The Neoclassical Growth Model Under a Constant Savings Rate

2

2.1 Introduction

We present in this chapter the first growth model, introduced almost simultaneously by R. Solow and S. Swan in two different papers published in 1956. In fact, as we will see, the assumptions embedded in this model imply that, in the long run, and in the absence of technological growth, economies do not grow in per-capita terms. The possibility of aggregate growth arises only from either population growth or growth in factor productivity. Since neither factor is supposed to depend on the decisions of economic agents, this is known as an *exogenous growth* model. There are model economies for which there are steady-states with constant, non-zero growth rates determined by some decisions made by economic agents, like the level of education, or by some policy choices, like a given tax rate. These are known as *endogenous growth models* and will be studied in later chapters.

Per capita income, the most obvious indicator of the state of a given economy, displays two different characteristics in most developed countries: (a) it increases over time, and (b) it experiences cyclical fluctuations around its long-term trend over relatively short periods of time. The Solow–Swan model focuses on explaining the first characteristic, long-term growth, even though, as we have already mentioned, the long-run equilibrium growth rate will be zero unless some conditions are met. Even in versions of the Solow–Swan model implying zero long-run growth, the economy will experience non-zero rates of change in the capital stock per worker or in the level of per-capita income over short periods of time, called *transition* periods. To characterize general conditions under which an economy may display non-zero long-term growth is the goal of the next section.

A stochastic version of growth models is needed if we want the model to reproduce the statistical characteristics of business cyclical fluctuations in actual economies. We will also consider a stochastic version of the Solow–Swan growth model, even though this will still be too simple a model to explain many interesting empirical observations.

A. Novales et al., *Economic Growth*, Springer Texts in Business and Economics,
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2.2 Returns to Scale and Sustained Growth

We start by discussing an important fact: the returns displayed by productive factors in the available aggregate technology will condition the possibilities for the economy to display sustained long-run growth. This initial discussion is of a general nature, although it is made under a set of assumptions defining the Solow–Swan model, to which it applies as a special case.

Assumption 1: The relationship between total output Y_t, and the two production inputs, the stock of physical capital K_t, and labour L_t, at the aggregate level of the economy, can be interpreted as coming from a Cobb–Douglas technology,

$$Y_t = AK_t^{\beta} L_t^{\alpha}, \alpha, \beta \geq 0,$$

with unrestricted numerical values for the elasticities of the production factors, except that they must be non-negative. A denotes a production scale factor, which affects the productivity of both factors. Changes in A will shift the production frontier. Physical capital tends to accumulate over time through investment. Gross investment I_t has two components: (a) net investment, defined as the variation in the stock of capital, $\dot{K}_t$, and (b) the loss by depreciation D_t:

$$\textit{Gross Investment} \equiv I_t = \dot{K}_t + D_t. \tag{2.1}$$

In the absence of depreciation, the change in capital would be equal to investment. Under positive depreciation, net investment may be positive, or negative, when investment is not enough to replace the loss by depreciation.

Assumption 2: The rate of depreciation of physical capital is constant, δ, so that: $D_t = \delta K_t$.

Assumption 3: Each worker has a unit of time available each period that is supplied inelastically in the labor market. This allows us to identify the number of workers and the supply of labor each period.

Assumption 4: We assume that there is full employment in the economy, so that employment, L_t, and labor supply, N_t, coincide. These first two assumptions allow us to use in what follows total population, N_t, as an input in the production function and write the technology in terms of per capita variables or per-worker variables,

$$\frac{Y_t}{N_t} = A\left(\frac{K_t}{N_t}\right)^{\beta} N_t^{\alpha+\beta-1}, \;\; \alpha, \beta \geq 0 \;\Rightarrow\; y_t = Ak_t^{\beta} N_t^{\alpha+\beta-1}, \tag{2.2}$$

where $y_t = \frac{Y_t}{N_t}, k_t = \frac{K_t}{N_t}$ denote per capita income and physical capital. As we will see, the capital–labor ratio k_t is the key variable determining the evolution over time of this economy.

Assumption 5: There is no government in the economy, which is supposed to be closed to financial or commodity trading with other countries, which implies that aggregate savings and investment are equal to each other every period, $S_t = I_t, \forall t$.

Assumption 6: Additionally, and this is a significant restriction, we assume savings to evolve over time as a constant fraction s of output,

$$Savings \equiv S_t = sY_t.$$

Using Assumptions 5 and 6 in (2.1) and dividing by N_t, and using (2.2), we have,

$$sy_t = \frac{\dot{K}_t}{N_t} + \delta k_t = sAk_t^{\beta} N_t^{\alpha+\beta-1}. \tag{2.3}$$

Assumption 7: We assume that labor force and employment (which are equal to each other at each point in time, by Assumption 3) grow at a constant rate of n,

$$N_t = N_0 e^{nt}.$$

We can now use these assumptions to obtain some properties of Growth models. Taking derivatives with respect to time in the definition of k_t, we have,

$$\dot{k}_t = \frac{\dot{K}_t}{N_t} - \frac{\dot{N}_t K_t}{N_t^2} = \frac{\dot{K}_t}{N_t} - nk_t. \tag{2.4}$$

From Eqs. (2.3) and (2.4), we get,

$$\dot{k}_t = sAk_t^{\beta} N_t^{\alpha+\beta-1} - (n + \delta)\, k_t,$$

and, dividing by k_t we obtain the growth rate of the per-worker stock of physical capital, γ_{k_t}:

$$\gamma_{k_t} \equiv \frac{\dot{k}_t}{k_t} = sAk_t^{\beta-1} N_t^{\alpha+\beta-1} - (n + \delta)\,, \tag{2.5}$$

which will change over time with population and with the level of the capital–labor ratio. We also have,

$$\frac{\gamma_{k_t} + (n + \delta)}{sA} = k_t^{\beta-1} N_t^{\alpha+\beta-1}.$$

Taking logs, we get,

$$\ln\left(\frac{\gamma_{k_t} + (n + \delta)}{sA}\right) = (\beta - 1)\ln k_t + (\alpha + \beta - 1)\ln N_t, \tag{2.6}$$

and taking derivatives with respect to time t, we have,

$$\frac{\dot{\gamma}_{k_t}}{\gamma_{k_t} + (n+\delta)} = (\beta - 1)\,\frac{\dot{k}_t}{k_t} + (\alpha + \beta - 1)\,n, \tag{2.7}$$

where we have used Assumption 7 to imply: $\frac{\dot{N}_t}{N_t} = n$.

We are particularly interested in characterizing a possible state of the economy in which the growth rate of per capita variables[1] can be maintained constant forever. In such a situation, which we will later define more precisely as *steady-state*, the left hand side at (2.6) would be constant. Notice that it is not the levels, but the growth rates of variables like k_t and y_t, that remain constant in steady-state. We will denote them by $\gamma_{k_{ss}}, \gamma_{y_{ss}}$.

Evaluating (2.7) at such steady-state, we get,

$$0 = (\beta - 1)\gamma_{k_{ss}} + (\alpha + \beta - 1)\,n, \tag{2.8}$$

a condition that any possible steady-state will have to fulfill. It is important to bear in mind that at this point we have not shown existence of such a steady-state, and much less its possible uniqueness. We have only shown that (2.8) is a necessary condition for a steady-state to exist.

We now take logs in (2.3), an expression which is valid at any point in time, to get,

$$\ln s + \ln y_t = \ln(sA) + \beta \ln k_t + (\alpha + \beta - 1)\ln N_t,$$

where $\ln s$ is constant. That taking derivatives with respect to time,

$$\frac{\dot{y}_t}{y_t} = \beta \frac{\dot{k}_t}{k_t} + (\alpha + \beta - 1)\,n \quad \Rightarrow \quad \gamma_{y_t} = \beta\gamma_{k_t} + (\alpha + \beta - 1)\,n,$$

so that, in steady-state,

$$\gamma_{y_{ss}} = \beta\gamma_{k_{ss}} + (\alpha + \beta - 1)\,n, \tag{2.9}$$

which describes the relationship between the growth rates of per capita income and physical capital in a steady-state.

To obtain the relationship with the rate of growth of consumption, we use the global constraint of resources of the economy to show the proportionality between

[1] In fact, a steady-state is defined by constant rates of growth of appropriately chosen ratios of variables. In this introductory discussion, it is convenient to define it in terms of per capita variables, although in a later section of this same chapter we need to define it differently.

per capita consumption and output:

$$C_t + S_t = Y_t \Rightarrow C_t + sY_t = Y_t \Rightarrow C_t/N_t = (1-s)Y_t/N_t \Rightarrow c_t = (1-s)y_t,$$

which implies that both variable grow at the same rate: $\gamma_{c_{ss}} = \gamma_{y_{ss}}$.

Let us now consider some possibilities:

Case 1: Economy with decreasing returns to scale in each production factor, but constant returns to scale on the aggregate,

$$Y_t = AK_t^{\beta} L_t^{\alpha}, \ \ 0 < \alpha, \beta < 1, \ \ \alpha + \beta = 1,$$

In this case, the second term at (2.8) is zero, so that,

$$0 = (\beta - 1)\gamma_{k_{ss}},$$

and since $\beta < 1$, we will necessarily have,

$$\gamma_{k_{ss}} = 0.$$

Hence, if there is any steady state, it will necessarily have to display a zero growth rate for the stock of physical capital per worker. As a consequence of the previous relationships between growth rates, in such an economy all per-capita variables will remain constant in such a steady-state. The constant returns to scale assumption, together with $\gamma_{k_{ss}} = 0$, imply in (2.9) that per-capita income does not grow in steady-state, i.e., $\gamma_{y_{ss}} = 0$ and, as a consequence, $\gamma_{c_{ss}} = 0$. Even though the steady-state condition only allows for a zero steady-state growth rate, that could still be obtained for different levels of per capita variables (k_{ss}, c_{ss}, y_{ss}), leading to multiple steady-states.

Figure 2.1 shows the values of the growth rate of the capital–labor ratio, by illustrating the two functions involved in (2.5). The gap between the two curves provides the growth rate of the capital–labor ratio, which will be positive to the left of the **crossing point**, k_{ss}, and negative to the right of it. That intersection characterizes the steady-state level of the capital–labor ratio. A monotonically decreasing marginal productivity of capital implies uniqueness of that steady-state ratio. To the left of k_{ss} the k_t-ratio will increase, with growth being higher the farther away to the left is the level of k_t. Something similar can be said about the decrease in k_t to the right of k_{ss}.

In fact, this graph shows the *existence and uniqueness* of a zero-growth steady-state in an economy with the assumptions described above. It is particularly important that we have assumed a constant returns to scale production technology together with diminishing returns on the cumulative input, the stock of capital. The graph also illustrates the *stability* of such steady-state, since the economy will converge to it from any position above or below the steady-state capital–labor ratio.

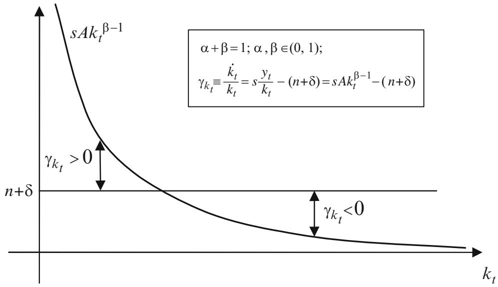

Fig. 2.1 Growth rate of capital–labor ratio: Cobb–Douglas technology with constant returns to scale

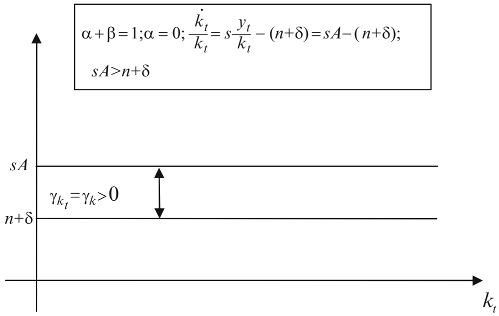

Fig. 2.2 Growth rate of capital–labor ratio: unit elasticity in cumulative factor

This analysis might suggest that it is not possible to obtain positive growth in steady-state if the technology is of the constant returns to scale type. The next case shows that the opposite is true.

Case 2: Let us now consider constant returns to scale in the aggregate, $\alpha+\beta=1$, as well as in the cumulative factor, physical capital, $\beta=1$. We then have $\alpha=0$, and a linear technology,

$$Y_t = AK_t,$$

usually known as an AK-technology, which will be studied in detail in Chap. 5. The second term in (2.8) again becomes zero but, since $\beta=1$, it is possible to find steady-state situations with $\gamma_{k_{ss}}>0$ (actually, with $\gamma_{k_{ss}}\neq 0$) as it can be seen in Fig. 2.2. Notice again that this argument does not show existence of a non-zero growth steady-state, but only that such a state is possible.

As we will see in later chapters, a linear technology like this one can generate *endogenous growth*. A possible interpretation of this structural feature comes by considering a second cumulative productive factor, human capital,

$$Y_t = AK_t^{\beta}H_t^{1-\beta},$$

where H_t is a variable including the quality as well as the quantity of labor, i.e., not only the number of workers, but their education level, work experience, and so on. If the two types of capital are assumed to be perfect substitutes, then we would end up with an AK-technology.[2]

This second case has not considered labor as a second input different from physical capital. In the next case we show the possibility of positive steady-state growth in the presence of both inputs: physical capital and labor.

Case 3: Let us consider constant returns to scale in the cumulative factor, $\beta = 1$, and non-zero returns in the labor factor, $\alpha > 0$, so that we have increasing returns to scale in the aggregate. As shown by (2.7), under these assumptions, steady-state will only be possible in an economy without population growth, $n = 0$. The second term in (2.8) then again disappears and, since $\beta = 1$, there is the possibility of non-zero growth steady-states, although we cannot prove their existence on general grounds.

An (unproven) general message of this section is that to produce non-zero long-run growth it is necessary to have either constant or increasing returns to scale in the cumulative inputs. Although in this section we have just considered one type of capital, there are interesting models including physical as well as human capital, both accumulating over time. The condition there is that the elasticities of the two capital inputs add up to at least 1, as we already saw in our interpretation of the AK-technology in Case 2.

2.3 The Neoclassical Growth Model of Solow and Swan

This model, introduced by Solow [2] and Swan [3], describes the time evolution of an economy in which there is growth from some initial, known conditions. The model incorporates the assumptions introduced in the previous section, in a case of decreasing returns in physical capital, but constant returns to scale on the aggregate. As shown in Case 1 above, this economy has a single, stable zero-growth steady-state.

Hence, we consider in this chapter a *closed economy*, *without government*, so that savings and investment are equal to each other every period, $S_t = I_t$. Firms use physical capital and labor to produce the single consumption commodity, which can either be consumed or accumulated in the form of physical capital. Output is only used as consumption or investment, since there is no public consumption or any exchange with the foreign sector. Physical capital depreciates at a constant rate δ. Consumers are endowed with *a unit of time which supply inelastically* in the labor market.[3] Population N_t grows over time at a constant rate n, so that from an initial population N_0 we have, $N_t = N_0 e^{nt}$. Prices and salaries are fully flexible, so that the

[2]See Barro and Sala-i-Martin [1], Chap. 4.

[3]That would be the case, for instance, if leisure does not enter as an argument in their utility function, which we will not specify in this chapter.

economy is always in a state of *full employment*. The full employment assumption, together with eliminating any age structure in the population,[4] makes the labor force and employment to be equal to each other at each point in time so that we will also have, $L_t = L_0 e^{nt}$, which implies $\dot{L}_t = nL_t$. When incorporating to the labor force, each consumer/worker receives an amount of physical capital equal to that owned by each person already in the labor force.

Aggregate savings are a constant proportion of income each period, $S_t = sY_t$ or, in per capita terms, $s_t = sy_t$. There is no reason to believe that this should be an optimal behavior on the part of consumers. In fact, we do not consider any optimizing behavior on the part of economic agents or government in the Solow–Swan model so, the analysis is more positive than normative in character. In the next chapter, we analyze a model where consumption/savings decisions are taken optimally.

2.3.1 Description of the Model

2.3.1.1 Technology

We assume that at the aggregate level, the available technology can be represented by a first-degree homogeneous production function $Y = F(K_t, N_t)$. As explained above, we identify employment with total population. Derivatives are: $F_{K_t}, F_{N_t}, F_{K_t N_t} > 0$, $F_{N_t N_t}, F_{K_t K_t} < 0$ and the Hessian is negative definite, so that F is concave. We further assume: $F(K_t, 0) = F(0, N_t) = 0$, so that we cannot produce anything without using positive amounts of the two inputs, and $\lim_{K_t \to 0} F_{K_t} = \lim_{N_t \to 0} F_{N_t} = \infty$, $\lim_{K_t \to \infty} F_{K_t} = \lim_{N_t \to \infty} F_{N_t} = 0$. These are usually known as Inada conditions.

The more restrictive aspect of this technology is the existence of decreasing returns to scale in each input, which, as we saw in Case 1 in the previous section, precludes the possibility of positive steady-state growth. The aggregate constant returns to scale assumption allows us to write,

$$Y_t = F(K_t, N_t) = N_t F(K_t/N_t, 1) = N_t f(k_t), \tag{2.10}$$

where $k_t = K_t/N_t$ denotes the *per capita stock of productive capital* or *capital–labor ratio,* and $f(k_t) = F(K_t/N_t, 1)$. The assumptions on F imply: $f'(k_t) > 0$, $f''(k_t) < 0$, $f(0) = 0$, $\lim_{k_t \to 0} f' = \infty$, $\lim_{k_t \to \infty} f' = 0$.

The *capital–labor ratio* determines output produced per worker Y_t/N_t and hence, income per worker, so it is reasonable to expect that consumption will also be determined by this capital–labor ratio, which is the key variable in this economy.

[4]Consumers are able to work from the moment they are born.

The marginal productivity for each input is related to the derivatives of $f(k_t)$. First, taking derivatives with respect to K_t,

$$F_{K_t} = N_t f'(k_t) \frac{\partial k_t}{\partial K_t} = N_t f'(k_t) \frac{1}{N_t} = f'(k_t) > 0, \tag{2.11}$$

where subindices denote partial derivatives. On the other hand, taking derivatives at (2.10) with respect to N_t we get,

$$F_{N_t} = f(k_t) + N_t f'(k_t) \left(\frac{-K_t}{N_t^2} \right) = f(k_t) - k_t f_t'(k_t). \tag{2.12}$$

Even though it is not implied by the properties of $f(k_t)$, the marginal product of labor must also be positive: $f(k_t) - k_t f_t'(k_t) > 0$ since otherwise, it would be in the benefit of the firm to reduce employment. Finally, it is simple to check that the concavity of $f(k_t)$ is implied by that of F.

A particular technology satisfying the assumptions above is a Cobb–Douglas production function,

$$F(K_t, N_t) = AK_t^{\alpha} N_t^{1-\alpha} \text{ with } 0 < \alpha < 1,$$

where $A>0$ indicates the level of technology. Aggregate output can be written,

$$Y_t = AK_t^{\alpha} N_t^{1-\alpha} = AN_t k_t^{\alpha}, \tag{2.13}$$

so we are in the setup above, with $f(k_t) = Ak_t^{\alpha}$. Per-capita output is in this case,

$$y_t = \frac{Y_t}{N_t} = Ak_t^{\alpha}, \ \ 0 < \alpha < 1.$$

Marginal factor productivity for both factors is positive under this technology,

$$\begin{aligned} F_{K_t} &= f'(k_t) = A\alpha k_t^{\alpha-1} > 0, \\ F_{N_t} &= f(k_t) - k_t f_t'(k_t) = Ak_t^{\alpha} - k_t A\alpha k_t^{\alpha-1} = (1-\alpha)\, Ak_t^{\alpha} > 0. \end{aligned}$$

2.3.2 The Dynamics of the Economy

In this simple economy, output (or, equivalently, income) is used either as consumption or in the form of gross investment. The later is used in part to compensate for depreciated capital, and also as net additions to the stock of capital,

$$\begin{aligned} \textit{Net investment} = \dot{K}_t = \frac{dK_t}{dt} &= \textit{Gross investment} - \textit{Depreciation} \\ &= I_t - D_t = I_t - \delta K_t, \end{aligned}$$

where we have used the assumption on a constant rate δ of physical capital depreciation, independent of the stock of capital, $D_t = \delta K_t$.

So, we have the global constraint of resources:

$$Y_t = C_t + I_t = C_t + \dot{K}_t + \delta K_t,$$

that is,

$$\dot{K}_t = F(K_t, N_t) - C_t - \delta K_t.$$

Dividing by employment,

$$\frac{\dot{K}_t}{N_t} = \frac{F(K_t, N_t)}{N_t} - \frac{C_t}{N_t} - \delta \frac{K_t}{N_t} = f(k_t) - c_t - \delta k_t,$$

and, taking into account that

$$\dot{k}_t = \frac{\dot{K}_t}{N_t} - \frac{\dot{N}_t}{N_t} k_t = \frac{\dot{K}_t}{N_t} + n k_t,$$

we obtain,

$$f(k_t) = c_t + \dot{k}_t + (n + \delta)\, k_t, \tag{2.14}$$

the identity that describes the uses of income, in per-capita terms: each worker's output is used in part as consumption and as a net addition to the stock of capital, which may be positive or negative. The rest reflects the need to recover the capital lost by depreciation, as well as to provide each new worker with the same units of capital associated to each old worker. The number of workers grows at a rate n, and population growth acts as some sort of depreciation. In fact, it is impossible to disentangle in this model the effects of δ and n.

Since $C_t = (1 - s)Y_t$, we can divide by N_t to obtain, in per capita terms,

$$c_t = (1 - s)\, f(k_t),$$

and finally,

$$\dot{k}_t = s f(k_t) - (n + \delta)\, k_t, \tag{2.15}$$

which is the *law of motion* of the economy, showing how the stock of capital per worker increases in those periods in which savings $s f(k_t)$ exceeds from capital depreciation $(\delta + n)\, k_t$.

2.3.2.1 Technological Growth

Maintaining the above assumptions on savings, capital formation, population growth and full employment, let us now consider the possibility that there is *exogenous technological growth*, in the form of a variable productivity factor Γ_t, that grows at a constant rate γ:

$$\frac{\dot{\Gamma}_t}{\Gamma_t} = \gamma \ , \ \forall t.$$

We assume now that the available technology can be represented by an aggregate production function $Y_t = F(K_t, \Gamma_t N_t)$, with $F_{K_t}, F_{\Gamma_t N_t} > 0$, second derivatives: $F_{K_t,\Gamma_t N_t} > 0$, $F_{\Gamma_t N_t,\Gamma_t N_t} < 0$, $F_{K_t,K_t} < 0$ and a negative definite Hessian, so that F is concave. Additionally, $F(K_t, 0) = F(0, \Gamma_t N_t) = 0$, so that we cannot produce anything without using positive amounts of the two inputs, and $\lim_{K_t \to 0} F_{K_t} = \lim_{\Gamma_t N_t \to 0} F_{\Gamma_t N_t} = \infty$, $\lim_{K_t \to \infty} F_{K_t} = \lim_{\Gamma_t N_t \to \infty} F_{\Gamma_t N_t} = 0$.

Introduced this way, technological progress, represented by Γ_t is said to be of the *labor-saving* type, because as Γ_t grows, we will be able to produce a given output with a lower amount of the labour input.[5] The second input in the production function, $\Gamma_t N_t$, is then known as *effective labor*. The more restrictive aspect of this technology is again the existence of decreasing returns to scale in each input, which precludes the possibility of positive steady-state growth.

The aggregate constant returns to scale assumption allows us to write,

$$Y_t = F(K_t, \Gamma_t N_t) = \Gamma_t N_t F\left(\frac{K_t}{\Gamma_t N_t}, 1\right) = \Gamma_t N_t f(k_t), \tag{2.16}$$

where $k_t = \frac{K_t}{\Gamma_t N_t}$ denotes *now* the stock of *capital per unit of effective labor*, and $f(k_t) = F(\frac{K_t}{\Gamma_t N_t}, 1)$. The main variables in the economy can be represented in terms of this ratio. For instance, from the last equation, we have output per unit of effective labor[6]:

$$y_t = \frac{Y_t}{\Gamma_t N_t} = f(k_t).$$

An example of such a production function is, $F(K_t, \Gamma_t N_t) = AK_t^{\alpha}(\Gamma_t N_t)^{1-\alpha}$, with output:

$$Y_t = AK_t^{\alpha}(\Gamma_t N_t)^{1-\alpha} = A\Gamma_t N_t k_t^{\alpha} = \Gamma_t N_t f(k_t)$$

with $0 < \alpha < 1$ and $f(k_t) = Ak_t^{\alpha}$,

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[5]It is also sometimes known as *neutral* in the sense defined by Harrod.

[6]The argument in Sect. 2.2, suggests that, under our maintained assumption of decreasing returns to scale, the ratios of physical capital and output per unit of effective labour will experience zero growth in steady-state. In turn, that would imply that per-capita variables like $\frac{K_t}{N_t}$ or $\frac{Y_t}{N_t} = \Gamma_t f(k_t)$ will grow in steady-state at a rate γ_A. These results are shown in the next section.

so that, *output per unit of effective labor* is,

$$y_t = \frac{Y_t}{\Gamma_t N_t} = Ak_t^{\alpha}, \;\; 0 < \alpha < 1.$$

Marginal productivity for each input is again related to the derivatives of $f(k_t)$. First, taking derivatives in (2.16) with respect to K_t,

$$F_{K_t} = \Gamma_t N_t f'(k_t)\frac{\partial k_t}{\partial K_t} = \Gamma_t N_t f'(k_t)\frac{1}{\Gamma_t N_t} = f'(k_t) > 0.$$

On the other hand, taking derivatives in (2.16) with respect to N_t we get,

$$F_{N_t} = \Gamma_t f(k_t) + \Gamma_t N_t f'(k_t)\left(\frac{-\Gamma_t K_t}{(\Gamma_t N_t)^2}\right) = \Gamma_t \left[f(k_t) - k_t \; f'(k_t)\right].$$

Output is again either consumed or used as gross investment, and we have the same global constraint of resources as before,

$$Y_t = C_t + I_t = C_t + \dot{K}_t + \delta K_t,$$

that is,

$$\dot{K}_t = F(K_t, \Gamma_t N_t) - C_t - \delta K_t.$$

Dividing by the number of effective units of labor, we have,

$$\frac{\dot{K}_t}{\Gamma_t N_t} = \frac{F(K_t, \Gamma_t N_t)}{\Gamma_t N_t} - \frac{C_t}{\Gamma_t N_t} - \delta \frac{K_t}{\Gamma_t N_t} = f(k_t) - c_t - \delta k_t,$$

where we have used the fact that the homogeneity of degree one of $F(.,.)$ allows us to write: $\frac{F(K_t,\Gamma_t N_t)}{\Gamma_t N_t} = F(\frac{K_t}{\Gamma_t N_t}, \frac{\Gamma_t N_t}{\Gamma_t N_t}) = F(\frac{K_t}{\Gamma_t N_t}{}_t, 1) = f(k_t)$. We denote consumption per unit of effective labor by $c_t = \frac{C_t}{\Gamma_t N_t}$. Taking into account that

$$\dot{k}_t = \frac{\dot{K}_t}{\Gamma_t N_t} - \frac{\Gamma_t \dot{N}_t}{\Gamma_t N_t}k_t - \frac{\dot{\Gamma}_t N_t}{\Gamma_t N_t}k_t = \frac{\dot{K}_t}{\Gamma_t N_t} - (n+\gamma)\, k_t,$$

we get,

$$f(k_t) = c_t + \dot{k}_t + (n + \delta + \gamma)\, k_t, \tag{2.17}$$

the identity that explores the uses of income, in per-capita terms: each worker's output is used in part as consumption and net additions to the stock of capital. The rest reflects the need to recover the capital lost by depreciation, as well as the need to provide to each new worker with the same capital per units of effective labor owned by each old worker. The number of workers grows at a rate n, while the general level of productivity grows at a rate γ. Again in this model, population growth acts as some sort of depreciation.

Finally,

$$Y_t = C_t + I_t = C_t + S_t = C_t + sY_t,$$

so that, $C_t = (1-s)Y_t$ and, dividing through by $\Gamma_t N_t$ we get, in effective units of labor,

$$c_t = (1-s)\, f(k_t), \tag{2.18}$$

and

$$\dot{k}_t = sf(k_t) - (n+\delta+\gamma)\, k_t, \tag{2.19}$$

which is the law of motion of the economy, showing how the stock of capital per unit of effective labor increases in those periods in which per capita savings $sf(k_t)$ exceeds total capital depreciation $(n+\delta+\gamma)\, k_t$.

2.3.3 Steady-State

Definition 1. In an exogenous growth economy, a *steady-state* is a vector of values for the rates of growth of the main variables (physical capital, output and consumption) in units of effective labor, that if it is ever reached, it can be maintained constant forever.

A steady-state is often referred to as a long-run equilibrium, because of the characteristic of having a constant rate of growth for appropriately defined variables.

Let us consider again the economy's *law of motion* (2.19), from which the growth rate of capital can be written,

$$\gamma_{k_t} = \frac{\dot{k}_t}{k_t} = s\frac{f(k_t)}{k_t} - (n+\delta+\gamma)\,. \tag{2.20}$$

In steady state, γ_{k_t} must be constant, so that $\frac{f(k_t)}{k_t}$ must also be constant. Its time derivative is,

$$\frac{d\left[\frac{f(k_t)}{k_t}\right]}{dt} = \frac{k_t f'(k_t) - f(k_t)}{k_t}\frac{\dot{k}_t}{k_t} \underset{\text{in steady state}}{=} 0.$$

Since $k_t f'(k_t) - f(k_t)$ is the negative of the marginal product of labor, which we assumed to be positive, then we will have in steady state $\frac{\dot{k}_t}{k_t} = 0$, which implies $\dot{k}_t = 0$, and the stock of capital per unit of effective labor will remain constant in steady-state. That, in turn, implies that the stock of productive capital per worker will grow at a rate γ. To see the relationship between the growth rates of income and capital, notice that,

$$\frac{Y_t}{N_t} = F\left(\frac{K_t}{N_t}, \Gamma_t\right) = \frac{K_t}{N_t} F\left(1, \frac{\Gamma_t}{K_t/N_t}\right),$$

and, since $k_t = \frac{K_t}{N_t \Gamma_t}$ is constant in steady-state, output and capital will grow at the same rate. In units of effective labor, these variables grow at a zero rate, while in per capita units they grow at a rate γ. In aggregate terms, they grow at a rate $n + \gamma$. Since consumption is proportional to income, consumption per-capita will also grow at a rate γ, while remaining constant in steady-state in units of effective labor. Even though per-capita variables experience growth in steady-state, since the common growth rate, γ, is exogenous to the model, we say this is an *exogenous growth model*.

Summarizing, steady state is characterized in this economy by $\dot{k}_t = 0$ so that, from (2.19), steady state levels of k_t are solutions to,

$$sf(k_{ss}) - (n + \delta + \gamma)k_{ss} = 0, \tag{2.21}$$

which defines the value of the stock of capital per unit of effective labor in steady state, k_{ss}. The properties of the solution to this equation like its existence and uniqueness, or the way how it is affected by structural parameters, depend on the specific production function assumed. Figure 2.3a shows the possibility of multiple steady-states. The upper graph presents them by the intersection between the $sf(k_{ss})$ curve and the $(n + \delta + \gamma)k_{ss}$ straight line. The lower graph displays the associated time derivatives of the stock of capital per unit of effective labor, as defined by (2.19). However, for standard production functions satisfying the Inada conditions above, (2.21) will have a single non-zero solution, the steady state then being uniquely defined (Fig. 2.3b). The stock of capital increases to the left of the steady-state, while decreasing to the right of it.

Figure 2.3b shows how $k_{ss} = 0$ is another steady-state. It solves Eq. (2.21) because $f(0) = 0$. At that point, there is zero physical capital, so production is zero and consumption is also zero. There can be no investment, and savings will be zero no matter what the savings rate is, since there are no resources. The economy never leaves this situation, although it has no economic interest.

As an example, let us consider again the Cobb–Douglas production technology $Y_t = F(K_t, \Gamma_t N_t) = AK_t^{\alpha}(\Gamma_t N_t)^{1-\alpha}, 0 < \alpha < 1$, which can also be represented: $y_t = Ak_t^{\alpha}, 0 < \alpha < 1$. Steady state is then characterized by,

$$sAk_{ss}^{\alpha} = (n + \delta + \gamma)\, k_{ss}.$$

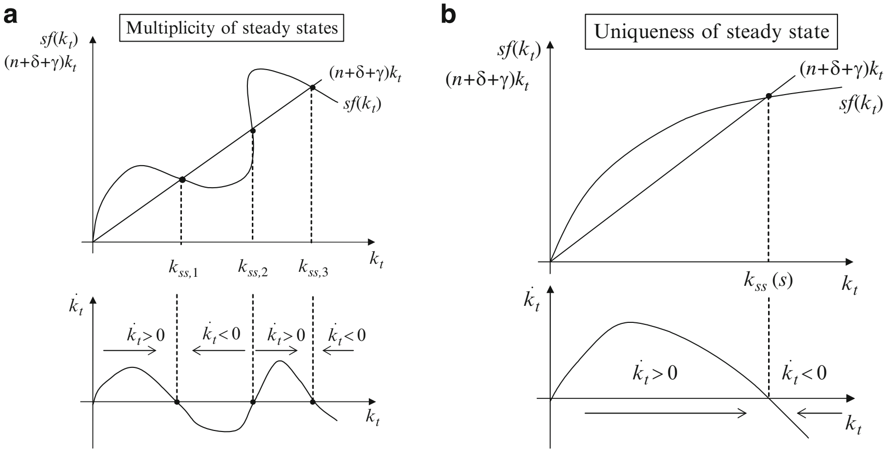


Fig. 2.3 The steady-state in the Solow-Swan model

The single solution[7] to that equation is,

$$k_{ss} = \left(\frac{sA}{n + \delta + \gamma} \right)^{\frac{1}{1-\alpha}}, \tag{2.22}$$

so that the steady-state level of physical capital, in units of efficient labor is higher for higher values of the constant savings rate, while being lower for higher values of either the rate of population growth, the depreciation rate of physical capital, or the rate of growth of productivity. It is also higher the higher the value of the elasticity of physical capital in the production function representing the aggregate technology.

A higher savings rate allows for a more important capital accumulation, leading to a higher stock of physical capital. On the other hand, a higher rate of depreciation detracts more resources from net capital accumulation. Higher population requires more resources to be devoted to provide newborn consumers with the same stock of physical capital as the already existing consumers. Since we are working with variables in terms of efficient units of labor, technological growth enters the model symmetrically with population growth, so the dependence of steady-state levels with respect to this variable is also negative. Finally, a higher elasticity of physical capital creates a higher incentive for capital accumulation, leading to a higher steady-state level of physical capital.

[7] The equation has another root: $k_{ss} = 0$. This would be a steady-state with zero capital, output and consumption.

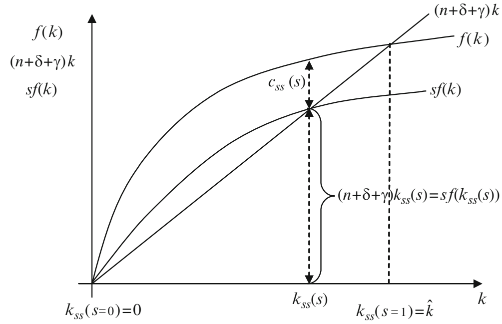


Fig. 2.4 Steady-state as a function of savings rate

Output is increasing on the level of physical capital, so that the steady-state levels of output and consumption, in efficient units of labor, y_{ss}, c_{ss}, will also depend on the values of structural parameters, $s, n, \delta, \gamma, \alpha$ as described for k_{ss}. The reader must be careful not to use the (2.18) representation to extrapolate a similar dependence of consumption on the values of structural parameters, because of the presence of the savings rate in that expression. We will get back to this issue in Sect. 2.3.8.

Figure 2.4 shows the dependence of the steady-state on the level of the constant savings rate. An increase in savings rate will raise the slope of the $sf(k_t)$-curve, which will intersect the straight line to the right of the current steady-state. So, the stock of capital per unit of efficient labor will rise and so will do income, investment and consumption. The figure shows that there is a limit to such a process. When $s = 1$, the $sf(k_t)$-curve coincides with the production function $f(k_t)$, and we have what is known as the *subsistence* steady-state, $\hat{k}$, that in which

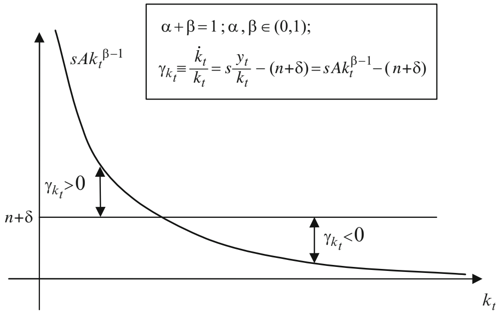
$$f\left(\hat{k}\right) = (n + \delta + \gamma)\,\hat{k}.$$

In the *subsistence* steady-state, so much physical capital has been accumulated, that all output is needed to replace what is lost to physical depreciation as well as to provide new workers with the same stock of physical capital than older workers. There are no resources left for consumption, which is hence equal to zero. Each value of the constant savings rate between 0 and 1 is associated with a steady-state level of capital per unit of labor between 0 and $\hat{k}$. Situations with $k_{ss} > \hat{k}$ are not sustainable as steady states, since they would imply negative consumption.

2.3.4 The Transition Towards Steady-State

Outside the steady-state the growth rate of the economy is not constant but, rather, it behaves according to (2.20), changing with the level of k_t. We call *transition* the process that unfolds from the starting situation, with a capital stock of k_0, towards the steady-state level.

Fig. 2.5 Steady-state determination under Cobb–Douglas technology



The first term in that expression, $sf(k_t)/k_t$, is a continuous, decreasing function of k_t which starts at infinity for $k_t = 0$, converging to zero for $k_t = \infty$, as can easily be seen by taking limits. The second term is a constant, represented by a horizontal straight line in Fig. 2.5. Hence, there is some single value of the capital stock for which $sf(k_{ss})/k_{ss} = \delta + n + \gamma$ and so, $\gamma_{k_t} = 0$. The point at which the growth rate of capital per unit of effective labor becomes zero is the single steady-state of the economy, k_{ss}. Since the growth rate γ_{k_t} becomes positive for any stock of capital below the steady-state level, and negative for any capital stock above steady-state, the model implies a monotonic convergence to steady-state so, the *steady-state is globally stable*.

The gap in Fig. 2.5 between the two lines is precisely the growth rate γ_{k_t}, which can be seen to reduce in size as the economy approaches steady-state from either side. As pointed out in Barro and Sala-i-Martin [1], when k_t is relatively low, the average product of capital, $f(k_t)/k_t$, is relatively large, due to the law of diminishing returns. Since consumers save a constant proportion of that product, gross investment per unit of capital, $sf(k_t)/k_t$, which is proportional to the average product of capital, will also be large. With a constant depreciation rate, that will make $\dot{k}_t/k_t$ to be relatively high, and the opposite happens for high levels of k_t. Analytically, changes in γ_{k_t} as the stock of capital changes are given by,

$$\frac{\partial \gamma_{k_t}}{\partial k_t} = s\frac{k_t f'(k_t) - f(k_t)}{k_t^2} < 0,$$

which is negative, since the numerator is equal to minus the marginal product of labor.

2.3.5 The Duration of the Transition to Steady-State

To have an idea of how fast the economy approaches steady-state, we focus on analyzing $\dot{k}_t$ rather than γ_{k_t}. If we construct the linear approximation of the law of motion for capital around steady-state, we get,

$$\begin{aligned}
\dot{k}_t &\simeq [sf(k_{ss}) - (\delta + n + \gamma)\, k_{ss}] + \left[sf'(k_{ss}) - (\delta + n + \gamma)\right](k_t - k_{ss}) \\
&= \left[\frac{(\delta + n + \gamma)\, k_{ss} s f'(k_{ss})}{sf(k_{ss})} - (\delta + n + \gamma)\right](k_t - k_{ss}) \\
&= (\alpha_k(k_{ss}) - 1)\,(\delta + n + \gamma)\,(k_t - k_{ss})\,,
\end{aligned}$$

where to obtain the first equality, we have used the fact that, in steady-state $sf(k_{ss}) = (\delta + n + \gamma)\, k_{ss}$ and where we have defined the elasticity of output with respect to the stock of capital,

$$\alpha_k(k_t) = \frac{k_t f'(k_t)}{f(k_t)} \in (0, 1)\,.$$

Under constant returns to scale, $\alpha_k(k_t)$ is also physical capital's share in income distribution. In the Cobb–Douglas production function, $\alpha_k(k_t) = \alpha$, constant. Borrowing from competitive equilibrium ideas, capital would be rented by firms at a price equal to its marginal product, and $\alpha_k(k_t)$ would the proportion of output that would be devoted to pay back to the owners of capital.

Changes in k_t will then be explained by,

$$\dot{k}_t = -(1 - \alpha_k(k_{ss}))\,(\delta + n + \gamma)\,(k_t - k_{ss}) \tag{2.23}$$

which depends negatively on the distance to steady-state k_{ss}. Hence, the stock of capital per unit of effective labor changes faster initially, when the economy is far from steady state, moving more gradually as the economy approaches its steady-state.[8]

The solution to the differential equation (2.23) is,

$$k_t - k_{ss} = e^{-(1-\alpha_k(k_{ss}))(\delta+n+\gamma)t}\,(k_0 - k_{ss}) = e^{-\mu t}\,(k_0 - k_{ss})\,, \tag{2.24}$$

with $\mu = (1 - \alpha_k(k_{ss}))\,(\delta + n + \gamma)$. For instance, if we assume that $\alpha_k(k_{ss}) = 1/3$, and $n + \delta + \gamma = 6\,\%$, then $\mu = 4\,\%$, so that 4 % of the difference between k_t and k_{ss} is closed each period. Half of the initial distance to steady-state would then be closed after 17 periods.

[8]Notice that this is a result on absolute changes in the stock of capital per unit of effective labor, while the result above was on its rate of growth.

2.3.6 The Growth Rate of Output and Consumption

Because of the global stability of the Solow–Swan model, the model predicts that any economy is either at steady-state, or converging to it. We consider in this section an economy outside steady-state. Because of the global stability of the model, that economy will be in a transition phase towards steady-state. Along the transition, the behavior of output is characterized by,

$$\gamma_{y_t} = \frac{\dot{y}_t}{y_t} = \frac{f'(k_t)}{f(k_t)}\dot{k}_t = k_t \frac{f'(k_t)}{f(k_t)}\gamma_{k_t} = \alpha_k(k_t)\gamma_{k_t}. \tag{2.25}$$

As an example, if the aggregate technology is of the Cobb–Douglas type, then capital's share is $\alpha_k(k_t) = \alpha$, and, along the transition,

$$\gamma_{y_t} = \alpha\gamma_{k_t},$$

the growth rates of income and capital behave similarly, decreasing in magnitude as the economy approaches steady-state.

More generally, we can use (2.20) for γ_{k_t} in (2.25) to get,

$$\gamma_{y_t} = sf'(k_t) - (n + \delta + \gamma)\,\alpha_k\,(k_t)\,,$$

so that,

$$\frac{\partial\gamma_{y_t}}{\partial k_t} = \frac{f''(k_t)k_t}{f(k_t)}\gamma_{k_t} - \frac{(n+\delta+\gamma)\,f'(k_t)}{f(k_t)}\,(1-\alpha_k(k_t))\,,$$

and since $0 \leq \alpha_k(k_t) \leq 1$, then $\frac{\partial\gamma_{y_t}}{\partial k_t} < 0$ at those points at which $\gamma_{k_t} \geq 0$. If, on the contrary, $\gamma_{k_t} < 0$, then the sign of $\frac{\partial\gamma_{y_t}}{\partial k_t}$ is ambiguous. However, in the proximity of the steady-state, γ_{k_t} will be small, and $\frac{\partial\gamma_{y_t}}{\partial k_t} < 0$. This means that if the economy starts with a capital stock below k_{ss}, both, k_t and y_t will increase, but the rate of growth of income per unit of effective labor, γ_{y_t}, will fall down as we approach steady state, as it is the case with γ_{k_t}. If, on the contrary, the initial stock of capital is above k_{ss}, then k_t and y_t will decrease, but we cannot say anything in general about the behavior of γ_{y_t}. However, once we get close enough to steady-state, γ_{y_t} will gradually increase as the stock of capital keeps falling towards k_{ss}. It may surprise to see that the rate of growth of y_t is increasing in spite of the fact that the stock of capital is falling down to k_{ss}, but it is a negative rate of growth. So, what we have is that as the stock of capital falls down towards steady-state, income per unit of effective labor is falling towards the new steady-state at a decreasing rate. For a relatively high k_t, depreciation is so high that savings and investment are not enough to replace depreciation and hence, the stock of capital decreases and output falls. As the stock of capital decreases from its initially high level, less resources need to be devoted to compensate for depreciation, and income per unit of effective labor falls by a lesser amount, until it stabilizes in its new sustainable steady-state.

On the other hand, since the maintained assumption of this model is,

$$c_t = (1 - s)\, y_t,$$

then,

$$\gamma_{c_t} = \gamma_{y_t}, \quad \forall t,$$

at any point outside the steady-state. Growth rates of per-capita variables will be equal to the growth rates calculated in this section added by γ, while growth rates for economy-wide aggregates will be the previous ones added by n.

2.3.7 Convergence in the Neoclassical Model

We have so far analyzed the implications of the neoclassical growth model on the evolution of a specific economy. We have characterized the existence of a single steady-state or long-run equilibrium and its dependence on the values of some structural parameters. But the previous discussion has also implications on the comparative evolution of economies from different countries, so long as these can be assumed to fulfill the assumptions characterizing the Solow–Swan model. We are particularly interested on possible implications on whether any two different economies will tend to be more similar to each other over time or rather, differences between them will tend to increase.

We say that two economies *converge in absolute terms* if, starting from a different initial situation in terms of the endowment of physical capital per unit of effective labor, k_0, k_0', and, hence, in terms of their levels of income per unit of effective labor, the difference between them narrows over time. Let us consider two economies sharing the same values of the structural parameters, s, n, δ, γ, but differing in their initial stocks of capital. The long-run equilibrium (steady-state) levels of physical capital, consumption and income per unit of efficient labor will be the same in both economies. Let us assume that one of them, the *poor economy*, has an initial capital stock k_0^p lower than that of the *rich economy*, k_0^r. Figure 2.6, that presents the determination of both growth rates, shows that the growth rate of the poor economy will be higher than that of the rich economy, so that the respective stocks of capital and, hence, the levels of output (or income) per-unit of effective labor, will become more similar over time, as both converge to the same steady-state level. As a consequence, the neoclassical model implies absolute convergence among countries.

This suggests that a regression like,

$$\gamma_{k_t} = \beta_0 + \beta_1 \ln k_t + u_t, \ \beta_1 < 0,$$

Fig. 2.6 Absolute Convergence

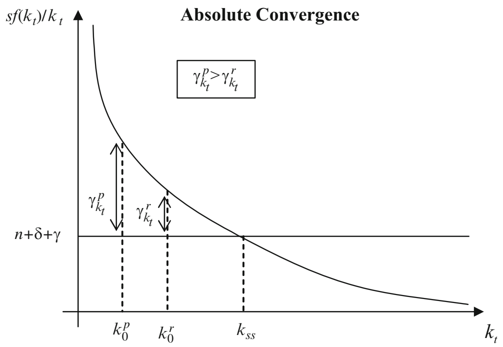
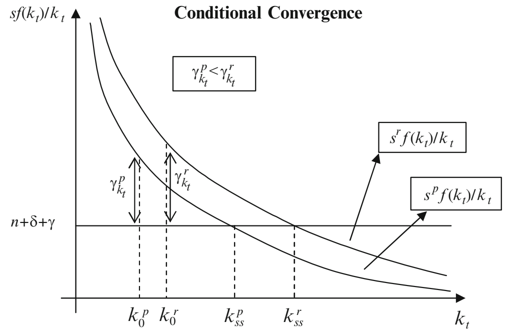


Fig. 2.7 Conditional Convergence



which explains the growth rate of the economy as a function of its current situation, would be an adequate representation of the time series produced by a neoclassical growth model with either time series or cross-section data. Actually, what we have seen as an implication of the Solow–Swan model is that the growth rate depends on the relative distance of income or productive capital from their steady-state values. Hence, a more appropriate representation would be,

$$\gamma_{k_t} = \beta_0 + \beta_1 (\ln k_t - \ln k_{ss}) + u_t, \tag{2.26}$$

where k_{ss} could be estimated from its expression,[9] after having some estimates of the values of structural parameters.

Empirical analysis does not show evidence on this type of convergence, unless we limit our consideration to a set of homogeneous economies (states in the US, OECD countries, province economies in a given country, etc.). One possible reason for that is that a broader set of economies may display substantial differences among their savings rates. In Fig. 2.7, we have labelled as *poor* the economy with the lower

[9]It is clear that, being a constant, the correction on physical capital data would not need to be done to estimate the regression, so long as we are careful when interpreting the estimated intercept, although estimates of (2.26) would have a more direct interpretation.

savings rate, which implies, as we already know, a lower capital stock and lower per capita income in steady-state. This figure shows that when economic structures are different, it is perfectly possible that the rich country may grow faster then the poor country, if the former is relatively farther away from its steady-state.

This means that empirical analysis should take into account the fact that different countries may have a different steady-state. This is done by conditioning the time evolution of γ_{k_t} on the determinants of steady-state. The result is then known as *conditional convergence*. The neoclassical growth model we have discussed implies that countries with different structural characteristics will experience conditional convergence: once we correct for the fact that the two economies have a different long-run equilibrium, poorer economies should be seen to experience faster growth than richer ones.

The correction is made by adding to the econometric model a vector z_t of variables determining steady-state k_{ss},

$$\gamma_{k_t} = \beta_0 + \beta_1 \ln k_t + \phi \ln z_t + u_t,$$

with ϕ being a vector of the same dimension as z_t. In the neoclassical Solow–Swan model z_t could include the savings rate, depreciation rate, population growth or the output elasticity of physical capital. Sometimes, other indicators as the level of education in the population, expenditures in infrastructures, and so on, are included in z_t, although these are not justified by the Solow–Swan model. In more elaborated models where the savings rate and the rate of technological progress are endogenous, and the role of the government is explicitly considered, there will be an even richer set of variables in z_t.

Similar regressions could be estimated for output per unit of effective labor or for per capita output, if we assume a given value for γ.

2.3.8 A Special Steady-State: The Golden Rule of Capital Accumulation

We remember that steady-state is defined by the relationships,

$$sf(k_{ss}) = (n + \delta + \gamma)\, k_{ss},$$
$$c_{ss} = f(k_{ss}) - (n + \delta + \gamma)\, k_{ss},$$

which we have used in the previous section to show that the steady-state stock of productive capital moves in parallel with the level of the savings rate s. That is, for given values of structural parameters $n, \delta, \gamma, \alpha$, the implied steady-state levels of physical capital, output and consumption will depend on the constant value chosen for the savings rate. Since the savings rate affects the stock of capital, and this influences consumption [see (2.18)], it makes sense to ask about the value of the savings rate that would maximize the steady-state level of consumption. That level

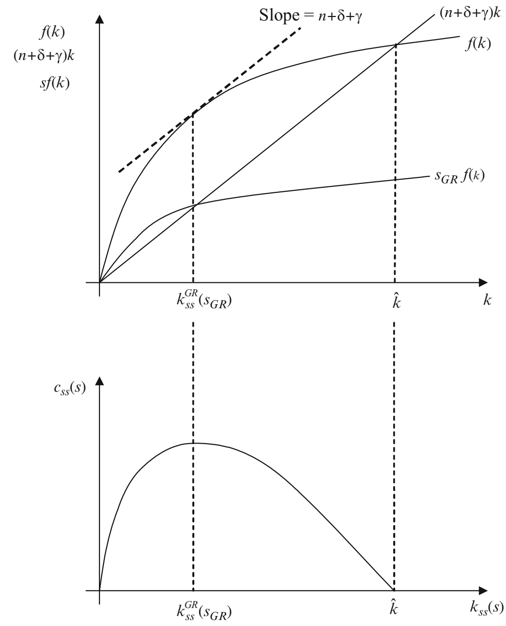


Fig. 2.8 Golden Rule determination

of savings, and the associated steady-state, are known as the Golden-Rule of capital accumulation.

From the last equation, we see that steady-state consumption will be maximum when $\partial c_{ss}/\partial k_{ss} = 0,\ \partial^2 c_{ss}/\partial k_{ss}{}^2 < 0$. That happens at the point where,

$$f'(k_{ss}^{GR}) = n + \delta + \gamma, \tag{2.27}$$

that is, the point at which the slope to $f(k)$ is parallel to the straight line $(n + \delta + \gamma)\, k$. That determines the Golden-Rule level of physical capital in units of efficient labor, k_{ss}^{GR}. The Golden Rule savings rate, s_{GR}, is the value of s for which the function $sf(k)$ intersects the $(n + \delta + \gamma)\, k$ straight line at k_{ss}^{GR} (see Fig. 2.8).

In the Cobb–Douglas case, $y_t = Ak_t^{\alpha}$, the Golden Rule condition takes the form,

$$A\alpha k^{\alpha-1} = \delta + n + \gamma,$$

leading to,

$$k_{ss}^{GR} = \left(\frac{\alpha A}{n + \delta + \gamma}\right)^{\frac{1}{1-\alpha}},$$

which, by comparison with (2.22) shows that, under this technology, the Golden Rule is *the steady-state arising for a constant savings rate equal to the output elasticity of capital*. Since constant returns to scale lead to a competitive equilibrium allocation with zero profits, and output being distributed to each factor according to their output elasticities, the Golden Rule can also be interpreted as the result of following either the rule: “Save all capital income” or, alternatively, “Consume all labor income”.

In fact, we now show that this is a general result that does not depend on the available technology, beyond the assumptions made in the Solow–Swan economy. First, notice, that the two statements above are equivalent in this economy because (a) being a closed economy with no government, what is not consumed it is saved, and (b) because the constant returns to scale assumption implies that all output (income) is distributed between the production factors, with no residual profit. Indeed, if we make savings equal to capital income, we have,

$$sY_t = F_{K_t} K_t \Rightarrow sf(k_t) = F_{K_t} \frac{K_t}{\Gamma_t N_t} = f'(k_t)k_t,$$

and, since any steady-state satisfies: $sf(k_{ss}) = (n + \delta + \gamma)\, k_{ss}$, the condition above implies,

$$f'(k_{ss}^{GR}) = n + \delta + \gamma,$$

so that the only steady state satisfying the described condition is the Golden Rule. This means that in the Golden Rule there are no income transfers between the capital and labor factors. To maintain a steady state with capital above k_{ss}^{GR} there would be a need for a high level of investment, to recover the capital lost to depreciation. That way, it will not be enough with capitalists investing all income they receive as owners of capital, and workers will also have to devote part of their labor income to investment. There will then be an income transfer from workers to the owners of capital. The opposite result would arise in a steady-state below k_{ss}^{GR}.

It would be wrong to interpret the Golden Rule of capital accumulation as an optimal allocation of resources.[10] Since the Golden Rule is the steady-state or long-run equilibrium offering the maximum consumption, it is clear that, unless the utility function of consumers presents a bliss point, the Golden Rule should be preferred to any other possible steady-state. But that is only true if we could place the economy initially at a steady-state of our choice. Unfortunately, that is not the case. The economy is endowed with a given stock of capital per unit of efficient labor, k_0, and its structural characteristics, together with a chosen rate of savings s, will determine the long-run equilibrium. However, to bring the economy to that equilibrium, the economy will go through a transition process, with physical capital converging from k_0 to k_{ss}.

[10]The following argument rests on utility comparisons, and we have not specified consumer preferences in this chapter. It is nevertheless interesting as an introduction to the type of normative analysis that is done in subsequent chapters. In fact, we will address again the suboptimality of the Golden Rule in Chap. 3.

So, suppose that, starting from k_0, consumers choose a savings rate of precisely s_{GR}, the level at which the $sf(k)$-curve intersects the straight line $(n+\delta+\gamma)\,k$ at k_{ss}^{GR}. The long-run equilibrium or steady-state stock of capital will we k_{ss}^{GR}, but the economy will enter into a *transition* phase towards k_{ss}^{GR} along which it is quite likely that it will have to make some sacrifices in terms of consumption. Once the economy reaches the Golden Rule, consumers will enjoy a higher level of consumption than can be enjoyed at any other steady-state, but it is unclear that the time aggregate level of utility along the whole trajectory would be maximized, precisely because of the initial sacrifice in consumption.

For instance, we could compare utility along the trajectory converging from k_0 to k_{ss}^{GR}, with the one that would be obtained with a savings rate of s_0, the one that would have allowed for maintaining the initial stock of capital k_0, unchanged forever. The result of such comparison is far from obvious, since it depends on: (a) the magnitude of short-run sacrifices needed to implement a savings rate of s_{GR}, (b) the differences between the level of utility provided by the Golden Rule level of consumption, and that corresponding to maintaining a steady-state of k_0, (c) the discount applied to future utility, and (d) how long it takes for the economy to be in the neighborhood of the Golden Rule, when the savings rate of s_{GR} is implemented.

These effects are far from trivial. To analyze whether consumers' would be better off by staying at their current steady-state or by starting a transition trajectory taking them to the Golden Rule, we need to be able to compute the time series representing the paths followed by the main variables under each scenario, with which to evaluate specific utility functions, as it is done in future chapters.

2.4 Solving the Continuous-Time Solow–Swan Model

2.4.1 Solution to the Exact Model

As in many other models that will be reviewed in future chapters, the time evolution of the stock of capital per worker obeys a nonlinear, first order differential equation, for which a closed form analytical solution generally does not exist. Such a solution exists in the Solow–Swan model, however, and we can find continuous functions of time: $k_t \equiv k\,(t)$, $y_t \equiv y\,(t)$, $c_t \equiv c\,(t)$, $s_t \equiv i_t \equiv s\,(t) = sy_t$, describing the exact time paths for the capital stock, output, consumption and savings or investment.

We start from the law of motion under a Cobb–Douglas technology,

$$\dot{k}_t = sAk_t^{\alpha} - (n+\delta+\gamma)k_t, \tag{2.28}$$

with a steady state defined by $\dot{k}_t = 0$, which leads to,

$$k_{ss} = \left(\frac{sA}{n+\delta+\gamma}\right)^{\frac{1}{1-\alpha}}. \tag{2.29}$$

If we introduce a new variable $z_t = k_t^{1-\alpha}$, we have $\dot{z}_t = (1-\alpha)\, k_t^{-\alpha} \dot{k}_t$, and multiplying through (2.28) by $(1-\alpha)\, k_t^{-\alpha}$, we get,

$$\dot{z}_t = (1-\alpha)\, sA - (1-\alpha)\,(n+\delta+\gamma) z_t,$$

a linear differential equation, with solution $z_t = Me^{\mu t} + J$. To find the values of the constants M, μ, J we first write the time derivative $\dot{z}_t = M\mu e^{\mu t}$ which, taken to the equation, yields $\mu = -(1-\alpha)\,(n+\delta+\gamma)$, $J = \frac{sA}{n+\delta+\gamma}$, so that $z_t = Me^{-(1-\alpha)(n+\delta+\gamma)t} + \frac{sA}{n+\delta+\gamma} = k_t^{1-\alpha}$. The remaining constant will be determined from a boundary condition. In this case, since the starting capital stock k_0 is given, we have at $t = 0$, $k_0^{1-\alpha} = M + \frac{sA}{n+\delta+\gamma}$, so that $M = k_0^{1-\alpha} - \frac{sA}{n+\delta+\gamma}$, and the solution to the original law of motion, finally, satisfies

$$k_t^{1-\alpha} = \left(k_0^{1-\alpha} - \frac{sA}{n+\delta+\gamma}\right) e^{-(1-\alpha)(n+\delta+\gamma)t} + \frac{sA}{n+\delta+\gamma}, \tag{2.30}$$

from which output, consumption, and investment/savings would be obtained through $y_t = k_t^{\alpha}$, $c_t = (1-s)y_t$, $i_t = s_t = sy_t$. Notice that, as time passes, we have $\lim_{t\to\infty} k_t = \left(\frac{sA}{n+\delta+\gamma}\right)^{\frac{1}{1-\alpha}}$, and the economy converges to steady-state, reflecting the *global stability* of the exact system.

2.4.2 The Linear Approximation to the Solow–Swan Model

Even the simpler growth models have a complex enough structure that prevents from computing an exact analytical solution. As we have just seen, the continuous-time version of the Solow–Swan is an exception. Since we will more often find the opposite situation, we familiarize now the reader with the standard approach of finding an approximation to the model, for which an exact solution can often be found.

Using Taylor's expansion, we can find the linear approximation to (2.19) around steady state k_{ss}. To do so, we need to consider that equation as a function: $\dot{k}_t = \Psi(k_t;\theta)$, where $\theta = (s, A, n, \delta, \alpha)$ is the vector of structural parameters, with a linear approximation:

$$\dot{k}_t \simeq \Psi(k_{ss};\theta) + \left(\frac{\partial \Psi(k_t;\theta)}{\partial k_t}\right)_{ss} (k_t - k_{ss}) \;\Rightarrow$$

$$\begin{aligned}\dot{k}_t &\simeq [sf(k_{ss}) - (n+\delta+\gamma)k_{ss}] + \left[sf'(k_{ss}) - (n+\delta+\gamma)\right](k_t - k_{ss}) \\ &= \left[sf'(k_{ss}) - (n+\delta+\gamma)\right](k_t - k_{ss}),\end{aligned} \tag{2.31}$$

since the constant term is equal to zero. The coefficient of $k_t - k_{ss}$, $sf'(k_{ss}) - (n + \delta + \gamma)$, is negative, since the $sf(k_t)$-curve crosses the $(n+\delta+\gamma)k_t$-line from above. Hence, if we start from below steady-state, the difference $k_t - k_{ss}$ will be negative, and $\dot{k}_t$ will be positive, indicating that physical capital will accumulate and the economy will converge to steady state. If we start from above steady state, the difference $k_t - k_{ss}$ will be positive, so $\dot{k}_t$ will be negative, indicating that physical capital will diminish while the economy gradually converges to steady state. So,

$$k_t < k_{ss} \Rightarrow \dot{k}_t > 0,$$
$$k_t > k_{ss} \Rightarrow \dot{k}_t < 0,$$

and the linearized model is also *globally stable,* the stock of capital converging towards its steady-state level, no matter whether its initial endowment of physical capital, k_0, is above or below steady-state level, k_{ss}.

2.4.2.1 Analytical Solution for the Cobb–Douglas Case

We examine now the special case of a Cobb–Douglas technology. We will have the law of motion for the stock of capital,

$$\begin{aligned}\dot{k}_t &= sAk_t^{\alpha} - (n+\delta+\gamma)k_t \\ &\simeq \left[sAk_{ss}^{\alpha} - (n+\delta+\gamma)k_{ss}\right] + \left[s\alpha Ak_{ss}^{\alpha-1} - (n+\delta+\gamma)\right](k_t - k_{ss}),\end{aligned}$$

which, using the steady state level of the capital–labor ratio k_{ss} characterized in (2.22), leads to,

$$\dot{k}_t \simeq \left[s\alpha Ak_{ss}^{\alpha-1} - (n+\delta+\gamma)\right](k_t - k_{ss}) = D(k_t - k_{ss}), \tag{2.32}$$

with $D = s\alpha Ak_{ss}^{\alpha-1} - (n+\delta+\gamma) = -(1-\alpha)(n+\delta+\gamma) < 0$, so that the coefficient of $k_t - k_{ss}$ in the linear approximation to the law of motion of the economy is negative, guaranteeing stability of the implied solution, as we have seen in the previous paragraph for the more general case.

This linear approximation in the Cobb–Douglas case (2.32) can be solved analytically. To that end, we try with a linear solution: $k_t = a + be^{\mu t}$ which, plugged into the differential equation (2.32), together with a given initial condition $k\,(t=0) = k_0$, leads to,[11]

$$k_t = k_{ss} + e^{Dt}\,(k_0 - k_{ss}) = (1 - e^{Dt})k_{ss} + e^{Dt}k_0, \tag{2.33}$$

showing that the stock of capital converges to steady state at a rate D, since taking time derivatives in this expression, we get: $\dot{k}_t/k_t = \frac{d(k_t - k_{ss})/dt}{k_t - k_{ss}} = D$.

[11]Substitution of the proposed solution yields, $b\mu e^{\mu t} = Da + Dbe^{\mu t} - Dk_{ss}$which can hold only if $\mu = D, a = k_{ss}$. Hence, we have: $k_t = k_{ss} + be^{Dt}$. To determine the value of the constant b we use the initial condition: $k_0 = k_{ss} + b$, so that: $b = k_0 - k_{ss}$.

2.4.3 Changes in Structural Parameters

This section is devoted to analyzing the long-run effects, i.e., the effects on steady-state levels of the main variables, of permanent changes in the values of structural parameters. We start by paying special attention to a change in the savings rate, since that is the parameter more easily linked to a policy intervention in this model, and extend the discussion to the remaining structural parameters later on.

2.4.3.1 A Change in Savings Rate

Let us assume that, starting at steady-state, with constant levels of the main variables in units of efficient labor, and a physical capital ratio k_{ss}^1, there is an increase in s, the constant savings rate. Then, the steady-state level of the physical capital ratio would increase to a new level k_{ss}^2, since its level depends positively on the value of the savings rate. A higher savings rate shifts the $sf(k_t)$ upwards, while leaving the $(n+\delta+\gamma)k_t$ function unchanged. Therefore, at k_{ss}^1 we will no longer be at steady-state but rather, to the left of it. As a consequence, right after the increase in savings rate, the stock of capital starts a gradual increase. A similar process is followed by income, $y_t = f(k_t)$, its rate of growth instantaneously jumping and becoming positive at the time of the increase in the rate of savings, and gradually decreasing back to zero as capital and income converge to their new steady-state levels. Later on, when the level k_{ss}^2 is attained, income per unit of efficient labor will again remain constant. Consumption $c_t = (1-s)f(k_t)$ experiences a discontinuity, with an initial fall due to the increase in s. These effects are shown in Fig. 2.9.

With respect to steady-state effects on consumption, we have from (2.17),

$$c_{ss} = f(k_{ss}) - (\delta + n + \gamma)\, k_{ss},$$

so that,

$$\frac{\partial c_{ss}}{\partial s} = \left[f'(k_{ss}) - (\delta + n + \gamma) \right] \frac{\partial k_{ss}}{\partial s},$$

which will be positive so long as,

$$f'(k_{ss}) > \delta + n + \gamma,$$

because $\frac{\partial k_{ss}}{\partial s}$ is always positive, as can be seen in (2.29). Initial consumption will always experience a jump down if a higher savings rate is implemented, but steady-state consumption can be either above or below the steady-state level of consumption with the old savings rate, as we will show in a numerical exercise in Sect. 2.5.4. In fact, an examination of (2.27) shows that steady-state consumption will increase following a rise in savings rate if the initial steady-state had a stock of capital below that associated to the Golden Rule, decreasing otherwise.

Effects following a fall in savings rate are just the opposite of those discussed above.

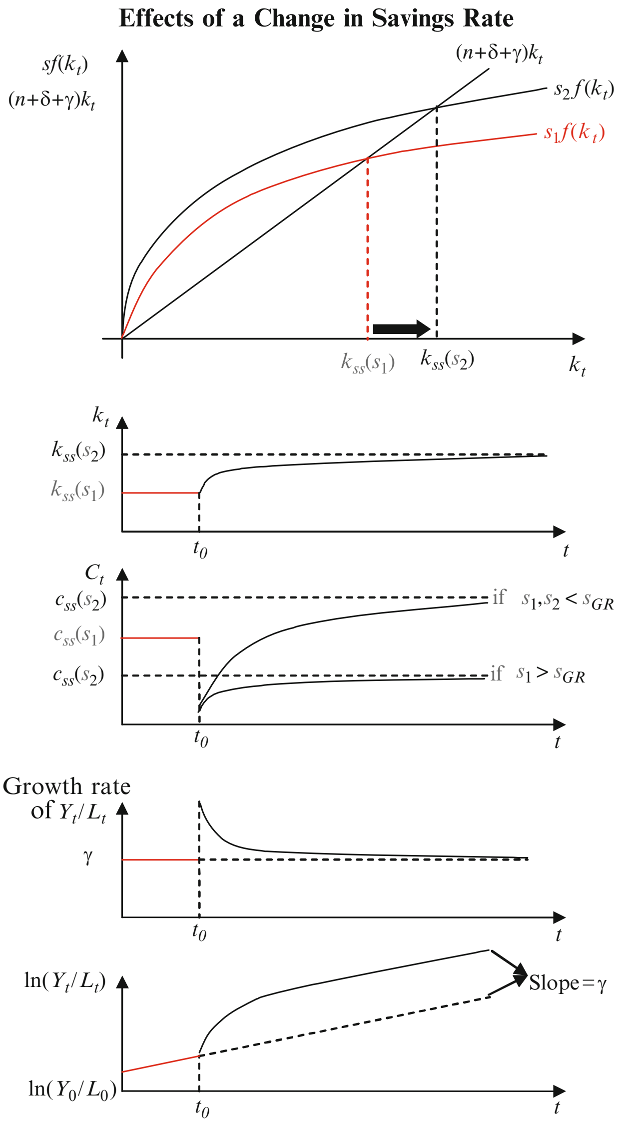


Fig. 2.9 Effects of a Change in Savings Rate

2.4.3.2 Structural Changes

We extend now the analysis of the previous paragraph, to consider the effects of changes in the values of the savings rate, s, the rate of growth of population, n, the rate of depreciation of physical capital, δ, the rate of technological growth, γ, and the output elasticity of capital, α. These effects can be summarized,

$$\left(\begin{array}{ccccccccc} & k_{ss} & c_{ss} & y_{ss} & \omega_{ss} & r_{ss} & Y/K & Y/N & \dot{Y}/Y \\ s & + & ? & + & + & - & - & + & 0 \\ n & - & - & - & - & + & + & - & + \\ \delta & - & - & - & - & + & + & - & 0 \\ \gamma & - & - & - & - & + & + & - & + \\ \alpha & + & + & + & + & - & - & + & 0 \end{array}\right)$$

where ω_{ss}, r_{ss} denote steady state values of the real wage and the real rate of interest. The reader may be familiar with the standard result that, when a firm takes factor prices as determined outside their control, profit maximization leads to use the production factors to the point where their marginal products equal their respective price. Even though we do not enter at this point in any detailed assumption on the structure of markets for production factors, we use the mentioned properties to justify considering real wages and interest rates defined by,[12]

$$\omega_t = f(k_t) - k_t f'(k_t),$$
$$r_t = f'(k_t),$$

with similar relationships holding in steady-state. The real rate of interest is inversely related to the steady-state stock of capital, while the real wage is positively related to it:

$$\frac{\partial r_{ss}}{\partial \eta} = \frac{\partial r_{ss}}{\partial k_{ss}}\frac{\partial k_{ss}}{\partial \eta} = f''(k_{ss})\frac{\partial k_{ss}}{\partial \eta}$$
$$\Rightarrow \ sign\left(\frac{\partial r_{ss}}{\partial \eta}\right) = -sign\left(\frac{\partial k_{ss}}{\partial \eta}\right), \ \eta = n, \delta, \gamma, s, \alpha$$
$$\frac{\partial \omega_{ss}}{\partial \eta} = \frac{\partial \omega_{ss}}{\partial k_{ss}}\frac{\partial k_{ss}}{\partial \eta} = -k_{ss} f''(k_{ss})\frac{\partial k_{ss}}{\partial \eta}$$
$$\Rightarrow \ sign\left(\frac{\partial \omega_{ss}}{\partial \eta}\right) = sign\left(\frac{\partial k_{ss}}{\partial \eta}\right), \ \eta = n, \delta, \gamma, s, \alpha.$$

To analyze the effect of a parameter change on consumption and output we use the relationships:

$$\frac{\partial c_{ss}}{\partial \xi} = (1-s) f'(k_{ss})\frac{\partial k_{ss}}{\partial \xi} \Rightarrow sign\left(\frac{\partial c_{ss}}{\partial \xi}\right) = sign\left(\frac{\partial k_{ss}}{\partial \xi}\right); \xi = n, \delta, \gamma, \alpha$$
$$\frac{\partial c_{ss}}{\partial s} = (1-s) f'(k_{ss})\frac{\partial k_{ss}}{\partial s} - f\,(k_{ss}),$$

[12]This assumption is not a proper element of the Solow–Swan model, which does not leave any role for a profit maximizing behavior on the part of producers of the single good in the economy.

$$\frac{\partial y_{ss}}{\partial \eta} = f'(k_{ss})\frac{\partial k_{ss}}{\partial \eta};\ \eta = n, \delta, \gamma, s, \alpha \Rightarrow sign\left(\frac{\partial y_{ss}}{\partial \eta}\right) = sign\left(\frac{\partial k_{ss}}{\partial \eta}\right).$$

The average product of capital $Y_t/K_t = \frac{f(k_t)}{k_t}$ satisfies: $\frac{\partial(Y_t/K_t)}{\partial k_t} = -\frac{f(k_t)-k_t f'(k_t)}{k_t^2}$ which is negative, since the numerator is equal to the real wage. Hence, average productivity of capital moves contrary to the *capital–labor* ratio. On the other hand, the average product of labor, $\frac{Y_t}{\Gamma_t N_t} = f(k_t)$, moves in the same direction as the capital–labor ratio. Finally, the rate of growth of output (or income) can be written: $\dot{Y}_t/Y_t = \dot{y}_t/y_t + n + \gamma$, its steady-state value being affected just by population growth and the rate of technological progress, since the rate of growth of income per unit of effective labor is zero in steady-state.

As an example, we have already seen that an increase in savings rate raises the steady-state stock of capital and output. The effect on the steady-state level of consumption depends on whether the initial stock of physical capital is above or below the Golden Rule level. The real rate of interest and the average productivity of capital will be lower while the real wage and the marginal product of labor will increase.

A change in savings rate could be thought of as being an economic policy intervention, specially since a higher rate will take the economy to a steady-state with higher per capita income. However, as discussed in the section devoted to the Golden Rule, it is far from clear that the sacrifices needed to place the economy on the path converging to the higher income steady-state are desirable in terms of time aggregate welfare. There is no much more room for policy analysis in the Solow–Swan setup, since it is hard to believe that the depreciation rate of physical capital or the rate of growth of population could be controlled by the government.[13]

2.4.4 Dynamic Inefficiency

If we consider an economy at a steady-state situation under a given savings rate, and we want that economy to converge to the Golden Rule, all we need to do is to set the savings rate equal to s_{GR}, since the global stability of the Solow–Swan model guarantees that any economy will converge to the steady-state associated to the prevailing savings rate. Following such change in savings rate, the economy would start a transition, along which the level of consumption will be changing every period, eventually converging to the level achieved at the Golden Rule. However, single-period consumption along the transition might be not only lower than the Golden Rule level, but also lower than the level of consumption at the initial steady-state. This is important, since it is then unclear that consumers' would prefer

[13]Even though in some European countries, tax incentives have recently been introduced in an attempt to increase the birthrate.

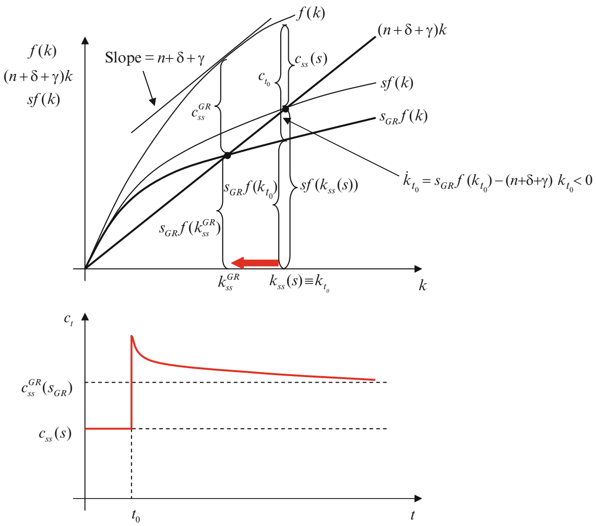


Fig. 2.10 Dynamically inefficient steady-state

entering into the transition trajectory taking the economy to the Golden Rule, to staying at the initial steady-state.[14]

As we pointed out at Sect. 2.3.8, factors influencing that comparison are: the magnitude of the utility loss along the transition, the difference in the utility levels at the Golden Rule and at the initial steady-state, the number of periods needed to reach the Golden Rule, the time discount factor applied to future utility. Let us now see how all these effects aggregate. Steady-states to the right of k_{ss}^{GR}, between k_{ss}^{GR} and $\hat{k}$, are *dynamically inefficient*, since starting from either one of them, a decrease in the savings rate starts a trajectory along which, at any time period, per-capita consumption is higher than at the initial state. Starting from either one of these steady-states, consumers would be happy to change the prevailing savings rate to s_{GR} forever.

In Fig. 2.10, suppose we start from a savings rate of s and a steady-state stock of capital equal to $k_{ss}(s)$. If we reduce the savings rate to s_{GR}, then per capita consumption will *immediately jump* from $c_{ss}(s)$ to c_{t_0}, which is higher than c_{ss}^{GR}. This dynamics implies a *gradual decrease* in the stock of capital, from $k_{ss}(s)$ towards k_{ss}^{GR}, which will imply, in turn, that per capita consumption will *gradually decrease* from c_{t_0} towards c_{ss}^{GR}. But c_{ss}^{GR} is still higher than $c_{ss}(s)$, since the Golden Rule is the steady-state with the highest consumption. Therefore, the decrease in the savings rate will have produced a path along which, at each point in time, per capita consumption is higher than the initial consumption level, before the change in

[14]The reader should not have much problem thinking about an economy which starts outside steady-state and changes its savings rate to s_{GR}.

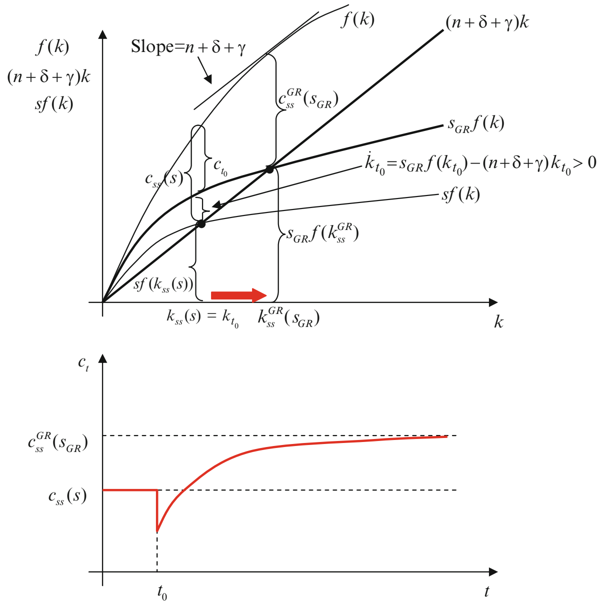


Fig. 2.11 Dynamically non-inefficient steady state

savings rate. This is true for any steady state to the right of the Golden Rule, which is why they are said to be *dynamically inefficient*.

The opposite is the case for steady-states to the left of k_{ss}^{GR}. Starting from $k_{ss}(s)$ in Fig. 2.11, a permanent increase in savings rate from s to s_{GR} will produce an immediate fall in consumption from $c_{ss}(s)$ to c_{t_0}. This is lower than initial consumption, $c_{ss}(s)$, and also lower than the level of consumption at the Golden Rule. The new steady-state is given by k_{ss}^{GR}, and the stability of the model implies that the economy starts a trajectory with the stock of capital gradually increasing from $k_{ss}(s)$ to k_{ss}^{GR}. The graph shows how along that trajectory, given the savings rate of s_{GR}, the level of consumption will gradually increase towards c_{ss}^{GR}. We know that c_{ss}^{GR} will be higher than $c_{ss}(s)$, since that is the characteristic defining the Golden Rule among all feasible steady-states. However, along the transition, consumption would have spent some periods below the level of the initial steady state. Hence, it is unclear that when we compute the associated period-by-period utility and aggregate over time its discounted value, we will reach a higher or a lower level than the one that would be obtained at the initial steady-state. We cannot say whether these steady-states are inefficient or not. An informal argument suggests that it is those steady-states with a low rate of savings which may be inefficient, since the associated level of consumption might then be very low. As a consequence, even though an increase in savings rate will require consumption sacrifices in the short-run, the opportunity to accumulate physical capital and reach higher levels of output and consumption may compensate for the short-run sacrifices. The elements mentioned in Sect. 2.3.8 will help determine which is the range of values for the saving rate for which a permanent increase may be welfare improving. A numerical examination of this issue is performed in a section below for the discrete time version of the Solow–Swan model, that we introduce next.

2.5 The Deterministic, Discrete-Time Solow–Swan Model

2.5.1 The Exact Solution

Theoretical models are built not only to analyze a variety of positive and normative issues, but also to be confronted with actual data, in an attempt to validate their implications. The continuous-time version of the Solow–Swan model can be used to produce time series for physical capital, output, consumption and investment by sampling at discrete points in time, from the continuous time processes obtained from (2.30) and the implied expressions for the remaining variables. Discrete sampling amounts to giving discrete values: $t = 1, 2, 3, \ldots$ to the time index in those expressions. This apparently innocuous procedure is subject, however, to potential pitfalls, that will be illustrated numerically in the next chapter.

An alternative method consists on analyzing directly the discrete version of the Solow–Swan model. To do so, we could think of directly translating the law of motion into discrete time by substituting a time difference $k_{t+1} - k_t$ for the time derivative $\dot{k}_t$, like in:

$$k_{t+1} - k_t = sf(k_t) - (n + \delta + \gamma)\, k_t. \tag{2.34}$$

Unfortunately, we are about to see that this procedure is also subject to some flaws. If we start from the discrete time analytical representation of all the assumptions characterizing the model, we will end up with a fully justified equation somewhat different from (2.34).

Maintaining the same assumptions on savings, capital formation, population growth and full employment as in the continuous time version of the model, let us now consider the possibility that there is *exogenous technological growth*, in the form of a variable productivity factor Γ_t, that grows at a constant rate γ:

$$\Gamma_t = (1 + \gamma)\Gamma_{t-1} \ ,$$

from an initial Γ_0 level. The aggregate production function is of the form $Y_t = F(K_t, \Gamma_t N_t)$, with the same assumptions on first and second order derivatives as in the continuous time model. Inada conditions are also assumed to hold. *Effective labor* is again defined as $\Gamma_t N_t$.

Because of the aggregate constant returns to scale assumption we again have,

$$Y_t = F(K_t, \Gamma_t N_t) = \Gamma_t N_t F\left(\frac{K_t}{\Gamma_t N_t}, 1\right) = \Gamma_t N_t f(k_t),$$

where $k_t = \frac{K_t}{\Gamma_t N_t}$ is the stock of *capital per unit of effective labor*, and $f(k_t) = F(\frac{K_t}{\Gamma_t N_t}, 1)$. Output per unit of effective labor is: $y_t = \frac{Y_t}{\Gamma_t N_t} = f(k_t)$.With the Cobb–Douglas specification, $F(K_t, \Gamma_t N_t) = AK_t^{\alpha}(\Gamma_t N_t)^{1-\alpha}$, $0 < \alpha < 1$, we have the same expressions as in continuous time: $Y_t = AK_t^{\alpha}(\Gamma_t N_t)^{1-\alpha} = A\Gamma_t N_t k_t^{\alpha} = \Gamma_t N_t f(k_t)$, with $f(k_t) = Ak_t^{\alpha}$ and *output per unit of effective labor*: $y_t = \frac{Y_t}{\Gamma_t N_t} = Ak_t^{\alpha}$.

In the discrete time version of the model investment is defined by: $I_t = K_{t+1} - (1-\delta)K_t$, so the National Income identity becomes,

$$C_t + I_t = C_t + [K_{t+1} - (1-\delta)K_t] = F(K_t, \Gamma_t N_t) = Y_t \Rightarrow$$
$$\Rightarrow \frac{C_t}{\Gamma_t N_t} + \left[\frac{K_{t+1}}{\Gamma_{t+1} N_{t+1}} \frac{\Gamma_{t+1} N_{t+1}}{\Gamma_t N_t} - (1-\delta)\frac{K_t}{\Gamma_t N_t}\right] = \frac{Y_t}{\Gamma_t N_t},$$

which, maintaining the assumption of constant population growth,[15] $N_t = (1+n)^t N_0$, and constant technological growth, $\Gamma_t = (1+\gamma)^t \Gamma_0$, leads to the law of motion in per capita variables,

$$c_t + [(1+n)(1+\gamma)k_{t+1} - (1-\delta)k_t] = f(k_t). \tag{2.35}$$

If we again consider a closed economy in which no external sector or government could finance private investment, we will have equality between savings and investment each period $S_t = I_t$, and if we add the crucial assumption of the Solow–Swan model that the savings rate is constant, we have, $S_t = sY_t$,

$$C_t + sY_t = Y_t \Rightarrow C_t = (1-s)Y_t,$$

with a similar relationship in per capita terms, $c_t = (1-s)y_t = (1-s)f(k_t)$, which allows us to write (2.35) as,

$$k_{t+1} = \frac{1}{(1+n)(1+\gamma)} sf(k_t) + \frac{1-\delta}{(1+n)(1+\gamma)} k_t. \tag{2.36}$$

Now we can see the point we raised before. This equation can be written,

$$k_{t+1} - k_t = sf(k_t) - [n + (1+n)\gamma]k_{t+1} - \delta k_t, \tag{2.37}$$

which shows some differences with respect to (2.34). The latter was just a rough approximation to the continuous time model, expression (2.37) being the correct discrete-time version of the model.

[15] Notice the different analytical representation for growth rates, relative to the exponential functions used in the continuous-time version of the model.

This difference equation allows us to obtain a numerical solution to the model given an initial condition on the single state variable in the economy, the stock of capital, k_0, a specific functional form for the available technology, $f(k_t)$, and a given parameterization. Indeed, if we assume, for instance, $f(k_t) = Ak_t^{\alpha}$, then we could substitute the numerical value defining the initial condition on k_0 for k_t in (2.37) to obtain the level of k_1. We would then use k_1 as k_t in the equation, to obtain the level of k_2, and so on. The time series for output would be obtained from $y_t = f(k_t) = Ak_t^{\alpha}$, investment, which is equal to savings in this closed economy without government would be given by $i_t = s_t = sy_t$, while the time series for consumption would be obtained by: $c_t = (1-s)y_t = y_t - i_t$. This is the *exact solution* to the deterministic, discrete-time version of the Solow–Swan model.

An argument similar to the one we made in the continuous time case, shows that zero is the only possible steady-state rate of growth of the stock of capital per worker. The steady state of this economy is found by making $k_{t+1} = k_t = k_{ss}$,

$$\begin{aligned} k_{ss} &= \frac{1}{(1+n)(1+\gamma)} sf(k_{ss}) + \frac{1-\delta}{(1+n)(1+\gamma)} k_{ss} \Rightarrow \\ &\Rightarrow [n+\delta+(1+n)\gamma]\, k_{ss} = sf\,(k_{ss})\,. \end{aligned} \tag{2.38}$$

Once again, we have one such expression for each possible constant value of the savings rate, each one leading to a different steady-state. For instance, with a Cobb–Douglas technology, $y_t = Ak_t^{\alpha}$, we would get,

$$k_{ss} = \left(\frac{sA}{n+\delta+(1+n)\gamma}\right)^{\frac{1}{1-\alpha}}, \tag{2.39}$$

slightly different from the expression we obtained in the continuous time formulation of the model. In general, the product $n\gamma$ will be small, so both expressions will lead to a similar steady-state. Since the power is positive, (2.39) shows that the steady-state level of the stock of capital is higher for higher savings rates or higher technology levels, as well as for lower depreciation rates or lower rates of population growth, as in the continuous time case.

2.5.2 Approximate Solutions to the Discrete-Time Model

As the continuous-time model, the discrete-time version of the Solow–Swan economy can be solved exactly through the use of (2.36), as we will show in a section below. That is an exception, since nonlinearities in growth models will usually preclude the existence of an exact solution. To familiarize the reader with that practice, we proceed in this section to obtain the solution to the *linear* and the *quadratic approximations* to the model.

Considering the nonlinear difference equation in (2.36) as a function $k_{t+1} = \Psi(k_t;\theta)$ and using Taylor's expansion and (2.38), the *linear approximation* to that equation around steady-state is,

$$
\begin{aligned}
k_{t+1} - k_{ss} &= \Psi(k_{ss}) + \left(\frac{\partial \Psi(k_t;\theta)}{\partial k_t} \right)_{ss} (k_t - k_{ss}) \\
\Rightarrow k_{t+1} &\simeq \left(\frac{1}{(1+n)(1+\gamma)} sf(k_{ss}) + \frac{1-\delta}{(1+n)(1+\gamma)} k_{ss} \right) \\
&\quad + \left(\frac{1}{(1+n)(1+\gamma)} sf'(k_{ss}) + \frac{1-\delta}{(1+n)(1+\gamma)} \right) (k_t - k_{ss}) \\
&= k_{ss} + \frac{sf'(k_{ss}) + (1-\delta)}{(1+n)(1+\gamma)} (k_t - k_{ss}),
\end{aligned}
$$

which, in the special case of a Cobb–Douglas technology, $f(k_t) = Ak_t^{\alpha}$, $0 < \alpha < 1$, becomes,

$$
k_{t+1} \simeq k_{ss} + \frac{s\alpha A(k_{ss})^{\alpha-1} + (1-\delta)}{(1+n)(1+\gamma)} (k_t - k_{ss}) = k_{ss} + D(k_t - k_{ss}), \tag{2.40}
$$

with

$$
\begin{aligned}
D &= \frac{s\alpha A(k_{ss})^{\alpha-1} + (1-\delta)}{(1+n)(1+\gamma)} \\
&= \alpha \frac{n + \delta + (1+n)\gamma}{(1+n)(1+\gamma)} + \frac{1-\delta}{(1+n)(1+\gamma)},
\end{aligned} \tag{2.41}
$$

where we have used (2.39) to obtain the last expression and, finally, the linear approximation,

$$
\begin{aligned}
k_{t+1} - k_{ss} &\cong D(k_t - k_{ss}) \\
&= \left[\frac{(1+\alpha n) - (1-\alpha)\delta}{(1+n)(1+\gamma)} + \alpha \frac{\gamma}{1+\gamma} \right] (k_t - k_{ss}).
\end{aligned} \tag{2.42}
$$

Iterating from an initial condition k_0, we get,

$$
k_t = k_{ss} + D^t (k_0 - k_{ss}), \tag{2.43}
$$

which will converge to steady state so long as $| D | < 1$, i.e., if:

$$
(1-\alpha)(n + \delta + (1+n)\gamma) > 0,
$$

which is clearly the case, since $0 < \alpha < 1$. Therefore, under this condition, the linearized system is *stable*. As time passes, the capital stock converges to its steady-state level, k_{ss}, with independence of the initial stock of capital, as we have already shown to happen in the continuous time version of the model.

For a better approximation, we could also use a second order Taylor's expansion to (2.36), by adding to the linear approximation a second order term

$$\frac{1}{2}\left(\frac{\partial^2 \Psi(k_t;\theta)}{\partial (k_t)^2}\right)_{ss} (k_t - k_{ss})^2 = \frac{1}{2}\frac{1}{(1+n)(1+\gamma)} s f''(k_{ss})(k_t - k_{ss})^2,$$

which, in the case of a Cobb–Douglas technology, leads to the approximation,

$$k_{t+1} \simeq k_{ss} + \left[\frac{(1+\alpha n) - (1-\alpha)\delta}{(1+n)(1+\gamma)} + \alpha\frac{\gamma}{1+\gamma}\right](k_t - k_{ss}) + \frac{1}{2}\frac{\alpha(\alpha-1)}{(1+n)(1+\gamma)} sAk_{ss}^{\alpha-2}(k_t - k_{ss})^2. \tag{2.44}$$

In the numerical exercise in the next section, this approximation is compared to the linear approximation above.

2.5.3 Numerical Exercise: Solving the Deterministic Solow–Swan Model

In the *Discrete* spreadsheet in the *Solow_deterministic.xls* file, time series are obtained for a deterministic, discrete-time version of the Solow–Swan economy from an initial capital stock of $k_0 = 20$. Aggregate technology is supposed to be of the Cobb–Douglas type, with a capital share of $\alpha = 0.36$, and a technological constant $A = 5.0$. Depreciation of physical capital is $\delta = 7.5\,\%$, savings are $36.0\,\%$ of output each period, and we assume zero population growth, $n = 0$. Since the savings rate is equal to the output elasticity of capital, the steady-state in this economy will be the Golden Rule.[16] With these parameter values, steady state levels turn out to be: $k_{ss} = 117.94, y_{ss} = 27.85, c_{ss} = 17.82, s_{ss} = i_{ss} = 10.02$. Therefore, the economy starts to the left of the steady-state, with a stock of capital well below the steady-state level. The constant savings rate is relatively high, and capital accumulates quickly because the level of savings initially exceeds from total depreciation expenditures.[17] After 16 periods, the economy has covered half the initial distance to steady-state, with a stock of capital above 70 units. The *Discrete* spreadsheet presents time series for 260 periods, and the discrete time model is solved using the exact solution (2.36), as well as using the solutions to the linear and quadratic approximations (2.43), (2.44) to the discrete-time model. The resulting time series for the stock of capital under the different approaches are reported in the first panel. The time series for output, savings and consumption that are obtained

[16] This is not necessary for the exercise, as the reader may see by changing the value of either the savings rate or the output share of capital.

[17] Which are obtained by adding the depreciation loss to the need to provide new workers with the same stock of capital than the older ones.

under the exact solution are shown in Panel 2, while Panels 3 and 4 display the similar time series obtained under the linear and quadratic approximations to the model. Notice that, according to the model, output is obtained each period from the stock of capital accumulated at the end of the previous period. As in subsequent exercises, this is organized in the spreadsheet by making output to be a function of the stock of capital in the previous row. That is, in the row corresponding to time t we have k_{t+1} and variables like y_t, c_t. (The same exercise can be reproduced by Matlab file: *Solow_stochastic.m* by setting the variance parameter *sigmae* to zero.) Consumers' preferences do not play any role in this exercise. Nevertheless, to familiarize the reader with the type of welfare evaluation that will often be performed in the next chapters, consumers are supposed to have a constant relative risk aversion utility function, $U(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$, with risk aversion coefficient of σ =3.0, and a time discount factor $\beta = 0.95$, and we compute single-period as well as time-aggregate, discounted utility.

We also present percent errors from the linear and the quadratic approximation, both for the stock of capital and for consumption. The approximation error for the capital stock starts around 17 % in the initial periods, when the economy is far away from steady-state, increasing during the first periods up to 40 % of the actual value, and quickly going to zero over time. These clearly excessive errors steam from the fact that the initial condition is far away from the steady-state, the point around which we have done the approximations to the law of motion of the economy. The approximation error for consumption starts at around 6 %, and increases in the initial phase of the transition to steady-state, decreasing to zero as time passes. As can be seen in the reported time series and the accompanying graph (*Comparing solutions* spreadsheet), approximation errors for the linear and the quadratic approximations are very similar, so that the contribution of the quadratic term to the linear approximation is minor.

For the sake of comparison, we also compute in Panel 1 the time series that would be obtained by observing the continuous process at regular intervals of time. We report time series obtained from the exact solution to the continuous-time model (2.30), as well as those obtained form the solution to the linear approximation to that model (2.32). Unfortunately, as we already mentioned, and it will be discussed in the next chapter, this latter approach of extracting discrete numerical observations from a continuous process is potentially subject to significant pitfalls. In this case, however, the exact continuous and discrete solutions are very similar to each other, while the continuous linear approximation is very close to the discrete linear approximation.

The *Increasing time path* and *Decreasing time path* spreadsheets present two transition economies. Both share the same parameter values: $\alpha = 0.36$, $A = 3.0$. $\delta = 7.5\,\%$, $s = 0.30$, $n = 0.01$, $\gamma = 0.01$. The implied steady-state is: $k_{ss} = 33.504$, $y_{ss} = 10.621$, $c_{ss} = 7.435$, $s_{ss} = i_{ss} = 3.186$. Since the savings rate is lower than the output elasticity of capital, this steady-state falls below the Golden Rule, which is in this case: $k_{GR} = 44.547$. In the first economy, initial capital is $k_0 = 30.0$, converging to steady-state from below, as it was the case with the economy in the *Discrete* spreadsheet. The second economy starts from $k_0 = 45.0$, converging to steady-state from above. In these two exercises, we present in Panel 1 the time series for the stock of capital, investment, consumption, output and output growth, as well as single period utility and its discounted value using the exact solution. The last column shows the time series for the stock of capital that would be obtained observing the continuous solution at discrete intervals of time. In Panel 2 we show the full solution obtained from the linear approximation (2.42) to the discrete-time problem, while Panel 3 displays the solution obtained from the discrete quadratic approximation (2.44). Approximation errors are much smaller in these two economies, as a consequence of their relative proximity to steady-state.

2.5.4 Numerical Exercise: A Permanent Change in the Savings Rate

The discrete-time version of the Solow–Swan economy is numerically solved in the *Change_savings.xls* file to simulate the effects of a *permanent increase* in the constant savings rate. The analytical details of this structural change were described in Sect. 2.4.3 [Matlab file: *change_savings.m* performs the same exercise]. Two different parameter structures are analyzed, and in each of the two implied model economies we consider a permanent increase in the savings rate. The exercise is performed twice, to analyze the effects of changes of different size in the savings rate. Effects from a permanent fall could be discussed similarly.

Consumption always falls immediately after the jump in savings rate. In one of the two economies, long-run consumption ends up above its steady-state level before the rise in savings rate, while in the other economy, steady-state consumption after the increase in savings rate is below the steady-state level of consumption for the initial, lower savings rate. As we saw in Sect. 2.4.3, the long-run effect on steady-state consumption of a permanent change in savings rate depends on whether the initial steady-state is above or below the Golden Rule. Steady-state consumption may end up being higher under a higher savings rate because that may allow for a more intense accumulation of capital stock, leading to higher output, which may leave more resources available for consumption, even after providing for the reposition of the stock of capital lost to depreciation.

Assuming a Cobb–Douglas technology, parameter values for the first economy are $\delta = 0.075, n = 0.01, A = 3.0, \alpha = 0.36, \gamma = 0.0$. The population starts at $t = 0$ from an initial value of 100. In the $C - increases(large)$ spreadsheet, the initial savings rate is $s = 0.20$, which is in place until period $t = 11$, when it increases to $s = 0.35$. The steady state stock of capital under the initial savings rate is $k_{ss} = 21.19$, which allows for steady-state output: $y_{ss} = Ak_{ss}^{\alpha} = 9.006$. A percentage of 20 % of this, 1.801 units of commodity, are devoted to investment, the remaining 7.205 units of commodity being consumed. The 1.801 units of commodity being invested allow for recovering the depreciation loss of 7.5 % of k_{ss}, in addition to providing the 1 % new consumers/workers being born every period, with the 21.19 units of steady state capital. In other words, 1.801 is precisely equal to 8.5 % of steady-state capital ($n + \delta + \gamma = 0.085$), as we know it should be the case.

Under the new savings rate of $s = 0.35$, the steady-state level of physical capital is 50.80 units, with output: $y_{ss} = Ak_{ss}^{\alpha} = 12.338$. Investment is 35 % of output, or 4.318 units of commodity, with consumption equal to 8.020 units of commodity every period in the new steady-state. So, the new, higher savings rate, allows for such an increase in the stock of capital that resources left for consumption after the reposition of depreciated capital are higher than those that could be consumed under the old, lower savings rate of 20 %. We assume the representative consumer in the economy has a constant, relative risk aversion utility function on current consumption: $U(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$, with $\sigma = 3.0$, and a discount factor on future utility of $\beta = 0.95$.

We solve the economy in three ways: first, we provide in Panel 1 the exact solution, obtained from the difference equation (2.36). The second method uses the linear approximation (2.43) to steady-state to obtain the stock of capital as a function of the distance between the previous period stock of capital and the steady-state level [Panel 2]. The third solution approach uses the second order approximation around steady-state (2.44) [Panel 3].

The savings rate is supposed to change at $t = 11$. It is central to the exercise to examine how the stock of physical capital is computed at that period. At that point in time, the economy is no longer in steady state. The new value of the savings rate must be used in Eqs. (2.36), (2.43), (2.44), when computing the exact solution, or the solutions to the linear and quadratic approximations to the model, respectively. Additionally, in (2.43) and (2.44), the steady-state level of capital under the new, higher savings rate must replace the steady-state level obtained under the old savings rate. The value of the D-constant in the linear approximation does not need to be updated in this case, since it is not affected by changes in savings rate. Changes in the rate of depreciation, the output elasticity of capital or population growth would change the value of D.

Graphs under the *Comparing solutions* and *Approximation error* spreadsheets shows that numerical differences among solution methods can be relatively large if the change in savings rate is sizeable. In particular, the quadratic term does not add anything significant to the linear approximation, both being very similar. That is the case in this first simulation, in which the savings rate jumps from 20 to 35 %, and the percent approximation error approaches 4 % for a few periods after the change, to then gradually decrease towards zero.

In all cases, output is obtained using the analytical representation for the Cobb–Douglas production function, savings is obtained as a proportion of income, investment is equal to savings, and consumption is the proportion of output which is not saved. Growth in per-capita output is also computed under the three solution approaches, and it is displayed in the *Output growth* spreadsheet for the first experiment. Numerical values for single period utility are also reported. These are also discounted and aggregated over time. The resulting level of welfare is 9.804 under the linear approximation and to 9.802 under the exact solution.

Graphs to the right of the simulated data display the time behavior of the main variables after the savings rate increases from 20 to 35 %. Growth of output per unit of efficient labor jumps from 0 to 2.2 % the period when savings rate increases, smoothly decreasing to zero afterwards.

The C – *increases* (*small*) spreadsheet presents an experiment in the same economic structure as above, but with a smaller increase in savings rate, which moves from 30 to 35 % at $t = 11$. For the sake of comparison, we have maintained the same ranges in the graphs displaying the responses of the main variables in the spreadsheets that contain the two changes considered in savings rate. It is quite evident that the effects of the 5-point increase in savings considered in the second case are rather smaller than those of the 10-point increase considered in the first analysis.

The C – *decreases* (*small*) spreadsheet presents a case in which steady-state consumption decreases following an increase in the savings rate from 30 to 35 %. Remaining parameters are $\delta = 2.5\,\%, n = 1.0\,\%, A = 5.0, \alpha = 0.25$. The steady state stock of capital under the initial savings rate is $k_{ss} = 149.98$, which allows for steady-state output: $y_{ss} = Ak_{ss}^{\alpha} = 17.50$. A percentage of 30 % of these, 5.249 units of commodity, are devoted to investment, the remaining 12.248 units of commodity being consumed. The resources being saved allow for recovering the depreciation loss of 2.5 % of k_{ss}, in addition to providing the 1 % new consumers/workers being born every period with the 149.98 units of steady state capital. In other words, 5.249 is equal to 3.5 % of steady-state capital ($n + \delta + \gamma = 0.035$). Under the new savings rate of $s = 35\,\%$, the steady-state level of physical capital is 184.20 units, with output: $y_{ss} = Ak_{ss}^{\alpha} = 18.42$. Investment is 35 % of output, or 6.447 units of commodity, with consumption equal to 11.973 units of commodity every period in the new steady-state. So, in this case, the higher savings rate leads to an increase in capital accumulation, but the implied growth in per capita income is not enough to allow for higher steady-state consumption once capital depreciation is accounted for. We maintain the same preferences but consider a discount factor $\beta = 0.90$.

The C – *decreases* (*large*) spreadsheet presents the same exercise above, except for a somewhat increase in savings rate, from $s = 30\,\%$ to $s = 40\,\%$.

2.5.5 Numerical Exercise: Dynamic Inefficiency

The *Dynamic_inefficiency.xls* file [Matlab file: *Dynamic_inefficiency.m* performs the same exercise] presents the transition trajectories for a number of economies differing in the level of their savings rate. Growth in technology is not considered in this exercise, so $\gamma = 0$. Each economy is supposed to be initially at steady-state. At some point, the savings rate experiences a permanent change, jumping to the level corresponding to the Golden Rule, where it stays forever. As we already know, that level is equal to the output elasticity of physical capital, which is taken to be 0.36 in this exercise. After the change in savings rate, the stock of capital quickly approximates the level corresponding to the Golden Rule. If the savings rate was initially above 0.36, the stock of capital will exponentially decrease after the fall in savings rate, the opposite being the case if the savings rate increases from an initial steady value below 0.36.

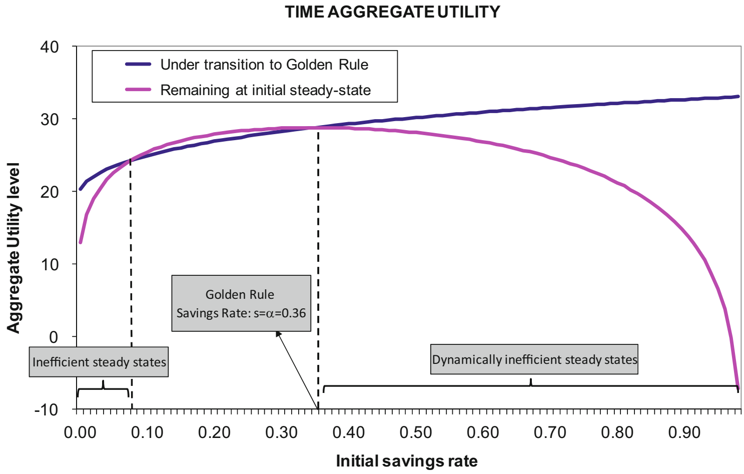


Fig. 2.12 Inefficient steady-states

After presenting the parameter values in the *Simulations* spreadsheet, we provide the different levels of the savings rate considered, together with their associated steady-state levels of physical capital and consumption, the single period utility in the steady-state prior to the change in savings rate and the time aggregate utility that would obtain by staying at that steady-state, i.e., the time aggregate level of utility with no change in savings rate. Steady-state consumption is zero for $s = 0.0$ or $s = 1.0$, so the level of utility cannot be computed in this case for some utility functions.

Below that, we present time series over 250 periods along the convergence trajectories for the stock of capital. To compute them, we have used the law of motion for capital after a permanent switch to the Golden Rule of savings, starting from a stock of capital equal to the steady-state level before the change in savings rate. The panel below the trajectories for the stock of capital presents the consumption trajectories in their convergence to the Golden Rule steady-state: $c_t = (1-s_{GR})y_t = (1-\alpha)Ak_t^{\alpha}$. Below them, we show the discounted levels of utility along the transition, under constant relative risk aversion (CRRA) preferences, $U(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}, \sigma > 0$. A value $\sigma = 1.00$ is chosen as default to approximate logarithmic differences. Finally, we aggregate over time the discounted utility series, to compare those sums with the utility consumers would have by staying at the initial steady-states, with no change in savings rate. As we can see in Fig. 2.12, the former is higher for all economies that start with a savings rate above the Golden Rule level. For these economies, changing from the old savings rate to the Golden Rule rate of savings would be preferable. The same would be the case for economies starting with a low savings rate, between 0.0 and 0.10 in our numerical exercise. All these are the *dynamically inefficient steady-states*. Economies with a constant savings rate between 0.10 and 0.36 are not dynamically inefficient.

We should bear in mind that what we have shown in this section is that there are steady-states which are dominated, in terms of welfare, by trajectories that start when the savings rate experiences a once-and-for-all change from its initial level to the level associated with the Golden Rule steady-state. We have not shown in any sense that such trajectories leading to the Golden Rule are optimal in any sense. That is, converging to the Golden Rule is not necessarily the best an economy can do, although we have shown that it is sometimes preferable to staying at the current steady-state. To conclude on optimality, we need an specific analysis which is the object of the next chapter. There, we will characterize the optimal trajectory from any given initial situation. We will also show that, possibly against a first impression, converging to *the Golden Rule* is a suboptimal strategy, in the sense that *it involves too much capital accumulation* early on. The optimal trajectory takes the economy into a trajectory converging to a steady-state with a level of capital below that of the Golden Rule.

2.6 The Stochastic, Discrete Time Version of the Solow–Swan Model

To end the presentation of the constant savings rate growth model, we consider a stochastic version of the Solow–Swan economy that incorporates a random productivity factor. This is only one of the possibilities to make the model stochastic. We consider a technology, $f(k_t) = \theta_t A k_t^{\alpha}, 0 < \alpha < 1$, where θ_t denotes a *stochastic process* with a known probability distribution. Following the same argument as in the deterministic version of the economy, we find the law of motion,

$$k_{t+1} = \frac{1}{(1+n)(1+\gamma)} s\theta_t A k_t^{\alpha} + \frac{1-\delta}{(1+n)(1+\gamma)} k_t. \qquad (2.45)$$

We assume $E(\theta_t) = 1$ and $Var(\theta_t) = \sigma^2$, although a more general case, with time-varying moments could also be considered. The stochastic properties of the θ_t-process will determine those of the main variables in the economy: output, consumption and investment. In particular, if θ_t displays cycles, as it would be the case if it obeys a second order autoregression with complex roots in its characteristic equation, so will output and consumption.

The same analysis we made of the deterministic, discrete-time version of the model applies to this stochastic case. Hence, we just need to combine the same law of motion for capital (2.36) with the new, stochastic functional form for the technology.

The steady-state in a stochastic economy is obtained assuming that each stochastic processes takes its mean value every single period. In our case, the single stochastic productivity shock would take its mean value of 1, producing the same condition (2.39) characterizing steady state as in the deterministic case. Hence, the steady state levels of the stock of capital, output and consumption in units of efficient labor will be the same as in the deterministic case.

Then, the law of motion of this stochastic economy (2.45) can be approximated around steady state, to obtain,

$$\begin{aligned}k_{t+1} &= k_{ss} + \left(\frac{1}{(1+n)(1+\gamma)} s\theta_{ss} A\alpha k_{ss}^{\alpha-1} + \frac{1-\delta}{(1+n)(1+\gamma)}\right)(k_t - k_{ss}) \\ &\quad + \frac{1}{(1+n)(1+\gamma)} s\theta_{ss} A k_{ss}^{\alpha} (\theta_t - \theta_{ss}) \\ &= k_{ss} + \left(\frac{n+\delta+\gamma}{(1+n)(1+\gamma)}\alpha + \frac{1-\delta}{(1+n)(1+\gamma)}\right)(k_t - k_{ss}) \\ &\quad + \frac{n+\delta+\gamma}{(1+n)(1+\gamma)} k_{ss} (\theta_t - 1) . \end{aligned} \tag{2.46}$$

2.6.1 Numerical Exercise: Solving the Stochastic Solow–Swan Model

Excel file *Solow_stochastic.xls* presents a numerical solution for a stochastic version of the Solow–Swan model (Matlab file *Solow_stochastic.m* performs the same numerical exercise). We assume that randomness comes in the economy through a productivity shock with a first-order autoregressive structure,

$$\ln\theta_t = \rho \ln\theta_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2),\ 0 < \rho < 1, \quad \theta_0 = 1,$$

which is consistent with $\theta_{ss} = 1$. ε_t is the *innovation* in the logged-productivity shock. We consider in this simulation: $\rho = 0.90, \sigma_\varepsilon = 0.10$, which imply $E(\ln\theta_t) = 0, Var(\ln\theta_t) = (0.229)^2$. Parameter values are: $s = 0.36, \delta = 0.075, n = 0.0, A = 5.0, \alpha = 0.36$, so the steady-state is the Golden Rule. Technological growth is not considered.[18] This is not necessary for the exercise, and can be changed without any problem. We assume the representative consumer in the economy has a constant, relative risk aversion utility function on current consumption: $U(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$, with $\sigma = 3.0$, and a discount factor on future utility of $\beta = 0.90$. Deterministic steady-state levels are computed following the expressions in the text. The steady-state stock of capital is 143.41 units, which allows for steady-state production oscillating around 29.88. Two-thirds of this amount is devoted to consumption, as it is approximately the case in developed economies, while the remaining one-third is devoted to investment. The solution starts with a time series realization for the innovation ε_t from a Normal distribution with zero mean and $\sigma_\varepsilon = 0.10$, obtained with the random number generator included in the *Tools/Data Analysis* tab of EXCEL. Then the implied time series for

[18] It would be simple to incorporate it into the simulation, but it would not change the qualitative aspects of the discussion.

the logged productivity shock $\ln\theta_t$ is obtained using the autoregressive structure, from an initial condition $\ln\theta_0 = 0$.

The time series for the productivity shock θ_t is then taken to either (2.46) or (2.45), to obtain either an *approximate solution* or an *exact solution* to the model for the stock of physical capital starting from an initial condition k_0. We take as initial condition the steady-state stock of capital, so the generated numerical solution will display fluctuations around steady-state for all variables: stock of physical capital, output, investment and consumption. The production technology is then used to obtain a time series for output, while consumption and savings/investment emerge from the constant-savings rate assumption. The fact that we can generate all the time observations for θ_t without need of computing a single value for k_t reflects the fact that the productivity shock is *exogenous* in this economy.

It is interesting to bear in mind that the structure of the productivity shock will also determine the volatilities of these variables, as well as the correlations among them. Ratios to output, or deviations from an estimated cyclical component can be computed on this simulated data the same way it is usually done in time series analysis of actual data. Sometimes, standard deviations and correlations using these transformations are used to see how a theoretical model matches the data. Main statistics are shown below the simulated time series. The linear approximation is seen to produce time series with statistical properties very similar to those obtained under the exact solution. The relative volatility of consumption to output is similar to the one usually observed in actual data for most economies, which is not the case for the investment volatility, which is well higher than that of output in actual time series data. We also present correlation coefficients between interest rates, consumption and investment, with output.

Unfortunately, this model, where no agent takes any optimal decision, is so simple that the linear correlation coefficients between either consumption or investment and output are 1.0, as a consequence of the fact that the two variables are an exact proportion of output each period, with independence of the fluctuations experienced by the latter variable. For the correlation coefficient to depart from one, we would need different sources of randomness in the two variables considered, which is not the case in this model.

Regression models between some variables, like consumption and output, or investment and output, could also be estimated using the set of time series provided by a numerical solution, the same way it is done with actual data. However, the simplicity of the random element in this model economy would also lead to trivial regressions. An exception is a relationship attempting to relate investment to the real rate of interest. This would be defined by the marginal product of capital, as it has been calculated in the spreadsheet. The nonlinear functions of capital defining these two variables allow for a non-trivial regression, $\mathit{Investment}_t = \alpha + \beta.\ \mathit{Real\ interest\ rate}_t + u_t$, which is shown below the table of correlation coefficients.

The important point, however, is that although the EXCEL file presents a single time series realization for the endogenous variables, we could conceivably compute as many of these realizations as we wished. The reason is that dealing with a stochastic economy, we could repeat the process starting from a new, different realization for the productivity time series, by using again the random number generator tool of EXCEL. In fact, the *Stochastic*(2),(3) and (4) spreadsheets are identical to the *Stochastic* spreadsheet except by the realization of the productivity shock.[19] By sampling repeatedly from the stochastic process for productivity, we could get a large number of realizations for each statistic of interest, like the relative volatility of consumption to output. A simple example would be the four values for the estimated slope of the investment regression in the different spreadsheets. Computing the numerical value of this statistic for each of 10,000 realizations, say, we could approximate arbitrarily well its probability distribution through the obtained frequency distribution. This should not be surprising. Everything in the model is stochastic, even each sample statistic. The model can be seen as a mapping from the probability structure for the innovation in the productivity shock to the probability distribution of any model characteristic. With actual time series data we have a single sample available, so we can compute a single numerical value for any given statistic, and the interesting point becomes how to compare the single value obtained from actual time series data to the probability distribution estimated from the theoretical model.

2.7 Exercises

Exercise 1. In the (2.29) expression, fix numerical values for three of the parameters A, n, s, α, δ, and discuss how the steady-state value of k_{ss} changes with changes in the remaining parameter. Draw a graph summarizing each of these analyses.

Exercise 2. In the deterministic, discrete-time version of the Solow–Swan economy, assume a Cobb–Douglas technology, with parameter values $\delta = 0.10, n = 0.02,\ A = 1, \alpha = 0.33, s = 0.25$, and compute the steady state value of capital. Take an initial value for capital $k_0 < k_{ss}$ and compute the converging path towards steady state. Repeat the exercise for an initial condition $k_0 > k_{ss}$. Repeat the exercise changing the value of one parameter, and draw the trajectories that obtain for different values of that parameter. Numerically obtain the rate of convergence to steady state in each case.

Exercise 3. For a given parameterization, including an initial value of the stock of capital, k_0, and a Cobb–Douglas technology, compare the time series for k_t obtained from propagating the linear approximation (2.40) as well as the exact, nonlinear mechanism.

[19]The reader can copy the spreadsheet and use the random number generator to write a different realization on top of the old one. All the calculations in the spreadsheet will change, providing a different set of time series for all the variables in the economy. We need to be careful about the fact that EXCEL does not automatically update the regression results.

Exercise 4. Show that the second order linear approximation to the law of motion of the discrete time, deterministic version of the Solow–Swan model around steady state is,

$$k_{t+1} = k_{ss} + \left(\frac{1}{1+n} s f'(k_{ss}) + \frac{1-\delta}{1+n} \right) (k_t - k_{ss}) + \frac{1}{1+n} s f''(k_{ss}) (k_t - k_{ss})^2$$

Solve the model assuming a Cobb–Douglas technology under a given parameterization using this approximation, and compare the implied time series with those obtained from the first order approximation. Would the second order approximation still be the same for the deterministic and stochastic versions of the model?

References

1. Barro, R., and X. Sala-i-Martin. 2003. *Economic growth*, 2nd ed. Cambridge: MIT.
2. Solow, R.M. 1956. A contribution to the theory of economic growth. *Quarterly Journal of Economics* 70(1): 65–94.
3. Swan, T.W. 1956. Economic growth and capital accumulation. *Economic Record* 32: 334–361.

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