} - \mathcal{D}(\hat{y}) \subset \{y\} - \mathcal{D}(A) - \mathcal{D}(\hat{y}) \\ &\subset \{y\} - (\mathcal{D}(A) + \mathcal{D}(A)) \subset \{y\} - \mathcal{D}(A),\end{aligned}$$

in contradiction to $\hat{y} \notin A_y$.
(ii) Analogously to (i).

□

Example 2.43. Let Y, A and $\mathcal{D}$ be defined as in Example 2.10, compare also Example 2.39. Then $\mathcal{D}(A) = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 + z_2 \geq 0,\ z_1 \geq 0\}$ is convex. For instance, the section $A_y = (\{y\} - K) \cap A$, see Fig. 2.9, defined by $y = (0, 0)$ and $K = \mathcal{D}(A)$ contains all nondominated elements of A w.r.t. $\mathcal{D}$ and according to Lemma 2.42 all nondominated elements of A_y w.r.t. $\mathcal{D}$ are also nondominated elements of A w.r.t. $\mathcal{D}$.

We conclude this section by a result about the nondominated elements of the sum of two sets.

Lemma 2.44. *Let $A,\ B \subset Y$ be nonempty subsets and let $\mathcal{D}$ satisfy*

$$\mathcal{D}(y^A) + \mathcal{D}(y^B) \subset \mathcal{D}(y^A + y^B) \qquad \textit{for all } y^A \in A,\ y^B \in B. \tag{2.10}$$

If $\bar{y} = \bar{y}^A + \bar{y}^B \in A + B$ with $\bar{y}^A \in A$, $\bar{y}^B \in B$ is a nondominated element of $A + B$ w.r.t. the ordering map $\mathcal{D}$, then $\bar{y}^A$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ and $\bar{y}^B$ is a nondominated element of B w.r.t. the ordering map $\mathcal{D}$.

Proof. We assume $\bar{y}^A$ is not a nondominated element of A w.r.t. the ordering map $\mathcal{D}$. Then there exists $y \in A$ with $\bar{y}^A \in \{y\} + (\mathcal{D}(y) \setminus \{0_Y\})$ and thus

$$\bar{y} = \bar{y}^A + \bar{y}^B \in \{y + \bar{y}^B\} + (\mathcal{D}(y) \setminus \{0_Y\}).$$

With $0_Y \in \mathcal{D}(\bar{y}^B)$ and because $\mathcal{D}$ satisfies (2.10) we get

$$\mathcal{D}(y) \subset \mathcal{D}(y) + \mathcal{D}(\bar{y}^B) \subset \mathcal{D}(y + \bar{y}^B).$$

Thus

$$\bar{y} \in \{\underbrace{y + \bar{y}^B}_{\in A+B}\} + (\mathcal{D}(y + \bar{y}^B) \setminus \{0_Y\})$$

in contradiction to $\bar{y}$ a nondominated element of $A + B$ w.r.t. the ordering map $\mathcal{D}$. The same for $\bar{y}^B$. $\square$

For a variable ordering structure satisfying (2.10) see the following example:

Example 2.45. Let $\mathcal{D}$ be specified as in Example 2.10. Let arbitrary sets A and B be given with $A, B \subset \mathbb{R}^2_+$. We get for any $y^A \in A$, $y^B \in B$ that $y_1^A + y_1^B \geq y_1^A$ and also $y_1^A + y_1^B \geq y_1^B$. Because $\mathcal{D}(y') \supset \mathcal{D}(y)$ for any $y, y' \in \mathbb{R}^2_+$ with $y_1' \geq y_1$, we conclude, using the convexity of $\mathcal{D}(y)$ for any $y \in \mathbb{R}^2$:

$$\mathcal{D}(y^A) + \mathcal{D}(y^B) \subset \mathcal{D}(y^A + y^B) + \mathcal{D}(y^A + y^B) = \mathcal{D}(y^A + y^B).$$

But note that on $\mathbb{R}^2_+$ the map $\mathcal{D}$ is constant.

If the set-valued map $\mathcal{D}$ is subadditive w.r.t. the cone $\{0_Y\}$ on Y, then condition (2.10) is satisfied. For the definition of subadditivity of a set-valued map see (3.9) in Chap. 3. However, we show in Lemma 3.23 that subadditivity is a too strong assumption for a cone-valued map as it implies that it is a constant-valued map.

The converse statement of Lemma 2.44 does not hold in general even not in the case of a partially ordered space [134, Remark 3.1.3].

2.4 Characterization of Minimal Elements

This section is devoted to basic properties of minimal elements of a set A w.r.t. a variable ordering structure. As before, let Y be a real linear space equipped with a variable ordering structure defined by the ordering map $\mathcal{D}: Y \to 2^Y$ with $\mathcal{D}(y)$ a pointed convex cone for all $y \in Y$. Additionally, let A be a nonempty subset of Y.

First we state that all (weakly) minimal elements are a subset of the algebraic boundary ∂A of the set A.

Lemma 2.46. *(i) Let $cor(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. If $\bar{y} \in A$ is a weakly minimal element of A w.r.t. $\mathcal{D}$, then $\bar{y} \in \partial A$.*
(ii) If $\bar{y} \in A$ is a minimal element of A w.r.t. $\mathcal{D}$ and $\mathcal{D}(\bar{y}) \neq \{0_Y\}$, then $\bar{y} \in \partial A$.

Proof. (i) Follows directly from Lemma 2.15(ii) and the known fact that in partially ordered spaces all weakly efficient elements are a subset of the boundary of the set [42, Theorem 1.13]. However, it can also be easily shown by choosing any $d \in \text{cor}(\mathcal{D}(\bar{y}))$. Then the proof is analogous to the proof of Lemma 2.34(i).
(ii) Follows directly from Lemma 2.15(i) and the known fact that in partially ordered spaces all efficient elements are a subset of the boundary of the set [70, Theorem 2.9].

□

Again, $A_1 \subset A_2$ implies that an element $\bar{y} \in A_1$, which is a (weakly) minimal element of A_2 w.r.t. $\mathcal{D}$ is also a (weakly) minimal element of A_1 w.r.t. $\mathcal{D}$. The following remark relates minimal elements w.r.t. two ordering maps $\mathcal{D}_1$ and $\mathcal{D}_2$.

Remark 2.47. Let $\mathcal{D}_1, \mathcal{D}_2 \colon Y \to 2^Y$ be cone-valued maps with $\mathcal{D}_1(y)$ and $\mathcal{D}_2(y)$ a pointed convex cone for all $y \in Y$. According to the Definitions 2.7 and 2.12, if $\bar{y}$ is a minimal element of the set A w.r.t. the ordering map $\mathcal{D}_2$ and $\mathcal{D}_1(\bar{y}) \subset \mathcal{D}_2(\bar{y})$, then $\bar{y}$ is also a minimal element of the set A w.r.t. the ordering map $\mathcal{D}_1$. If $\text{cor}(\mathcal{D}_1(\bar{y})) \neq \emptyset$, then this also holds if we replace minimal by weakly minimal.

We can relate the minimal elements of the set A with the minimal elements of the set $M = \bigcup_{y \in A} \{y\} + \mathcal{D}(y)$ only under strong assumptions on the variable ordering.

Lemma 2.48. *Define the set*

$$M := \bigcup_{y \in A} \{y\} + \mathcal{D}(y).$$

(i) If $\bar{y} \in A$ is a minimal element of M w.r.t. $\mathcal{D}$, then it is also a minimal element of A w.r.t. $\mathcal{D}$.
(ii) If $\bar{y} \in A$ is a minimal element of A w.r.t. $\mathcal{D}$ and if $\mathcal{D}(y) \subset \mathcal{D}(\bar{y})$ for all $y \in A$, then $\bar{y}$ is also a minimal element of M w.r.t $\mathcal{D}$.

Proof. (i) The assertion follows from $A \subset M$, compare the proof of Lemma 2.38(i).
(ii) We assume that $\bar{y}$ is a minimal element of A but not of M w.r.t. $\mathcal{D}$, i.e. there exist $y \in A$ and $d_y \in \mathcal{D}(y) \setminus \{0_Y\}$ with $y + d_y \in \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\})$. As $\mathcal{D}(\bar{y})$ is a pointed convex cone, this implies

$$\begin{aligned} y &\in \{\bar{y}\} - (\mathcal{D}(y) \setminus \{0_Y\}) - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}) \\ &\subset \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}) - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}) \\ &\subset \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}), \end{aligned}$$

in contradiction to $\bar{y}$ being a minimal element of A w.r.t. $\mathcal{D}$.

□

The transitivity of the binary relation $\leq_2$ defined in (1.2) by the ordering map $\mathcal{D}$ is not sufficient for any minimal element of A being also a minimal element of M w.r.t. $\mathcal{D}$, in contrast to the result for nondominated elements. This is illustrated in the following example.

Example 2.49. Let Y be the Euclidean space $\mathbb{R}^2$, $A = \{(1,0),(1,1)\}$ and let the ordering map $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathcal{D}(y_1, y_2) := \begin{cases} \operatorname{cone}\{(-1,1)\} & \text{if } (y_1,y_2) \in \{(1,0) + \lambda(1,-1) \mid \lambda \geq 0\}, \\ \operatorname{cone}\{(1,1)\} & \text{if } (y_1,y_2) \in \{(1,1) + \lambda(1,1) \mid \lambda \in \mathbb{R}\}, \\ \{0_Y\} & \text{otherwise.} \end{cases}$$

Then it is easy to verify that the relation $\leq_2$ is transitive according to Lemma 1.10(iii). The point $(1,1)$ is a minimal element of A w.r.t. $\mathcal{D}$, but the set M as defined in Lemma 2.48 equals

$$M = \{(1,0) + \lambda(-1,1) \mid \lambda \geq 0\} \cup \{(1,1) + \lambda(1,1) \mid \lambda \geq 0\},$$

and thus $(1/2, 1/2) \in M \cap (\{(1,1)\} - \mathcal{D}((1,1)))$. So $(1,1)$ is not a minimal element of M w.r.t. $\mathcal{D}$.

Note that in Lemma 2.48(i) above we have assumed that the minimal element $\bar{y}$ of M is also an element of A. Under additional assumptions we can show that any minimal element $\bar{y}$ of M is always an element of the set A.

Lemma 2.50. *Define M as in Lemma 2.48. If $\bar{y} \in M$ is a minimal element of M w.r.t. $\mathcal{D}$ and if*

$$d \in \mathcal{D}(y + d) \text{ for all } y \in A \text{ and for all } d \in \mathcal{D}(y), \tag{2.11}$$

then $\bar{y}$ is also a minimal element of A w.r.t. $\mathcal{D}$.

Proof. With Lemma 2.48 it remains to be shown that $\bar{y} \in A$. For this, we assume that $\bar{y} \in M \setminus A$, i.e. there exist $\bar{a} \in A$ and $\bar{d}_a \in \mathcal{D}(\bar{a}) \setminus \{0_Y\}$ with $\bar{y} = \bar{a} + \bar{d}_a$. Then $\bar{d}_a \in \mathcal{D}(\bar{y})$ and for $y := \bar{a} + \frac{1}{2}\bar{d}_a \in M \setminus \{\bar{y}\}$ we obtain

$$y = \bar{y} - \frac{1}{2}\bar{d}_a \in \{\bar{y}\} - (\mathcal{D}(\bar{y}) \setminus \{0_Y\}),$$

in contradiction to the minimality of $\bar{y}$ for the set M w.r.t. $\mathcal{D}$. □

The condition (2.11) is satisfied if

$$\mathcal{D}(y+d) \supset \mathcal{D}(y) \quad \text{for all } \; y \in A \; \text{ and for all } \; d \in \mathcal{D}(y). \tag{2.12}$$

The ordering map presented in Example 1.12 satisfies also condition (2.12) and thus condition (2.11). We reconsider Example 2.10 where the ordering map $\mathcal{D}$ does not satisfy (2.11).

Example 2.51. Let Y, A and $\mathcal{D}$ be specified as in Example 2.10. For this set and ordering map we have already presented the set M, which is illustrated in Fig. 2.7, in Example 2.39. The set of minimal elements of M w.r.t. $\mathcal{D}$ is

$$\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in [-1, 0), y_2 = -\sqrt{1 - y_1^2}\}$$
$$\cup\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > 0, y_2 = -1 - y_1\}.$$

Hence the set of minimal elements of M w.r.t. $\mathcal{D}$ intersected with A equals the set of minimal elements of A w.r.t. $\mathcal{D}$. But for instance $(1, -2)$ is a minimal element of M w.r.t. $\mathcal{D}$ but not of A.

The condition (2.11) does not hold: for $y = (0, -1)$, $d = (1, -1) \in \mathcal{D}(y)$ we obtain

$$d \notin \mathcal{D}(y+d) = \mathcal{D}((1, -2)) = \mathbb{R}^2_+.$$

Analogously to Lemma 2.42, we can relate minimal elements of a section of A w.r.t. $\mathcal{D}$ to minimal element of the set A w.r.t. $\mathcal{D}$.

Lemma 2.52. *Let $K \subset Y$ be a convex cone and consider for some $y \in Y$ the set $A_y = (\{y\} - K) \cap A$.*

(i) If $\bar{y}$ is a minimal element of A_y w.r.t. $\mathcal{D}$ and $\mathcal{D}(\bar{y}) \subset K$, then $\bar{y}$ is also a minimal element of A w.r.t. $\mathcal{D}$.
(ii) Let $cor(\mathcal{D}(y)) \neq \emptyset$ for all $y \in A$. If $\bar{y}$ is a weakly minimal element of A_y w.r.t. $\mathcal{D}$ and $\mathcal{D}(\bar{y}) \subset K$, then $\bar{y}$ is also a weakly minimal element of A w.r.t. $\mathcal{D}$.

Proof. (i) $\bar{y}$ minimal for A_y w.r.t. $\mathcal{D}$ is equivalent to $(\{\bar{y}\} - \mathcal{D}(\bar{y})) \cap A_y = \{\bar{y}\}$, thus to

$$(\{\bar{y}\} - \mathcal{D}(\bar{y})) \cap A \cap (\{y\} - K) = \{\bar{y}\}.$$

As $\{\bar{y}\} - \mathcal{D}(\bar{y}) \subset \{y\} - K - \mathcal{D}(\bar{y}) \subset \{y\} - K$ this implies $(\{\bar{y}\} - \mathcal{D}(\bar{y})) \cap A = \{\bar{y}\}$.

(ii) $\bar{y}$ weakly minimal for A_y w.r.t. $\mathcal{D}$ is equivalent to $(\{\bar{y}\} - \mathrm{cor}(\mathcal{D}(\bar{y}))) \cap A_y = \emptyset$, thus to

$$(\{\bar{y}\} - \mathrm{cor}(\mathcal{D}(\bar{y}))) \cap A \cap (\{y\} - K) = \emptyset.$$

As $\{\bar{y}\} - \operatorname{cor}(\mathcal{D}(\bar{y})) \subset \{y\} - K - \operatorname{cor}(\mathcal{D}(\bar{y})) \subset \{y\} - K$ this implies $(\{\bar{y}\} - \operatorname{cor}(\mathcal{D}(\bar{y}))) \cap A = \emptyset$. □

Choosing $K = \mathcal{D}(A)$ (assuming $\mathcal{D}(A)$ to be convex) in the above lemma, then any (weakly) minimal element of the section w.r.t. $\mathcal{D}$ is also a (weakly) minimal element of A w.r.t. $\mathcal{D}$.

We conclude this section by a result about the minimal elements of the sum of two sets.

Lemma 2.53. *Let $A, B \subset Y$ be nonempty subsets and let $\mathcal{D}$ satisfy (2.10). If $\bar{y} = \bar{y}^A + \bar{y}^B \in A + B$ with $\bar{y}^A \in A$, $\bar{y}^B \in B$ is a minimal element of $A + B$ w.r.t. the ordering map $\mathcal{D}$, then $\bar{y}^A$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ and $\bar{y}^B$ is a minimal element of B w.r.t. the ordering map $\mathcal{D}$.*

Proof. We assume $\bar{y}^A$ is not a minimal element of A w.r.t. the ordering map $\mathcal{D}$. Then there exists $y \in A$ with $\bar{y}^A \in \{y\} + (\mathcal{D}(\bar{y}^A) \setminus \{0_Y\})$ and thus

$$\bar{y} = \bar{y}^A + \bar{y}^B \in \{y + \bar{y}^B\} + (\mathcal{D}(\bar{y}^A) \setminus \{0_Y\}).$$

With (2.10) we conclude $\mathcal{D}(\bar{y}^A) \subset \mathcal{D}(\bar{y})$ and thus

$$\bar{y} \in \{\underbrace{y + \bar{y}^B}_{\in A+B}\} + (\mathcal{D}(\bar{y}) \setminus \{0_Y\})$$

in contradiction to $\bar{y}$ a minimal element of $A + B$ w.r.t. the ordering map $\mathcal{D}$. The same for $\bar{y}^B$. □

For a variable ordering structure satisfying (2.10) see Example 2.45. The converse statement of Lemma 2.53 does in general not hold even not in the case of a partially ordered space [134, Remark 3.1.3].

2.5 Notes on the Literature

The definition of a nondominated element was first given by Yu in 1974 [158], see also the work [19] by him and his colleagues Bergstresser and Charnes from 1976 and his book [159]. It is also shortly mentioned in the book by Sawaragi et al. [134], by Chew in [32] and by Weidner in [149] as well as in a more recent survey about advances in preference modeling by Wiecek [154]. Note that the original definition by Yu is incorrectly cited in [30, p. 98], [29, Definition 1.11]. There, (2.5) is replaced by $A \cap (\{\bar{y}\} - \mathcal{D}(y)) = \{\bar{y}\}$ for all $y \in A$. In the definition of the nondominated elements, the cone

$$\mathcal{D}(y) = \{d \in Y \mid y + d \text{ is dominated by } y\} \cup \{0_Y\},$$

is also called domination cone or the set of dominated directions or domination factors for each element $y \in Y$, cf. [19, 154, 159]. In the definition of the minimal elements, the cone

$$\mathcal{D}(y) = \{d \in Y \mid y - d \text{ is preferred to } y\} \cup \{0_Y\}$$

can be interpreted as the cone of preferred directions, compare page 10.

The concept of minimal elements, denoted there as nondominated-like elements, was introduced by Chen in 1992 [28], see also [29, 30]. This notion is also used by Huang et al. [87] and by Li and Li [109]. The minimal elements are denoted as nondominated elements w.r.t. $\mathcal{D}$ by Engau in [57, 58] and as efficient elements by Xiao et al. in [155]. In addition to the Examples 2.9 and 2.10, further examples comparing the concepts of minimal and nondominated elements w.r.t. a variable ordering structure are given by Chen [30], Eichfelder and Ha [52] and Gebhardt [61].

The definition of an efficient element in a partially ordered space can be found—under different names—in many books on vector optimization, see for instance the books by Jahn [94] or by Eichfelder [42], where these optimal elements are denoted as minimal elements, or the books by Göpfert and Nehse [70], Luc [113] and Sawaragi et al. [134], where the name efficient elements is used. For a survey about the different names used by different authors see the survey given by Ehrgott in [39, Table 2.4].

The concept of an efficient solution of a multiobjective optimization problem was probably first introduced in the applied sciences by Edgeworth in 1881 [38] and Pareto in 1906 [127]. Therefore, efficient elements are also called *Edgeworth-Pareto optimal points*. This naming was first suggested by Stadler [137]. For a brief historical sketch of the early works of Edgeworth and Pareto we refer to the survey in [53]. Condition (2.2) is also used to define optimal elements for an arbitrary set K with $0_Y \in K$, see Bergstresser et al. [19] and Weidner [153], or even with $0_Y \notin K$, see Weidner [150]. For a literature survey about the usage of the concept given in (2.2) for K as a set, a cone or the nonnegative orthant we refer to Engau [58].

The definitions of weaker and stronger optimality notions in partially ordered spaces can also be found in most introductory books on vector optimization. Ha provides in [77, Definition 21.3] an extensive collection of the different weaker and stronger concepts known in the literature. A survey about the various notions of proper efficiency and results on the contingent cone are given for instance in [94]. The definition of proper efficiency according to Henig is (of course) by Henig [83]. The definition according to Benson is from [16] while the definition according to Borwein is from [23]. For instance in [134] several examples are provided for showing the differences between the various proper optimality notions. For studies on the relations between the different notions of proper optimality in a partially ordered space we refer to [75, 111].

The concepts of weakly nondominated, weakly minimal and strongly minimal elements, some of the provided examples as well as some of the characterizations were first provided in [46], for some results see also [47] and [49]. The notion of strongly nondominated elements was introduced in [52]. The various discussed concepts of properly minimal and properly nondominated elements were given in [54]. There, one can also find some basic results on relations between the different notions. A more detailed study is provided in [55].

In [140, 141], Soleimani and Tammer extend notions of approximate solutions from vector optimization with a partially ordered linear space to problems with variable ordering structures. Thereby, they do not assume the sets $\mathcal{D}(y)$ to be convex or cones but closed sets containing the zero element. The definitions given here can also be used under these assumptions

In [155], Xiao, Xiao and Liu define strongly minimal solutions, denoted there as strong efficient solutions, of a vector optimization problem with a vector-valued objective map $f: X \to Y$ in a different way than proposed here: they denote by that minimal solutions $\bar{x}$ with the additional property that there is no other feasible x with $f(\bar{x}) = f(x)$. Note that for instance in [155] the variable ordering structure is not defined on Y but on X by a cone-valued map $\mathcal{C}: X \to 2^Y$. Of course, a direct relation exists to the ordering map as presented here: If $\mathcal{D}$ is given, then we can set $\mathcal{C}(x) := \mathcal{D}(f(x))$ for all x.

For defining an optimal solution of the set-valued vector optimization problem we have chosen here the so-called vector approach, see for instance [94, Chap. 17] or [53, Sect. 1.4.1]. Also another approach exists, denoted set-approach, which is based on binary set-relations in the power set of the space. For instance, there is a binary relation known as set less or KNY order relation which has been independently introduced by Young [157] and Nishnianidze [124] and has been presented by Kuroiwa [106] in a slightly modified form. Recently, several new binary relations for defining solutions to set-valued optimization problems have been proposed by Jahn and Ha [95].

Lemma 2.17 was already given by Yu in [158, Lemma 5.1(iv)]. Results on the set $A+K$ for some set $A \subset Y$ and an ordering cone $K \subset Y$ in a partially ordered linear space Y as well as on sections of sets are collected for instance in the book by Jahn [94]. Examinations for A convex-like, i.e. $A+K$ convex, as well as for A satisfying other concepts of generalized convexity, are given for instance by Adán and Novo in [1,2]. If A is not a convex set but $A+K$ is, for instance necessary linear scalarization results can be formulated which cannot be derived directly for A. The notion of external stability is taken from the book by Sawaragi et al. [134]. This notion is also denoted domination property, cf. the book by Luc [113] and the references therein. In this book and also in [134] several sufficient conditions ensuring the external stability of a set of efficient or nondominated elements are given. For the special case $Y = \mathbb{R}^m$, $\mathbb{R}^m_+ \subset K$ and compact sets A, the results of Theorems 2.20 and 2.22 were already given by Hirsch et al. in [86]. There, these results were used to solve vector optimization problems with a variable ordering structure by applying multiobjective evolutionary algorithms, cf. Sect. 9.3. The results in the way they are given here are due to [49].

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