

2 Chapter Two: Literature Review

Recently, an effort has been made to measure mathematics teachers' knowledge needed for teaching even though there is more than 20 years of history in studying teachers' knowledge. Shulman's (1986) classification of subject matter knowledge, pedagogical knowledge, and curriculum knowledge laid a foundation for the study of teacher knowledge. Drawing on Shulman's framework, researchers have further refined and developed models to better describe and measure teacher knowledge needed for teaching (e.g., Ball et al., 2005; Krauss et al., 2008; Schmidt et al., 2007; Silverman & Thompson, 2008; Tatto et al., 2012). This literature review consists of two sections: teacher knowledge needed for teaching, and mathematics preparation of teachers in China and the U.S. In the first section, I reported the conceptualization of teacher knowledge for teaching in general and discussed the models for describing teacher knowledge for teaching mathematics in particular. Then, I discussed relevant studies on teachers' knowledge needed for teaching algebra. In the second section, I analyzed the mathematics education systems and mathematics teacher preparation in China and the U.S., and summarized relevant studies on mathematics teachers' knowledge for teaching in China and the U.S. Finally, grounded in the literature review, a framework for this study was proposed.

2.1 Knowledge Needed for Teaching

Great efforts have been made to seek what kind of knowledge a teacher needs to know in order to teach students effectively. In Shulman's (1986) seminal work on teachers' knowledge, he identified three categories, namely, *content knowledge*, *curriculum knowledge*, and *pedagogical knowledge*. The first, *content knowledge*, includes knowledge of the subject and its organizing structures. The teacher needs not only to understand *that* something is so; the teacher must further understand *why* it is so. The second, *curricular knowledge*, is "represented by the full range of programs designed for the teaching of particular subjects and

topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances” (p. 9). The third, *pedagogical content knowledge* (PCK), is described as follows:

The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the most useful ways of representing and formulating the subject that makes it comprehensible to others. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

Since Shulman (1986) coined the term PCK, many researchers have attempted to illustrate and clarify the nature of PCK and its implications for teacher education (e.g., Gess-Newsome, 1999). However, pedagogical content knowledge is often not clearly distinguished from other forms of teacher knowledge. For example, pedagogical content knowledge has been defined as “the intersection of knowledge of the subject with knowledge of teaching and learning” (Niess, 2005, p. 510) or as “that domain of teachers’ knowledge that combines subject matter knowledge and knowledge of pedagogy” (Lowery, 2002, p. 69). An even more careful description of PCK is still unclear:

Pedagogical content knowledge is a teacher’s understandings of how to help students understand specific subject matter. It includes knowledge of how particular subject matter topics, problems, and issues can be organized, represented and adapted to the diverse interests and abilities of learners, and then presented for instruction. The defining feature of pedagogical content knowledge is its conceptualization as the result of a *transformation* of knowledge from other domains (Magnusson, Krajcik, & Borko, 1999, p. 96).

In summary, PCK includes multiple components: (1) knowledge of students’ (mis)conceptions and difficulties, (2) knowledge of instructional strategies, (3) knowledge of mathematical tasks and cognitive demands, (4) knowledge of educational ends, (5) knowledge of curriculum and

media, (6) context knowledge, (7) content knowledge, and (8) pedagogical knowledge (Depaepe, Verschaffel, & Kelchtermans, 2013).

2.2 Mathematics Knowledge for Teaching

There is a widespread agreement that mathematics teachers need to have a deep understanding of mathematics (Ball, 1993; Grossman, Wilson, & Shulman, 1989; Ma, 1999). However, teachers' knowledge of mathematics alone is insufficient to support their attempts to teach mathematics effectively. There are various ways to define PCK in mathematics. While Ball (1990) differentiated two dimensions of teachers' content knowledge: teachers' ability to execute an operation (division by a fraction) and their ability to represent that operation accurately for students, Ma (1999) described "profound understanding of fundamental mathematics" in terms of the connectedness, multiple perspectives, fundamental ideas, and longitudinal coherence. Moreover, the National Research Council [NRC] suggested that mathematics teachers need specialized knowledge that "includes an integrated knowledge of mathematics, knowledge of the development of students' mathematical understanding, and a repertoire of pedagogical practices that take into account the mathematics being taught and the students learning it." (Kilpatrick, Swafford, & Findell, 2001, p. 428)

A research team at the University of Michigan has focused on understanding and measuring mathematical knowledge that was considered pertinent and important for teaching (Ball & Bass, 2000; Ball et al., 2005; Hill, Ball, & Schilling, 2008). Furthermore, Hill and her colleagues further explored the relationship between mathematics knowledge needed for teaching (MKT) and students' achievement (Hill et al., 2004), and classroom instruction (Hill, Ball, & Schilling, 2008). Growing attention has been given to studying characteristics of teacher's knowledge needed for teaching specific content areas (e.g., Ball et al., 2005; Even, 1990, 1993; Ferrini-Mundy et al., 2006; Ma, 1999), which will be discussed in the following sections.

Researchers have attempted to understand *what* mathematical knowledge is entailed in teaching, how to assess it (Ball & Bass, 2000; Ball et al., 2005; Hill, Schilling, & Ball, 2004), and how to develop and

refine ways to effectively promote *mathematical knowledge for teaching* (MKT) in teacher education and teacher professional development programs (NMAP, 2008; Stylianides & Stylianides, 2006). Ball and her colleagues have developed a specific framework describing MKT (Ball et al., 2005). According to this model, subject matter knowledge is divided into two categories: *Common Content Knowledge* (CCK), which can be developed in anyone who has had school mathematics education, and *Specialized Content Knowledge* (SCK), which is used mainly by teachers. Meanwhile, the model makes a distinction between two main categories within pedagogical content knowledge: *Knowledge of Content and Students* (KCS) and *Knowledge of Content and Teaching* (KCT). This model highlights the kind of mathematical content knowledge that is the specialty of teachers, and recognizes that knowledge of mathematics for teaching is partially the product of content knowledge interacting with students in their learning processes and with teachers in their teaching practices.

Grounded in the concept of mathematics proficiency (Kilpatrick et al., 2001), Kilpatrick, Blume and Allen (2006) proposed a framework for Mathematical Proficiency for Teaching. It suggests that *mathematical proficiency with content* (MPC) and *mathematical proficiency in teaching* (MPT) should be the main components for teachers to teach for mathematics proficiency. The *mathematical proficiency with content* (MPC) includes conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, cultural and historical knowledge, knowledge of structure and conventions, and knowledge of connections within and outside the subject. The *mathematical proficiency in teaching* (MPT) consists of knowing students as learners, assessing one's teaching, selecting or constructing examples and tasks, understanding and translating across representations, understanding and using classroom discourse, knowing and using the curriculum, and knowing and using instructional tools and materials. This model illustrates Shulman's (1986) subject matter knowledge and pedagogical knowledge with a focus on mathematics proficiency.

Simon (2006) adopted the idea of a *Key Developmental Understanding* (KDU) in mathematics, namely, understanding a topic from multiple perspectives, building a well-structured knowledge web sur-

rounding the topic as a way to think about understandings. KDUs are regarded as powerful springboards for learning and useful goals of mathematics instruction. Silverman and Thompson (2008) argued that developing MKT involves transforming these personal KDUs of a particular mathematical concept to an understanding of: (1) how this KDU could empower their students' learning of related ideas; (2) actions a teacher might take to support students' development of KDU and reasons why those actions might work. They further suggested a framework of mathematical knowledge for teaching as follows: A teacher has developed knowledge that supports conceptual understanding of a particular mathematical topic when he or she (1) has developed a KDU within which that topic exists, (2) has constructed models of the variety of ways students may understand the content, (3) has an image of how someone else might come to think of the mathematical idea in a similar way, (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea, and (5) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn related mathematical ideas. This framework opens up the possibility for the goal of mathematics teacher education to shift from positioning prospective teachers to develop particular MKT to developing professional practices that would support teachers' abilities to continually develop MKT.

The Teacher Education Development Study in Mathematics (TEDS-M) (Tatto et al., 2012) identified two components of MKT: *Mathematics content knowledge* (MCK) and *mathematics pedagogical content knowledge* (MPCK). MCK, in the TEDS-M framework, includes not only basic factual knowledge of mathematics but also the conceptual knowledge of structuring and organizing principles of mathematics as a discipline. MPCK consists of three domains: mathematics curricular knowledge, knowledge of planning, and knowledge for enacting mathematics.

The COACTIV (Professional Competence of Teachers, **C**ognitively **A**ctivating Instruction, and the Development of Students' Mathematical Literacy) project in Germany (Baumert et al., 2010; Krauss et al., 2008) examined the relationship between content knowledge (CK) and peda-

gogical content knowledge (PCK). It was found that although CK and PCK are highly correlated, they are statistically distinct and CK alone is not sufficient, but a necessary condition for PCK. In addition, it was noticed that CK and PCK are more correlated in teachers who had a more thorough education in mathematics (Krauss et al., 2008) and instructional quality is significantly more correlated to PCK than CK (Baumert et al., 2010).

The previously described studies enrich and/or extend Shulman's taxonomy with a focus on mathematics subject, and provide insights into the construct of CK and PCK and the relationship between them. In the next section, specific research on teacher's knowledge needed for teaching algebra will be analyzed.

2.3 Teachers' Knowledge of Algebra for Teaching

Algebra is an important part of school mathematics but is challenging for students to learn (Blume & Heckman, 1997; NCTM, 2000; NMAP, 2008). As more students take algebra, more teachers teach algebra, yet the preparation of algebra teachers has not been widely researched (Stein, Kaufman, Sherman, & Hillen, 2011). Very little research has examined teachers' knowledge of algebra for teaching (Artigue, Assude, Grugeon, & Lenfant, 2001; Even, 1990, 1993; Ferrini-Mundy et al., 2006; Li, 2007). Artigue and colleagues differentiated three dimensions of knowledge of algebra for teaching as follows: (1) epistemological dimension; (2) cognitive dimension; and (3) didactic dimension. The *epistemological dimension* includes: (a) the complexity of the algebraic symbolic system and the difficulties of its historical development, and (b) how to flexibly use algebraic tools in solving different kinds of problems that are internal or external to the field of mathematics. The *cognitive dimension* deals with knowledge about learning processes in algebra, which includes knowing (a) the development of the student's algebraic thinking and (b) students' interpretations of algebraic concepts and notations. The *didactic dimension* involves knowledge of (a) the algebra curriculum, and (b) the specific goals of algebraic teaching at a given grade.

Even (1990) identified and illustrated seven dimensions of subject matter knowledge based on an in-depth examination of the concept of

function: (1) essential features, (2) different representations, (3) alternative ways of approaching, (4) the strength of the concept, (5) basic repertoire, (6) knowledge and understanding of a concept, and (7) knowledge about mathematics. *Essential features* are referred to as concept images by Vinner (1983) as the mental pictures of this concept, together with the set of properties associated with the concept (in the person's mind). It is crucial for teachers to judge if an instance belongs to a concept family by using an analytical judgment as opposed to a mere use of a prototypical judgment. It is necessary that teachers are able to correctly distinguish between concept examples and non-examples. *Different representations* give different insights which allow a better, deeper, more powerful and more complete understanding of a concept. When dealing with a mathematical concept in different representations, one may abstract the concept by grasping the common properties of the concept while ignoring the irrelevant characteristics that are imposed by the specific representation at hand. *Alternative ways of approaching the same concept* are used to deal with complex concepts in various forms, representations, labels and notations. *The strength of the concept* refers to the importance or power to open new possibilities, understand new concepts and capture the essence of the definition, as well as a more sophisticated formal mathematical knowledge. *Basic repertoire* includes powerful examples that illustrate important principles, properties, and theorems. The basic repertoire should be well known and familiar in order to be readily available for use. *Knowledge and understanding of a concept* means to achieve procedural proficiency and conceptual knowledge. The learning of a new concept or relationship implies the addition of a node or link to the existing cognition structure; thus making the whole more stable than before. *Knowledge about mathematics* includes knowledge about the nature of mathematics. This is a more general knowledge about a discipline, which guides the construction and use of conceptual and procedural knowledge.

Comparing Artigue et al. (2001) and Evens' (1990) models, categories (1), (4), (6) and (7) of Even's category belong to the epistemological dimension while the others belong to the didactic dimension.

Ferrini-Mundy and her colleagues (2006, 2012) have developed a two-dimensional framework that describes mathematics knowledge of

algebra for teaching. In their model, the horizontal dimension indicates the fundamental *categories of knowledge* involved in teaching algebra, while the vertical dimension identifies several *tasks of teaching* in which teachers may apply their mathematical knowledge. The three overarching categories, *decompressing*, *trimming*, and *bridging*, are more sophisticated mathematical practices that utilize multiple elements of knowledge of algebra for teaching and involve multiple tasks of teaching. *Categories of knowledge* include core content knowledge, representation, content trajectory, application and context, language and convention, and mathematical reasoning and proof. *Tasks of teaching* consist of analyzing students' work and thinking, designing, modifying and selecting mathematical tasks; establishing and revising mathematical goals for students; accessing and using tools and resources for teaching; explaining mathematical ideas and solving mathematical problems; building and supporting mathematical community and discourse. The framework illustrates the overall landscape of knowledge of algebra for teaching: the major types of knowledge that may be used and contexts in which they may be used. *Decompressing* refers to the decompression of knowledge in the practice of teaching. For secondary school algebra teachers, what needs to be decompressed may include algorithms for solving equation, systems of equations, for simplifying expressions, and for moving among representations. *Trimming* refers to forming the mathematical content in a way that matches students' current levels of sophistication while treating the mathematics with integrity (McCrory et al., 2012). *Bridging* includes connecting and linking mathematics across topics, courses, concepts, and goals, including connecting the ideas of school algebra to those of abstract algebra and real analysis, linking one area of school mathematics to another (McCrory et al., 2012).

Flodden and McCrory (2007) have further created a three dimensional construct, as illustrated in Figure 2.1, to guide the measuring of teachers' knowledge of algebra for teaching.

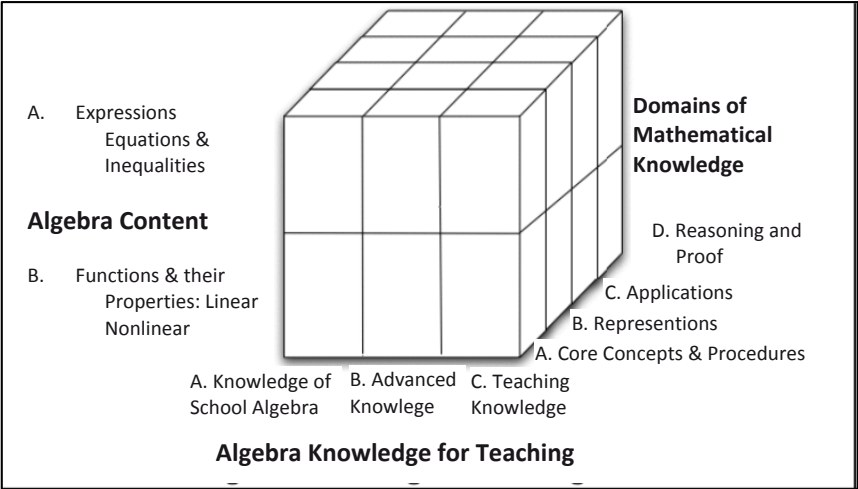


Figure 2.1. A framework for assessing knowledge of algebra for teaching.

In this framework, the base of the matrix consists of three types of algebra knowledge for teaching including knowledge of school algebra, advanced algebra knowledge and teaching knowledge. *School Algebra Knowledge* refers to the algebra covered in the curriculum from K-12. *Advanced Algebra Knowledge* includes other mathematical knowledge such college-level mathematics that provides perspectives on the trajectory and growth of mathematical ideas beyond school algebra. Typically, those areas of mathematics are found in calculus, linear algebra, real and complex analysis, and mathematical modeling. Similar to specialized content knowledge in the Hill et al. (2008) model, *Teaching Knowledge* includes “knowledge that is mathematical, that is intuitively useful for teaching, and that is unlikely to be taught explicitly, except to teachers” (McCrory et al., 2012, p. 608). Teaching knowledge also includes typical errors, canonical uses of school math, and curriculum trajectories among others. There are four *domains of mathematical knowledge* or aspects of algebra teaching and learning (core concepts and procedures, representations, applications and contexts, and reasoning and proof on the Y-axis). The Z-axis contains two major themes in school *algebra content*: expressions, equations and inequalities, and

functions and their properties. Assessment items can be specifically written for each cell in the matrix; for instance, knowledge of school algebra that is related to a core procedure for solving equations. Each assessment item would be uniquely located in Figure 2.1 as a coordinated system.

Based on the existing frameworks, Li (2007) reported a refined framework for investigating teachers' mathematical knowledge of algebra for teaching linear equations. His framework consists of three domains of knowledge: knowledge of the subject matter, knowledge of learners' conceptions, and knowledge of didactic representations. *Knowledge of the subject matter* refers to mathematics subject matter as a system of established definitions, properties, facts, relations and connections; use of notations and representations; and methods for reasoning and problem solving. *Knowledge of learners' conceptions* includes the subject matter as understood by learners, including typical pre-conceptions, misconceptions, mistakes, questions, difficulties, strategies, reasoning, and factors that make a particular concept or procedure easy or difficult. *Knowledge of didactic representations* refers to how the subject matter is unpacked, linked, organized, and tailored through purposeful sequencing of topics and choices of examples, models, explanations, tasks, metaphors, and technological presentations.

2.4 Mathematics Knowledge for Teaching Some Key Concepts in Algebra

Recent research has examined teachers' knowledge needed for teaching several important concepts in school algebra, such as function, expressions, and equations.

2.4.1 Teaching and Learning of the Concept of Function

There have been different approaches to developing meaningful algebra (i.e., Bednarz, Kieran, & Lee, 1996; Hart, 1981; Usiskin, 1988). Usiskin (1988) summarized the following four approaches: (1) algebra as generalized arithmetic, (2) algebra as a study of procedures for solving certain kinds of problems, (3) algebra as the study of relationships among quan-

tities, and (4) algebra as the study of structures. Learning algebra should include the following three core activities: generational, transformational and global/meta-level (Kieran, 2004). Function concept is one of the most important but difficult concepts across middle and high school levels (NCTM, 2000, 2006). Based on the theory of process-object duality of mathematics concept development (Sfard, 1991, 1992), a model of developing function concept is described as four stages: pre-function, action, process and object (Briedenbach, Dubinsky, Hawks, & Nichols, 1992). According to this model, *pre-function* indicates that the student does not display very much of a function concept. An *action* is the repeatable mental or physical manipulation of an object; such a conception of function would involve, for example, the ability to substitute numbers into an algebraic expression and evaluate. *Process* involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. A function is conceived as an *object*, if it is possible to perform action on it, namely transformations.

In general, three representations are used for presenting functions: (1) geometrical representations including chart, graph, and histogram; (2) numerical representations including numbers, table, and ordered number pairs; and (3) algebraic representations including letters, formula, and mapping (Verstappen, 1982). However, different representations play different roles in helping students understand the concept of function (Schwartz & Yerushalmy, 1992). For example, the algebraic representations benefit the understanding of function as a *process*, while the graphical representations help in understanding functions as an *object*. Moreover, certain operations are easier to understand in particular representations. For example, composition functions performed on algebraic representations are easy to understand, while transformations performed on graphical representations will be much easier to understand. Thus, it is critical to select appropriate representations with regard to different contexts.

There are a large number of studies on teachers' knowledge for teaching the concept of function (e.g., Even, 1990, 1992, 1993, 1998; Norman, 1992, 1993). For example, based on a study on 10 secondary teachers' knowledge of the function concept, Norman (1992) found that

the secondary teachers tended to have *inflexible images* of the concept of function that restricted their abilities to identify functions in unusual contexts and to shift among representations of functions. The sampled teachers were able to give formal definitions of function, distinguish functions from relations, and correctly identify whether or not a given situation was functional. However, the teachers did not show strong connections between their informal notions of function and formal definitions, and were not comfortable with generating contexts for functions. Hitt (1994) investigated 117 mathematics teachers' ideas on function and found that the teachers had difficulty in constructing functions that were not continuous or piece-wise defined. Consistent with Norman's findings, Chinnappan and Thomas (2001) found that all four prospective secondary teachers in their study had a preference of thinking about function graphically, had a weak understanding of the connections of representations, and a limited ability to describe applications of functions.

Based on a survey of teachers' knowledge about function with 152 prospective secondary teachers, Even (1993) found that many prospective secondary teachers did not hold a modern conception of a function as a univalent correspondence between two sets. These teachers tended to believe that functions are always represented by equations and that their graphs are well behaved. None of the teachers had a reasonable explanation of the need for functions to be univalent and over-emphasized the procedure of the "vertical line test" without concern for understanding. Given the often weak and fragile understanding of secondary mathematics teachers about the concept of function, it is not surprising to find that the knowledge of an experienced fifth grade teacher was missing several key ideas (such as univalence and unclear notions of dependency) and lacked a notion of the connectivity among representations (Leinhardt, Zaslavsky, & Stein, 1990; Stein, Baxter, & Leinhardt, 1990).

In summary, teachers have difficulties (1) in understanding the concept of functions as a univalent correspondence between two sets (sometimes, it is not presented by a formula, or it is not of a discontinuous graph), (2) shifting different representations flexibly, and (3) relating formal function notion to contextual situations that produce the function.

2.4.2 Teaching and Learning of Expressions and Equations

Expressions. An algebraic expression can be seen as a string of symbols, a computational process, or a representation of a number. An expression can also become a function representing change if the context changes (Sfard & Linchevski, 1994). Sfard and Linchevski also clarify a potential issue in understanding algebraic expressions when they note:

...the difficulty lies ... in the necessity to imbue the symbolic formulae with the double meanings: that of computational procedures and that of the objects produced.... To those who are well versed in algebraic manipulation (teachers among them), it may soon become totally imperceptible. (pp.198-199)

The duality (procedure vs. structure) of algebraic expression results in student learning difficulties. For example, when knowing $x=3$, $y=2$, find out the value of $3x+y$, a procedure perspective is adopted. While simplifying the expression of $3x+y+8x$, it is necessary to adopt an object (structure) perspective. In addition, it was found that extrapolating some manipulation rules to other contexts inappropriately is a common mistake (Matz, 1982). For example, applying the distribution law $a(AnB)=oAnoB$ to an inappropriate situation: $\sin(\alpha + \beta) = \sin(\alpha) + \sin(\beta)$ or applying " $\frac{AX}{A} = X$ " to " $\frac{AX + BY}{X + Y} = A + B$ " are common mistakes. For another example, applying a known rule (such as if $ab=0$ then $a=0$ or $b=0$) to an unfamiliar situation: $(X-A)(X-B)=K \Rightarrow (X-A)=K$ or $(X-B)=K$ is also a common mistake.

Equation. An equation is a combination of letters, operations, and an equal sign such that if numbers are substituted for the letters, either a true or false proposition results. Aspects of student difficulty within this topic are well documented in the literature on students' algebra knowledge (Booth, 1984; Kieran, 1992; Wagner, Rachlin, & Jensen, 1984; Wagner & Kieran, 1989).

First of all, it is not easy to understand the meaning of the sign "=". In algebra, equal sign "=" means equivalence of two algebraic expressions (equation), or presents one expression by another one (computation). Alibali, Knuth, Hattikudur, Mcneil and Stephens (2007) investigated

81 students at grades 6, 7 and 8 about their understanding of equal sign and equation for three years. They found that overall the students increased their understanding of the two concepts, but some eighth students still did not understand the equal sign completely.

Second, when solving equations, students face two challenges: the meaning of equal sign and the reverse computation relationship between addition and subtraction (Booth, 1984; Sfard & Linchevski, 1994). For example, Sfard and Linchevski (1994) found that students at the ages of 14 and 15 were able to solve the equation $7x+157=248$, but failed to solve the equation $112=12x+247$. They attribute the students' difficulty to two issues: the position and meaning of the equal sign "=", and the subtraction of a larger number from a smaller number. Moreover, even though students understood the reverse computation relationship between addition and subtraction, they still did not understand that the order of computation cannot be changed arbitrarily. For them, it is still a big challenge to solve equations including combination computation (Piaget & Moreau, 2001; Ronbing, Ninowski, & Gray, 2006).

2.4.3 Two Perspectives about the Concept of Function: A Case Study of Quadratic Function

Process-product dichotomy is a widely accepted theory of mathematics concept development (Briedenbach et al., 1992; Schwartz & Yerushalmy, 1992; Sfard, 1992). With regard to the development of the function concept, according to the process perspective, a function is perceived as linking x and y values: for each value of x , the function has only one corresponding y value. On the other hand, the object perspective regards functions or relations and any of its representation as entities. For example, functions could be regarded algebraically as members of parameterized classes, or in the plane graphs could be thought of as being rotated or translated (Moschkovich, Schoenfeld, & Arcavi, 1993).

Researchers have further provided more detailed descriptions and illustrations of these two perspectives (Breidenback et al., 1992; Even, 1990; Moschkovick et al., 1993; Schwartz & Yerushalmy, 1992; Sfard, 1992). For example, Schwartz and Yerushalmy (1992) illustrated the process-product distinction as follows:

Consider the two functions: $x+3$ and $4+x-1$.

From the point of view of the process that is carried out with the recipe, these are two different recipes. If, however, one was to plot the output of each of these recipes again its input on a Cartesian plane then the two recipes would be indistinguishable. We see that the symbolic representation of function makes its process nature salient, while the graphical representation suppresses the process nature of the function and thus helps to make the function more entity-like. A proper understanding of algebra requires that students be comfortable with both of these aspects of function (p.265).

Furthermore, Moschkovick et al. (1993) not only extensively illustrated the distinction between the process and object perspectives, but also emphasized the importance of the connection between these two perspectives, and the flexibility in switching from different perspectives. They discussed multiple methods of solving the following question:

Why is the graph of $y=3x$ steeper than the graph of $y=2x$? What about $y=4x$, $y=5x$, $y=10x$?

In one method, by considering the equation of line L : $y = mx + b$ and two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, an algebraic formula was resulted in: $m = \frac{y_2 - y_1}{x_2 - x_1}$. If taking any two points on L whose x co-

ordinates differ by 1, then $y_2 = y_1 + m$. If m is positive, the graph of L raises m units for each unit change in x . Thus, large (positive) m corresponds to steeper slope. This interpretation basically adopted a process perspective.

In another method, letting $L_1: y = m_1x$, and $L_2: y = m_2x$ pass respectively through the points $(1, m_1)$, and $(1, m_2)$, hence $m_2 > m_1$, L_2 rise more steeply. In this solution, two aspects of both the process and object perspective were adopted in the algebraic and graphical representations. From the object perspective, the individual equations and lines are considered as members of the parametric family $\{y = mx : m \in R\}$. But using points on the graphs and determining their coordinates using the equations of the lines, employs the process perspective.

2.4.4 *Flexibility in Learning the Concept of Function: A Case Study of Quadratic Function.*

In this section, I review the meanings of flexibility in using representations, and summarize the studies on teachers' knowledge of algebra for teaching with regard to the flexibility in using representations.

Flexibility in using representations. Learning algebra with understanding requires students to “understand the meaning of equivalent forms of expressions, equations, inequalities, and relations” (NCTM, 2000, p. 296). In order to create that understanding, teachers have to organize a classroom discussion to open questions about the equivalence. For example, with regard to quadratic equations or functions, students need opportunities to discuss questions across equations, expressions and functions as follows:

When solving $3x^2 + 3x + 3 = 0$, I can think of the task as finding the zeros of the function $y = 3x^2 + 3x + 3$. In the context of finding zeros, I can divide 3 in the equation. However, when working with the function of $y = 3x^2 + 3x + 3$, we cannot divide all coefficients by 3. Why is that? (Chazan & Yerushalemy, 2003, p.124)

In the *Focal Points from pre-K to Grade 8* (NCTM, 2006), students (grade 8) are suggested to use linear functions, linear equations, and systems of linear equations to represent, analyze, and solve a variety of problems. Students are expected to

1. Recognize a proportion ($y/x = k$, or $y = kx$) as a special case of a linear equation of the form $y = mx + b$, understanding that the constant of proportionality (m) is the slope and the resulting graph is a line through the origin.
2. Understand that the slope (m) of a line is a constant rate of change, so if the input, or x-coordinate, changes by a specific amount, a , the output, or y-coordinate, changes by the amount ma .
3. Translate among verbal, tabular, graphical, and algebraic representations of functions (recognizing that tabular and graphical representations are usually only partial representations).

4. Describe how such aspects of a function as slope and y-intercept appear in different representations. (NCTM, 2006, p. 20).

In the *Focus of High School Mathematics* (NCTM, 2009), sense making and reasoning is the core value of learning mathematics in general, and algebra in particular. It was suggested that key elements of reasoning and sense making with algebraic symbols should include the following:

1. *Meaningful use of symbols.* Choosing variables and constructing expressions and equations in context; interpreting the form of expressions and equations; manipulating expression so that interesting interpretations can be made.
2. *Mindful manipulation.* Connecting manipulation with the laws of arithmetic; anticipating the results of manipulations; choosing procedures purposefully in context; picturing calculations mentally.
3. *Reasoned solving.* Seeing solution steps as logical deductions about equality; interpreting solutions in context.
4. *Connecting algebra with geometry.* Representing geometric situations algebraically and algebraic situations geometrically; using connections in solving problems.
5. *Linking expressions and functions.* Using multiple algebraic representations to understand functions; working with function notation. (NCTM, 2009, p. 31)

In order to develop algebra fluency, attention should be paid to interpret expressions both at the formal level and as statements about real-world situations. At the outset, the reasons and justifications for forming and manipulating expressions should be a major emphasis of instruction (Kaput et al., 2008). As comfort with expressions grows, constructing and interpreting them require less and less effort and gradually become almost subconscious. For example, students should know which is most useful for finding the maximum value of the quadratic function:

$$1\frac{3}{16} + 18t - 16t^2, (t - \frac{19}{16})(t + \frac{1}{16}), (t - \frac{9}{16})^2 + \frac{100}{16}.$$

Although multiple representations of functions—symbolic, graphical, numerical, and verbal—are commonly seen, the idea of multiple *alge-*

braic representations of functions is not often made explicit. Different but equivalent ways of writing the same function can reveal different properties of the function (as illustrated by the above example). Building fluency in working with algebraic notation that is grounded in reasoning and sense making will ensure “that students can flexibly apply the powerful tools of algebra in a variety of contexts both within and outside mathematics” (NCTM, 2009, p. 37).

Function is one of the most important tools for helping students make sense of the world around them and prepare them for further study in mathematics as well. Students’ continuing development of the concept of function must be rooted in reasoning, and conversely, functions serve as an important tool for reasoning. Key elements of reasoning and sense making with functions include the following (NCTM, 2009, p. 41):

1. *Using multiple representations of functions.* Representing functions in various ways, including tabular, graphic, symbolic (explicit and recursive), visual, and verbal; making decisions about which representations are most helpful in problem-solving circumstances; and moving flexibly among those representations.
2. *Modeling by using families of functions.* Working to develop a reasonable mathematical model for a particular contextual situation by applying knowledge of the characteristic behaviors of different families of functions.
3. *Analyzing the effects of parameters.* Using a general representation of a function in a given family (e.g., the vertex form of a quadratic, $f(x) = a(x - h)^2 + k$ to analyze the effects of varying coefficients or other parameters; converting between different forms of functions (e.g., the standard form of a quadratic and its factored form) according to the requirements of the problem-solving situation (e.g., finding the vertex of a quadratic or finding its zeros).

In summary, these documents suggested that the following aspects are important in algebra learning, particular with the learning of function: (1) building the connection among expressions, equations/inequality, and functions; (2) flexible use of multiple representations of a function and shift among different representations, and (3) flexible use of multiple

expressions of a function. As Star and Rittle-Johnson (2009) argued, “understanding in algebra can be considered to consist of two complementary capacities, which we refer to as between and within representation fluency. The first concerns the ability to operate fluently between and across multiple representations, while the second is about facility within each individual representation” (p. 11). Thus, it is critical to have a flexible and adaptive use of representations and expressions.

The representation flexibility should include the following abilities: (1) Having the necessary diagrammatic knowledge to interact with the representations (de Jong et al. 1998; Roth & Bowen, 2001); (2) Being able to coordinate the translation and switching between representations within the same domain (de Jong et al. 1998; Gagatsis & Shiakalli, 2004; Lesh, Post, & Behr 1987); and (3) Having the necessary strategic knowledge and skills to choose the most appropriate representation for each occasion (Uesaka & Manalo, 2006).

With regard to the concept of functions, it is necessary to consider flexibility in two aspects. One is the flexibility in selecting perspectives of function: process and object, and shifting between these two perspectives (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Sfard, 1991). Another is the flexibility in using appropriate representations of functions: tabular, graphic, symbolic, and verbal representations, and shifts between them (Even, 1998; Moschkovick et al., 1993).

Teachers’ knowledge of representational flexibility. Studies have found that teachers do not have the appropriate knowledge of using representation flexibly (Black, 2007; Even, 1998). To investigate prospective mathematics teachers’ subject knowledge, Even (1998) examined teachers’ knowledge of using representations. For example, she presented teachers with the following question:

If you substitute 1 for x in expression $ax^2 + bx + c$ (a , b and c are real numbers), you get a positive number, while substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

Only 14% of the 152 subjects correctly solved the problem. These subjects considered the function corresponding to $y = ax^2 + bx + c$, switched representations, and either referred to a graph mentally or actually

sketched a graph. Most of the subjects (about 80%) did not show any attempt to look at another representation of the problem, and did not solve the problem. A large number of subjects were stuck in the manipulation of inequalities: $a + b + c > 0$, $36a + 6b + c < 0$. As a result, they were not able to reach correct solutions.

In addition, she also found that the subjects who used a point-wise approach (e.g., process perspective) were more successful in solving problems that involved different representations of function than subjects who used a global approach (e.g., object). For example, in the following question:

This is the graph (Figure 2.2) of the function $f(x) = ax^2 + bx + c$. State whether a , b , and c are positive, negative or zero. Explain your decision.

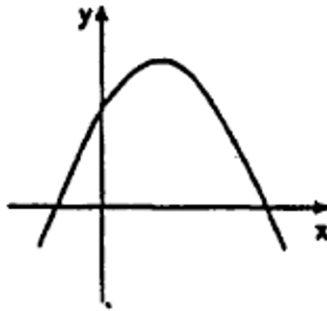


Figure 2.2. Graph of quadratic function

Only subjects who used a point-wise approach and looked at the y-intercept found correctly the sign of “ c ”. They explained as follows: c is positive because when $x=0$, $f(x)=c$ is positive).

In Black’s (2007) study, he asked in-service high school mathematics teachers to explain their choice to the following question to students. 15 of 67 participants arrived at the correct answer.

Mr. Seng's algebra class is studying the graph of $y = ax^2 + bx + c$ and how changing the parameters of a , b , and c will cause different translations of the original graph (Figure 2.3).

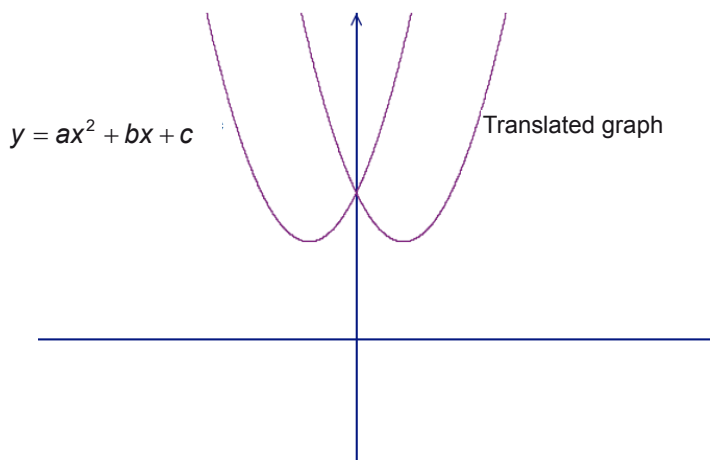


Figure 2.3. Graphs of original and translated quadratic functions

Which of the following is an appropriate explanation of the translation of the original graph $y = ax^2 + bx + c$ to the translated graph?

- A. Only the **a** value changed.
- B. Only the **c** value changed.
- C. Only the **b** value changed.
- D. At least two of the parameters hanged.
- E. You cannot generate the translated graph by changing any of the parameters.

He concluded that the participants had “a lot of difficulty in both their answer selection, as well as in the explanations provided for those answers” (Black, 2007, p. 134). Additionally he found that 13 of the 67 participants did not even attempt to answer the problem, and many more did not explain their answers. The finding seemed to suggest those

mathematics teachers had difficulty in performing on functions as an *objects* (i.e., translation of graph) and shift between different representations (Using algebraic representation to explain graphic changes).

As argued by Moschkovick et al. (1993),

“Competence in the domain (i.e., line equation) consists of being able to move flexibly across representations and perspectives, where warranted: to be able to ‘see’ lines in the plane, in their algebraic form, or in tabular form, as objects when any of those perspective is useful, but also to switch to the process perspective, where that perspective is appropriate” (p. 97).

Given the importance of developing students’ flexibility in learning the concept of function, and the weakness of teachers’ knowledge for teaching functions that promote this flexibility, I focus on teachers’ flexibility in shifting between different perspectives (process vs. object), and selecting multiple representations (verbal, tabular, symbolic, and graphic). Further, I will illustrate the concept of flexibility in solving problems related to quadratic function in the methodology section.

2.5 Mathematics Teacher Education Systems in China and the U.S.

In this section, I briefly introduce teacher preparation systems in China and the U.S. to provide a background for understanding teacher knowledge growth.

2.5.1 Mathematics Teacher Education in China

Since adopting the “nine-year compulsory education system” in 1986, teacher education has become a daunting task. Through approximately 20 years of effort, a three-stage “normal” education has been established and has made significant contribution to educating teachers from elementary to secondary levels in China. It means: (1) primary school teachers are trained in secondary normal schools; (2) junior high school teachers are trained in three-year teacher colleges; and (3) senior high school teachers are trained in four-year teacher colleges and normal universities. However, with the rapid development of the economy and

technology in China, it has been an urgent agenda to upgrade and foster teachers' quality. In order to meet this challenge, the Ministry of Education (1998) documented an action plan for revitalizing education in the 21st century. Two projects were launched; one is referred to as the "Gardener Project" and aims to establish continuing teacher education systems for practicing teachers. Meanwhile, the Ministry of Education (1999) enacted a *decision on deepening education reform and whole advancing quality education* in which comprehensive universities and non-normal Universities were encouraged to engage in educating elementary and secondary teachers. This meant that the privilege of normal universities for teacher education changed. The Ministry of Education (2001a) put forward a process to improve an open teacher education system based on the existing Normal universities and supported by other universities, and to integrate prospective teacher preparation and practicing teachers' professional development.

Through five years of development and research, teacher education has seen some changes:

1. Integration of education of prospective and practicing teachers;
2. Opening of teacher education in all qualified universities rather than just Normal universities;
3. Forming a new three-staged teacher education: where primary school teachers are trained in the three-year teacher colleges or four-year teacher colleges; the junior and senior high school teachers are trained in four-year teacher colleges and Normal universities, and some of the senior high school teachers are required to attain postgraduate level studies (Gu, 2006).

In 2004, there were more than 400 institutes conducting teacher education programs and about 280 of them were teacher education universities or colleges. It was also found that one third of graduates who became teachers were from non-teacher education institutes (Yuan, 2004). This proportion has steadily increased in recent years. According to the *Educational Statistics Yearbook of China 2008* (Ministry of Education, 2009), there are only 188 normal institutions where preparing teachers at

different levels is the main purpose. In addition, there is now a flexible and encouraging accreditation system for teacher recruitment in China. University degree holders who wish to become secondary school teachers can pass the related examinations, usually pedagogy, psychology, and subject didactics.

Middle and high school mathematics teacher preparation programs were typically hosted in mathematics departments. Through analysis of the course design in a mathematics department at a normal university (Li, Huang, & Shin, 2008), it was found that prospective mathematics teachers were required to complete 63 credit hours in mathematics content, 6 credit hours in pedagogical courses, and 14 credit hours in student teaching and thesis. In addition, there were 20 credit hours in elective mathematics and mathematics education courses. Li and colleagues (2008) concluded that the secondary mathematics preparation program exhibits the following characteristics:

1. Providing prospective teachers with a strong foundation in profound mathematics knowledge and advanced mathematics literacy
2. Emphasizing review and study of elementary mathematics. It was believed that a profound understanding of elementary mathematics and strong ability of solving problems in elementary mathematics were crucial to being a qualified mathematics teacher at secondary schools. Due to the tradition of examination oriented teaching, a high level of problem solving ability is necessary for a qualified teacher;
3. Teaching practicum is limited. A six-week teaching practicum can only provide prospective teachers with a preliminary experience of teaching in secondary schools. (Li et al., 2008, p. 70)

This reflects the belief that a solid mathematics base is vital for teacher preparation. Furthermore, taking more advanced mathematics courses is on a higher priority since prospective teachers will not have many opportunities to learn them in their career. The main goal is to foster prospective teachers with a bird's eye view of understanding elementary mathematics deeply, even though there are special courses such as Modern Mathematics and School Mathematics, which connect advanced mathematics to elementary mathematics. In contrast with the high requirement

in mathematics content, teacher educators normally believe that prospective teachers can develop pedagogical skills and knowledge later during their teaching.

2.5.2 *Mathematics Teacher Education in the U.S.*

In the United States, there are three levels of pre-college education: elementary school (Grades K-5 or K-6); middle school or junior high school (Grades 6-8 or 7-8); and high school (Grades 9-12). In 2007, forty states had either a middle school or junior high school certification or endorsement requirement (National Middle School Association, 2007). Many of these states have special mathematics requirements for that certification or endorsements by the teachers' selected area of content expertise. In mathematics, these special requirements range from passing a test to completing the equivalent of an undergraduate minor in mathematics. The Conference Board of Mathematical Science [CBMS] (2001) recommendations call for the teaching of mathematics in middle school (Grades 5-8) to be conducted by mathematics specialists; teachers specially educated to teach mathematics to the students of the grade levels they instruct. These teachers should have at least twenty-one semester hours in mathematics, including at least twelve semester hours of fundamental ideas of mathematics appropriate for middle school teachers.

Middle school math teachers. Based on National NAEP survey (Smith, Arbaugh, & Fi, 2007), 85% of the nation's eighth graders are taught by teachers who were certified by their state. When examined by teachers' degrees, 30% of the nation's eighth graders had teachers with an undergraduate degree in mathematics; 26 % had teachers with an undergraduate degree in mathematics education; and the remaining students were taught by a teacher with a degree in some other discipline. Thus, at least one-third of the nation's eighth-grade students were being taught mathematics by teachers without substantial mathematics training. According to National Report Card 2009 (National Assessment of Education Progress [NAEP], 2009), the situation gets worse: 27% of the nation's eighth graders had teachers with an undergraduate degree in mathematics; 30 % had teachers with an undergraduate degree in

mathematics education. This is a major concern of U.S. mathematics teacher education.

High school mathematics teachers. For high school mathematics teacher certification, states require from eighteen to forty-five semester hours of mathematics, equivalent to six to fifteen semester courses, or they require a major in the subject. When a number of credit hours of mathematics are specified for the certificate, almost half require thirty credit hours. The specific courses mentioned include three courses in calculus (two single-variable and one multivariable), linear algebra, geometry, and abstract algebra, plus a number of electives.

The 2000 National Survey of Science and Mathematics Education (Whittington, 2002) was designed to identify trends in the areas of teacher background and experience, curriculum and instruction, and the availability and use of instructional resources. A total of 5,728 science and mathematics teachers in schools (1,367 of them were high school mathematics teachers) across the United States participated in this survey. According to this survey, 58% of mathematics teachers in grades 9-12 in their sample had an undergraduate major in mathematics, 22% had a degree in mathematics education, 10% had a degree in some other education field, and 10% had a degree in a field other than education or mathematics. In this sample, 96% of teachers had completed a course in calculus, 86% in probability and statistics, 83% in geometry, 82 % in linear algebra, 70% in advanced calculus, and 65% in differential equations and so on. See table 2.1.

Table 2.1. High School Mathematics Teachers Completing Various College Courses

Course	Percent of teachers
General methods of teaching	90
Methods of teaching mathematics	77
Supervised student teaching in mathematics	70
Instructional uses of computers/other technologies	43
Mathematics for middle school teachers	26
Geometry for elementary/middle school teachers	17
Calculus	96

Course	Percent of teachers
Probability and statistics	86
Geometry	83
Linear algebra	82
College algebra/trigonometry/ elementary functions	80
Advanced calculus	70
Computer science course	68
Differential equations	65
Abstract algebra	65
Computer programming	62
Other upper division mathematics	60
Number theory	56
History of mathematics	41
Real analysis	38
Discrete mathematics	38
Applications of mathematics/ problem solving	37

The CBMS report (2001) recommends that high school teachers of mathematics have a major in mathematics; that includes a six-hour capstone course connecting their college mathematics course with high school mathematics. This recommendation stems from the view that teachers need to know the subject they will teach, they need to understand the broad range of the mathematical sciences their students will encounter in their careers, and they need to develop the habits of mind and dispositions towards doing mathematics that characterize effective workers in the field.

2.6 Comparative Studies on Teachers' Knowledge for Teaching between China and the U.S.

Ma's (1999) revealed Chinese elementary teachers had a profound understanding of fundamental mathematics concepts in four areas: subtraction with regrouping, multi-digit multiplication, division by fractions,

and the relationship between perimeter and area, in comparison with U.S. counterparts. After that, several studies have focused on mathematics teacher knowledge in China and the U.S. (An et al., 2004; Cai, 2005; Cai & Wang, 2006; She, Lan, & Wilhelm, 2011; Zhou, Peverly, & Xin, 2006). By comparing pedagogical content knowledge of middle school mathematics teachers between the U.S. and China, An et al. (2004) found that the Chinese mathematics teachers emphasized gaining correct conceptual knowledge by relying on a more rigid development of procedures, while the United States teachers emphasized a variety of activities designed to promote creativity and inquiry in order to develop concept mastery.

A study investigating Chinese and US middle school teachers' ways of solving algebraic problems (She, Lan, & Wilhlem, 2011) revealed that U.S. teachers were more likely to use concrete models and practical approaches in problem solving, but they seemed to lack a deep understanding of underlying mathematical theories. The Chinese teachers were inclined to utilize general roles/strategies and standard procedures for teaching, and they demonstrated an interconnected knowledge network when solving problems.

In addition, a study comparing 162 U.S. and Chinese third grade mathematics teachers' expertise in teaching fractions (Zhou et al., 2006) found that Chinese teachers significantly outperformed their U.S. counterparts in subject matter knowledge, but they performed poorly in comparison to their U.S. counterparts on a test designed to measure general pedagogical knowledge. However, in pedagogical content knowledge there are no determinative patterns found.

Cai and his colleagues (Cai, 2000, 2005; Cai & Wang, 2006) have conducted a series of comparative studies on students' problem solving and teachers' construction of representations between China and the U.S. It was found that Chinese students preferred using symbolic representations while U.S. students tended to use pictorial representations (Cai, 1995, 2000). In addition, both Chinese and U.S. teachers used concrete representations for developing the concepts of ratio and average, but Chinese teachers tended to use symbolic representations for solving problems while U.S. teachers still preferred to use concrete representations (Cai, 2005; Cai & Wang, 2006). Furthermore, Huang and

Cai (2011) found that the U.S teachers tended to develop multiple representations simultaneously while the Chinese teachers tend to selectively use representations hierarchically.

These studies seem to suggest Chinese elementary mathematics teachers have a stronger subject matter knowledge, probably pedagogical content knowledge, compared with their U.S. counterparts. Meanwhile, Chinese mathematics teachers value symbolic representation more than U.S. mathematics teachers when solving problems.

However, there are no comparative studies on teachers' knowledge needed for teaching special areas at middle and high school levels between China and the U.S. Also, teachers' representational flexibility, which is closely related to teachers' beliefs and teaching has not been explored appropriately. Thus, in this study, I examined prospective secondary school teachers' knowledge of algebra for teaching with a focus on the concept of function and representation flexibility in China and the U.S. Therefore, the current study will contribute to the understanding of mathematics teacher's knowledge of algebra for teaching in China and U.S. at middle and high school levels in addition to shedding light on improvement of teacher preparation programs in China and the U.S.

2.7 Conclusion

This chapter provided a review of relevant literature, laying a theoretical foundation for this study. First, the development of the notion of teacher knowledge in general, and ways of defining and measuring mathematics teacher knowledge needed for teaching in particular were summarized. Second, the specific frameworks for studying mathematics knowledge of algebra for teaching were analyzed and compared. Third, relevant studies on algebra teaching and learning, and teacher knowledge needed for teaching algebra were summarized. Fourth, the literature review focused on teacher knowledge for teaching the concept of function promoting flexibility in adapting appropriate perspectives and representations of function. Fifth, a brief summary of mathematics teacher preparation in China and the U.S. was presented. Finally, comparative studies on teachers' knowledge for teaching between China and the U.S. were summarized.

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