Preface

In mathematical optimization one aims at minimizing or maximizing an objective function over some feasible set. In case the objective function is scalar-valued it is straightforward how to define an optimal solution: a feasible element with the smallest (or largest) scalar as objective function value is an optimal solution.

However, in case the objective function is vector-valued, i.e. it maps in a real linear space, it is not obvious anymore how to compare the values of the objective function. In the applied sciences Edgeworth [38] and Pareto [127] were probably the first who introduced an optimality concept for vector optimization problems, i.e., for optimization problems with such a vector-valued objective function, cf. [53, 113]. Both have studied the so-called *multiobjective optimization* problems which are vector optimization problems with an objective function mapping in the m-dimensional Euclidean space $\mathbb{R}^m$ for $m \geq 2$. For comparing elements in $\mathbb{R}^m$ they used the natural ordering cone, the nonnegative orthant, which corresponds to the componentwise partial ordering. Based on that, in vector optimization it is often assumed that the linear space is partially ordered by a convex cone and an element is an optimal element (referred to as an efficient element) of a set if it is not dominated by (i.e., worse than) any other reference element w.r.t. the partial ordering of the linear space.

But already in the first publications in mathematics in the 1970s related to the definition of optimal elements in vector optimization [19, 158] also the idea of variable ordering structures was given: it was assumed that there is a set-valued map with cone values that associates with each element of the linear space and ordering. A candidate element is called a nondominated element if it is not dominated by other reference elements w.r.t. the corresponding ordering of these other ones. In addition to the notion of nondominated elements, in [28–30] they also consider another notion of optimal elements in case of a variable ordering structure. Namely, a candidate element is called a minimal element (also called nondominated-like) if it is not dominated by any other reference element w.r.t. the ordering of the candidate element.

This book aims at providing an introduction to vector optimization with a variable ordering. Application problems are presented which motivate the study of these optimization problems. Next to a comprehensive basic theory, numerical approaches are discussed which allow to solve such problems in practice. Throughout this book we assume that the variable ordering structure is defined in the objective space by a set-valued map which associates with each element of the space a cone of preferred or dominated directions.

The recent interest in vector optimization problems with a variable ordering structure started some years ago with an application problem in image registration in medical engineering [146]: there it is the aim to merge several medical images gained by different imaging methods as, for instance, computer tomography, magnetic resonance tomography, positron emission tomography, or ultrasound. One searches for a best transformation map, also called registration. To measure the quality of such a transformation, a multitude of similarity measures is known which all possess different properties and advantages. The values of the different measures can be interpreted as objectives which have to be minimized all at the same time.

However, depending on the objective values, it is advantageous to put a higher weight on some of these objectives than on others. To each element in the objective space one can thus relate a weight vector and consider a related cone which depends on the weight vector and hence on the considered element. Then one searches for an optimal solution of a vector optimization problem with a variable ordering structure. We discuss this application in more detail in Sect. 1.3.1.

Already in [99] Karaskal and Michalowski recognized that the importance of criteria may change during the decision-making process and that it may depend on the current objective function values. Wiecek gives in [154] an example for that. In [9] Baatar and Wiecek examine the concept of equitability, which has applications in portfolio optimization and location problems. They show that this minimality notion is related to a finite number of ordering cones or polyhedral sets instead of a unique ordering cone. The objective space is partitioned into sections and the related ordering is variable and depends on the section in which an element lies. More details are given in Sect. 1.3.2.

Engau examines in [58] the role of variable ordering structures in preference modeling. He gives examples showing the limitations of preference modeling using only one ordering cone. Variable structures defined by convex cones containing the nonnegative orthant and being some kind of symmetric are studied. These convex cones are special Bishop-Phelps cones (BP cones) and BP cones play an important role for some scalarization results in Chap. 6.

We start in Chap. 1 by recalling basic concepts of partially ordered spaces and basic results on ordering cones and dual cones. Then we introduce variable ordering structures. We assume that the variable ordering structure is defined by a set-valued map called ordering map. Two binary relations defined by such an ordering map are introduced and their properties are examined. We also focus on special ordering maps where the images are BP cones because such ordering maps are important in applications and also for theoretical results. Moreover, we discuss the applications mentioned above which illustrate that partial orderings are not always an adequate tool for modeling real-world problems.

In Chap. 2 we collect several optimality notions based on the two binary relations introduced in Chap. 1. We compare these new concepts with known concepts in partially ordered spaces, and we show that the main two different optimality concepts w.r.t. a variable ordering structure, the minimal and the nondominated elements, are in general not related in the sense that one does not imply the other. We continue the chapter by providing first results on characterizations of the optimal elements.

The variable ordering structures are defined by ordering maps which are cone-valued maps. For that reason we study in Chap. 3 cone-valued maps. We examine classical properties, formerly introduced for arbitrary set-valued maps, like convexity, cone-convexity, linearity, or monotonicity. It turns out that some of these properties like convexity directly imply that the cone-valued map is constant. This is important to know for the study of scalarization functionals. Convexity of the ordering map would imply convexity of the scalarization functionals studied in Chap. 5 but is thus a too strong assumption. In case of non-appropriateness of the classical notions we propose new concepts.

In Chaps. 4–6 linear and nonlinear scalarization functionals are proposed and their properties are studied. With these functionals at hand a vector optimization problem can be replaced by a scalar-valued optimization problem which allows, for instance, the formulation of optimality conditions of Fermat and Lagrange type or can be used as the base of numerical solution methods. In Chap. 4 we start with linear scalarization functionals which turn out to be appropriate in case of convexity of the considered set only. But also under convexity assumptions the necessary conditions for nondominated elements are in general too weak and the sufficient conditions are too strong, and a complete characterization of the optimal elements is not possible.

For that reason we discuss nonlinear scalarization functionals in Chap. 5 which allow a complete characterization of nondominated and minimal elements. We consider a modification of the so-called signed distance functional which was introduced by Hiriart-Urruty, and of a second functional called translative functional, which generalizes a functional known in the literature as Gerstewitz or Tammer-Weidner functional or Pascoletti-Serafini scalarization. For both scalarization functionals it can in general not be assumed that they are convex in case of a variable ordering structure. However, convexity is required for the formulation of sufficient optimality conditions of Fermat and Lagrange type for the vector optimization problems.

This leads to Chap. 6 in which we concentrate on variable ordering structures which are defined by ordering maps with images being BP cones. This additional structure allows to introduce a new scalarization functional which is also new in partially ordered spaces. This functional allows a complete characterization of the nondominated and also the minimal elements and, what is more, is convex at least under strong assumptions.

We provide in Chap. 7 subdifferential information for the scalarization functionals introduced in Chap. 6. Then we are able to formulate necessary and sufficient optimality conditions of Fermat and Lagrange type for unconstrained and constrained vector optimization problems with (set-valued) objective maps mapping in a real linear space equipped with a variable ordering structure. For defining optimal solutions of an optimization problem with a set-valued objective map we choose here the vector approach, i.e., optimal solutions are defined as pre-images of optimal elements of the image set of the feasible set under the objective map.

These new scalarization functionals based on the structure of BP cones are also used in Chap. 8 for obtaining duality results, i.e., for defining a dual set to the original vector optimization problem with the optimal elements of the dual set being related to the optimal solutions of the original problem. We provide in Chap. 8 also duality results based on linear scalarizations as well as results concerning general duality for a primal and a dual set. It is interesting to see that the two optimality concepts, the nondominated and the minimal elements, which are in general not related in the sense that one does not imply the other, are related by duality results.

Chapter 9 gives a survey on numerical approaches for solving vector optimization problems with a variable ordering structure. We provide algorithms for solving finite discrete as well as continuous vector optimization problems without any significant restrictions.

In the final chapter we give a short outlook on the appearance of variable ordering structures in vector variational inequalities, vector complementarity, and equilibrium problems. We show that also the theory of consumer demand in economics is related to variable ordering structures. Finally we discuss an application in the treatment planning in intensity-modulated radiation therapy which shows that a cone-valued ordering map might in some applications be an adequate concept only locally which gives rise to future examinations.

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