

Chapter 2

Robust Static Output Feedback H_∞ Control

Abstract This chapter will focus on robust static output feedback H_∞ control design for linear systems with polytopic uncertainties and norm bounded uncertainties. First, new H_∞ performance analysis criteria are proposed for the systems by an LMI decoupling approach. Then, sufficient conditions for designing static output feedback H_∞ controllers are given in terms of solutions to a set of linear matrix inequalities (LMIs). In contrast to the existing methods for designing the static output feedback H_∞ controllers, the input matrices and output matrices of the considered systems are allowed to have uncertainties. Moreover, theoretical proof is given to show that the proposed design conditions include the existing results as special cases. Simulation examples are provided to show the effectiveness of the proposed design method.

Keywords Uncertain linear systems · Static output feedback · H_∞ controllers · Linear matrix inequalities (LMIs)

2.1 With Time-Invariant Polytopic Uncertainties

2.1.1 Discrete-Time Systems

Consider a linear discrete-time system with time-invariant polytopic uncertainties described by state-space equations

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B(\theta)u(k) + E(\theta)w(k), \\ z(k) &= C_1(\theta)x(k) + D(\theta)u(k) + F(\theta)w(k), \\ y(k) &= C_2(\theta)x(k) + H(\theta)w(k), \end{aligned} \quad (2.1)$$

where $x(k) \in \mathcal{R}^n$ is the state variable, $u(k) \in \mathcal{R}^m$ is the control input, $w(k) \in \mathcal{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathcal{R}^q$

is the controlled output variable, $y(k) \in \mathcal{R}^p$ is the measurement output. The matrices $A(\theta)$, $B(\theta)$, $E(\theta)$, $C_1(\theta)$, $D(\theta)$, $F(\theta)$, $C_2(\theta)$, and $H(\theta)$ are constant matrices of appropriate dimensions and belong to the following uncertainty polytope [4]:

$$\begin{aligned} \Omega &= \{ [A(\theta), B(\theta), E(\theta), C_1(\theta), D(\theta), F(\theta), C_2(\theta), H(\theta)] \\ &= \sum_{i=1}^r \theta_i [A_i, B_i, E_i, C_{1i}, D_i, F_i, C_{2i}, H_i], \sum_{i=1}^r \theta_i = 1, \theta_i \geq 0 \}. \end{aligned} \quad (2.2)$$

Our aim is to design a static output feedback controller

$$u(k) = Ky(k), \quad (2.3)$$

such that the resulting following closed-loop system (2.4) is robustly stable or simultaneously meets H_∞ performance bound requirement.

$$\begin{aligned} x(k+1) &= (A(\theta) + B(\theta)KC_2(\theta))x(k) + (E(\theta) + B(\theta)KH(\theta))w(k), \\ z(k) &= (C_1(\theta) + D(\theta)KC_2(\theta))x(k) + (F(\theta) + D(\theta)KH(\theta))w(k). \end{aligned} \quad (2.4)$$

2.1.1.1 Case A: $D(\theta) = 0$

First, based on the parameter-dependent Lyapunov function approach, the H_∞ performance analysis problem of the closed-loop system (2.4) with $D(\theta) = 0$ is concerned. A new H_∞ performance analysis criterion is given, which will play a key role in static output feedback H_∞ controller design. The following preliminary lemma is needed to prove our results.

Lemma 2.1 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if there exist matrices $P(\theta)$, $G(\theta)$, and K such that the following matrix inequality holds*

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta) & G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \quad (2.5)$$

where $\mathcal{G}(\theta) = -G(\theta) - G^T(\theta) + P(\theta)$.

Proof Construct a parameter-dependent Lyapunov function as

$$V(k) = x^T(k)P(\theta)x(k), \quad P(\theta) > 0. \quad (2.6)$$

The difference of $V(k)$ can be given by

$$V(k+1) - V(k) = x^T(k+1)P(\theta)x(k+1) - x^T(k)P(\theta)x(k). \quad (2.7)$$

From (2.7) and recalling (2.4) with $D(\theta) = 0$, it can be verified that

$$\begin{aligned}
 & V(k+1) - V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\
 &= x^T(k+1)P(\theta)x(k+1) - x^T(k)P(\theta)x(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\
 &= \left((A(\theta) + B(\theta)KC_2(\theta))x(k) + (E(\theta) + B(\theta)KH(\theta))w(k) \right)^T P(\theta) \\
 &\quad \times \left((A(\theta) + B(\theta)KC_2(\theta))x(k) + (E(\theta) + B(\theta)KH(\theta))w(k) \right) \\
 &\quad - x^T(k)P(\theta)x(k) + (C_1(\theta)x(k) + F(\theta)w(k))^T (C_1(\theta)x(k) + F(\theta)w(k)) \\
 &\quad - \gamma^2 w^T(k)w(k) \\
 &= \zeta^T(k) \left(\begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \\ C_1(\theta) & F(\theta) \end{bmatrix}^T \right. \\
 &\quad \times P(\theta) \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \\ C_1(\theta) & F(\theta) \end{bmatrix} \\
 &\quad \left. + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \right) \zeta(k), \tag{2.8}
 \end{aligned}$$

where $\zeta(k) = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$.

Thus, $V(k+1) - V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) < 0$ for any $\zeta(k) \neq 0$ if

$$\begin{aligned}
 & \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \\ C_1(\theta) & F(\theta) \end{bmatrix}^T \\
 & \times P(\theta) \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \\ C_1(\theta) & F(\theta) \end{bmatrix} \\
 & + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{2.9}
 \end{aligned}$$

By using Schur complement to (2.9), we have

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) & -P^{-1}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{2.10}$$

Pre- and post-multiplying (2.10) by $\begin{bmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & G(\theta) & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose,

respectively, we verify that (2.10) is equivalent to the following matrix inequality:

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta) & G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta) & \Lambda & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{2.11}$$

where $\Lambda = -G(\theta)P^{-1}(\theta)G^T(\theta)$.

Note that $-(G(\theta) - P(\theta))^T P^{-1}(\theta)(G(\theta) - P(\theta)) \leq 0$, $P(\theta) > 0$ implies that $-G(\theta)P^{-1}(\theta)G^T(\theta) \leq -G(\theta) - G^T(\theta) + P(\theta)$, then, the inequality (2.11) can

be verified by (2.5). If the condition (2.5) is satisfied, we have $V(k+1) - V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) < 0$ for any $\zeta(k) \neq 0$, which implies that

$$V(\infty) - V(0) + \sum_{k=0}^{\infty} z^T(k)z(k) - \sum_{k=0}^{\infty} w^T(k)w(k) < 0.$$

With zero initial condition $\psi(0) = 0$ and $V(\infty) > 0$, we obtain $\sum_{k=0}^{\infty} z^T(k)z(k) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k)$ for any nonzero $w(k) \in l_2[0, \infty)$. Thus, the proof is completed. $\square$

In order to obtain LMI-based conditions for designing static output feedback H_∞ controllers, the existing results [3, 5] have to impose some constraints on the system matrices, which require that the input (or output) matrix $B(\theta)$ (or $C_2(\theta)$) is fixed (is without uncertainties i.e., $B(\theta) = B$ and $C_2(\theta) = C_2$) and B (C_2) is of full column (low) rank. Obviously, those results are limited and cannot be applied to general control systems. In our study, the constraints on the input and output matrices have been avoided. Thus, our results have more advantages than the ones in [3, 5].

Remark 2.1 It is noted that the results given by [6] are only applicable to that the system input matrix (or output matrix) is with time-varying polytopic uncertainties for linear systems. See Remark 2.6 for details.

In this following, a new H_∞ performance analysis criterion is presented in the following theorem.

Theorem 2.1 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices $P(\theta)$, $G(\theta)$, $J(\theta)$, M , N , V , and U , scalar β such that the following matrix inequality holds*

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathcal{G}(\theta) + J(\theta) & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.12)$$

where

$$\begin{aligned} \mathcal{G}(\theta) &= -G(\theta) - G^T(\theta) + P(\theta), \\ \Sigma_1 &= -\beta NU - \beta U^T N^T, \\ \Sigma_2 &= G(\theta)B(\theta) - MU. \end{aligned}$$

Proof We are about to prove the conclusion using Lemma 2.1. Obviously, the matrix inequality (2.5) can be rewritten as follows:

$$\begin{aligned}
& \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} K [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T K^T \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T < 0. \tag{2.13}
\end{aligned}$$

By defining $UK = V$ and considering matrices M and N , where U and N are nonsingular without loss of generality, we have

$$\begin{aligned}
& \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} K [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T K^T \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T \\
& = \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} U^{-1}N^{-1}NV [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T \\
& = \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}NV [C_2(\theta) \ H(\theta) \ 0 \ 0]
\end{aligned}$$

$$\begin{aligned}
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
& + \begin{bmatrix} 0 \\ 0 \\ M \\ 0 \end{bmatrix} V [C_2(\theta) \ H(\theta) \ 0 \ 0] + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T \begin{bmatrix} 0 \\ 0 \\ M \\ 0 \end{bmatrix}^T \\
& = \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + M V C_2(\theta) & G(\theta)E(\theta) + M V H(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1} N^{-1} N V [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T < 0.
\end{aligned} \tag{2.14}$$

Based on Lemma 1.3, for a positive matrix $J(\theta)$ one gives

$$\begin{aligned}
& \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1} N^{-1} N V [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
& = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} (G(\theta)B(\theta) - MU) U^{-1} N^{-1} N V [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T N^T N^{-T} U^{-T} (G(\theta)B(\theta) - MU)^T \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T
\end{aligned}$$

$$\begin{aligned}
&\leq \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} J(\theta) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} (G(\theta)B(\theta) - MU)^T J^{-1}(\theta) \\
&\times (G(\theta)B(\theta) - MU) U^{-1} N^{-1} NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}. \tag{2.15}
\end{aligned}$$

Then, (2.14) holds if the following condition is satisfied:

$$\begin{aligned}
&\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + M V C_2(\theta) & G(\theta)E(\theta) + M V H(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} J(\theta) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} (G(\theta)B(\theta) - MU)^T J^{-1}(\theta) \\
&\times (G(\theta)B(\theta) - MU) U^{-1} N^{-1} NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} < 0. \tag{2.16}
\end{aligned}$$

Without loss of generality, we assume that matrix $G(\theta)B(\theta) - MU$ is of full rank. By Schur complement to (2.16), which leads to

$$\begin{aligned}
&\begin{bmatrix} -P(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ G(\theta)A(\theta) + M V C_2(\theta) & G(\theta)E(\theta) + M V H(\theta) & \mathcal{G}(\theta) + J(\theta) & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * \\ N V C_2(\theta) & N V H(\theta) & 0 & 0 & \Xi_1 \end{bmatrix} \\
&= \begin{bmatrix} -P(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ G(\theta)A(\theta) + M V C_2(\theta) & G(\theta)E(\theta) + M V H(\theta) & \mathcal{G}(\theta) + J(\theta) & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * \\ N V C_2(\theta) & N V H(\theta) & 0 & 0 & \Xi_2 \end{bmatrix} < 0, \tag{2.17}
\end{aligned}$$

where

$$\Xi_1 = -\left(N^{-T} U^{-T} (G(\theta)B(\theta) - MU)^T J^{-1} (G(\theta)B(\theta) - MU) U^{-1} N^{-1}\right)^{-1},$$

$$\Xi_2 = -NU \left((G(\theta)B(\theta) - MU)^T J^{-1} (G(\theta)B(\theta) - MU) \right)^{-1} U^T N^T.$$

For a scalar β , note that $-(V - \beta Q)Q^{-1}(V - \beta Q)^T \leq 0$, $Q > 0$ implies that $-VQ^{-1}V^T \leq -\beta V - \beta V^T + \beta^2 Q$. Then, one has

$$\begin{aligned}
\Xi_2 &= -NU \left((G(\theta)B(\theta) - MU)^T J^{-1} (G(\theta)B(\theta) - MU) \right)^{-1} U^T N^T \\
&\leq -\beta NU - \beta U^T N^T + \beta^2 (G(\theta)B(\theta) - MU)^T J^{-1}(\theta) (G(\theta)B(\theta) - MU) \\
&= \Xi_3.
\end{aligned} \tag{2.18}$$

Then, (2.17) can be guaranteed by

$$\begin{bmatrix}
-P(\theta) & * & * & * & * \\
0 & -\gamma^2 I & * & * & * \\
G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathcal{G}(\theta) + J(\theta) & * & * \\
C_1(\theta) & F(\theta) & 0 & -I & * \\
NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Xi_3
\end{bmatrix} < 0. \tag{2.19}$$

Applying Schur complement to (2.19) yields (2.12). Thus, the proof is complete. $\square$

Remark 2.2 In the proof of Theorem 2.1, it should be noted that the procedure from (2.16) to (2.17) needs a constraint condition, which requires that the matrix $G(\theta)B(\theta) - MU$ is of full rank. When the matrix $G(\theta)B(\theta) - MU$ is of full rank, we can know that the matrix $(G(\theta)B(\theta) - MU)^T J^{-1} (G(\theta)B(\theta) - MU)$ is invertible. In fact, the constraint condition is not necessary. For the matrix inequality (2.16), by using Lemma 1.4 with

$$T = \begin{bmatrix}
-P(\theta) & * & * & * \\
0 & -\gamma^2 I & * & * \\
G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathcal{G}(\theta) + J(\theta) & * \\
C_1(\theta) & F(\theta) & 0 & -I
\end{bmatrix},$$

$$A = U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix},$$

$$L = NU,$$

$$P = (G(\theta)B(\theta) - MU)^T J^{-1}(\theta) (G(\theta)B(\theta) - MU),$$

we can also obtain the following matrix condition (2.19).

By the LMI decoupling approach, the appearance of crossing terms between $G(\theta)$ and K has been avoided in (2.12), it enables us to obtain strict LMI conditions for designing static output feedback H_∞ controllers.

In this following, based on the analysis result in Theorem 2.1, we proposed sufficient conditions for designing the static output feedback H_∞ controller in the form of (2.3), that is, to compute the gains K in (2.3) such that the closed-loop system (2.4) with $D(\theta) = 0$ is asymptotically stable with the prescribed H_∞ performance γ . As can be seen from (2.12), the appearance of crossing terms has been avoided, it makes the LMI formulation of design conditions easier. Of course, in order to obtain

LMI-based design conditions, the analysis criterion in Theorem 2.1 dependent on the premise that is the matrix parameters M , N and scalar parameter β should be known, it may lead to conservative design. However, when the system input matrices are with time-invariant polytopic uncertainties, the new condition given by Theorem 2.1 is undoubtedly effective for dealing with this case.

When the input matrix is with polytopic uncertainties and not full column rank, we choose $M = B(\theta)$ and $N = I$ in (2.12). Then, the corresponding design condition is given in the following theorem.

Theorem 2.2 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give scalars $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if for known scalar β , exist matrices U , V , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold*

$$\begin{aligned}
 & \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_i + B_i V C_{2i} & G_i E_i + B_i V H_i & \mathcal{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_i B_i - B_i U & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \\
 & i = 1, 2, \dots, r, \\
 & \begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + B_i V C_{2j} & G_j E_i + B_i V H_j & \mathcal{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2j} & V H_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_j B_i - B_i U & -\frac{J_j}{\beta^2} \end{bmatrix} \\
 & + \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_j + B_j V C_{2i} & G_i E_j + B_j V H_i & \mathcal{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_i B_j - B_j U & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \\
 & i, j = 1, 2, \dots, r, i < j,
 \end{aligned} \tag{2.20}$$

$$\tag{2.21}$$

where $\mathcal{G}_j = -G_j - G_j^T + P_j$.

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) can be given by

$$K = U^{-1}V. \tag{2.22}$$

Proof First, in this case, the matrix inequality (2.12) becomes

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta) + B(\theta)VC_2(\theta) & G(\theta)E(\theta) + B(\theta)VH(\theta) & \mathcal{G}(\theta) + J(\theta) & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ VC_2(\theta) & VH(\theta) & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.23)$$

where

$$\Sigma_1 = -\beta U - \beta U^T,$$

$$\Sigma_2 = G(\theta)B(\theta) - B(\theta)U.$$

From Theorem 2.1, we know that the prescribed H_∞ performance γ can be ensured if there exist matrices $P(\theta)$, $J(\theta)$, and $G(\theta)$ satisfying (2.23). Now, assume that the aforementioned matrices have the following form:

$$\begin{aligned} P(\theta) &= \sum_{j=1}^r \theta_j P_j, \quad P_j > 0, \quad j = 1, 2, \dots, r, \\ G(\theta) &= \sum_{j=1}^r \theta_j G_j, \\ J(\theta) &= \sum_{j=1}^r \theta_j J_j, \quad J_j > 0, \quad j = 1, 2, \dots, r. \end{aligned} \quad (2.24)$$

Then, inequality (2.23) is equivalent to

$$\begin{aligned} & \sum_{i=1}^r \sum_{j=1}^r \theta_i \theta_j \begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + B_i V C_{2j} & G_j E_i + B_i V H_j & \mathcal{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2j} & V H_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_j B_i - B_i U & -\frac{J_j}{\beta^2} \end{bmatrix} \\ &= \sum_{i=1}^r \theta_i^2 \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_i + B_i V C_{2i} & G_i E_i + B_i V H_i & \mathcal{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_i B_i - B_i U & -\frac{J_i}{\beta^2} \end{bmatrix} \\ &+ \sum_{i=1}^r \sum_{i < j}^r \theta_i \theta_j \end{aligned}$$

$$\begin{aligned}
& \times \left(\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + B_i V C_{2j} & G_j E_i + B_i V H_j & \mathcal{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2j} & V H_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_j B_i - B_i U & -\frac{J_j}{\beta^2} \end{bmatrix} \right. \\
& + \left. \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_j + B_j V C_{2i} & G_i E_j + B_j V H_i & \mathcal{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_i B_j - B_j U & -\frac{J_i}{\beta^2} \end{bmatrix} \right) < 0. \quad (2.25)
\end{aligned}$$

If LMIs (2.20) and (2.21) are satisfied, the inequality (2.25) holds. $\square$

Remark 2.3 Theorem 2.2 presents a new condition for designing static output feedback H_∞ controllers for discrete-time linear systems with time-invariant polytopic uncertainties which is of LMIs and can be effectively solved via LMI Control Toolbox [7].

In Theorem 2.2, a significant result is proposed to design static output feedback H_∞ controllers for uncertain discrete-time linear systems. The new result overcomes the deficiencies of the existing ones, it is able to handle this case that the system input matrices are nonfixed. In addition to this, the proposed result can give less conservative design than the existing LMI methods. In order to clarify this issue thoroughly, in the following, we consider the same system input matrix with [3, 5] as $B(\theta) = B$ (B is of full column rank). In this case, three H_∞ performance analysis conclusions with different values of matrices M and N are given based on Theorem 2.1. The first conclusion chooses $M = B$ and $N = B^T B$ in Theorem 2.1, the second chooses $M = \begin{bmatrix} I \\ 0 \end{bmatrix} R_1$, where R_1 is a known matrix parameter and $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $N = I$ in Theorem 2.1, while the third chooses $M = Y^T \begin{bmatrix} I \\ 0 \end{bmatrix}$, $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $N = I$ in Theorem 2.1.

Remark 2.4 Here, there is a description of this matrix Y . Because of this matrix B is full column rank, there exist a nonsingular matrix Y such that $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$. It should be noted that for each matrix B , the corresponding Y generally is not unique. A special Y can be obtained by the following formula:

$$Y = \begin{bmatrix} (B^T B)^{-1} B^T \\ B^{T\perp T} \end{bmatrix},$$

where $B^{T\perp}$ denotes an orthogonal basis for the null space of B^T .

Theorem 2.3 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices $P(\theta)$, $G(\theta)$, $J(\theta)$, V , and U , scalar β such that the following matrix inequality holds

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta) + BVC_2(\theta) & G(\theta)E(\theta) + BVH(\theta) & \mathcal{G}(\theta) + J(\theta) & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ B^T BVC_2(\theta) & B^T BVH(\theta) & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.26)$$

where

$$\mathcal{G}(\theta) = -G(\theta) - G^T(\theta) + P(\theta),$$

$$\Sigma_1 = -\beta B^T BU - \beta U^T (B^T B)^T,$$

$$\Sigma_2 = G(\theta)B - BU.$$

Theorem 2.4 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices $P(\theta)$, $G(\theta)$, $J(\theta)$, V , R_1 , and U , scalar β such that the following matrix inequality holds

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & * & * & * & * & * \\ G(\theta)A(\theta) + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 VC_2(\theta) & * & * & * & * & * \\ C_1(\theta) & * & * & * & * & * \\ VC_2(\theta) & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix} < 0, \quad (2.27)$$

$$\begin{bmatrix} * & * & * & * & * \\ -\gamma^2 I & * & * & * & * \\ G(\theta)E(\theta) + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 VH(\theta) & \mathcal{G}(\theta) + J(\theta) & * & * & * \\ F(\theta) & 0 & -I & * & * \\ VH(\theta) & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & G(\theta)B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 U & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0,$$

where $\mathcal{G}(\theta) = -G(\theta) - G^T(\theta) + P(\theta)$.

Theorem 2.5 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices $P(\theta)$, $G(\theta)$, $J(\theta)$, M , N , V ,

and U , scalar β such that the following matrix inequality holds

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta) + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V C_2(\theta) & G(\theta)E(\theta) + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V H(\theta) & \Xi & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ V C_2(\theta) & V H(\theta) & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.28)$$

where

$$\Xi = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),$$

$$\Sigma_1 = -\beta U - \beta U^T,$$

$$\Sigma_2 = G(\theta)B - Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} U.$$

Based on the three H_∞ performance analysis conclusions, the corresponding static output feedback H_∞ controller design results are given in the following corollaries.

Corollary 2.1 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β , exist matrices U , V , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_i + B V C_{2i} & G_i E_i + B V H_i & \mathcal{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ B^T B V C_{2i} & B^T B V H_i & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & G_i B - B U & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.29)$$

$$\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + B V C_{2i} & G_j E_i + B V H_i & \mathcal{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ B^T B V C_{2j} & B^T B V H_j & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & G_j B - B U & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$+ \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_j + BVC_{2j} & G_i E_j + BVH_j & \mathcal{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ B^T BVC_{2i} & B^T BVH_i & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & G_i B - BU & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \quad (2.30)$$

$i, j = 1, 2, \dots, r, i < j,$

where

$$\mathcal{G}_j = -G_j - G_j^T + P_j,$$

$$\Sigma_1 = -\beta B^T BU - \beta U^T (B^T B)^T.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

Corollary 2.2 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known matrix R_1 and scalar β , exist matrices U, V, P_j, J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\Pi_{ii} < 0, \quad i = 1, 2, \dots, r, \quad (2.31)$$

$$\Pi_{ij} + \Pi_{ji} < 0, \quad i < j, \quad i, j = 1, 2, \dots, r, \quad (2.32)$$

with

$$\Pi_{ij} = \begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 VC_{2i} & G_j E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 VH_i & \mathcal{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ VC_{2i} & VH_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_{2j} & -\frac{J_j}{\beta^2} \end{bmatrix}$$

and

$$\mathcal{G}_j = -G_j - G_j^T + P_j,$$

$$\Sigma_{2j} = G_j B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 U.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

Corollary 2.3 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β , exist matrices U , V , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V C_{2i} & G_i E_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V H_i & \mathcal{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.33)$$

$$\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V C_{2i} & G_j E_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V H_i & \mathcal{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2j} & V H_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_{2j} & -\frac{J_j}{\beta^2} \end{bmatrix} + \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_j + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V C_{2j} & G_i E_j + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V H_j & \mathcal{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \quad i, j = 1, 2, \dots, r, \quad i < j, \quad (2.34)$$

where

$$\begin{aligned} \mathcal{G}_j &= -G_j - G_j^T + P_j, \\ \Sigma_{2j} &= G_j B - Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} U. \end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

In order to show the contrasts between the existing static output feedback H_∞ control results and the proposed ones, we also recall the following existing results, which can directly be obtained from [3] and [6].

Lemma 2.2 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices $\hat{P}$ and N such that the following matrix equations hold*

$$\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \hat{P}A_i + BNC_{2i} & \hat{P}E_i + BNH_i & -\hat{P} & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.35)$$

$$\hat{P}B = BU. \quad (2.36)$$

Remark 2.5 By regulating $P(\theta) = \hat{P}$ and using matrix inequality congruence

property with $\begin{bmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & \hat{P} & * \\ 0 & 0 & 0 & I \end{bmatrix}$, the matrix inequality (2.10) is equivalent to

$$\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \hat{P}A_i + \hat{P}BK C_{2i} & \hat{P}E_i + \hat{P}BK H_i & -\hat{P} & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r. \quad (2.37)$$

By considering the condition (2.36) and defining a new variable $N = UK$, it can be known that LMIs (2.35) can guarantee the negative-definiteness of (2.37).

In addition to this, it should be noted that the single quadratic Lyapunov function approach had been used to ensure the matrix condition (2.36) in Lemma 2.2.

Remark 2.6 Obviously, the condition in Lemma 2.2 can be relaxed further with introducing a slack matrix variables G . In this case, (2.35) and (2.36) become, respectively, as follows:

$$\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ GA_i + BNC_{2i} & GE_i + BNH_i & -G - G^T + \hat{P} & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.38)$$

$$GB = BU. \quad (2.39)$$

Lemma 2.3 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable*

with the H_∞ performance γ if exist matrices L , $\hat{P}_j$, and R_j , $j = 1, 2, \dots, r$ such that the following matrix equations hold

$$\begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_i Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} F_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \quad (2.40)$$

$i = 1, 2, \dots, r,$

$$\begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_j Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} F_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} \\ + \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j} & R_i Y E_j + \begin{bmatrix} L \\ 0 \end{bmatrix} F_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} < 0, \quad (2.41)$$

$i, j = 1, 2, \dots, r, i < j,$

$$R_j = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix}, \quad j = 1, 2, \dots, r, \quad (2.42)$$

where $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

Remark 2.7 Define $P(\theta) = \hat{P}(\theta)$, $L = R_1 K$ and use matrix inequality congruence

property with $\begin{bmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & \sum_{j=1}^r \theta_j R_j Y & * \\ 0 & 0 & 0 & I \end{bmatrix}$ for the matrix inequality (2.10), the LMIs (2.40) and (2.41) can derived easily.

Remark 2.8 This design condition given in Lemma 2.3 is a simple extension of the results given in [6] for discrete-time linear system with time-invariant polytopic uncertainties. In fact, [6] is concerned with the problem of designing robust static output feedback controllers for discrete-time linear systems with time-varying polytopic uncertainties. The proposed technique in [6] is applicable for linear systems with the time-varying polytopic uncertainties, which may simultaneously emerge on system output and input matrices (nonfixed). However, it should be noted that

this results in [6] are not enough to be used directly for system output or input matrices with time-invariant polytopic uncertainties. This is because in the product terms $R_{ij}Y_i(\sum_{v=1}^r \theta_v P_v)^{-1}(R_{ij}Y_i)^T$, $Y_i B_i = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $i, j = 1, 2, \dots, r$, it does not allow $v = i$ or $v = j$. Therefore, in Lemma 2.3, we assume that the system input matrix $B(\theta)$ is fixed, it leads to the matrix Y is also fixed.

In addition to these two conditions given in Lemmas 2.2 and 2.3, which can directly be obtained from [3] and [6], the other significant result should be mentioned. The mentioned result is given is based on the study in [5] and [8].

Lemma 2.4 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices L , Q_j , and R_j , $j = 1, 2, \dots, r$ such that the following matrix equations hold*

$$\begin{bmatrix} -Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i Y^{-1} + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} Y^{-1} & R_i Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} F_i & -R_i - R_i^T + Q_i & * \\ C_{1i} Y^{-1} & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.43)$$

$$\begin{aligned} & \begin{bmatrix} -Q_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_j Y^{-1} + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j} Y^{-1} & R_j Y E_j + \begin{bmatrix} L \\ 0 \end{bmatrix} F_j & -R_j - R_j^T + Q_j & * \\ C_{1j} Y^{-1} & F_j & 0 & -I \end{bmatrix} \\ & + \begin{bmatrix} -Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j Y^{-1} + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j} Y^{-1} & R_i Y E_j + \begin{bmatrix} L \\ 0 \end{bmatrix} F_j & -R_i - R_i^T + Q_i & * \\ C_{1j} Y^{-1} & F_j & 0 & -I \end{bmatrix} < 0, \\ & i, j = 1, 2, \dots, r, i < j, \end{aligned} \quad (2.44)$$

$$R_j = \begin{bmatrix} R_1 & 0 \\ 0 & R_{2j} \end{bmatrix} \text{ or } R_j = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix}, \quad j = 1, 2, \dots, r, \quad (2.45)$$

where $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

Proof In fact, the design result presented in Lemma 2.4 is an extension, which puts the dynamic output feedback H_∞ controllers design method given in [8] to robust static output feedback H_∞ controller design for linear discrete-time system with time-invariant polytopic uncertainties.

Consider the matrix inequality (2.10), pre- and post-multiplying it by M and its transpose, respectively, we have:

$$\begin{bmatrix} -Y^{-T}P(\theta)Y^{-1} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ YA(\theta)Y^{-1} + YBK C_2(\theta)Y^{-1} & YE(\theta) + YBK H(\theta) & -YP^{-1}(\theta)Y^T & * \\ C_1(\theta)Y^{-1} & F(\theta) & 0 & -I \end{bmatrix} < 0, \quad (2.46)$$

where $M = \begin{bmatrix} Y^{-T} & * & * & * \\ 0 & I & * & * \\ 0 & 0 & Y & * \\ 0 & 0 & 0 & I \end{bmatrix}$.

Again, pre- and post-multiplying it by $\begin{bmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & R(\theta) & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose,

respectively, it follows that:

$$\begin{bmatrix} -Y^{-T}P(\theta)Y^{-1} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Delta_{31} & \Delta_{32} & -R(\theta)(YP^{-1}(\theta)Y^T)R^T(\theta) & * \\ C_1(\theta)Y^{-1} & F(\theta) & 0 & -I \end{bmatrix} < 0. \quad (2.47)$$

where

$$\Delta_{31} = R(\theta)YA(\theta)Y^{-1} + R(\theta)YBK C_2(\theta)Y^{-1},$$

$$\Delta_{32} = R(\theta)YE(\theta) + R(\theta)YBK H(\theta).$$

Based on a fact $-R(\theta)(YP^{-1}(\theta)Y^T)R^T(\theta) \leq -R(\theta) - R^T(\theta) + Y^{-T}P(\theta)Y^{-1}$ and defining $Y^{-T}P(\theta)Y^{-1} = Q(\theta)$, we can obtain the matrix inequality (2.48) to verify (2.47)

$$\begin{bmatrix} -Q(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Delta_{31} & \Delta_{32} & -R(\theta) - R^T(\theta) + Q(\theta) & * \\ C_1(\theta)Y^{-1} & F(\theta) & 0 & -I \end{bmatrix} < 0. \quad (2.48)$$

In order to solve the controller design problem, we partition matrices $Q(\theta)$ and $R(\theta)$ as

$$\begin{aligned} Q(\theta) &= \sum_{j=1}^r \theta_j Q_j, \quad Q_j > 0, \quad j = 1, 2, \dots, r, \\ R(\theta) &= \sum_{j=1}^r \theta_j R_j, \quad R_j = \begin{bmatrix} R_1 & 0 \\ 0 & R_{2j} \end{bmatrix} \text{ or } R_j = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix}, \quad j = 1, 2, \dots, r. \end{aligned} \quad (2.49)$$

By substituting the corresponding partitioned parts into (2.48), the matrix inequality is equivalent to

$$\sum_{i=1}^r \sum_{j=1}^r \theta_i \theta_j \begin{bmatrix} -Q_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i Y^{-1} + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} Y^{-1} & R_j Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} F_i & -R_j - R_j^T + Q_j & * \\ C_{1i} Y^{-1} & F_i & 0 & -I \end{bmatrix} < 0. \quad (2.50)$$

Choosing a new variable $L = R_1 K$, it is observed that the LMIs (2.43) and (2.44) can guarantee the negative-definiteness of (2.50). $\square$

Remark 2.9 It should be pointed out that the result given in Lemma 2.4 can also obtain by using a linear transformation approach. Consider the linear transformation on the system state

$$\bar{x}(k) = Yx(k), \quad (2.51)$$

where $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$, then we can obtain the following transformed closed-loop uncertain system:

$$\begin{aligned} Yx(k+1) &= Y(A(\theta) + BKC_2(\theta))Y^{-1}Yx(k) + Y(E(\theta) + BKH(\theta))w(k), \\ z(k) &= C_1(\theta)Y^{-1}Yx(k) + F(\theta)w(k). \end{aligned} \quad (2.52)$$

i.e.,

$$\begin{aligned} \bar{x}(k+1) &= (YA(\theta)Y^{-1} + YBKC_2(\theta)Y^{-1})\bar{x}(k) + (YE(\theta) + YBKH(\theta))w(k), \\ z(k) &= C_1(\theta)Y^{-1}\bar{x}(k) + F(\theta)w(k). \end{aligned} \quad (2.53)$$

Choose the Lyapunov matrix as $Y^{-T}P(\theta)Y^{-1}$, $P(\theta) > 0$, then we can obtain easily the H_∞ performance analysis conclusion (2.46).

In what follows, we will study the relationships among the proposed results and the LMI conditions in Lemmas 2.2–2.4. The following theorem shows that the results suggested herein include the ones given by [3, 5, 6, 8] as special cases, it is helpful to obtain a conclusion that the new results are less conservative than the existing LMI conditions for designing static output feedback H_∞ controllers.

Theorem 2.6 *If the condition given in Lemma 2.2 holds, the condition in Corollary 2.1 also holds.*

Proof If (2.35) and (2.36) in Lemma 2.2 are satisfied, which imply that $\hat{P}B = BU$ ($\hat{P}B - BU = 0$) and $\hat{P} > 0$. Since the matrix B is of full column rank, we have $B^T BU + U^T (B^T B)^T = B^T \hat{P}B + B^T \hat{P}B > 0$. Then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that the following matrix inequalities hold

$$\begin{aligned}
& \begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \hat{P}A_i + BNC_{2i} & \hat{P}E_i + BNH_i & -\hat{P} & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} + \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
& + \frac{1}{\beta} (B^T B N [C_{2i} \ H_i \ 0 \ 0])^T \\
& \times (B^T B U + U^T (B^T B)^T - \beta (\hat{P}B - BU)^T \frac{1}{\rho} I (\hat{P}B - BU))^{-1} \\
& \times (B^T B N [C_{2i} \ H_i \ 0 \ 0]) < 0, \ i = 1, 2, \dots, r.
\end{aligned} \tag{2.54}$$

By defining $G_j = P_j = \hat{P}$, $J_j = \rho I$, $j = 1, 2, \dots, r$, $V = N$ and applying Schur complement, the inequalities (2.29) and (2.30) in Corollary 2.1 can be obtained. $\square$

Theorem 2.7 *If the condition given in Lemma 2.3 holds, the condition in Corollary 2.2 also holds.*

Proof First, from $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$, we can know that

$$\begin{aligned}
R_j Y B &= R_j \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} R_1, \ j = 1, 2, \dots, r,
\end{aligned} \tag{2.55}$$

i.e.,

$$R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 = 0, \ j = 1, 2, \dots, r. \tag{2.56}$$

On the other hand, the matrix inequalities (2.40) and (2.41) in Lemma 2.3 can be rewritten as follows:

$$\begin{aligned}
& \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2i} & R_i Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} L H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} \\
& = \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2i} & R_i Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix}
\end{aligned}$$

$$R_i Y E + \begin{bmatrix} -\gamma^2 I \\ I \\ 0 \\ F_i \end{bmatrix} R_1 R_1^{-1} L H_i \quad \begin{matrix} * & * \\ * & * \\ * & * \\ 0 & -I \end{matrix} \quad \begin{matrix} * \\ * \\ * \\ * \end{matrix} < 0,$$

$$i = 1, 2, \dots, r, \quad (2.57)$$

and

$$\begin{aligned} & \begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2i} & R_j Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} L H_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} \\ & + \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j + \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2j} & R_i Y E_j + \begin{bmatrix} I \\ 0 \end{bmatrix} L H_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} \\ & = \begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2i} & R_j Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L H_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} \\ & + \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2j} & R_i Y E_j + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L H_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} < 0, \\ & i, j = 1, 2, \dots, r, i < j. \quad (2.58) \end{aligned}$$

Note that if the above condition is satisfied, which implies that $R_j Y + (R_j Y)^T > 0$, $j = 1, 2, \dots, r$. Because of the matrix B is of full column rank, we only can know that

$$\begin{aligned}
 B^T R_j Y B + B^T Y^T R_j^T B &= B^T R_j \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix}^T R_j^T B \\
 &= B^T \begin{bmatrix} R_1 \\ 0 \end{bmatrix} + \begin{bmatrix} R_1 \\ 0 \end{bmatrix}^T B = B^T \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_1^T \begin{bmatrix} I \\ 0 \end{bmatrix}^T B \\
 &= B^T Y B R_1 + R_1^T B^T Y^T B > 0, \quad j = 1, 2, \dots, r.
 \end{aligned} \tag{2.59}$$

However, we are not sure that $R_1 + R_1^T > 0$.

Combining (2.55)–(2.58), then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that the following matrix inequality holds:

$$\begin{aligned}
 &\begin{bmatrix} -\hat{P}_i \\ 0 \\ R_i Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2i} \\ C_{1i} \end{bmatrix} \\
 &\quad \begin{bmatrix} * & * & * \\ -\gamma^2 I & * & * \\ R_i Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ F_i & 0 & -I \end{bmatrix} \\
 &+ \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta} (R_1^{-1} L [C_{2i} \quad H_i \quad 0 \quad 0])^T \\
 &\times \left(2I - \beta \left(R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right)^T \frac{1}{\rho} I \left(R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right) \right)^{-1} (R_1^{-1} L [C_{2i} \quad H_i \quad 0 \quad 0]) \\
 &< 0, \quad i = 1, 2, \dots, r,
 \end{aligned} \tag{2.60}$$

and

$$\begin{bmatrix} -\hat{P}_j \\ 0 \\ R_j Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2i} \\ C_{1i} \end{bmatrix}$$

$$\begin{aligned}
& R_j Y E_i + \begin{bmatrix} -\gamma^2 I \\ I \\ 0 \\ F_i \end{bmatrix} R_1 R_1^{-1} L H_i \quad \begin{matrix} * & * & * \\ * & * & * \\ 0 & & -I \end{matrix} \\
& + \begin{bmatrix} -\hat{P}_i \\ 0 \\ I \\ C_{1j} \end{bmatrix} R_i Y A_j + \begin{bmatrix} I \\ 0 \\ C_{1j} \end{bmatrix} R_1 R_1^{-1} L C_{2j} \\
& R_i Y E_j + \begin{bmatrix} -\gamma^2 I \\ I \\ 0 \\ F_j \end{bmatrix} R_1 R_1^{-1} L H_j \quad \begin{matrix} * & * & * \\ * & * & * \\ 0 & & -I \end{matrix} \\
& + 2\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta} (R_1^{-1} L [C_{2i} + C_{2j} \quad H_i + H_j \quad 0 \quad 0])^T \\
& \times \left(4I - \beta \left(R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right)^T \right. \\
& \times \left. \frac{1}{2\rho} I \left(R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right) \right)^{-1} (R_1^{-1} L [C_{2i} + C_{2j} \quad H_i + H_j \quad 0 \quad 0]) \\
& < 0, \quad i, j = 1, 2, \dots, r, \quad i < j.
\end{aligned} \tag{2.61}$$

By defining $G_j = R_j Y$, $P_j = \hat{P}_j$, $J_j = \rho I$, $j = 1, 2, \dots, r$, $V = R_1^{-1} L$, $U = I$ and applying Schur complement, the inequalities (2.31) and (2.32) in Corollary 2.2 can be obtained. $\square$

Theorem 2.8 *If the condition given in Lemma 2.4 holds, the condition in Corollary 2.3 also holds.*

Proof First, pre- and post-multiplying (2.43) and (2.44) and by $\begin{bmatrix} Y^T & * & * & * \\ 0 & I & * & * \\ 0 & 0 & Y^T & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, we have

$$\begin{bmatrix} -Y^T Q_i Y & * & * & * \\ 0 & -\gamma^2 I & * & * \\ Y^T R_i Y A_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2i} & Y^T R_i Y E_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L H_i & \Omega_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.62)$$

$$\begin{bmatrix} -Y^T Q_j Y & * & * & * \\ 0 & -\gamma^2 I & * & * \\ Y^T R_j Y A_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2i} & Y^T R_j Y E_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L H_i & \Omega_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} + \begin{bmatrix} -Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ Y^T R_i Y A_j + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2j} & Y^T R_i Y E_j + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L H_j & \Omega_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} < 0, \quad i, j = 1, 2, \dots, r, \quad i < j, \quad (2.63)$$

where $\Omega_j = -Y^T R_j Y - Y^T R_j^T Y + Y^T Q_j Y$. Obviously, these LMIs imply that $R_1 + R_1^T > 0$.

On the other hand, from (2.45) and $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$, one can be given

$$Y^T R_j Y B = Y^T \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} R_1, \quad j = 1, 2, \dots, r. \quad (2.64)$$

Similar to the proof of Theorem 2.6, by choosing $G_j = Y^T R_j Y$, $P_j = Y^T Q_j Y$, $J_j = \rho I$, $j = 1, 2, \dots, r$, $V = L$, $U = R_1$ and applying Schur complement, the inequalities (2.62) and (2.63) generate (2.33) and (2.34) in Corollary 2.3, respectively. $\square$

2.1.1.2 Case B: $H(\theta) = 0$

For this case, Lemma 2.1 is changed as the following lemma.

Lemma 2.5 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if there exist matrices $P(\theta)$, $G(\theta)$, and K such that the following matrix inequality holds:

$$\begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) + B(\theta)KC_2(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1G(\theta) + D(\theta)KC_2(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \quad (2.65)$$

Proof Choose a parameter-dependent Lyapunov function to be

$$V(k) = x^T(k)P^{-1}(\theta)x(k), \quad P(\theta) > 0. \quad (2.66)$$

Similar to the proof Lemma 2.1, the H_∞ performance of closed-loop system (2.4) with $H(\theta) = 0$ can be guaranteed by the following matrix inequality:

$$\begin{bmatrix} -P^{-1}(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta) + B(\theta)KC_2(\theta) & E(\theta) & -P(\theta) & * \\ C_1(\theta) + D(\theta)KC_2(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \quad (2.67)$$

Pre- and post-multiplying (2.67) by $\begin{bmatrix} G^T(\theta) & * & * & * \\ 0 & I & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose,

respectively, we have:

$$\begin{bmatrix} -G^T(\theta)P^{-1}(\theta)G(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) + B(\theta)KC_2(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1G(\theta) + D(\theta)KC_2(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \quad (2.68)$$

Obviously, the inequality (2.68) can be ensured by (2.65). $\square$

Based on Lemma 2.5, the following theorem proposes another form of H_∞ performance analysis criterion.

Theorem 2.9 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices $M, N, U, L, P(\theta), G(\theta)$, and $J(\theta)$, scalar β such that the following matrix inequality holds

$$\begin{bmatrix} \mathcal{G}(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * & * & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I & * & * \\ 0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.69)$$

where

$$\mathcal{G}(\theta) = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),$$

$$\Sigma_1 = -\beta U N - \beta N^T U^T,$$

$$\Sigma_2 = (C_2(\theta)G(\theta) - U M)^T.$$

Proof Rewrite matrix inequality (2.65) in the following form:

$$\begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} K [C_2(\theta)G(\theta) \ 0 \ 0 \ 0] + [C_2(\theta)G(\theta) \ 0 \ 0 \ 0]^T K^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0. \quad (2.70)$$

Let us define $KU = L$ and consider two matrix M and N , where the matrices U and N are nonsingular, from (2.70), we have

$$\begin{bmatrix} \Omega(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} L N N^{-1} U^{-1} [C_2(\theta)G(\theta) \ 0 \ 0 \ 0] + [C_2(\theta)G(\theta) \ 0 \ 0 \ 0]^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0. \quad (2.71)$$

where $\Omega(\theta) = -G(\theta) - G^T(\theta) + P(\theta)$.

Furthermore, one gives

$$\begin{bmatrix} \Omega(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} L N N^{-1} U^{-1} [C_2(\theta)G(\theta) - U M \ 0 \ 0 \ 0] + [C_2(\theta)G(\theta) - U M \ 0 \ 0 \ 0]^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T$$

$$\begin{aligned}
& + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B(\theta)LM & 0 & 0 & 0 \\ D(\theta)LM & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B(\theta)LM & 0 & 0 & 0 \\ D(\theta)LM & 0 & 0 & 0 \end{bmatrix}^T \\
& = \begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1} [C_2(\theta)G(\theta) - UM \quad 0 \quad 0 \quad 0] \\
& + [C_2(\theta)G(\theta) - UM \quad 0 \quad 0 \quad 0]^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0. \quad (2.72)
\end{aligned}$$

By Lemma 1.3 with a positive matrix $J(\theta)$, one can be given

$$\begin{aligned}
& \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1} (C_2(\theta)G(\theta) - UM) [I \quad 0 \quad 0 \quad 0] \\
& + [I \quad 0 \quad 0 \quad 0]^T (C_2(\theta)G(\theta) - UM)^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T \\
& \leq \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} J(\theta) [I \quad 0 \quad 0 \quad 0] + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1} (C_2(\theta)G(\theta) - UM) J^{-1}(\theta) \\
& \times (C_2(\theta)G(\theta) - UM)^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T \\
& = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} J(\theta) [I \quad 0 \quad 0 \quad 0] + \begin{bmatrix} 0 \\ 0 \\ B(\theta)LN \\ D(\theta)LN \end{bmatrix} N^{-1}U^{-1} (C_2(\theta)G(\theta) - UM) J^{-1}(\theta)
\end{aligned}$$

$$\times (C_2(\theta)G(\theta) - UM)^T U^{-T} N^{-T} \begin{bmatrix} 0 \\ 0 \\ B(\theta)LN \\ D(\theta)LN \end{bmatrix}^T. \quad (2.73)$$

Then, the inequality (2.72) holds if the following inequality is satisfied:

$$\begin{bmatrix} \mathcal{G}(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B(\theta)LN \\ D(\theta)LN \end{bmatrix} N^{-1} U^{-1} \\ \times (C_2(\theta)G(\theta) - UM) J^{-1}(\theta) (C_2(\theta)G(\theta) - UM)^T U^{-T} N^{-T} \begin{bmatrix} 0 \\ 0 \\ B(\theta)LN \\ D(\theta)LN \end{bmatrix}^T < 0, \quad (2.74)$$

where $\mathcal{G}(\theta)$ is defined in (2.69).

Without loss of generality, we assume that matrix $C_2(\theta)G(\theta) - UM$ is of full rank. Applying Schur complement to (2.74) yields

$$\begin{bmatrix} \mathcal{G}(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I & * \\ 0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & \Omega_1 \end{bmatrix} < 0, \quad (2.75)$$

where $\Omega_1 = -N^T U^T \left((C_2(\theta)G(\theta) - UM) J^{-1}(\theta) (C_2(\theta)G(\theta) - UM)^T \right)^{-1} UN$.

At the same time, one can be known that

$$\begin{aligned} & -N^T U^T \left((C_2(\theta)G(\theta) - UM) J^{-1}(\theta) (C_2(\theta)G(\theta) - UM)^T \right)^{-1} UN \\ & \leq -\beta UN - \beta N^T U^T + \beta^2 (C_2(\theta)G(\theta) - UM) J^{-1}(\theta) (C_2(\theta)G(\theta) - UM)^T \\ & = \Omega_2. \end{aligned} \quad (2.76)$$

The above result (2.76) implies that the inequality (2.75) can be verified by

$$\begin{bmatrix} \mathcal{G}(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I & * \\ 0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & \Omega_2 \end{bmatrix} < 0. \quad (2.77)$$

By using Schur complement, the inequality (2.77) becomes (2.69). $\square$

In the following, we consider several combinations about the known matrices M and N to design static output feedback H_∞ controllers.

$$\left\{ \begin{array}{l} \mathbf{A}: M = C_2(\theta) \text{ and } N = I, \text{ the output matrix is with polytopic uncertainties} \\ \\ \mathbf{B}: M = C_2 \text{ and } N = C_2 C_2^T \\ \text{or} \\ \mathbf{C}: M = S_1 [I \ 0] \text{ } (S_1 \text{ is known}) \text{ and } N = I \text{ fixed and} \\ \text{or} \\ \mathbf{D}: M = [I \ 0] T^T \text{ and } N = I \end{array} \right\} \begin{array}{l} \\ \\ \\ \text{of full row rank} \end{array} \quad (2.78)$$

where $C_2 T = [I \ 0]$.

Remark 2.10 As Remark 2.4, there is a description of this matrix T . Because of this matrix C_2 is full row rank, there exist a nonsingular matrix T such that $C_2 T = [I \ 0]$. It should be noted that for each matrix C_2 , the corresponding T generally is not unique. A special T can be obtained by the following formula:

$$T = [C_2^T (C_2 C_2^T)^{-1} \ C_2^\perp],$$

where $C_2^\perp$ denotes an orthogonal basis for the null space of C_2 .

For the cases A–D, the H_∞ performance analysis condition (2.69) are rewritten as follows, respectively

A:

$$\left[\begin{array}{cccccc} \mathcal{G}(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A(\theta)G(\theta) + B(\theta)LC_2(\theta) & E(\theta) & -P(\theta) & * & * & * \\ C_1(\theta)G(\theta) + D(\theta)LC_2(\theta) & F(\theta) & 0 & -I & * & * \\ 0 & 0 & L^T B^T(\theta) & L^T D^T(\theta) & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{array} \right] < 0, \quad (2.79)$$

where $\Sigma_2 = (C_2(\theta)G(\theta) - UC_2(\theta))^T$.

B:

$$\left[\begin{array}{cccccc} \mathcal{G}(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A(\theta)G(\theta) + B(\theta)LC_2 & E(\theta) & -P(\theta) & * & * & * \\ C_1(\theta)G(\theta) + D(\theta)LC_2 & F(\theta) & 0 & -I & * & * \\ 0 & 0 & (C_2 C_2^T)^T L^T B^T(\theta) & (C_2 C_2^T)^T L^T D^T(\theta) & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{array} \right] < 0, \quad (2.80)$$

where

$$\Sigma_1 = -\beta U C_2 C_2^T - \beta (C_2 C_2^T)^T U^T,$$

$$\Sigma_2 = (C_2 G(\theta) - U C_2)^T.$$

C:

$$\begin{bmatrix} \mathcal{G}(\theta) & & & & \\ 0 & & & & \\ A(\theta)G(\theta) + B(\theta)LS_1[I & 0] & & & \\ C_1(\theta)G(\theta) + D(\theta)LS_1[I & 0] & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} \begin{bmatrix} * & * & * & * & * \\ -\gamma^2 I & * & * & * & * \\ E(\theta) & -P(\theta) & * & * & * \\ F(\theta) & 0 & -I & * & * \\ 0 & L^T B^T(\theta) & L^T D^T(\theta) & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & \Lambda & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.81)$$

where $\Lambda = (C_2 G(\theta) - U S_1[I \ 0])^T$.

D:

$$\begin{bmatrix} \mathcal{G}(\theta) & & & & & \\ 0 & & & & & \\ A(\theta)G(\theta) + B(\theta)L[I \ 0]T^T & E(\theta) & -P(\theta) & * & * & * \\ C_1(\theta)G(\theta) + D(\theta)L[I \ 0]T^T & F(\theta) & 0 & -I & * & * \\ 0 & 0 & L^T B^T(\theta) & L^T D^T(\theta) & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.82)$$

where

$$\Sigma_1 = -\beta U - \beta U^T,$$

$$\Sigma_2 = (C_2 G(\theta) - U[I \ 0]T^T)^T.$$

Based on these analysis conditions (2.79–2.82), the corresponding design results are given in the following corollaries.

Corollary 2.4 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β , exist matrices U , L , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_i + B_i L C_{2i} & E_i & -P_i & * & * & * \\ C_{1i} G_i + D_i L C_{2i} & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_{2i} G_i - U C_{2i})^T & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.83)$$

$$\begin{bmatrix} -G_j - G_j^T + P_j + J_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_j + B_j L C_{2j} & E_j & -P_j & * & * & * \\ C_{1j} G_j + D_j L C_{2j} & F_j & 0 & -I & * & * \\ 0 & 0 & L^T B_j^T & L^T D_j^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_{2j} G_j - U C_{2j})^T & -\frac{J_j}{\beta^2} \end{bmatrix} + \begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_i + B_j L C_{2j} & E_j & -P_i & * & * & * \\ C_{1j} G_i + D_j L C_{2j} & F_j & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_{2j} G_i - U C_{2j})^T & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \quad i, j = 1, 2, \dots, r, \quad i < j. \quad (2.84)$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by

$$K = L U^{-1}. \quad (2.85)$$

Corollary 2.5 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β , exist matrices U , L , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_i + B_i L C_2 & E_i & -P_i & * & * & * \\ C_{1i} G_i + D_i L C_2 & F_i & 0 & -I & * & * \\ 0 & 0 & (C_2 C_2^T)^T L^T B_i^T & (C_2 C_2^T)^T L^T D_i^T & \Delta_1 & * \\ 0 & 0 & 0 & 0 & \Delta_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.86)$$

$$\begin{aligned}
& \begin{bmatrix} -G_j - G_j^T + P_j + J_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_j + B_i L C_2 & E_i & -P_j & * & * & * \\ C_{1i} G_j + D_i L C_2 & F_i & 0 & -I & * & * \\ 0 & 0 & (C_2 C_2^T)^T L^T B_i^T & (C_2 C_2^T)^T L^T D_i^T & \Delta_1 & * \\ 0 & 0 & 0 & 0 & \Delta_{2j} & -\frac{J_j}{\beta^2} \end{bmatrix} \\
& + \begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_i + B_j L C_2 & E_j & -P_i & * & * & * \\ C_{1j} G_i + D_j L C_2 & F_j & 0 & -I & * & * \\ 0 & 0 & (C_2 C_2^T)^T L^T B_j^T & (C_2 C_2^T)^T L^T D_j^T & \Delta_1 & * \\ 0 & 0 & 0 & 0 & \Delta_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \\
& i, j = 1, 2, \dots, r, i < j,
\end{aligned} \tag{2.87}$$

where

$$\begin{aligned}
\Delta_1 &= -\beta U C_2 C_2^T - \beta (C_2 C_2^T)^T U^T, \\
\Delta_{2j} &= (C_2 G_j - U C_2)^T.
\end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.85).

Corollary 2.6 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known matrix S_1 and scalar β , exist matrices U, V, P_j, J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\begin{aligned}
& \begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_i + B_i L S_1 [I \ 0] & E_i & -P_i & * & * & * \\ C_{1i} G_i + D_i L S_1 [I \ 0] & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & \Psi_1 & * \\ 0 & 0 & 0 & 0 & \Psi_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, i = 1, 2, \dots, r, \\
& \tag{2.88}
\end{aligned}$$

$$\begin{bmatrix} -G_j - G_j^T + P_j + J_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_j + B_i L S_1 [I \ 0] & E_i & -P_j & * & * & * \\ C_{1i} G_j + D_i L S_1 [I \ 0] & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & \Psi_1 & * \\ 0 & 0 & 0 & 0 & \Psi_{2j} & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$\begin{aligned}
& + \begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_i + B_j L S_1 [I \ 0] & E_j & -P_i & * & * & * \\ C_{1j} G_i + D_j L S_1 [I \ 0] & F_j & 0 & -I & * & * \\ 0 & 0 & L^T B_j^T & L^T D_j^T & \Psi_1 & * \\ 0 & 0 & 0 & 0 & \Psi_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \\
& i, j = 1, 2, \dots, r, i < j,
\end{aligned} \tag{2.89}$$

where

$$\begin{aligned}
\Psi_1 &= -\beta U - \beta U^T, \\
\Psi_{2j} &= (C_2 G_j - U S_1 [I \ 0])^T.
\end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.85).

Corollary 2.7 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β , exist matrices U , V , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\begin{aligned}
& \begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_i + B_i L [I \ 0] T^T & E_i & -P_i & * & * & * \\ C_{1i} G_i + D_i L [I \ 0] T^T & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_2 G_i - U [I \ 0] T^T)^T & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \\
& i = 1, 2, \dots, r,
\end{aligned} \tag{2.90}$$

$$\begin{aligned}
& \begin{bmatrix} -G_j - G_j^T + P_j + J_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_j + B_j L [I \ 0] T^T & E_j & -P_j & * & * & * \\ C_{1j} G_j + D_j L [I \ 0] T^T & F_j & 0 & -I & * & * \\ 0 & 0 & L^T B_j^T & L^T D_j^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_2 G_j - U [I \ 0] T^T)^T & -\frac{J_j}{\beta^2} \end{bmatrix} \\
& + \begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_i + B_j L [I \ 0] T^T & E_j & -P_i & * & * & * \\ C_{1j} G_i + D_j L [I \ 0] T^T & F_j & 0 & -I & * & * \\ 0 & 0 & L^T B_j^T & L^T D_j^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_2 G_i - U [I \ 0] T^T)^T & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \\
& i, j = 1, 2, \dots, r, i < j.
\end{aligned} \tag{2.91}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.85).

For the case $H(\theta)=0$, we also make a comparison with the existing results to reflect advantage of the proposed results. With $H(\theta)=0$, several existing design conditions can be described by the following lemmas.

Lemma 2.6 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices $\hat{P}$ and N such that the following matrix equations hold

$$\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i \hat{P} + B_i N C_2 & E_i & -\hat{P} & * \\ C_{1i} \hat{P} + D_i N C_2 & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.92)$$

$$C_2 \hat{P} = U C_2. \quad (2.93)$$

Lemma 2.7 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices L_1 , $\hat{P}_j$, and S_j , $j = 1, 2, \dots, r$ such that the following matrix equations hold

$$\begin{bmatrix} -T S_i - (T S_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i T S_i + B_i [L_1 \ 0] & E_i & -\hat{P}_i & * \\ C_{1i} T S_i + D_i [L_1 \ 0] & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.94)$$

$$\begin{aligned} & \begin{bmatrix} -T S_j - (T S_j)^T + \hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j T S_j + B_j [L_1 \ 0] & E_j & -\hat{P}_j & * \\ C_{1j} T S_j + D_j [L_1 \ 0] & F_j & 0 & -I \end{bmatrix} \\ & + \begin{bmatrix} -T S_i - (T S_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j T S_i + B_j [L_1 \ 0] & E_j & -\hat{P}_i & * \\ C_{1j} T S_i + D_j [L_1 \ 0] & F_j & 0 & -I \end{bmatrix} < 0, \quad i, j = 1, 2, \dots, r, \quad i < j, \end{aligned} \quad (2.95)$$

$$S_j = \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix}, \quad j = 1, 2, \dots, r, \quad (2.96)$$

where $C_2 T = [I \ 0]$.

Lemma 2.8 Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices L_1 , P_j , and S_j , $j = 1, 2, \dots, r$ such that the following matrix equations hold

$$\begin{bmatrix} -S_i - S_i^T + Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ T^{-1}A_i T S_i + T^{-1}B_i [L_1 \ 0] & T^{-1}E_i & -Q_i & * \\ C_{1i} T S_i + D_i [L_1 \ 0] & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r, \quad (2.97)$$

$$\begin{bmatrix} -S_j - S_j^T + Q_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ T^{-1}A_j T S_j + T^{-1}B_j [L_1 \ 0] & T^{-1}E_j & -Q_j & * \\ C_{1j} T S_j + D_j [L_1 \ 0] & F_j & 0 & -I \end{bmatrix} \\ + \begin{bmatrix} -S_i - S_i^T + Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ T^{-1}A_j T S_i + T^{-1}B_j [L_1 \ 0] & T^{-1}E_j & -Q_i & * \\ C_{1j} T S_i + D_j [L_1 \ 0] & F_j & 0 & -I \end{bmatrix} < 0, \quad i, j = 1, 2, \dots, r, \quad i < j, \quad (2.98)$$

$$S_j = \begin{bmatrix} S_1 & 0 \\ 0 & S_{3j} \end{bmatrix} \quad \text{or} \quad S_j = \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix}, \quad j = 1, 2, \dots, r, \quad (2.99)$$

where $C_2 T = [I \ 0]$.

Remark 2.11 The design conditions given in Lemmas 2.6 and 2.7 can directly be obtained from [3] and [6], respectively. And the condition in Lemma 2.8 can be derived by applying the homologous matrix inequality technique with Lemma 2.4 or consider a linear transformation on the system state as $\bar{x}(k) = T^{-1}x(k)$, with $C_2 T = [I \ 0]$.

Similar to the Case A ($D(\theta) = 0$), the following theorems show also that the proposed results include the ones given by Lemmas as special cases.

Theorem 2.10 If the condition given in Lemma 2.6 hold, the condition in Corollary 2.5 also holds.

Proof If (2.92) and (2.93) in Lemma 2.6 are satisfied, which implies that $C_2 \hat{P} = UC_2$, $\hat{P} > 0$. Since the matrix C_2 is of full row rank, we have $UC_2 C_2^T + C_2 C_2^T U^T = C_2 \hat{P} C_2^T + C_2 \hat{P} C_2^T > 0$. Then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that the LMIs (2.86) and (2.87) hold. $\square$

Theorem 2.11 If the condition given in Lemma 2.7 hold, the condition in Corollary 2.6 also holds.

Proof First, the LMIs (2.94) and (2.95) in Lemma 2.7 can be rewritten as follows:

$$\begin{aligned}
 & \begin{bmatrix} -TS_i - (TS_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i TS_i + B_i L_1 [I \ 0] & E_i & -\hat{P}_i & * \\ C_{1i} TS_i + D_i L_1 [I \ 0] & F_i & 0 & -I \end{bmatrix} \\
 = & \begin{bmatrix} -TS_i - (TS_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i TS_i + B_i L_1 S_1^{-1} S_1 [I \ 0] & E_i & -\hat{P}_i & * \\ C_{1i} TS_i + D_i L_1 S_1^{-1} S_1 [I \ 0] & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \dots, r,
 \end{aligned} \tag{2.100}$$

and

$$\begin{aligned}
 & \begin{bmatrix} -TS_j - (TS_j)^T + \hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j TS_j + B_j L_1 [I \ 0] & E_j & -TS_j - (TS_j)^T + \hat{P}_j & * \\ C_{1j} TS_j + D_j L_1 [I \ 0] & F_j & 0 & -I \end{bmatrix} \\
 + & \begin{bmatrix} -TS_i - (TS_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j TS_i + B_j L_1 [I \ 0] & E_j & -TS_i - (TS_i)^T + \hat{P}_i & * \\ C_{1j} TS_i + D_j L_1 [I \ 0] & F_j & 0 & -I \end{bmatrix} \\
 = & \begin{bmatrix} -TS_j - (TS_j)^T + \hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i TS_j + B_i L_1 S_1^{-1} S_1 [I \ 0] & E_i & -\hat{P}_j & * \\ C_{1i} TS_j + D_i L_1 S_1^{-1} S_1 [I \ 0] & F_i & 0 & -I \end{bmatrix} \\
 + & \begin{bmatrix} -TS_i - (TS_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j TS_i + B_j L_1 S_1^{-1} S_1 [I \ 0] & E_j & -\hat{P}_i & * \\ C_{1i} TS_i + D_i L_1 S_1^{-1} S_1 [I \ 0] & F_j & 0 & -I \end{bmatrix} < 0, \tag{2.101} \\
 & i, j = 1, 2, \dots, r, \quad i < j.
 \end{aligned}$$

If the above condition is satisfied, which implies that $TS_j + (TS_j)^T > 0$, $j = 1, 2, \dots, r$. Because C_2 is of full row rank, we have

$$\begin{aligned}
 C_2(TS_j + T^T S_j^T)C_2^T &= C_2 TS_j C_2^T + C_2 S_j^T T^T C_2^T \\
 &= [I \ 0] S_j C_2^T + C_2 S_j^T [I \ 0]^T \\
 &= [S_1 \ 0] C_2^T + C_2 [S_1 \ 0]^T
 \end{aligned}$$

$$\begin{aligned}
&= S_1 [I \ 0] C_2^T + C_2 [I \ 0]^T S_1^T \\
&= S_1 C_2 T C_2^T + C_2 T^T C_2^T S_1^T > 0, \quad j = 1, 2, \dots, r.
\end{aligned} \tag{2.102}$$

However, we are not sure that $S_1 + S_1^T > 0$.

On the other hand, from $C_2 T = [I \ 0]$, we can know that

$$\begin{aligned}
C_2 T S_j &= [I \ 0] S_j \\
&= [I \ 0] \begin{bmatrix} S_{1j} & 0 \\ S_{2j} & S_{3j} \end{bmatrix} = [S_{1j} \ 0] = S_1 [I \ 0], \quad j = 1, 2, \dots, r.
\end{aligned} \tag{2.103}$$

Then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that the following matrix inequality holds:

$$\begin{aligned}
&\begin{bmatrix} -TS_i - (TS_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i TS_i + B_i L_1 S_1^{-1} S_1 [I \ 0] & E_i & -\hat{P}_i & * \\ C_{1i} TS_i + D_i L_1 S_1^{-1} S_1 [I \ 0] & F_i & 0 & -I \end{bmatrix} \\
&+ \rho \begin{bmatrix} I & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta} ((L_1 S_1^{-1})^T [0 \ 0 \ B_i^T \ D_i^T])^T \\
&\times \left(2I - \beta (C_2 TS_i - S_1 [I \ 0]) \frac{1}{\rho} I (C_2 TS_i - S_1 [I \ 0])^T \right)^{-1} \\
&\times ((L_1 S_1^{-1})^T [0 \ 0 \ B_i^T \ D_i^T]) < 0, \quad i = 1, 2, \dots, r,
\end{aligned} \tag{2.104}$$

$$\begin{aligned}
&\begin{bmatrix} -TS_j - (TS_j)^T + \hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j TS_j + B_j L_1 S_1^{-1} S_1 [I \ 0] & E_j & -\hat{P}_j & * \\ C_{1j} TS_j + D_j L_1 S_1^{-1} S_1 [I \ 0] & F_j & 0 & -I \end{bmatrix} \\
&+ \begin{bmatrix} -TS_i - (TS_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j TS_i + B_j L_1 S_1^{-1} S_1 [I \ 0] & E_j & -\hat{P}_i & * \\ C_{1j} TS_i + D_j L_1 S_1^{-1} S_1 [I \ 0] & F_j & 0 & -I \end{bmatrix} \\
&+ 2\rho \begin{bmatrix} I & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta} ((L_1 S_1^{-1})^T [0 \ 0 \ B_i^T + B_j^T \ D_i^T + D_j^T])^T
\end{aligned}$$

$$\begin{aligned}
& \times \left(4I - \beta (C_2 T S_i - S_1 [I \ 0] + C_2 T S_j - S_1 [I \ 0]) \frac{1}{2\rho} I \right. \\
& \times \left. (C_2 T S_i - S_1 [I \ 0] + C_2 T S_j - S_1 [I \ 0])^T \right)^{-1} \\
& \times ((L_1 S_1^{-1})^T [0 \ 0 \ B_i^T + B_j^T \ D_i^T + D_j^T]) < 0, \ i < j, \ i, j = 1, 2, \dots, r.
\end{aligned} \tag{2.105}$$

By defining $L = L_1 S_1^{-1}$, $U = I$, $G_j = T S_j$, $P_j = \hat{P}_j$, $J_j = \rho I$, $j = 1, 2, \dots, r$, and applying Schur complement, the LMIs (2.88) and (2.89) in Corollary 2.6 can be obtained. $\square$

Theorem 2.12 *If the condition given in Lemma 2.8 holds, the condition in Corollary 2.7 also holds.*

Proof Pre- and post-multiplying (2.97) and (2.98) and by $\begin{bmatrix} T & * & * & * \\ 0 & I & * & * \\ 0 & 0 & T & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, we have

$$\begin{bmatrix} -T S_i T^T - T S_i^T T^T + T Q_i T^T & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i T S_i T^T + B_i L_1 [I \ 0] T^T & E_i & -T Q_i T^T & * \\ C_{1i} T S_i T^T + D_i L_1 [I \ 0] T^T & F_i & 0 & -I \end{bmatrix} < 0, \ i = 1, 2, \dots, r, \tag{2.106}$$

$$\begin{aligned}
& \begin{bmatrix} -T S_j T^T - T S_j^T T^T + T Q_j T^T & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i T S_j T^T + B_i L_1 [I \ 0] T^T & E_i & -T Q_j T^T & * \\ C_{1i} T S_j T^T + D_i L_1 [I \ 0] T^T & F_i & 0 & -I \end{bmatrix} \\
& + \begin{bmatrix} -T S_i T^T - T S_i^T T^T + T Q_i T^T & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j T S_i T^T + B_j L_1 [I \ 0] T^T & E_j & -T Q_i T^T & * \\ C_{1j} T S_i T^T + D_j L_1 [I \ 0] T^T & F_j & 0 & -I \end{bmatrix} < 0, \\
& i, j = 1, 2, \dots, r, \ i < j.
\end{aligned} \tag{2.107}$$

Obviously, these LMIs imply that $S_1 + S_1^T > 0$.

From (2.99) and $C_2 T = [I \ 0]$, we have

$$C_2 T S_j T^T = [I \ 0] S_j T^T = [S_1 \ 0] T^T = S_1 [I \ 0] T^T, \ j = 1, 2, \dots, r. \tag{2.108}$$

Similar to the proof of Theorem 2.11, by choosing $G_j = T S_j T^T$, $J_j = \rho I$, $P_j = T Q_j T^T$, $j = 1, 2, \dots, r$, $L = L_1$, $U = S_1$ and applying Schur complement, the

inequalities (2.107) and (2.108) can be changed into (2.90) and (2.91) in Corollary 2.7, respectively. $\square$

2.1.1.3 Several Further Studies

A: Application of Lemma 1.5

The proposed design results in Sects. 2.1.1.1 and 2.1.1.2 can further be relaxed by using LMI technique in Lemma 1.5, which adds more slack matrix variables. To mention a few:

For Theorem 2.2:

Theorem 2.13 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give scalars $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known scalar β , exist matrices $U, V, P_j, J_j, G_j, j = 1, 2, \dots, r, \hat{Y}_{ii}, i = 1, 2, \dots, r$, and $\hat{Y}_{ji}, i, j = 1, 2, \dots, r, i < j$ such that the following matrix inequalities hold

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_i + B_i V C_{2i} & G_i E_i + B_i V H_i & \mathcal{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_i B_i - B_i U & -\frac{J_i}{\beta^2} \end{bmatrix} < \hat{Y}_{ii},$$

$i = 1, 2, \dots, r,$ (2.109)

$$\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + B_j V C_{2j} & G_j E_i + B_j V H_j & \mathcal{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2j} & V H_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_j B_i - B_i U & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$+ \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_j + B_j V C_{2i} & G_i E_j + B_j V H_i & \mathcal{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_i B_j - B_j U & -\frac{J_i}{\beta^2} \end{bmatrix}$$

$< \hat{Y}_{ji} + \hat{Y}_{ji}^T, i, j = 1, 2, \dots, r, i < j,$ (2.110)

$$\begin{bmatrix} \hat{\Upsilon}_{11} & * & \dots & * \\ \hat{\Upsilon}_{21} & \hat{\Upsilon}_{22} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\Upsilon}_{r1} & \hat{\Upsilon}_{r2} & \dots & \hat{\Upsilon}_{rr} \end{bmatrix} < 0, \quad (2.111)$$

where $\mathcal{G}_j = -G_j - G_j^T + P_j$.

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

For Corollary 2.2:

Theorem 2.14 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give scalars $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known scalar β , exist matrices $U, V, P_j, J_j, G_j, j = 1, 2, \dots, r, \hat{\Upsilon}_{ii}, i = 1, 2, \dots, r$, and $\hat{\Upsilon}_{ji}, i, j = 1, 2, \dots, r, i < j$ such that the following matrix inequalities hold

$$\Pi_{ii} < \hat{\Upsilon}_{ii}, \quad i = 1, 2, \dots, r, \quad (2.112)$$

$$\Pi_{ij} + \Pi_{ji} < \hat{\Upsilon}_{ji} + \hat{\Upsilon}_{ji}^T, \quad i < j, \quad i, j = 2, \dots, r, \quad (2.113)$$

$$\begin{bmatrix} \hat{\Upsilon}_{11} & * & \dots & * \\ \hat{\Upsilon}_{21} & \hat{\Upsilon}_{22} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\Upsilon}_{r1} & \hat{\Upsilon}_{r2} & \dots & \hat{\Upsilon}_{rr} \end{bmatrix} < 0, \quad (2.114)$$

with

$$\Pi_{ij} = \begin{bmatrix} -P_j & & & & & \\ 0 & & & & & \\ G_j A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 V C_{2i} & & & & & \\ C_{1i} & & & & & \\ V C_{2i} & & & & & \\ 0 & & & & & \\ G_j E_i + \begin{bmatrix} -\gamma^2 I \\ I \\ 0 \end{bmatrix} R_1 V H_i & \mathcal{G}_j + J_j & * & * & * & \\ F_i & 0 & -I & * & * & \\ V H_i & 0 & 0 & \Sigma_1 & * & \\ 0 & 0 & 0 & \Sigma_2 & -\frac{J_j}{\beta^2} & \end{bmatrix},$$

and

$$\begin{aligned}\mathcal{G}_j &= -G_j - G_j^T + P_j, \\ \Sigma_1 &= -\beta U - \beta U^T, \\ \Sigma_2 &= G_j B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 U.\end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

Of course, the relaxed characteristic of Lemma 1.5 can also be applied to the existing design results. To Lemma 2.3 as an example, we have

Lemma 2.9 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices L , $\hat{P}_j$, R_j , $j = 1, 2, \dots, r$, Υ_{ii} , $i = 1, 2, \dots, r$, and Υ_{ji} , $i, j = 1, 2, \dots, r$, $i < j$ such that the following matrix equations hold

$$\begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_i Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} F_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < \Upsilon_{ii},$$

$i = 1, 2, \dots, r,$
(2.115)

$$\begin{aligned}& \begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_j Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} F_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} \\ & + \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j} & R_i Y E_j + \begin{bmatrix} L \\ 0 \end{bmatrix} F_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} < \Upsilon_{ji} + \Upsilon_{ji}^T, \\ & i, j = 1, 2, \dots, r, i < j,\end{aligned}$$

(2.116)

$$\begin{bmatrix} \Upsilon_{11} & * & \dots & * \\ \Upsilon_{21} & \Upsilon_{22} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r1} & \Upsilon_{r2} & \dots & \Upsilon_{rr} \end{bmatrix} < 0,$$

(2.117)

$$R_j = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix}, \quad j = 1, 2, \dots, r. \quad (2.118)$$

Next, we will study the relationship between the proposed results and the existing ones under considering the relaxed characteristic of Lemma 1.5 (i.e., Theorem 2.14 and Lemma 2.9).

Theorem 2.15 *If the condition given in Lemma 2.9 holds, the condition in Theorem 2.14 also holds.*

Proof The LMIs (2.115) and (2.116) in Lemma 2.9 can be rewritten as follows:

$$\begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_i Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} - \Upsilon_{ii} < 0, \quad i = 1, 2, \dots, r, \quad (2.119)$$

$$\begin{aligned} & \begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_j Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} \\ & + \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j} & R_i Y E_j + \begin{bmatrix} L \\ 0 \end{bmatrix} H_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} - \Upsilon_{ji} - \Upsilon_{ji}^T \\ & < 0, \quad i, j = 1, 2, \dots, r, \quad i < j. \end{aligned} \quad (2.120)$$

Recall (2.56), if the LMIs (2.119) and (2.120) are satisfied, then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that

$$\begin{aligned} & \Omega_{ii} + \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta} (R_1^{-1} L [C_{2i} \quad H_i \quad 0 \quad 0])^T \\ & \times \left(2I - \beta \left(R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right)^T \frac{1}{\rho} I \left(R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right) \right)^{-1} \\ & \times (R_1^{-1} L [C_{2i} \quad H_i \quad 0 \quad 0]) - \Upsilon_{ii} < 0, \quad i = 1, 2, \dots, r, \end{aligned} \quad (2.121)$$

and

$$\begin{aligned}
 & \Omega_{ij} + \Omega_{ji} + 2\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta} (R_1^{-1} L [C_{2i} + C_{2j} \quad H_i + H_j \quad 0 \quad 0])^T \\
 & \times \left(4I - \beta \left(R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right)^T \frac{1}{2\rho} I \left(R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right. \right. \\
 & \left. \left. + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 \right) \right)^{-1} (R_1^{-1} L [C_{2i} + C_{2j} \quad H_i + H_j \quad 0 \quad 0]) - \Upsilon_{ji} - \Upsilon_{ji}^T < 0, \\
 & i, j = 1, 2, \dots, r, i < j,
 \end{aligned} \tag{2.122}$$

with

$$\begin{aligned}
 \Omega_{ij} = & \begin{bmatrix} -\hat{P}_j \\ 0 \\ R_j Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2i} \\ C_{1i} \end{bmatrix} \\
 & \begin{bmatrix} * & * & * \\ -\gamma^2 I & * & * \\ R_j Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L H_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ F_i & 0 & -I \end{bmatrix}.
 \end{aligned}$$

Using Schur complement to (2.121) and (2.122) yields, respectively, it follows as:

$$\begin{bmatrix} \Omega_{ii} + \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} - \Upsilon_{ii} & * & * \\ R_1^{-1} L [C_{2i} \quad H_i \quad 0 \quad 0] & -\beta I - \beta I & * \\ 0 & R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 & -\frac{\rho I}{\beta^2} \end{bmatrix} < 0, i = 1, 2, \dots, r, \tag{2.123}$$

and

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} + 2\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} - \Upsilon_{ji} - \Upsilon_{ji}^T \\ R_1^{-1} L [C_{2i} + C_{2j} \quad H_i + H_j \quad 0 \quad 0] \\ 0 \end{bmatrix}$$

$$\begin{aligned}
 & \begin{bmatrix} * & * \\ -\beta I - \beta I - \beta I - \beta I & * \\ R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 & -2 \frac{\rho I}{\beta^2} \end{bmatrix} < 0, \\
 & i, j = 1, 2, \dots, r, i < j. \\
 & (2.124)
 \end{aligned}$$

If the above LMIs (2.123) and (2.124) are satisfied, then there exists a small enough $\sigma > 0$ such that

$$\begin{aligned}
 & \begin{bmatrix} \Omega_{ii} + \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} & * & * \\ R_1^{-1} L [C_{2i} \ H_i \ 0 \ 0] & -\beta I - \beta I & * \\ 0 & R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 & -\frac{\rho I}{\beta^2} \end{bmatrix} < \begin{bmatrix} \Upsilon_{ii} & * & * \\ 0 & -\sigma I & * \\ 0 & 0 & -\sigma I \end{bmatrix}, \\
 & i = 1, 2, \dots, r, \\
 & (2.125)
 \end{aligned}$$

$$\begin{aligned}
 & \begin{bmatrix} \Omega_{ij} + \Omega_{ji} + 2\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} & * & * \\ R_1^{-1} L [C_{2i} + C_{2j} \ H_i + H_j \ 0 \ 0] & -\beta I - \beta I & -\beta I - \beta I & * \\ 0 & R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 & -2 \frac{\rho I}{\beta^2} \end{bmatrix} \\
 & < \begin{bmatrix} \Upsilon_{ji} & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \Upsilon_{ji}^T & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}, \ i, j = 1, 2, \dots, r, \ i < j. \\
 & (2.126)
 \end{aligned}$$

Let us consider the following matrix:

$$\Pi = \begin{bmatrix} \Upsilon_{11} & * & * & * & * & * & * & * & * & \cdots & * & * & * \\ 0 & -\sigma I & * & * & * & * & * & * & * & \cdots & * & * & * \\ 0 & 0 & -\sigma I & * & * & * & * & * & * & \cdots & * & * & * \\ \Upsilon_{21} & 0 & 0 & \Upsilon_{22} & * & * & * & * & * & \cdots & * & * & * \\ 0 & 0 & 0 & 0 & -\sigma I & * & * & * & * & \cdots & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -\sigma I & * & * & * & \cdots & * & * & * \\ \Upsilon_{31} & 0 & 0 & \Upsilon_{32} & 0 & 0 & \Upsilon_{33} & * & * & \cdots & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma I & * & \cdots & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma I & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \Upsilon_{r1} & 0 & 0 & \Upsilon_{r2} & 0 & 0 & \Upsilon_{r3} & 0 & 0 & \cdots & \Upsilon_{rr} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\sigma I & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\sigma I \end{bmatrix}. \quad (2.127)$$

By applying the technique of elementary transformation of matrix, the matrix Π can be transformed as follows:

$$\Gamma = \begin{bmatrix} \Upsilon_{11} & * & * & \cdots & * & * & * & * & * & \cdots & * & * & * \\ \Upsilon_{21} & \Upsilon_{22} & * & \cdots & * & * & * & * & * & \cdots & * & * & * \\ \Upsilon_{31} & \Upsilon_{32} & \Upsilon_{33} & \cdots & * & * & * & * & * & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \Upsilon_{r1} & \Upsilon_{r2} & \Upsilon_{r3} & \cdots & \Upsilon_{rr} & * & * & * & * & \cdots & * & * & * \\ 0 & 0 & 0 & \cdots & 0 & -\sigma I & * & * & * & \cdots & * & * & * \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\sigma I & * & * & \cdots & * & * & * \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\sigma I & * & \cdots & * & * & * \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -\sigma I & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & -\sigma I & * & * \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\sigma I & * \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\sigma I \end{bmatrix}. \quad (2.128)$$

If the inequality (2.117) is satisfied, which leads to $\Gamma < 0$, i.e., $\Pi < 0$. Let us define

$$\hat{\Upsilon}_{ii} = \begin{bmatrix} \Upsilon_{ii} & * & * \\ 0 & -\sigma I & * \\ 0 & 0 & -\sigma I \end{bmatrix}, \quad i = 1, 2, \dots, r, \quad (2.129)$$

$$\hat{\Upsilon}_{ji} = \begin{bmatrix} \Upsilon_{ji} & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}, \quad i, j = 1, 2, \dots, r, \quad i < j,$$

and unite (2.125) and (2.126), the LMIs (2.112)–(2.114) in Theorem 2.14 can be obtained by defining $G_j = R_j Y$, $P_j = \hat{P}_j$, $J_j = \rho I$, $j = 1, 2, \dots, r$, $V = R_1^{-1} L$, and $U = I$. $\square$

B: Application of Lemma 1.6

When $D(\theta) = 0$, let us recollect the following H_∞ performance analysis criterion

$$\begin{aligned} & \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}^T \\ & \times P(\theta) \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix} \\ & + \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix} < 0. \end{aligned} \quad (2.130)$$

By using Lemma 1.6 with

$$\begin{aligned} T &= \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \\ A &= \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}, \\ P &= P(\theta), \end{aligned}$$

and introducing two auxiliary parameter-dependent matrix variables $G(\theta)$ and $M(\theta)$, the matrix inequality (2.130) can be guaranteed by

$$\begin{bmatrix} \Omega_1 & * \\ \Omega_2 & -G(\theta) - G^T(\theta) + P(\theta) \end{bmatrix} < 0. \quad (2.131)$$

where

$$\begin{aligned} \Omega_1 &= \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \\ &+ \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}^T M^T(\theta) \\ &+ M(\theta) \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}, \\ \Omega_2 &= -M^T(\theta) + G(\theta) \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}. \end{aligned}$$

By defining $M(\theta) = \begin{bmatrix} S(\theta) \\ 0 \end{bmatrix}$ and applying Schur complement to (2.131), it yields

$$\begin{bmatrix} \Phi_{11} & * & * & * \\ \Phi_{21} & -\gamma^2 I & * & * \\ \Phi_{31} & \Phi_{32} & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \quad (2.132)$$

where

$$\begin{aligned}
\Phi_{11} &= -P(\theta) + S(\theta)A(\theta) + S(\theta)B(\theta)KC_2(\theta) \\
&\quad + A^T(\theta)S^T(\theta) + C_2^T(\theta)K^TB^T(\theta)S^T(\theta), \\
\Phi_{21} &= E^T(\theta)S^T(\theta) + H^T(\theta)K^TB^T(\theta)S^T(\theta), \\
\Phi_{31} &= -S^T(\theta) + G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta), \\
\Phi_{32} &= G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta), \\
\mathcal{G}(\theta) &= -G(\theta) - G^T(\theta) + P(\theta).
\end{aligned}$$

Remark 2.12 Compared with the condition (2.5), the H_∞ performance analysis criterion (2.132) is more relaxed. In fact, when $M(\theta) = 0$, the condition (2.132) reduces to (2.5) of Lemma 2.1 that implies that the condition (2.5) is a special case of (2.132).

Next, we derive new static output feedback H_∞ controller design results based on the H_∞ performance analysis criterion (2.132).

Obviously, the matrix inequality (2.132) is equivalent to

$$\begin{aligned}
&\begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} K \\
&\times \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T < 0, \tag{2.133}
\end{aligned}$$

where $\Omega = -P(\theta) + S(\theta)A(\theta) + A^T(\theta)S^T(\theta)$.

By defining $UK = V$ and considering matrices M , R , and N , where U and N are nonsingular without loss of generality, we have

$$\begin{aligned}
&\begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \\
&\times \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
&+ \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
&+ \begin{bmatrix} R \\ 0 \\ M \\ 0 \end{bmatrix} V \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T \begin{bmatrix} R \\ 0 \\ M \\ 0 \end{bmatrix}^T \\
&= \begin{bmatrix} \Pi & * & * & * \\ E^T(\theta)S^T(\theta) + H^T(\theta)V^T R^T & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) + M V C_2(\theta) & G(\theta)E(\theta) + M V H(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
&+ \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0, \tag{2.134}
\end{aligned}$$

where $\Pi = -P(\theta) + S(\theta)A(\theta) + A^T(\theta)S^T(\theta) + RVC_2(\theta) + C_2^T(\theta)V^T R^T$.

By Lemma 1.3, for two positive matrices $J(\theta)$ and $X(\theta)$, we have

$$\begin{aligned}
& \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
& \leq \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} J(\theta) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T \\
& + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} (G(\theta)B(\theta) - MU)^T J^{-1}(\theta) \\
& \times (G(\theta)B(\theta) - MU) U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}, \tag{2.135}
\end{aligned}$$

and

$$\begin{aligned}
& \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
& \leq \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} X(\theta) \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
& + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} (S(\theta)B(\theta) - RU)^T X^{-1}(\theta) \\
& \times (S(\theta)B(\theta) - RU) U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}. \tag{2.136}
\end{aligned}$$

Based on (2.135) and (2.136), we can use the following matrix inequality to ensure (2.134):

$$\begin{aligned}
& \begin{bmatrix} \Pi + X(\theta) & * & * & * & * \\ E^T(\theta)S^T(\theta) + H^T(\theta)V^TR^T & -\gamma^2 I & * & * & * \\ -S^T(\theta) + G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathcal{G}(\theta) + J(\theta) & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * \end{bmatrix} \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T N^T N^{-T} U^{-T} (G(\theta)B(\theta) - MU)^T J^{-1}(\theta) \\
& \times (G(\theta)B(\theta) - MU)U^{-1} N^{-1} NV [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T N^T N^{-T} U^{-T} (S(\theta)B(\theta) - RU)^T X^{-1}(\theta) \\
& \times (S(\theta)B(\theta) - RU)U^{-1} N^{-1} NV [C_2(\theta) \ H(\theta) \ 0 \ 0] < 0. \tag{2.137}
\end{aligned}$$

Follow the same procedure of Theorem 2.1, strictly, the final H_∞ performance analysis criterion is summarized by

$$\begin{bmatrix} \Pi + X(\theta) & * & * & * & * & * & * & * \\ \Theta_{21} & -\gamma^2 I & * & * & * & * & * & * \\ \Theta_{31} & \Theta_{32} & \mathcal{G}(\theta) + J(\theta) & * & * & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * & * & * \\ NV C_2(\theta) & NV H(\theta) & 0 & 0 & \Sigma_1 & * & * & * \\ NV C_2(\theta) & NV H(\theta) & 0 & 0 & 0 & \Sigma_3 & * & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & \Sigma_4 & 0 & -\frac{X(\theta)}{\eta^2} \end{bmatrix} < 0, \tag{2.138}$$

where

$$\begin{aligned}
\Theta_{21} &= E^T(\theta)S^T(\theta) + H^T(\theta)V^TR^T, \\
\Theta_{31} &= -S^T(\theta) + G(\theta)A(\theta) + MVC_2(\theta), \\
\Theta_{32} &= G(\theta)E(\theta) + MVH(\theta), \\
\Sigma_1 &= -\beta NU - \beta U^T N^T, \\
\Sigma_2 &= G(\theta)B(\theta) - MU, \\
\Sigma_3 &= -\eta NU - \eta U^T N^T, \\
\Sigma_4 &= S(\theta)B(\theta) - RU.
\end{aligned}$$

Remark 2.13 In fact, the H_∞ performance analysis criterion (2.138) can be relaxed further by adding another matrix variable $\tilde{N}$. Let us rewrite the matrix inequality (2.133) as follows:

$$\begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix}$$

$$\begin{aligned}
& + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} K [C_2(\theta) \ H(\theta) \ 0 \ 0] + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T K^T \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T \\
& + \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix} K [C_2(\theta) \ H(\theta) \ 0 \ 0] + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T K^T \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
& = \begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathcal{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} U^{-1}N^{-1}NV [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T \\
& + \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1}\tilde{N}^{-1}\tilde{N}V [C_2(\theta) \ H(\theta) \ 0 \ 0] \\
& + [C_2(\theta) \ H(\theta) \ 0 \ 0]^T V^T \tilde{N}^T \tilde{N}^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix}^T.
\end{aligned}$$

Then, the H_∞ performance analysis criterion (2.138) becomes

$$\begin{bmatrix} \Pi + X(\theta) & * & * & * & * & * & * & * \\ \Theta_{21} & -\gamma^2 I & * & * & * & * & * & * \\ \Theta_{31} & \Theta_{32} & \mathcal{G}(\theta) + J(\theta) & * & * & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * & * & * \\ NV C_2(\theta) & NV H(\theta) & 0 & 0 & \Sigma_1 & * & * & * \\ \tilde{N}V C_2(\theta) & \tilde{N}V H(\theta) & 0 & 0 & 0 & \tilde{\Sigma}_3 & * & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & \Sigma_4 & 0 & -\frac{X(\theta)}{\eta^2} \end{bmatrix} < 0,$$

with $\tilde{\Sigma}_3 = -\eta \tilde{N}U - \eta U^T \tilde{N}^T$.

Based on (2.138), the static output feedback H_∞ controller design result is given in the following theorem.

Theorem 2.16 Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M , N , R and scalar β , η , exist matrices U , V , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\Delta_{ii} < 0, \quad i = 1, 2, \dots, r, \quad (2.139)$$

$$\Delta_{ij} + \Delta_{ji} < 0, \quad i < j, \quad i, j = 1, 2, \dots, r, \quad (2.140)$$

with

$$\Delta_{ij} = \begin{bmatrix} \Lambda_{11} & * & * & * & * & * & * & * \\ \Lambda_{21} & -\gamma^2 I & * & * & * & * & * & * \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & * & * & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * & * & * \\ NVC_{2i} & NVH_i & 0 & 0 & -\beta NU - \beta U^T N^T & * & * & * \\ 0 & 0 & 0 & 0 & G_j B_i - MU & -\frac{J_j}{\beta^2} & * & * \\ NVC_{2i} & NVH_i & 0 & 0 & 0 & 0 & \Xi_3 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Xi_4 & -\frac{X_j}{\eta^2} \end{bmatrix},$$

and

$$\begin{aligned} \Lambda_{11} &= -P_j + S_j A_i + A_i^T S_j^T + RVC_{2i} + C_{2i}^T V^T R^T + X_j, \\ \Lambda_{21} &= E_i^T S_j^T + H_i^T V^T R^T, \\ \Lambda_{31} &= -S_j^T + G_j A_i + MVC_{2i}, \\ \Lambda_{32} &= G_j E_i + MVH_i, \\ \Lambda_{33} &= -G_j - G_j^T + P_j + J_j, \\ \Xi_3 &= -\eta NU - \eta U^T N^T, \\ \Xi_4 &= S_j B_i - RU. \end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

C: An Extended Study for the Case $D(\theta) \neq 0$ and $H(\theta) \neq 0$

Approach 1:

In this case, the condition (2.5) is replaced to

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta) & G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta) & \Lambda_1 & * \\ S(\theta)C_1(\theta) + S(\theta)D(\theta)KC_2(\theta) & S(\theta)F(\theta) + S(\theta)D(\theta)KH(\theta) & 0 & \Lambda_2 \end{bmatrix} < 0, \quad (2.141)$$

where

$$\Lambda_1 = -G(\theta) + G^T(\theta) + P(\theta),$$

$$\Lambda_2 = -S(\theta) + S^T(\theta) + I.$$

By considering a similar process with the condition (2.138), we can obtain the following H_∞ performance analysis criterion:

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * & * & * \\ \Theta_{31} & \Theta_{32} & \Theta_{33} & * & * & * & * & * \\ \Theta_{41} & \Theta_{42} & 0 & \Theta_{44} & * & * & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & -\beta NU - \beta U^T N^T & * & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & 0 & \Sigma_3 & * & * \\ 0 & 0 & 0 & 0 & G(\theta)B(\theta) - MU & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_4 & 0 & -\frac{X(\theta)}{\eta^2} \end{bmatrix} < 0, \quad (2.142)$$

where

$$\Theta_{31} = G(\theta)A(\theta) + MVC_2(\theta),$$

$$\Theta_{32} = G(\theta)E(\theta) + MVH(\theta),$$

$$\Theta_{33} = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),$$

$$\Theta_{41} = S(\theta)C_1(\theta) + RVC_2(\theta),$$

$$\Theta_{42} = S(\theta)F(\theta) + RVH(\theta),$$

$$\Theta_{44} = -S(\theta) - S^T(\theta) + I + X(\theta),$$

$$\Sigma_3 = -\eta NU - \eta U^T N^T,$$

$$\tilde{\Sigma}_4 = S(\theta)D(\theta) - RU.$$

Theorem 2.17 Consider the closed-loop system (2.4) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M , N , R and scalars β , η , exist matrices U , V , P_j , J_j , S_j , X_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\Delta_{ii} < 0, \quad i = 1, 2, \dots, r, \quad (2.143)$$

$$\Delta_{ij} + \Delta_{ji} < 0, \quad i < j, \quad i, j = 1, 2, \dots, r, \quad (2.144)$$

with

$$\Delta_{ij} = \begin{bmatrix} -P_j & * & * & * & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * & * & * & * \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & * & * & * & * & * & * \\ \Lambda_{41} & \Lambda_{42} & 0 & \Lambda_{44} & * & * & * & * & * \\ NVC_{2i} & NVH_i & 0 & 0 & -\beta NU - \beta U^T N^T & * & * & * & * \\ NVC_{2i} & NVH_i & 0 & 0 & 0 & \Sigma_3 & * & * & * \\ 0 & 0 & 0 & 0 & G_j B_i - MU & 0 & -\frac{J_j}{\beta^2} & * & * \\ 0 & 0 & 0 & 0 & 0 & S_j D_i - RU & 0 & -\frac{X_j}{\eta^2} & * \end{bmatrix},$$

and

$$\begin{aligned} \Lambda_{31} &= G_j A_i + MVC_{2i}, \\ \Lambda_{32} &= G_j E_i + MVH_i, \\ \Lambda_{33} &= -G_j - G_j^T + P_j + J_j, \\ \Lambda_{41} &= S_j C_{1i} + RVC_{2i}, \\ \Lambda_{44} &= -S_j - S_j^T + I + X_j, \\ \Lambda_{42} &= S_j F_i + RVH_i, \\ \Sigma_3 &= -\eta NU - \eta U^T N^T. \end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

Approach 2:

Let us consider (2.65) with $H(\theta) \neq 0$, it follows that

$$\begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -S(\theta) - S^T(\theta) + \frac{1}{\gamma^2} I & * & * \\ A(\theta)G(\theta) + B(\theta)KC_2(\theta)G(\theta) & E(\theta)S(\theta) + B(\theta)KH(\theta)S(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) + D(\theta)KC_2(\theta)G(\theta) & F(\theta)S(\theta) + D(\theta)KH(\theta)S(\theta) & 0 & -I \end{bmatrix} < 0. \quad (2.145)$$

By defining $K = LU^{-1}$ and adding a nonsingular matrix N , the matrix inequality (2.145) can be rewritten as the following form:

$$\begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -S(\theta) - S^T(\theta) + \frac{1}{\gamma^2} I & * & * \\ A(\theta)G(\theta) & E(\theta)S(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) & F(\theta)S(\theta) & 0 & -I \end{bmatrix}$$

$$\begin{aligned}
& + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} L N N^{-1} U^{-1} \begin{bmatrix} C_2(\theta)G(\theta) & H(\theta)S(\theta) & 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} C_2(\theta)G(\theta) & H(\theta)S(\theta) & 0 & 0 \end{bmatrix}^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0.
\end{aligned} \tag{2.146}$$

Considering two matrices M and R , from (2.146), we have

$$\begin{aligned}
& \begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -S(\theta) - S^T(\theta) + \frac{1}{\gamma^2}I & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta)S(\theta) + B(\theta)LR & -P(\theta) & * \\ C_1G(\theta) + D(\theta)LM & F(\theta)S(\theta) + D(\theta)LR & 0 & -I \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} L N N^{-1} U^{-1} \begin{bmatrix} C_2(\theta)G(\theta) - UM & H(\theta)S(\theta) - UR & 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} C_2(\theta)G(\theta) - UM & H(\theta)S(\theta) - UR & 0 & 0 \end{bmatrix}^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0.
\end{aligned} \tag{2.147}$$

Then, we can give the following H_∞ performance analysis criterion:

$$\begin{bmatrix} \tilde{\Gamma}_{11} & * & * & * & * & * & * & * \\ 0 & \tilde{\Gamma}_{22} & * & * & * & * & * & * \\ \tilde{\Gamma}_{31} & \tilde{\Gamma}_{32} & -P(\theta) & * & * & * & * & * \\ \tilde{\Gamma}_{41} & \tilde{\Gamma}_{42} & 0 & -I & * & * & * & * \\ 0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & \Sigma_1 & * & * & * \\ 0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & 0 & \Sigma_2 & * & * \\ 0 & 0 & 0 & 0 & \tilde{\Sigma}_3 & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_4 & 0 & -\frac{X(\theta)}{\eta^2} \end{bmatrix} < 0, \tag{2.148}$$

where

$$\tilde{\Gamma}_{11} = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),$$

$$\tilde{\Gamma}_{22} = -S(\theta) - S^T(\theta) + \frac{1}{\gamma^2}I + X(\theta),$$

$$\tilde{\Gamma}_{31} = A(\theta)G(\theta) + B(\theta)LM,$$

$$\tilde{\Gamma}_{32} = E(\theta)S(\theta) + B(\theta)LR,$$

$$\tilde{\Gamma}_{41} = C_1(\theta)G(\theta) + D(\theta)LM,$$

$$\tilde{\Gamma}_{42} = F(\theta)S(\theta) + D(\theta)LR,$$

$$\Sigma_1 = -\beta UN - \beta N^T U^T,$$

$$\Sigma_2 = -\eta UN - \eta N^T U^T,$$

$$\tilde{\Sigma}_3 = (C_2(\theta)G(\theta) - UM)^T,$$

$$\tilde{\Sigma}_4 = (H(\theta)S(\theta) - UR)^T.$$

Theorem 2.18 Consider the closed-loop system (2.4) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M , N , R and scalar β , η , exist matrices U , L , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\Delta_{ii} < 0, \quad i = 1, 2, \dots, r, \quad (2.149)$$

$$\Delta_{ij} + \Delta_{ji} < 0, \quad i < j, \quad i, j = 1, 2, \dots, r, \quad (2.150)$$

with

$$\Delta_{ij} = \begin{bmatrix} \Gamma_{11} & * & * & * & * & * & * & * & * \\ 0 & \Gamma_{22} & * & * & * & * & * & * & * \\ \Gamma_{31} & \Gamma_{32} & -P(\theta) & * & * & * & * & * & * \\ \Gamma_{41} & \Gamma_{42} & 0 & -I & * & * & * & * & * \\ 0 & 0 & N^T L^T B_i^T & N^T L^T D_i^T & \Sigma_1 & * & * & * & * \\ 0 & 0 & N^T L^T B_i^T & N^T L^T D_i^T & 0 & \Sigma_2 & * & * & * \\ 0 & 0 & 0 & 0 & \Sigma_3 & 0 & -\frac{J_j}{\beta^2} & * & * \\ 0 & 0 & 0 & 0 & 0 & \Sigma_4 & 0 & -\frac{X_j}{\eta^2} & * \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix},$$

and

$$\Gamma_{11} = -G_j - G_j^T + P_j + J_j,$$

$$\Gamma_{22} = -S_j - S_j^T + X_j,$$

$$\Gamma_{31} = A_i G_j + B_i LM,$$

$$\Gamma_{32} = E_i S_j + B_i LR,$$

$$\Gamma_{41} = C_{1i} G_j + D_i LM,$$

$$\Gamma_{42} = F_i S_j + D_i LR,$$

$$\begin{aligned}
\Sigma_1 &= -\beta U N - \beta N^T U^T, \\
\Sigma_2 &= -\eta U N - \eta N^T U^T, \\
\Sigma_3 &= (C_{2i} G_j - U M)^T, \\
\Sigma_4 &= (H_i S_j - U R)^T.
\end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.85).

2.1.2 Continuous-Time Systems

Consider the following continuous-time linear systems system with polytopic uncertainties

$$\begin{aligned}
\dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) + E(\theta)w(t), \\
z(t) &= C_1(\theta)x(t) + D(\theta)u(t) + F(\theta)w(t), \\
y(t) &= C_2(\theta)x(t) + H(\theta)w(t),
\end{aligned} \tag{2.151}$$

where $x(t) \in \mathcal{R}^n$ is the state variable, $u(t) \in \mathcal{R}^m$ is the control input, $w(t) \in \mathcal{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $L_2[0, \infty)$, $z(t) \in \mathcal{R}^q$ is the controlled output variable, $y(t) \in \mathcal{R}^p$ is the measurement output. The system matrices $A(\theta)$, $B(\theta)$, $E(\theta)$, $C_1(\theta)$, $D(\theta)$, $F(\theta)$, $C_2(\theta)$, and $H(\theta)$ belong to (2.2).

Here, we consider the following static output feedback H_∞ controller

$$u(t) = K y(t). \tag{2.152}$$

By substituting (2.152) to (2.151), the closed-loop system is given by

$$\begin{aligned}
\dot{x}(t) &= (A(\theta) + B(\theta)K C_2(\theta))x(t) + (E(\theta) + B(\theta)K H(\theta))w(t), \\
z(t) &= (C_1(\theta) + D(\theta)K C_2(\theta))x(t) + (F(\theta) + D(\theta)K H(\theta))w(t).
\end{aligned} \tag{2.153}$$

2.1.2.1 Case A: $D(\theta) = 0$

First, we will study new H_∞ performance analysis criterions for the continuous-time closed-loop system (2.153) with $D(\theta) = 0$. In this following, a basic lemma is given.

Theorem 2.19 *Consider the closed-loop system (2.153) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices $P(\theta)$, $G(\theta)$, $X(\theta)$, and K such that the following matrix inequality holds*

$$\begin{bmatrix}
-G(\theta) - G^T(\theta) & * & * & * & * \\
(A(\theta) + B(\theta)KC_2(\theta))^T G^T(\theta) + P(\theta) & -2P(\theta) + X(\theta) & * & * & * \\
(E(\theta) + B(\theta)KH(\theta))^T G^T(\theta) & 0 & -\gamma^2 I & * & * \\
0 & C_1(\theta) & F(\theta) & -I & * \\
G^T(\theta) & 0 & 0 & 0 & -X(\theta)
\end{bmatrix}$$

< 0 .

(2.154)

Proof Construct a parameter-dependent Lyapunov function as

$$V(t) = x^T(t)P(\theta)x(t), \quad P(\theta) > 0. \quad (2.155)$$

The derivative of $V(t)$ can be given by

$$\dot{V}(t) = \dot{x}^T(t)P(\theta)x(t) + x^T(t)P(\theta)\dot{x}(t). \quad (2.156)$$

Then, we have

$$\begin{aligned}
& \dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\
&= \dot{x}^T(t)P(\theta)x(t) + x^T(t)P(\theta)\dot{x}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\
&= \left((A(\theta) + B(\theta)KC_2(\theta))x(t) + (E(\theta) + B(\theta)KH(\theta))w(t) \right)^T P(\theta)x(t) \\
&\quad + x^T(t)P(\theta) \left((A(\theta) + B(\theta)KC_2(\theta))x(t) + (E(\theta) + B(\theta)KH(\theta))w(t) \right) \\
&\quad + (C_1(\theta)x(t) + F(\theta)w(t))^T (C_1(\theta)x(t) + F(\theta)w(t)) - \gamma^2 w^T(t)w(t) \\
&= \begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix} \begin{bmatrix} \Omega & * \\ (E(\theta) + B(\theta)KH(\theta))^T P(\theta) & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} x^T(t) & w^T(t) \end{bmatrix} \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix},
\end{aligned} \quad (2.157)$$

where $\Omega = P(\theta)(A(\theta) + B(\theta)KC_2(\theta)) + (A(\theta) + B(\theta)KC_2(\theta))^T P(\theta)$.

If the following inequality

$$\begin{bmatrix} \Omega & * \\ (E(\theta) + B(\theta)KH(\theta))^T P(\theta) & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} < 0, \quad (2.158)$$

holds, we have $\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0$ for any $\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \neq 0$.

By using Schur complement, (2.158) leads to

$$\begin{bmatrix} \Omega & * & * \\ (E(\theta) + B(\theta)KH(\theta))^T P(\theta) & -\gamma^2 I & * \\ C_1(\theta) & F(\theta) & -I \end{bmatrix} < 0. \quad (2.159)$$

Then, based on Lemma 1.7 with considering a parameter-dependent matrix $X(\theta)$ and $V = G(\theta)$, the matrix inequality (2.159) can be guaranteed by (2.154). If the matrix inequality (2.154) is satisfied, we have $\dot{V}(t) + e^T(t)e(t) - \gamma^2 w^T(t)w(t) < 0$ for any $\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \neq 0$, which implies that

$$V(\infty) - V(0) + \int_0^\infty e^T(t)e(t)dt - \gamma^2 \int_0^\infty w^T(t)w(t)dt < 0.$$

With zero initial condition $x(0) = 0$ and $V(\infty) > 0$, we obtain $\int_0^\infty e^T(t)e(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt$ for any nonzero $w(t) \in L_2[0, \infty)$. Thus, the proof is completed. $\square$

Based on the discussion in Theorem 2.19, we introduce our main results on H_∞ performance analysis for the continuous-time closed-loop system (2.153) with $D(\theta) = 0$. Obviously, the matrix inequality (2.154) is equivalent to

$$\begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * \\ A^T(\theta)G^T(\theta) + P(\theta) & -2P(\theta) + X(\theta) & * & * & * \\ E^T(\theta)G^T(\theta) & 0 & -\gamma^2 I & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix} + \begin{bmatrix} G(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times K \begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} G(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0. \quad (2.160)$$

Then, by following the same derivation with Theorem 2.1, we give the following H_∞ performance analysis criterion:

Theorem 2.20 Consider the closed-loop system (2.153) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices $P(\theta)$, $G(\theta)$, $X(\theta)$, $J(\theta)$, M , N , V , and U , scalar β such that the following matrix inequality holds

$$\begin{bmatrix} \Xi_{11} & * & * & * & * & * & * \\ \Xi_{21} & -2P(\theta) + X(\theta) & * & * & * & * & * \\ \Xi_{31} & 0 & -\gamma^2 I & * & * & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * & * & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) & * & * \\ 0 & NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Lambda_1 & * \\ 0 & 0 & 0 & 0 & 0 & \Lambda_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.161)$$

where

$$\begin{aligned} \Xi_{11} &= -G(\theta) - G^T(\theta) + J(\theta), \\ \Xi_{21} &= A^T(\theta)G^T(\theta) + P(\theta) + C_2^T(\theta)V^T M^T, \\ \Xi_{31} &= E^T(\theta)G^T(\theta) + H^T(\theta)V^T M^T, \\ \Lambda_1 &= -\beta NU - \beta U^T N^T, \\ \Lambda_2 &= G(\theta)B(\theta) - MU. \end{aligned}$$

By the analysis condition (2.161), the corresponding static output feedback H_∞ controller design result is given in the following theorem.

Theorem 2.21 Consider the closed-loop system (2.153) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M , N and scalar β , exist matrices U , L , P_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\Upsilon_{ii} < 0, \quad i = 1, 2, \dots, r, \quad (2.162)$$

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad i < j, \quad i, j = 1, 2, \dots, r, \quad (2.163)$$

with

$$\Upsilon_{ij} = \begin{bmatrix} \Upsilon_{11} & * & * & * & * & * & * \\ \Upsilon_{21} & -2P_j + X_j & * & * & * & * & * \\ \Upsilon_{31} & 0 & -\gamma^2 I & * & * & * & * \\ 0 & C_{1i} & F_i & -I & * & * & * \\ G_j^T & 0 & 0 & 0 & -X_j & * & * \\ 0 & NVC_{2i} & NVH_i & 0 & 0 & \Phi_1 & * \\ 0 & 0 & 0 & 0 & 0 & \Phi_2 & -\frac{J_j}{\beta^2} \end{bmatrix},$$

and

$$\begin{aligned} \Upsilon_{11} &= -G_j - G_j^T + J_j, \\ \Upsilon_{21} &= A_i^T G_j^T + P_j + C_{2i}^T V^T M^T, \\ \Upsilon_{31} &= E_i^T G_j^T + H_i^T V^T M^T, \\ \Phi_1 &= -\beta NU - \beta U^T N^T, \end{aligned}$$

$$\Phi_2 = G_j B_i - MU.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

2.1.2.2 Case B: $H(\theta) = 0$

In this case, by choosing a parameter-dependent Lyapunov function as $V(t) = x^T(t)Q^{-1}(\theta)x(t)$, $Q(\theta) > 0$ and using Lemma 1.8, the basic lemma is given as follows.

Lemma 2.10 Consider the closed-loop system (2.153) with $H(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices $Q(\theta)$, $G(\theta)$, $X(\theta)$, and K such that the following matrix inequality holds

$$\begin{aligned} & \begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * \\ (A(\theta) + B(\theta)KC_2(\theta))G(\theta) + Q(\theta) & -2Q(\theta) + X(\theta) & * & * & * \\ 0 & E^T(\theta) & -\gamma^2 I & * & * \\ (C_1(\theta) + D(\theta)KC_2(\theta))G(\theta) & 0 & F(\theta) & -I & * \\ G(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix} \\ &= \begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * \\ A(\theta)G(\theta) + Q(\theta) & -2Q(\theta) + X(\theta) & * & * & * \\ 0 & E^T(\theta) & -\gamma^2 I & * & * \\ C_1(\theta)G(\theta) & 0 & F(\theta) & -I & * \\ G(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix} + \begin{bmatrix} 0 \\ B(\theta) \\ 0 \\ D(\theta) \\ 0 \end{bmatrix} \\ & \quad \times K [C_2(\theta)G(\theta) \ 0 \ 0 \ 0 \ 0] + [C_2(\theta)G(\theta) \ 0 \ 0 \ 0 \ 0]^T K^T \begin{bmatrix} 0 \\ B(\theta) \\ 0 \\ D(\theta) \\ 0 \end{bmatrix}^T < 0. \end{aligned} \quad (2.164)$$

By applying the same process with Theorem 2.9, we obtain a new H_∞ performance analysis criterion for the continuous-time closed-loop system with $H(\theta) = 0$, which is given in the following theorem.

Theorem 2.22 Consider the closed-loop system (2.153) with $H(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices M , N , U , L , $Q(\theta)$, $G(\theta)$, $X(\theta)$, and $J(\theta)$, scalar β such that the following matrix inequality holds

$$\begin{bmatrix} \Omega_{11} & * & * & * & * & * & * \\ \Omega_{21} & -2Q(\theta) + X(\theta) & * & * & * & * & * \\ 0 & E^T(\theta) & -\gamma^2 I & * & * & * & * \\ \Omega_{41} & 0 & F(\theta) & -I & * & * & * \\ G(\theta) & 0 & 0 & 0 & -X(\theta) & * & * \\ 0 & N^T L^T B^T(\theta) & 0 & N^T L^T D^T(\theta) & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \quad (2.165)$$

where

$$\Omega_{11} = -G(\theta) - G^T(\theta) + J(\theta),$$

$$\Omega_{21} = A(\theta)G(\theta) + B(\theta)LM + Q(\theta),$$

$$\Omega_{41} = C_1(\theta)G(\theta) + D(\theta)LM,$$

$$\Sigma_1 = -\beta UN - \beta N^T U^T,$$

$$\tilde{\Sigma}_2 = (C_2(\theta)G(\theta) - UM)^T.$$

Further, we have the following design result:

Theorem 2.23 Consider the closed-loop system (2.153) with $H(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M , N and scalar β , exist matrices U , L , Q_j , X_j , J_j , and G_j , $j = 1, 2, \dots, r$ such that the following matrix inequalities hold

$$\Lambda_{ii} < 0, \quad i = 1, 2, \dots, r, \quad (2.166)$$

$$\Lambda_{ij} + \Lambda_{ji} < 0, \quad i < j, \quad i, j = 1, 2, \dots, r, \quad (2.167)$$

with

$$\Lambda_{ij} = \begin{bmatrix} \Psi_{11} & * & * & * & * & * & * \\ \Psi_{21} & -2Q_j + X_j & * & * & * & * & * \\ 0 & E_i^T & -\gamma^2 I & * & * & * & * \\ \Psi_{41} & 0 & F_i & -I & * & * & * \\ G_j & 0 & 0 & 0 & -X_j & * & * \\ 0 & N^T L^T B_i^T & 0 & N^T L^T D_i^T & 0 & -\beta UN - \beta N^T U^T & * \\ 0 & 0 & 0 & 0 & 0 & (C_{2i}G_j - UM)^T & -\frac{J_j}{\beta^2} \end{bmatrix},$$

where

$$\Psi_{11} = -G_j - G_j^T + J_j,$$

$$\Psi_{21} = A_i G_j + B_i LM + Q_j,$$

$$\Psi_{41} = C_{1i} G_j + D_i LM.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.85).

Remark 2.14 Here, we discuss briefly the application of Lemma 1.9 to design static output feedback H_∞ controllers for the continuous-time closed-loop system (2.153). Let us rewrite the H_∞ performance analysis condition (2.158) as

$$\begin{aligned} & \begin{bmatrix} P(\theta) \\ 0 \end{bmatrix} \begin{bmatrix} A(\theta) + B(\theta)K C_2(\theta) & E(\theta) + B(\theta)K H(\theta) \end{bmatrix} \\ & + \begin{bmatrix} A(\theta) + B(\theta)K C_2(\theta) & E(\theta) + B(\theta)K H(\theta) \end{bmatrix}^T \begin{bmatrix} P(\theta) \\ 0 \end{bmatrix}^T \\ & + \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} 0 & * \\ 0 & -\gamma^2 I \end{bmatrix} < 0. \end{aligned} \quad (2.168)$$

By using Lemma 1.9 with

$$\begin{aligned} T &= \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} 0 & * \\ 0 & -\gamma^2 I \end{bmatrix}, \\ A &= \begin{bmatrix} A(\theta) + B(\theta)K C_2(\theta) & E(\theta) + B(\theta)K H(\theta) \end{bmatrix}, \\ P &= \begin{bmatrix} P(\theta) \\ 0 \end{bmatrix}, \end{aligned}$$

and Schur complement, we see that the matrix inequality (2.167) can be verified by

$$\begin{bmatrix} \begin{bmatrix} 0 & * \\ 0 & -\gamma^2 I \end{bmatrix} + M(\theta)\tilde{A} + \tilde{A}^T M^T(\theta) & * & * \\ \begin{bmatrix} P(\theta) & 0 \end{bmatrix} - M^T(\theta) + G(\theta)\tilde{A} & -G(\theta) - G^T(\theta) & * \\ \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} & 0 & -I \end{bmatrix} < 0, \quad (2.169)$$

where $\tilde{A} = \begin{bmatrix} A(\theta) + B(\theta)K C_2(\theta) & E(\theta) + B(\theta)K H(\theta) \end{bmatrix}$.

Now, select $M(\theta) = \begin{bmatrix} S(\theta) \\ 0 \end{bmatrix}$. Thus, (2.168) is equivalent to

$$\begin{bmatrix} \Omega_{11} & * & * & * \\ \Omega_{21} & -\gamma^2 I & * & * \\ \Omega_{31} & G(\theta)E(\theta) + G(\theta)B(\theta)K H(\theta) & -G(\theta) - G^T(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \quad (2.170)$$

where

$$\begin{aligned} \Omega_{11} &= S(\theta)A(\theta) + S(\theta)B(\theta)K C_2(\theta) + A^T(\theta)S^T(\theta) + C_2^T(\theta)K^T B^T(\theta)S^T(\theta), \\ \Omega_{21} &= E^T(\theta)S^T(\theta) + H^T(\theta)K^T B^T(\theta)S^T(\theta), \end{aligned}$$

$$\Omega_{31} = P(\theta) - S^T(\theta) + G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta).$$

Then, the remaining discussion can reference Theorem 2.16, and it is omitted.

Remark 2.15 For the case $D(\theta) \neq 0$ and $H(\theta) \neq 0$, we can apply the following two H_∞ performance analysis criterions to design static output feedback H_∞ controllers, which are obtained easily from (2.161) and (2.165).

$$\begin{bmatrix} \Xi_{11} & * & * & * & * & * & * & * & * \\ \Xi_{21} & -2P(\theta) + X(\theta) & * & * & * & * & * & * & * \\ \Xi_{31} & 0 & -\gamma^2 I & * & * & * & * & * & * \\ 0 & \Xi_{42} & \Xi_{43} & \Xi_{44} & * & * & * & * & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) & * & * & * & * \\ 0 & NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Lambda_1 & * & * & * \\ 0 & NVC_2(\theta) & NVH(\theta) & 0 & 0 & 0 & \Lambda_3 & * & * \\ 0 & 0 & 0 & 0 & 0 & \Lambda_2 & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_4 & 0 & -\frac{W(\theta)}{\eta^2} \end{bmatrix} < 0, \quad (2.171)$$

where Ξ_{11} , Ξ_{21} , Ξ_{31} , Λ_1 , and Λ_2 are defined in (2.161), and

$$\begin{aligned} \Xi_{42} &= S(\theta)C_1(\theta) + RVC_2(\theta), \\ \Xi_{43} &= S(\theta)F(\theta) + RVH(\theta), \\ \Xi_{44} &= -S(\theta) - S^T(\theta) + I + W(\theta), \\ \Lambda_3 &= -\eta NU - \eta U^T N^T, \\ \Lambda_4 &= S(\theta)D(\theta) - RU. \end{aligned}$$

and

$$\begin{bmatrix} \Omega_{11} & * & * & * & * & * & * & * & * & * \\ \Omega_{21} & -2Q(\theta) + X(\theta) & * & * & * & * & * & * & * & * \\ 0 & \Omega_{32} & \Omega_{33} & * & * & * & * & * & * & * \\ \Omega_{41} & 0 & \Omega_{43} & -I & * & * & * & * & * & * \\ G(\theta) & 0 & 0 & 0 & -X(\theta) & * & * & * & * & * \\ 0 & N^T L^T B^T(\theta) & 0 & N^T L^T D^T(\theta) & 0 & \Sigma_1 & * & * & * & * \\ 0 & N^T L^T B^T(\theta) & 0 & N^T L^T D^T(\theta) & 0 & 0 & \Sigma_3 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_2 & 0 & -\frac{J(\theta)}{\beta^2} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_4 & 0 & -\frac{W(\theta)}{\beta^2} & * \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad (2.172)$$

where Ω_{11} , Ω_{21} , Ω_{41} , Σ_1 , and Σ_2 are defined in (2.165), and

$$\Omega_{32} = S^T(\theta)E^T(\theta) + R^T L^T B^T(\theta),$$

$$\begin{aligned}
\Omega_{33} &= -S(\theta) - S^T(\theta) + W(\theta), \\
\Omega_{43} &= F(\theta)S(\theta) + D(\theta)LR, \\
\Sigma_3 &= -\eta UN - \eta N^T U^T, \\
\Sigma_4 &= (H(\theta)S(\theta) - UR)^T.
\end{aligned}$$

Remark 2.16 In the above study, either discrete-time or continuous-time, the proposed H_∞ performance analysis conditions are given are based on Lemma 1.3 with a matrix $J(\theta) > 0$ (or another $X(\theta) > 0$). In fact, we can use Lemma 1.10 to displace the application of Lemma 1.3. The displacement avoids the appearance of the auxiliary matrix variable $J(\theta) > 0$ (or another $X(\theta) > 0$), which reduces the dimension of LMIs in those design conditions. To the case of the closed-loop system (2.153) with $D(\theta) = 0$ as an example, we rewrite the matrix inequality (2.160) as follows:

$$\begin{aligned}
& \begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * \\ A^T(\theta)G^T(\theta) + C_2^T(\theta)V^T M^T + Q(\theta) & -2Q(\theta) + X(\theta) & * & * & * \\ E^T(\theta)G^T(\theta) + H^T(\theta)V^T M^T & 0 & -\gamma^2 I & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix} \\
& + \begin{bmatrix} G(\theta)B(\theta) - MU \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
& + \begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} G(\theta)B(\theta) - MU \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0.
\end{aligned} \tag{2.173}$$

Now, for (2.173), we use Lemma 1.10 with

$$\begin{aligned}
T &= \begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * \\ A^T(\theta)G^T(\theta) + Q(\theta) & -2Q(\theta) + X(\theta) & * & * & * \\ E^T(\theta)G^T(\theta) & 0 & -\gamma^2 I & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix}, \\
A &= U^{-1}N^{-1}NV \begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}, \\
P &= \begin{bmatrix} G(\theta)B(\theta) - MU \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\end{aligned}$$

$$S = NU.$$

Then, a new H_∞ performance analysis condition can be obtained.

$$\begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * & * \\ \Pi_1 & -2Q(\theta) + X(\theta) & * & * & * & * \\ E^T(\theta)G^T(\theta) + H^T(\theta)V^T M^T & 0 & -\gamma^2 I & * & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) & * \\ \beta(G(\theta)B(\theta) - MU)^T & NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Pi_2 \end{bmatrix} < 0, \quad (2.174)$$

where

$$\Pi_1 = A^T(\theta)G^T(\theta) + C_2^T(\theta)V^T M^T + Q(\theta),$$

$$\Pi_2 = -\beta NU - \beta U^T N^T.$$

From the H_∞ performance analysis result (2.174), it can easily be known that the corresponding static output feedback H_∞ controllers design conditions are of LMIs.

2.2 With Norm Bounded Uncertainties

To keep things simple, we just study discrete-time systems in this section. Consider the following linear discrete-time dynamic model with time-varying norm bounded uncertainties:

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + (B + \Delta B)u(k) + (E + \Delta E)w(k), \\ z(k) &= (C_1 + \Delta C_1)x(k) + (D + \Delta D)u(k) + (F + \Delta F)w(k), \\ y(k) &= (C_2 + \Delta C_2)x(k) + (H + \Delta H)w(k), \end{aligned} \quad (2.175)$$

where $x(k) \in \mathcal{R}^n$ is the state variable, $u(k) \in \mathcal{R}^m$ is the control input, $w(k) \in \mathcal{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathcal{R}^q$ is the controlled output variable, $y(k) \in \mathcal{R}^p$ is the measurement output. $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $E \in \mathcal{R}^{n \times f}$, $C_1 \in \mathcal{R}^{q \times n}$, $D \in \mathcal{R}^{q \times m}$, $F \in \mathcal{R}^{q \times f}$, $C_2 \in \mathcal{R}^{p \times n}$, and $H \in \mathcal{R}^{p \times f}$ are system matrices. ΔA , ΔB , ΔE , ΔC_1 , ΔD , ΔF , ΔC_2 , and ΔH are uncertainties formulated as [10]

$$\begin{aligned} \Delta A &= X_A \Delta(k) Y_A, & \Delta B &= X_B \Delta(k) Y_B, \\ \Delta E &= X_E \Delta(k) Y_E, & \Delta C_1 &= X_{C1} \Delta(k) Y_{C1}, \\ \Delta D &= X_D \Delta(k) Y_D, & \Delta F &= X_F \Delta(k) Y_F, \\ \Delta C_2 &= X_{C2} \Delta(k) Y_{C2}, & \Delta H &= X_H \Delta(k) Y_H, \end{aligned} \quad (2.176)$$

$$\Delta^T(k)\Delta(k) \leq I.$$

The model (2.175) is inferred as follows:

$$\begin{aligned} x(k+1) &= A_\Delta x(k) + B_\Delta u(k) + E_\Delta w(k), \\ z(k) &= C_{1\Delta} x(k) + D_\Delta u(k) + F_\Delta w(k), \\ y(k) &= C_{2\Delta} x(k) + H_\Delta w(k), \end{aligned} \quad (2.177)$$

where

$$\begin{aligned} A_\Delta &= A + \Delta A, & B_\Delta &= B + \Delta B, \\ E_\Delta &= E + \Delta E, & C_{1\Delta} &= C_1 + \Delta C_1, \\ D_\Delta &= D + \Delta D, & F_\Delta &= F + \Delta F, \\ C_{2\Delta} &= C_2 + \Delta C_2, & H_\Delta &= H + \Delta H. \end{aligned}$$

In this section, the following static output feedback controller will be designed

$$u(k) = Ky(k) = K(C_{2\Delta}x(k) + H_\Delta w(k)). \quad (2.178)$$

where K is the controller gain.

Substituting (2.178) into (2.177) yields the following closed-loop system:

$$\begin{aligned} x(k+1) &= (A_\Delta + B_\Delta K C_{2\Delta})x(k) + (E_\Delta + B_\Delta K H_\Delta)w(k), \\ z(k) &= (C_{1\Delta} + D_\Delta K C_{2\Delta})x(k) + (F_\Delta + D_\Delta K H_\Delta)w(k). \end{aligned} \quad (2.179)$$

Remark 2.17 Similar to the research on robust static output feedback H_∞ control design for linear systems with polytopic uncertainties, most of the literature claim that the system input or output matrix should be without uncertainties. In this study, the requirement is not necessary.

In this following, we will develop new LMI conditions to design the static output feedback H_∞ controller in the form of (2.178) such that the resulting closed-loop system (2.179) meets an H_∞ performance bound requirement. For frugality, we just consider the case $D = 0$. First, a new H_∞ performance analysis criterion is presented in the following theorem.

Theorem 2.24 *Consider the closed-loop system (2.179) with $D = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ*

if exist matrices P, G, J, M, N, V , and U , scalar β such that the following matrix inequality holds

$$\begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA_{\Delta} + MVC_{2\Delta} & GE_{\Delta} + MVH_{\Delta} & -G - G^T + P + J & * & * & * \\ C_{1\Delta} & F_{\Delta} & 0 & -I & * & * \\ NVC_{2\Delta} & NVH_{\Delta} & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J}{\beta^2} \end{bmatrix} < 0, \quad (2.180)$$

where

$$\Sigma_1 = -\beta NU - \beta U^T N^T,$$

$$\Sigma_2 = GB_{\Delta} - MU.$$

Proof Follow (2.12) with the Lyapunov matrix P , the inequality (2.180) can be obtained easily. $\square$

In this following, based on the analysis result in Theorem 2.24, we proposed sufficient conditions for designing the static output feedback H_{∞} controller in the form of (2.178). First, separate the certain terms and uncertain terms in (2.180), we have

$$\Omega + \Delta_{\Omega} < 0, \quad (2.181)$$

with

Ω

$$= \begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA + MVC_2 & GE + MVH & -G - G^T + P + J & * & * & * \\ C_1 & F & 0 & -I & * & * \\ NVC_2 & NVH & 0 & 0 & -\beta NU - \beta U^T N^T & * \\ 0 & 0 & 0 & 0 & GB - MU & -\frac{J}{\beta^2} \end{bmatrix},$$

and

$$\begin{aligned} \Delta_{\Omega} &= \begin{bmatrix} 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ G\Delta_A + MV\Delta_{C2} & G\Delta_E + MV\Delta_H & 0 & * & * & * \\ \Delta_{C1} & \Delta_F & 0 & 0 & * & * \\ NV\Delta_{C2} & NV\Delta_H & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & G\Delta_B & 0 \end{bmatrix} \\ &= \tilde{A} + \tilde{A}^T + \tilde{B} + \tilde{B}^T + \tilde{E} + \tilde{E}^T + \tilde{C}_1 + \tilde{C}_1^T + \tilde{F} + \tilde{F}^T + \tilde{C}_2 + \tilde{C}_2^T + \tilde{H} + \tilde{H}^T, \end{aligned}$$

where

$$\begin{aligned}
 \tilde{A} + \tilde{A}^T &= \begin{bmatrix} 0 \\ 0 \\ GX_A \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta(k) \begin{bmatrix} Y_A & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_A & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \Delta^T(k) \begin{bmatrix} 0 \\ 0 \\ GX_A \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \\
 \tilde{B} + \tilde{B}^T &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ GX_B \end{bmatrix} \Delta(k) \begin{bmatrix} 0 & 0 & 0 & 0 & Y_B & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & Y_B & 0 \end{bmatrix}^T \Delta^T(k) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ GX_B \end{bmatrix}^T, \\
 \tilde{E} + \tilde{E}^T &= \begin{bmatrix} 0 \\ 0 \\ GX_E \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta(k) \begin{bmatrix} 0 & Y_E & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_E & 0 & 0 & 0 & 0 \end{bmatrix}^T \Delta^T(k) \begin{bmatrix} 0 \\ 0 \\ GX_E \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \\
 \tilde{C}_1 + \tilde{C}_1^T &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_{C1} \\ 0 \\ 0 \end{bmatrix} \Delta(k) \begin{bmatrix} Y_{C1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_{C1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \Delta^T(k) \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_{C1} \\ 0 \\ 0 \end{bmatrix}^T, \\
 \tilde{F} + \tilde{F}^T &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_F \\ 0 \\ 0 \end{bmatrix} \Delta(k) \begin{bmatrix} 0 & Y_F & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_F & 0 & 0 & 0 & 0 \end{bmatrix}^T \Delta^T(k) \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_F \\ 0 \\ 0 \end{bmatrix}^T,
 \end{aligned}$$

$$\tilde{C}_2 + \tilde{C}_2^T$$

$$= \begin{bmatrix} 0 \\ 0 \\ MVX_{C2} \\ 0 \\ NVX_{C2} \\ 0 \end{bmatrix} \Delta(k) \begin{bmatrix} Y_{C2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_{C2} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \Delta^T(k) \begin{bmatrix} 0 \\ 0 \\ MVX_{C2} \\ 0 \\ NVX_{C2} \\ 0 \end{bmatrix}^T,$$

$$\tilde{H} + \tilde{H}^T$$

$$= \begin{bmatrix} 0 \\ 0 \\ MVX_H \\ 0 \\ NVX_H \\ 0 \end{bmatrix} \Delta(k) \begin{bmatrix} 0 & Y_H & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_H & 0 & 0 & 0 & 0 \end{bmatrix}^T \Delta^T(k) \begin{bmatrix} 0 \\ 0 \\ MVX_H \\ 0 \\ NVX_H \\ 0 \end{bmatrix}^T.$$

From Lemma 1.11 for positive scalars ε_A , ε_B , ε_E , ε_{C1} , ε_F , ε_{C2} , and ε_H , we can know that

$$\tilde{A} + \tilde{A}^T \leq \frac{1}{\varepsilon_A} \begin{bmatrix} 0 \\ 0 \\ GX_A \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ GX_A \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon_A \begin{bmatrix} Y_A & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} Y_A & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.182)$$

$$\tilde{B} + \tilde{B}^T \leq \frac{1}{\varepsilon_B} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ GX_B \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ GX_B \end{bmatrix}^T + \varepsilon_B \begin{bmatrix} 0 & 0 & 0 & 0 & Y_B & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & Y_B & 0 \end{bmatrix}, \quad (2.183)$$

$$\tilde{E} + \tilde{E}^T \leq \frac{1}{\varepsilon_E} \begin{bmatrix} 0 \\ 0 \\ GX_E \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ GX_E \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon_E \begin{bmatrix} 0 & Y_E & 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & Y_E & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.184)$$

$$\tilde{C}_1 + \tilde{C}_1^T \leq \frac{1}{\varepsilon_{C1}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_{C1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_{C1} \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon_{C1} \begin{bmatrix} Y_{C1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} Y_{C1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.185)$$

$$\tilde{F} + \tilde{F}^T \leq \frac{1}{\varepsilon_F} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_F \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_F \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon_F [0 \ Y_F \ 0 \ 0 \ 0 \ 0]^T [0 \ Y_F \ 0 \ 0 \ 0 \ 0], \quad (2.186)$$

$$\tilde{C}_2 + \tilde{C}_2^T \leq \frac{1}{\varepsilon_{C2}} \begin{bmatrix} 0 \\ 0 \\ MVX_{C2} \\ 0 \\ NVX_{C2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ MVX_{C2} \\ 0 \\ NVX_{C2} \\ 0 \end{bmatrix}^T + \varepsilon_{C2} [Y_{C2} \ 0 \ 0 \ 0 \ 0 \ 0]^T [Y_{C2} \ 0 \ 0 \ 0 \ 0 \ 0], \quad (2.187)$$

$$\tilde{H} + \tilde{H}^T \leq \frac{1}{\varepsilon_H} \begin{bmatrix} 0 \\ 0 \\ MVX_H \\ 0 \\ NVX_H \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ MVX_H \\ 0 \\ NVX_H \\ 0 \end{bmatrix}^T + \varepsilon_H [0 \ Y_H \ 0 \ 0 \ 0 \ 0]^T [0 \ Y_H \ 0 \ 0 \ 0 \ 0]. \quad (2.188)$$

So far, the design result can be summarized in the following theorem.

Theorem 2.25 Consider the closed-loop system (2.179) with $D = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrix M , N and scalar β , there exist matrices U , P , G , and L , scalars ε_A , ε_B , ε_E , ε_{C1} , ε_F , ε_{C2} , and ε_H such that the following LMI holds

$$\begin{bmatrix} M_{11} & * \\ M_{21} & M_{22} \end{bmatrix} < 0, \quad (2.189)$$

where

$$M_{11} = \begin{bmatrix} \Lambda_1 & * & * & * & * & * \\ 0 & \Lambda_2 & * & * & * & * \\ GA + MVC_2 & GE + MVH & -G - G^T + P + J & * & * & * \\ C_1 & F & 0 & -I & * & * \\ NVC_2 & NVH & 0 & 0 & \Lambda_5 & * \\ 0 & 0 & 0 & 0 & GB - MU & -\frac{J}{\beta^2} \end{bmatrix},$$

$$\Lambda_1 = -P + \varepsilon_A Y_A^T Y_A + \varepsilon_{C1} Y_{C1}^T Y_{C1} + \varepsilon_{C2} Y_{C2}^T Y_{C2},$$

$$\Lambda_2 = -\gamma^2 I + \varepsilon_E Y_E^T Y_E + \varepsilon_F Y_F^T Y_F + \varepsilon_H Y_H^T Y_H,$$

$$\Lambda_5 = -\beta N U - \beta U^T N^T + \varepsilon_B Y_B^T Y_B,$$

$$M_{21} = \begin{bmatrix} 0 & 0 & X_A^T G^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_B^T G^T \\ 0 & 0 & X_E^T G^T & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{C1}^T & 0 & 0 \\ 0 & 0 & 0 & X_F^T & 0 & 0 \\ 0 & 0 & X_{C2}^T V^T M^T & 0 & X_{C2}^T V^T N^T & 0 \\ 0 & 0 & X_H^T V^T M^T & 0 & X_H^T V^T N^T & 0 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} -\varepsilon_A I & * & * & * & * & * & * \\ 0 & -\varepsilon_B I & * & * & * & * & * \\ 0 & 0 & -\varepsilon_E I & * & * & * & * \\ 0 & 0 & 0 & -\varepsilon_{C1} I & * & * & * \\ 0 & 0 & 0 & 0 & -\varepsilon_F I & * & * \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon_{C2} I & * \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_H I \end{bmatrix}.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.178) can be given by (2.22).

Remark 2.18 When the system input or output matrices are without uncertainties, we can also get several different forms of LMI-based design conditions for the closed-loop system (2.179). Similar to the polytopic uncertainties, in contrast to the existing LMI conditions for designing the robust static output feedback H_∞ controllers, the improvement of the our results over the existing ones can shown by strict theoretical proof. A related study has been made in our preliminary work for uncertain discrete-time T-S fuzzy systems [2].

Remark 2.19 The used LMI decoupling approach brings new results for robust static output feedback H_∞ controllers design of uncertain linear systems. It is necessary to mention that when β is known parameter, the proposed design conditions are of LMIs that can be easily and effectively solved via LMI control toolbox [7]. Due to β is free parameter, the problem is then how to find the optimal value of β in order to minimize the H_∞ performance bound. One way to address the search issue is to first solve the feasibility problem of the corresponding LMI conditions using LMI control toolbox and obtain a set of initial scalar parameters. Then, applying a numerical optimization algorithm, such as the program “fminsearch” in the optimization toolbox of Matlab, a locally convergent solution to the problem is obtained. In [9], the algorithm has been used and proved to be effective.

2.3 Numerical Examples

2.3.1 Example 1

To show the less conservativeness of the presented design result on robust static output feedback H_∞ controller design, a simulation example is given with different design methods. For simplicity, here we only consider Corollary 2.1 and Lemma 2.2. Let us consider the uncertain system (2.1) with

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 1.1 & 0.8 & -0.4 \\ -0.5 & 0.4 & 0.5 \\ 1.2 & 1.1 & 0.8 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.8 & 0.1 & 0 \\ -0.3 & 0.1 & 0.6 \\ 1.2 & -1 & 1.1 \end{bmatrix}, \\
 B_1 = B_2 = B &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 1.3 \end{bmatrix}, \\
 C_{11} &= [-1 \quad 0 \quad 2], & C_{12} &= [-0.6 \quad 0.2 \quad 1], \\
 D_1 = D_2 &= 0, \\
 F_1 &= 0.3, & F_2 &= -0.4, \\
 C_{21} &= \begin{bmatrix} -1 & 1.2 & 1 \\ 0 & -3 & 1 \end{bmatrix}, & C_{22} &= \begin{bmatrix} -0.8 & 1 & 1 \\ 0 & -2 & 1.2 \end{bmatrix}, \\
 H_1 &= \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, & H_2 &= \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}.
 \end{aligned} \tag{2.190}$$

For the system (2.190), since the system input matrix is fixed, the conditions in Corollary 2.1 and Lemma 2.2 are applicable for designing robust static output feedback H_∞ controllers. Now applying two conditions to design static output feedback H_∞ controllers such that γ is minimized.

By (2.35) and (2.36), the minimum H_∞ performances $\gamma_{\min} = 11.4687$. However, apply (2.29) and (2.30) with $\beta = 6.16$, we can find the minimum H_∞ performance $\gamma_{\min} = 7.0441$. From this comparison, it can be seen that the design condition in Corollary 2.1 is much less conservative than the existing result in Lemma 2.2.

2.3.2 Example 2

In this example, to show the effectiveness of the proposed design approach, a static output feedback H_∞ control problem of the following discrete-time system is considered

$$\begin{aligned}
x(k+1) &= (A + \Delta A)x(k) + (B + \Delta B)u(k) + Ew(k), \\
z(k) &= C_1x(k) + Du(k) + Fw(k), \\
y(k) &= (C_2 + \Delta C_2)x(k),
\end{aligned} \tag{2.191}$$

with

$$\begin{aligned}
A &= \begin{bmatrix} 2.3 & 0.5 \\ 1 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
E &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & C_1 &= [-1 \quad 1], \\
D &= 0, & F &= 2, \\
C_2 &= [1 \quad 0], \\
X_A &= \begin{bmatrix} 0.3 \\ -0.1 \end{bmatrix}, & Y_A &= [0.1 \quad 0.2], \\
X_B &= \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, & Y_B &= -0.3, \\
X_{C_2} &= -0.1, & Y_{C_2} &= [0.3 \quad 0.2].
\end{aligned}$$

We need to note that the system input matrix and output matrices in the system (2.191) have uncertainties. As a result, the existing approaches in [3, 5, 6, 8] fail to design the static output feedback H_∞ controller for this example. However, the proposed design condition in Theorem 2.25 is feasible to find a static output feedback controller to stabilize the system (2.191) with the H_∞ performance index. For example, by using the MATLAB toolbox to solve LMI (2.189) in Theorem 2.25 with $\beta = 5.34$, $M = B$, $N = 1$, minimum H_∞ performances $\gamma_{\min} = 4.3174$ is obtained.

On the other hand, by LMI (2.189) in Theorem 2.25 with $\gamma = 4.3174$, the following computational results are obtained

$$U = 6.8776, \quad V = -15.7883. \tag{2.192}$$

Substituting U and V in to (2.22), the static output feedback H_∞ controller gain matrix can be given as follows:

$$K = -2.2956. \tag{2.193}$$

Fig. 2.1 State trajectories of the closed-loop systems

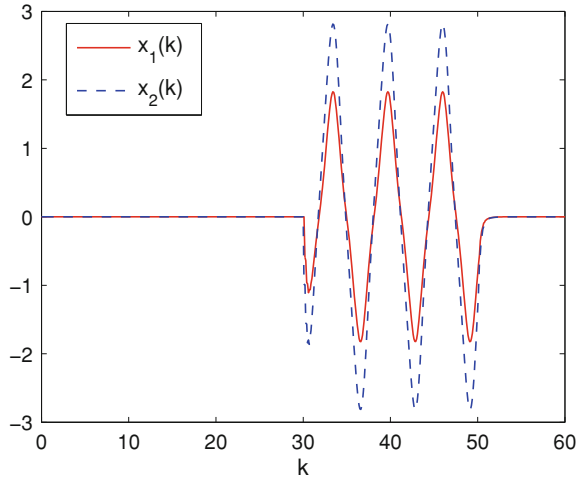
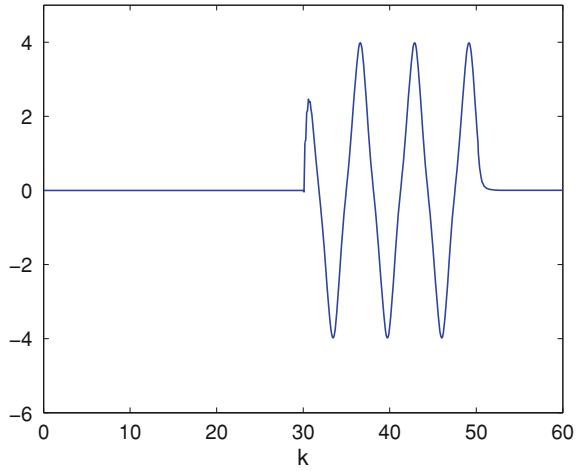


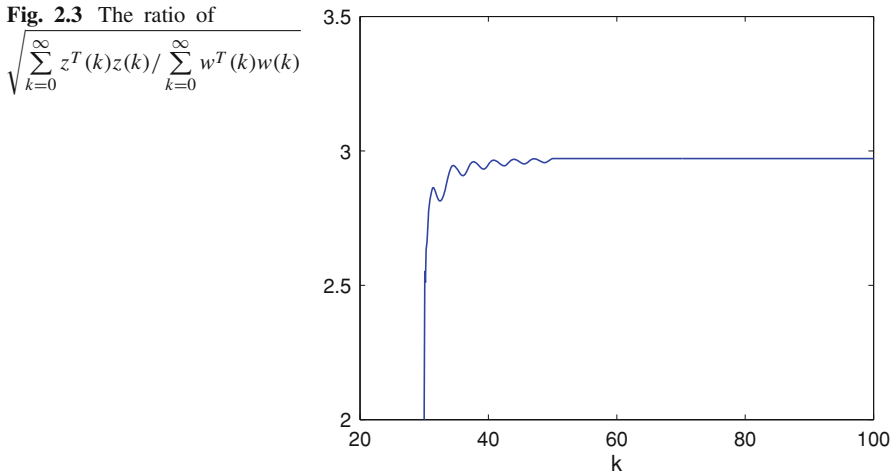
Fig. 2.2 System control signal $u(k)$



We assume the disturbance $w(k)$ as the following:

$$w(k) = \begin{cases} \sin(k), & 30 \leq k \leq 50, \\ 0, & \text{elsewhere.} \end{cases} \quad (2.194)$$

Under the initial conditions $x(0) = [0 \ 0]^T$, using the static output feedback controller gain (2.193) with $\Delta(k) = \sin^2(k)$, $k = 1, 2, \dots$, the state response of the system (2.191) is shown in Fig. 2.1. The control signal $u(k)$ is depicted in Fig. 2.2, it is able to stabilize the system with the H_∞ performance 4.3174. The ratio

Fig. 2.3 The ratio of

of $\sqrt{\sum_{k=0}^{\infty} z^T(k)z(k) / \sum_{k=0}^{\infty} w^T(k)w(k)}$ can show the influence of the disturbance $w(k)$ on the controlled output $z(k)$, and the plot of the ratio is shown in Fig. 2.3. It can be seen that the ratio tends to a constant value 2.9716, which is less than the prescribed value, i.e., 4.3174. From Figs. 2.1, 2.2, 2.3, it can be seen the H_{∞} performance is guaranteed for the system with the designed static output feedback H_{∞} controller.

2.4 Conclusion

In this chapter, the robust static output feedback H_{∞} control problem for the both discrete-time and continuous-time uncertain systems has been studied. Sufficient conditions for designing static output feedback H_{∞} controllers have been given based on an LMI decoupling approach. The design conditions are presented in the form of LMIs. In contrast to existing LMI methods for robust static output feedback H_{∞} controllers design of the uncertain linear systems, the proposed results not only allow the system input and output matrices to have uncertainties, but it can provide less conservative design. Numerical examples show the advantage of the proposed design method.

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