# Chapter 2

# Mathematical preliminaries

The following sections acquaint the reader with some mathematical background for the analytical part of this thesis. After the basic notations are given in Section 2.1, we will present some standard and nonstandard techniques for variational problems and since the presentation can, of course, not be comprehensive, we will refer to the monographs in functional analysis for more details at the corresponding parts.

A short introduction to measures and spaces of functions of bounded variations with values in a (possibly infinite dimensional) Banach space $X$ is given in Section 2.2. These spaces will appear as trajectory spaces in weak notions of system (1.1) in Chapter 6 and Chapter 7. In particular, the space of $SBV$-functions defined on a time-interval and with values in $X$ is introduced.

In Section 2.3, we firstly define the notion of $\Gamma$-convergence and summarize certain inequalities in Sobolev spaces and compact embedding results due to Aubin and Lions which are used in oder to obtain the proper convergence properties in the existence proofs. Beyond that, we introduce a new variational method in Subsection 2.3.3 and Subsection 2.3.4. More specifically, Lemma 2.3.19 gives in combination with an approximation scheme presented in Lemma 2.3.18 a new tool to deal with coupled variational inequalities arising from a weak formulation of the doubly nonlinear differential inclusion in (1.1c).

The analytical approach in Chapter 6 and Chapter 7 to complete damage systems requires certain spaces which do not seem to be well established in the mathematical literature so far. It turns out that the displacement field in a weak formulation of (1.1) exists only in a space-time local Sobolev space. This space is introduced in Section 2.4. Moreover, for the degenerate limit we employ covering and representation results for shrinking sets by families of Lipschitz domains which are proven in Section 2.4.

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## 2.1 Notation

In this work, we fix a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ of dimension $n$ and a time constant $T > 0$. We assume that the material in the reference configuration is located in $\Omega$. For the Dirichlet boundary $\Gamma_{\mathrm{D}}$ and the Neumann boundary $\Gamma_{\mathrm{N}}$ of $\partial\Omega$, we adopt the assumptions from [Ber11], i.e., $\Gamma_{\mathrm{D}}$ and $\Gamma_{\mathrm{N}}$ are non-empty and relatively open sets in $\partial\Omega$ with finitely many path-connected components such that $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$ and $\overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}} = \partial\Omega$. Note that $\mathcal{H}^{n-1}(\Gamma_{\mathrm{D}}) > 0$). The following table provides an overview of some elementary notation used in this thesis.

- Measures and sets:

| | |
|---|---|
| $\mathcal{L}^n$, $\mathcal{H}^n$ | *$n$-dimensional Lebesgue and Hausdorff measure* |
| $\mathbb{R}_+, \mathbb{R}_\infty$ | $[0, \infty)$, $\mathbb{R} \cup \{+\infty\}$ |
| $\mathbb{S}^n$ | *$n$-dimensional unit sphere in $\mathbb{R}^{n+1}$* |
| $\Omega_T$ | $\Omega \times (0, T)$ |
| $B_\varepsilon(A)$ | *$\varepsilon$-neighborhood of $A \subseteq \mathbb{R}^n$* |
| $Q_\varepsilon(x_0)$ | *open cube with center $x_0 \in \mathbb{R}^n$ and edge length $2\varepsilon$, i.e., $\big\{x \in \mathbb{R}^n \,\big|\, \|x - x_0\|_\infty < \varepsilon\big\}$* |
| $\overline{A}$, $\mathrm{int}(A)$, $\partial A$ | *closure, interior and boundary of $A \subseteq \mathbb{R}^n$* |
| $\{v = 0\}$, $\{v > 0\}$ | *level and super-level set of $v$, i.e., $\big\{x \in \overline{\Omega} \,\big|\, v(x) = 0\big\}$ and $\big\{x \in \overline{\Omega} \,\big|\, v(x) > 0\big\}$ for functions $f \in L^1(\Omega)$ defined up to a set of measure $0$ and defined uniquely if $v \in W^{1,p}(\Omega)$, $p > n$, as $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$* |
| $\mathrm{supp}(v)$ | *support of a function $v$, i.e., $\overline{\big\{x \,\big|\, v(x) \neq 0\big\}}$* |

- Spaces:

| | |
|---|---|
| $\mathcal{L}(X)$ | *space of linear and continuous functions from $X$ to $X$* |
| $\mathcal{C}^k(\overline{\Omega}; \mathbb{R}^N)$ | *space of $k$-times continuously differentiable functions on the open set $\Omega \subseteq \mathbb{R}^n$ where the $k$-th derivatives can be continuously extended to $\overline{\Omega}$* |
| $\mathcal{C}^{k,\alpha}(\overline{\Omega}; \mathbb{R}^N)$ | *space of $k$-times continuously differentiable functions on the open set $\Omega \subseteq \mathbb{R}^n$ where the $k$-th derivatives are Hölder continuous with exponent $\alpha$ and can be continuously extended to $\overline{\Omega}$* |

| | |
|---|---|
| $\mathcal{C}^k_x(\overline{\Omega_T};\mathbb{R}^N)$ | *space of k-times continuously differentiable functions with respect to the spatial variable on the set* $\Omega \times [0,T]$ *where the k-th spatial derivatives can be continuously extended to* $\overline{\Omega_T}$ |
| $W^{m,p}(\Omega;\mathbb{R}^N)$ | *standard Sobolev space of m-times weakly differentiable functions with weak derivatives in* $L^p(\Omega;\mathbb{R}^N)$ |
| $W^{1,r}_+(\Omega)$, $W^{1,r}_-(\Omega)$ | *space of non-negative and non-positive Sobolev functions, i.e.,* $\big\{\zeta \in W^{1,r}(\Omega) \,\big|\, \zeta \geq 0 \text{ a.e. in } \Omega\big\}$ *and* $\big\{\zeta \in W^{1,r}(\Omega) \,\big|\, \zeta \leq 0 \text{ a.e. in } \Omega\big\}$ |
| $W^{1,r}_{\Gamma_\mathrm{D}}(\Omega;\mathbb{R}^N)$ | *space of Sobolev functions vanishing on the Dirichlet boundary* $\Gamma_\mathrm{D}$*:* $\big\{\zeta \in W^{1,r}(\Omega;\mathbb{R}^N) \,\big|\, \zeta = 0 \text{ on } \Gamma_\mathrm{D} \text{ in the sense of traces}\big\}$ |
| $X^*$ | *dual space of the Banach space* $X$ |

- Functions and operators:

| | |
|---|---|
| $\mathbb{1}_A$, $I_A$ | *characteristic function and indicator function* $X \to \mathbb{R}_\infty$ *with respect to a subset* $A \subseteq X$ |
| $A : B$ | *Euclidean matrix product of* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times n}$ |
| $\langle\cdot,\cdot\rangle_{X^\star\times X}$ | *dual pairing of* $X^\star$ *and* $X$*, abbr.* $\langle\cdot,\cdot\rangle$ |
| $[f]^+$ | *non-negative part of* $f$*, i.e.,* $\max\{0,f\}$ |
| $f^+, f^-$ | *one-sided limits of* $f : I \to X$*,* $I \subseteq \mathbb{R}$ *interval, i.e.,* $f^\pm(a) = \lim_{x\to a^\pm} f(x)$ |
| $\Delta_p$ | *p-Laplacian* $\Delta_p := \operatorname{div}(\lvert\nabla z\rvert^{p-2}\nabla z)$ |
| $⨍_A f(x)\,\mathrm{d}x$ | *mean value of* $f$ *in* $A \subseteq \mathbb{R}^n$*, i.e.,* $\frac{1}{\mathcal{L}^n(A)}\int_A f(x)\,\mathrm{d}x$ |
| $\partial J$ | *subdifferential of a convex function* $J : X \to \mathbb{R}_\infty$ *i.e.,* $\partial J(x) = \big\{x^\star \,\big|\, J(x) + \langle x^\star, y-x\rangle \leq J(y) \text{ for all } y\big\}$ |
| $\mathrm{d}E$ | *Gâteaux differential of a functional* $E : X \to Y$ |
| $p^\star$ | *Sobolev critical exponent* $\frac{np}{n-p}$ *for* $n > p$ |
| $\mathrm{diam}(Q)$ | *diameter of a subset* $Q \subseteq \mathbb{R}^n$ |

- Binary relations:

| | |
|---|---|
| $A \subset\subset B$ | *if* $\overline{A} \subseteq B$ |

We adopt the convention that for two given functions $\zeta, \xi \in L^1(0,T;W^{1,p}(\Omega))$ with $p > n$ the inclusion $\{\zeta = 0\} \supseteq \{\xi = 0\}$ is an abbreviation for $\{\zeta(t) = 0\} \supseteq \{\xi(t) = 0\}$ for a.e. $t \in (0,T)$. Here, $\zeta(t), \xi(t) \in \mathcal{C}(\overline{\Omega})$ due to the embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$.

## 2.2 Vector measures and vector-valued functions of bounded variations

In the beginning, we will review some basic definitions from the theory of vector measures and standard Bochner spaces. For further readings on this topic, we would like to refer to [Din02, CV02].

Let $(X, \|\cdot\|)$ be a Banach space and $(S, \Sigma, \mu)$ be a measure space consisting of a set $S$, a $\sigma$-algebra over $S$ and a positive measure $\mu : \Sigma \to [0, +\infty]$. The measure space $(S, \Sigma, \mu)$ is called finite if $\mu(S) < \infty$. Furthermore, let $m : \Sigma \to X$ be a Banach space valued measure. We assume that all measures are $\sigma$-additive (finitely additive measures also studied in the literature; see [Din02, Chapter 1, §2.A]), i.e.,

$$\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mu\left(A_i\right) \text{ for all pairwise disjoint sets } A_i \in \Sigma,\ i \in \mathbb{N}.$$

As usual, $L^p(S, \mu; X)$ denotes the $p$-Bochner $\mu$-integrable functions with values in $X$ ($\mu$-essentially bounded for $p = \infty$, respectively), cf. [Zei90, Chapter 23.2]. In the following, we restrict ourselves to the case where $S$ is a finite interval $I \subseteq \mathbb{R}$. We write $L^p(I; X)$ for $L^p(I, \mathcal{L}^1; X)$. The subspace $H^q(I; X) \subseteq L^2(I; X)$, $q \in \mathbb{N}$, indicates the $L^2(I; X)$-functions $f$ which are $q$-times weakly differentiable with weak derivatives $\partial_t^s f$ in $L^2(I; X)$, $s = 1, \ldots, q$, i.e.,

$$\int_I f(t)\partial_t^s \xi(t)\,\mathrm{d}t = (-1)^s \int_I (\partial_t^s f(t))\xi(t)\,\mathrm{d}t \quad \text{for all } \xi \in C_0^\infty([0, T]).$$

The norm is given by $\|f\|_{H^q(I;X)} := \sum_{s=0}^{q} \|\partial_t^s f\|_{L^2(I;X)}$ (see [Zei90, Chapter 23.5]).

In the most literature, functions of bounded variations are usually considered with values in $\mathbb{R}$ or in a finite dimensional vector space [AFP00]. We will give the definition of $BV$ and $SBV$-functions defined on a time-interval and with values in a Banach space $X$. $X$-valued $BV$-functions are, for instance, also investigated in the monograph [Din66].

**Definition 2.2.1 ($BV$-space of Banach space-valued functions)**
*The subspace $BV(I; X) \subseteq L^1(I; X)$ consists of functions $f \in L^1(I; X)$ with the norm*

$$\operatorname{ess\,var}_I(f) := \inf\left\{\operatorname{var}_I(g) \,\middle|\, g = f \ \mathcal{L}^1\text{-a.e. in } I\right\} < +\infty,$$

*and*

$$\operatorname{var}_I(f) := \sup\left\{ \sum_{i=1}^{k-1} \|f(t_{i+1}) - f(t_i)\|_X \,\middle|\, t_1 < t_2 < \ldots < t_k \text{ with } t_1, t_2, \ldots t_k \in I \text{ for } k \geq 2 \right\}.$$

Since the weak derivatives of $BV$-functions are measures, we turn our attention to $X$-valued measures. To proceed, we define the variation of the vector-valued measure $m$ as (cf. [Din02, Chapter 1, §2.A])

$$|m| := \operatorname{var}(m) := \sup\left\{ \sum_{i\in I} \|m(A_i)\|_X \,\middle|\, \{A_i\} \text{ is a finite family of disjoint sets } A_i \in \Sigma \right\}.$$

We say that $m$ has finite variation if $|m| < \infty$. Furthermore, $m$ is said to be absolutely continuous with respect to $\mu$, abbr. $m \ll \mu$, if for all $A \in \Sigma$ with $\mu(A) = 0$ we have $m(A) = 0$ (cf. [Din02, Chapter 1, §3.C]).

Now, let $f \in L^1(I, \mu; X)$ and $B \subseteq I$ be measurable. we define the measures $f\mu$ and $\mu\lfloor B$ as follows:

$$(f\mu)(A) := \int_A f(t)\,\mathrm{d}\mu(t) \text{ for measurable } A \subseteq I,$$
$$(\mu\lfloor B)(A) := \mu(A \cap B) \text{ for measurable } A \subseteq I.$$

For finite dimensional spaces $X$, there exists always a decomposition $m = g\mu$ for a function $g \in L^1(S, \mu; X)$, i.e., $m(A) = \int_A g\,\mathrm{d}\mu$ for all $A \in \Sigma$, when $m \ll \mu$ holds (see [AFP00, Theorem 1.28]). In this case, we call $g$ the *Radon-Nikodým derivative.* This is, in general, not true when $X$ is an infinite dimensional Banach space and, therefore, motivates the subsequent definition.

**Definition 2.2.2 (Radon-Nikodým property, cf. [Din02, Chapter 1, §2.G])** *A Banach space $X$ has the Radon-Nikodým property if for every finite measure space $(S, \Sigma, \mu)$ and every measure $m : \Sigma \to X$ with finite variation $|m|$ such that $m \ll \mu$ there exists a function $g \in L^1(S, \mu; X)$ such that $m = g\mu$, i.e., $m(A) = \int_A g\,\mathrm{d}\mu$ for all $A \in \Sigma$.*

**Remark 2.2.3** *Reflexive spaces and separable duals of Banach spaces are examples for Banach spaces which possess the Radon-Nikodým property, see [Din02, Chapter 1, §2.G].*

To every $f \in BV(I; X)$, we can choose a representant (also denoted by $f$) with $\mathrm{var}_I(f) < +\infty$. Then the values $f(t^\pm) := \lim_{s\to t^\pm} f(s)$ exist for all $t \in \overline{I}$ (and are independent of the representant) by adopting the convention $f\big((\inf I)^-\big) := f\big((\inf I)^+\big)$ and $f\big((\sup I)^+\big) := f\big((\sup I)^-\big)$. The functions $f^+(t) := f(t^+)$ and $f^-(t) := f(t^-)$ are thus uniquely defined for every $t \in \overline{I}$ and do not coincide for at most countably many points, i.e., in the jump discontinuity set $J_f$. Furthermore, a regular measure $\mathrm{d}f$ with finite variation, i.e., $|\mathrm{d}f|(I) < \infty$, and with values in $X$ (also called *differential measure*) can be assigned such that $\mathrm{d}f((a, b]) = f^+(b) - f^+(a)$ for all $a, b \in \overline{I}$ with $a \leq b$, cf. [Din66, Chapter III, §17.2, Theorem 1].

If $X$ exhibits the Radon-Nikodým property the differential measure decomposes into $\mathrm{d}f = f'_\mu \mu$ for a positive Radon measure $\mu$ and a function $f'_\mu \in L^1(I, \mu; X)$ (see also [MV87]). We call $f$ a special function of bounded variations if $\mathrm{d}f$ even decomposes into an *absolutely continuous part* and a *jump part.* More precisely, we define the following subspace (cf. [AFP00, Chapter 4.1]).

**Definition 2.2.4 ($SBV^p$-space of vector-valued functions)** *The subspace $SBV(I; X) \subseteq BV(I; X)$ of special functions of bounded variations is defined as the space of functions $f \in BV(I; X)$ where the decomposition*

$$\mathrm{d}f = f'\mathcal{L}^1 + (f^+ - f^-)\mathcal{H}^0\lfloor J_f \tag{2.1}$$

*for an* $f' \in L^1(I;X)$ *exists. Here,* $\mathcal{H}^0$ *denotes the* 0*-dimensional Hausdorff measure, i.e.,* (2.1) *reads as*

$$\mathrm{d}f(A) = \int_A f'(t)\,\mathrm{d}t + \sum_{t\in J_f\cap A} (f^+(t) - f^-(t)).$$

*The function* $f'$ *is called the absolutely continuous part of the differential measure and we also write* $\partial_t^{\mathrm{a}} f$*. If, additionally,* $\partial_t^{\mathrm{a}} f \in L^p(I;X)$*,* $p \geq 1$*, we write* $f \in SBV^p(I;X)$*.*

**Theorem 2.2.5 (BV-chain rule [MV87, Theorem 3])** *Let* $I \subseteq \mathbb{R}$ *be an interval,* $X$ *be a Banach space with the Radon-Nikodým property,* $f \in BV(I;X)$ *with* $\mathrm{d}f = f'_\mu \mu$ *for a non-negative Radon measure* $\mu$ *on* $I$ *and* $f'_\mu \in L^1(I,\mu;X)$*. Moreover, let* $E : X \to \mathbb{R}$ *be continuously Fréchet-differentiable. Then* $E \circ f \in BV(I;\mathbb{R})$ *and* $\mathrm{d}(E\circ f)$ *admits as density relative to* $\mu$ *the function* $t \mapsto \langle \theta(t), f'_\mu(t)\rangle$*, where* $\theta : I \to X^\star$ *is defined as*

$$\theta(t) := \int_0^1 \mathrm{d}E((1-r)f(t^-) + rf(t^+))\,\mathrm{d}r.$$

**Corollary 2.2.6** *Suppose* $f \in SBV(0,T;X)$ *and* $E : X \to \mathbb{R}$ *is continuously Fréchet-differentiable. Then* $E \circ f \in SBV(0,T)$ *and for all* $0 \leq a \leq b \leq T$*:*

$$\mathrm{d}(E\circ f)((a,b]) = \int_a^b \langle \mathrm{d}E(f(s)), f'(s)\rangle\,\mathrm{d}s + \sum_{s\in J_f\cap(a,b]} \left(E(f(s^+)) - E(f(s^-))\right).$$

*Proof.* We apply Theorem 2.2.5. By assumption, we obtain the decomposition $\mathrm{d}f = f'_\mu \mu$ with $\mu = \mathcal{L}^1 + \mathcal{H}^0 \lfloor J_f$ and

$$f'_\mu(t) = \begin{cases} f'(t) & \text{if } t \in [0,T]\setminus J_f, \\ f(t^+) - f(t^-) & \text{if } t \in J_f. \end{cases}$$

Applying Theorem 2.2.5 yields

$$\begin{aligned} \mathrm{d}(E\circ f)((a,b]) &= \int_{(a,b]} \langle \theta(s), f'_\mu(s)\rangle\,\mathrm{d}\mu(s) \\ &= \int_{(a,b]} \langle \theta(s), f'(s)\rangle\,\mathrm{d}\mathcal{L}^1(s) + \sum_{t\in J_f\cap(a,b]} \langle \theta(s), f(s^+) - f(s^-)\rangle \end{aligned}$$

Since $f(s^+) = f(s^-) = f(s)$ for $\mathcal{L}^1-$a.e. $s \in (a,b)$, the first term on the right hand side becomes

$$\begin{aligned} \int_{(a,b]} \langle \theta(s), f'(s)\rangle\,\mathrm{d}\mathcal{L}^1(s) &= \int_{(a,b]} \Big\langle \int_0^1 \mathrm{d}E((1-r)f(s^-) + rf(s^+))\,\mathrm{d}r, f'(s)\Big\rangle\,\mathrm{d}\mathcal{L}^1(s) \\ &= \int_{(a,b]} \langle \mathrm{d}E(f(s), f'(s)\rangle\,\mathrm{d}s, \end{aligned}$$

where, as usual, $\mathrm{d}s := \mathrm{d}\mathcal{L}^1(s)$. Furthermore, by the classical chain rule,

$$\begin{aligned}
&\sum_{s\in J_f\cap(a,b]} \langle\theta(s), f(s^+)-f(s^-)\rangle \\
&\qquad= \sum_{s\in J_f\cap(a,b]} \Big\langle \int_0^1 \mathrm{d}E((1-r)f(s^-)+rf(s^+))\,\mathrm{d}r, f(s^+)-f(s^-)\Big\rangle \\
&\qquad= \sum_{s\in J_f\cap(a,b]} \int_0^1 \Big\langle \mathrm{d}E((1-r)f(s^-)+rf(s^+)), f(s^+)-f(s^-)\Big\rangle\,\mathrm{d}r \\
&\qquad= \sum_{s\in J_f\cap(a,b]} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}E((1-r)f(s^-)+rf(s^+))\,\mathrm{d}r \\
&\qquad= \sum_{s\in J_f\cap(a,b]} \Big(E(f(s^+))-E(f(s^-))\Big).
\end{aligned}$$

□

## 2.3 Variational methods

### 2.3.1 Γ-convergence

In the following, we give the definition of Γ-convergence and some basic properties taken from [Bra06]. For further details, we also refer to the monograph [Mas93, Bra02]. The complete damage approach in Chapter 6 as well as in Chapter 7 uses Γ-convergence methods to gain the energy estimate in the notion of weak solutions.

The definition of Γ-convergence can be given in the case of topological spaces. To this end, we fix a topological space $(X,\mathcal{T})$ and consider a sequence of functionals $\{f_\varepsilon\}$ with $f_\varepsilon : X \to [-\infty,+\infty]$, $\varepsilon \in (0,1)$, as well as $f : X \to [-\infty,+\infty]$. We say that $f_\varepsilon$ Γ-converges to the functional $f : X \to [-\infty,+\infty]$ as $\varepsilon \to 0^+$ if for all $x \in X$

$$f(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{\varepsilon\to0} \inf_{y\in U} f_\varepsilon(y) = \sup_{U\in\mathcal{N}(x)} \limsup_{\varepsilon\to0} \inf_{y\in U} f_\varepsilon(y)$$

is fulfilled, where $\mathcal{N}(x)$ denotes the set of all neighborhoods $U \in \mathcal{T}$ of $x$ in $X$. In this case, we write $f_\varepsilon \xrightarrow{\Gamma} f$.

For further studies, we switch to metric spaces $(X,d)$. We obtain a more convenient definition of Γ-convergence in this case. Note that bounded subsets of Banach spaces with separable duals are metrizable in the weak topology (see [AB07, Theorem 6.31]).

**Theorem 2.3.1 (Γ-convergence, cf. [Bra06, Theorem 2.1])**
*The following properties are equivalent:*

*(i)* $f_\varepsilon \xrightarrow{\Gamma} f$

*(ii)* $f = \Gamma - \liminf_{\varepsilon\to 0} f_\varepsilon = \Gamma - \limsup_{\varepsilon\to 0} f_\varepsilon$ *with*

$$\Gamma - \liminf_{\varepsilon\to 0} f_\varepsilon(x) := \inf\{\liminf_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \mid x_\varepsilon \to x\}$$
$$\Gamma - \limsup_{\varepsilon\to 0} f_\varepsilon(x) := \inf\{\limsup_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \mid x_\varepsilon \to x\}.$$

*(iii)* *(a) liminf estimate. For every sequence* $x_\varepsilon \to x$ *in* $X$*, it holds*

$$f(x) \le \liminf_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon).$$

*(b) limsup estimate. There exists a sequence (so-called recovery sequence)* $x_\varepsilon \to x$ *in* $X$ *such that*

$$f(x) \ge \limsup_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon).$$

At this point, we would like to mention the fundamental theorem of Γ-convergence which emphasizes its importance in variational methods (cf. [Bra06, Theorem 2.10]).

**Theorem 2.3.2 (Fundamental theorem of Γ-convergence)** *Assume* $f_\varepsilon \xrightarrow{\Gamma} f$*. Furthermore, let* $f_\varepsilon$ *satisfy the following condition (equi-coercivity)*

$$\forall t \in \mathbb{R},\ \exists K \subseteq X \ \text{compact},\ \forall \varepsilon \in (0,1) : \{f_\varepsilon \le t\} \subseteq K.$$

*Then* $f$ *has a minimum in* $X$ *and* $\min_{x\in X} f(x) = \lim_{\varepsilon\to 0} \inf_{x\in X} f_\varepsilon(x)$*.*

In order to draw some further conclusions, we introduce the lower semi-continuous envelope $\overline{f} : X \to [-\infty, +\infty]$ of a functional $f$ as

$$\overline{f}(x) := \liminf_{y\to x} f(y).$$

**Remark 2.3.3** *We have the following properties (cf. [Bra06, Proposition 2.4 and Remark 2.12]).*

*(i) If* $f_\varepsilon \xrightarrow{\Gamma} f$ *then* $f$ *is lower semi-continuous.*

*(ii) If a sequence* $\{f_k\}_{k\in\mathbb{N}}$ *is monotonically decreasing, i.e.,* $f_{k+1} \le f_k$ *for all* $k \in \mathbb{N}$*, then the* Γ*-limit exists and is given by* $\Gamma - \lim_{k\to\infty} f_k = \overline{\inf_{k\in\mathbb{N}} f_k}$*.*

### 2.3.2 Embedding theorems and inequalities

In the following, we give a short collection of some inequalities and compactness results which are extensively used in the successive chapters.

**Theorem 2.3.4 (Sobolev embedding theorem)**

(i) *Into Sobolev spaces [Alt99, Chapter 8.9]. Let $m_1, m_2, p_1, p_2 \in \mathbb{R}$ be constants with $m_1 > m_2 \geq 0$, $1 \leq p_1, p_2 < \infty$, $m_1 - \frac{n}{p_1} > m_2 - \frac{n}{p_2}$. Then, there exists the compact embedding*

$$\mathrm{Id} : W^{m_1,p_1}(\Omega) \hookrightarrow W^{m_2,p_2}(\Omega).$$

*In the limiting case $m_1 - \frac{n}{p_1} = m_2 - \frac{n}{p_2}$ or $m_1 = m_2$, the embedding also exists and is continuous.*

(ii) *Into Hölder spaces [Alt99, Chapter 8.5]. Let $m, p, k, \alpha \in \mathbb{R}$ be constants with $m \geq 1$, $1 \leq p < \infty$, $k \geq 0$, $0 < \alpha < 1$, $m - \frac{n}{p} > k + \alpha$. Then, there exists the compact embedding*

$$\mathrm{Id} : W^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega}).$$

*In the limiting case $m - \frac{n}{p} = k + \alpha$, the embedding also exists and is continuous.*

**Theorem 2.3.5 (Poincaré's inequalities)**
*Let $p \geq 1$. There exists a $C > 0$ such that:*

(i) *For functions with vanishing mean value [Zie89, Theorem 4.4.2].*

$$\int_\Omega |u - \fint_\Omega u \,\mathrm{d}x|^p \,\mathrm{d}x \leq C \int_\Omega |\nabla u|^p \,\mathrm{d}x \qquad \textit{for all } u \in W^{1,p}(\Omega).$$

(ii) *For functions vanishing on a Dirichlet boundary of positive measure [Dob07, Theorem 6.22].*

$$\int_\Omega |u|^p \,\mathrm{d}x \leq C \int_\Omega |\nabla u|^p \,\mathrm{d}x \qquad \textit{for all } u \in W^{1,p}_{\Gamma_\mathrm{D}}(\Omega).$$

Combining the Sobolev embedding theorem and the Poincaré's inequalities, we obtain the so-called Sobolev-Poincaré's inequalities. For our analysis, we will use the following versions.

**Theorem 2.3.6 (Sobolev-Poincaré's inequality)**
*Let $1 \leq p < n$. There exists a constant $C > 0$ such that*

(i) *for all rectangles $Q \subseteq \mathbb{R}^n$ and all $u \in W^{1,p}(Q)$:*

$$\left(\fint_Q |u - \fint_Q u|^{p^\star}\right)^{\frac{1}{p^\star}} \leq C \left(\fint_Q |\nabla u|^p\right)^{\frac{1}{p}} (\mathrm{diam} Q),$$

(ii) *for all rectangles $Q = \prod_{i=1}^n (a_i, b_i) \subseteq \mathbb{R}^n$ and all $u \in W^{1,p}(Q)$ with $u = 0$ on $\big\{(x_1, \ldots, x_{n-1}, a_n) \,|\, a_i \leq x_i \leq b_i,\ i = 1, \ldots, n-1\big\} \subseteq \partial Q$ (in the sense of traces):*

$$\left(\fint_Q |u|^{p^\star}\right)^{\frac{1}{p^\star}} \leq C \left(\fint_Q |\nabla u|^p\right)^{\frac{1}{p}} (\mathrm{diam} Q).$$

**Remark 2.3.7** *Theorem 2.3.6 can be obtained by establishing the corresponding inequalities on the unit cube $(0,1)^n$ (by applying Theorem 2.3.4 and Theorem 2.3.5) and then using a scaling argument. It should be remarked that the case $1 < p < n$ has been considered by Sobolev [Sob38] while Nirenberg [Nir59] has studied the case $p = 1$.*

**Theorem 2.3.8 (Korn's first inequality, cf. [Nef02])**
*There exists a $C > 0$ such that for all $u \in W^{1,p}_{\Gamma_\mathrm{D}}(\Omega)$:*

$$\int_\Omega (|u|^2 + |\nabla u|^2)\,\mathrm{d}x \le C \int_\Omega |\epsilon(u)|^2\,\mathrm{d}x,$$

*where $\epsilon(u)$ is defined as $\frac{1}{2}(\nabla u + (\nabla u)^\mathrm{t})$.*

**Theorem 2.3.9 (Aubin-Lions type embeddings, cf. [Sim86])**
*Let $X \subseteq B \subseteq Y$ be Banach spaces where $X$ compactly embeds into $B$.*

*(i) For $L^p$-spaces. Let a sequence $\{f_k\}$ be bounded in $L^p(0,T;X)$ with $1 \le p \le \infty$ and the derivatives $\{\partial_t f_k\}$ be bounded in $L^p(0,T;Y)$. Then $\{f_k\}$ is relatively compact in $L^p(0,T;B)$.*

*(ii) For $\mathcal{C}$-spaces. Let a sequence $\{f_k\}$ be bounded in $L^\infty(0,T;X)$ and the derivatives $\{\partial_t f_k\}$ be bounded in $L^2(0,T;Y)$. Then $\{f_k\}$ is relatively compact in $\mathcal{C}([0,T];B)$.*

The following theorem is an adaption from [Gia83, Chapter V.1, Proposition 1.1].

**Theorem 2.3.10 (Reverse Hölder inequality, cf. [Gar00, Proposition 8.1])**
*Let $Q \subseteq \mathbb{R}^n$ be a cube, $g \in L^q_{\mathrm{loc}}(Q)$ for some $q > 1$ and $g \ge 0$. Suppose that there exist a constant $b > 0$ and a function $f \in L^r_{\mathrm{loc}}(Q)$ with $r > q$ and $f \ge 0$ such that*

$$\fint_{Q_R(x_0)} g^q\,\mathrm{d}x \le b \left( \fint_{Q_{2R}(x_0)} g\,\mathrm{d}x \right)^q + \fint_{Q_{2R}(x_0)} f^q\,\mathrm{d}x$$

*for each $x_0 \in Q$ and all $R > 0$ with $2R < \operatorname{dist}(x_0, \partial Q)$. Then $g \in L^s_{\mathrm{loc}}(Q)$ for $s \in [q, q+\varepsilon)$ with some $\varepsilon > 0$ and*

$$\left( \fint_{Q_R(x_0)} g^s\,\mathrm{d}x \right)^{\frac{1}{s}} \le c \left( \left( \fint_{Q_{2R}(x_0)} g^q\,\mathrm{d}x \right)^{\frac{1}{q}} + \left( \fint_{Q_{2R}(x_0)} f^s\,\mathrm{d}x \right)^{\frac{1}{s}} \right)$$

*for all $x_0 \in Q$ and $R > 0$ such that $Q_{2R}(x_0) \subseteq Q$. The positive constants $c, \varepsilon > 0$ depend on $b$, $q$, $n$ and $r$.*

**Theorem 2.3.11 (Conical Poincaré inequality, cf. [BK98, Corollary 2])**
*Suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded and star-shaped domain, $r \ge 0$ and $1 \le p < \infty$. Then, there exists a constant $C = C(\Omega, p, r) > 0$ such that*

$$\int_\Omega |w(x) - w_{\Omega,\delta^t}|^p \delta^r(x)\,\mathrm{d}x \le C \int_\Omega |\nabla w(x)|^p \delta^r(x)\,\mathrm{d}x$$

*for all $w \in C^1(\Omega)$ where the $\delta^r$-weight $w_{\Omega,\delta^r}$ is given by*

$$w_{\Omega,\delta^r} := \int_\Omega w(x)\delta^r(x)\,\mathrm{d}x, \quad \delta(x) := \mathrm{dist}(x, \partial\Omega).$$

**Remark 2.3.12** *By a density argument, the statement is, of course, also true for all $w \in W^{1,p}(\Omega)$ which will be used in this paper.*

The inclusion $L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \subseteq \mathcal{C}(\overline{\Omega_T})$ for $p > n$ follows from Theorem 2.3.9 (ii). It can also be shown with the following generalized version of Poincaré's inequality.

**Theorem 2.3.13 (Generalized Poincaré inequality, cf. [Alt99, Theorem 6.15])**

*Let $M \subseteq W^{1,p}(\Omega;\mathbb{R}^m)$ be non-empty, convex and closed with $1 < p < \infty$. Furthermore, $M$ satisfies the property*

$$u \in M,\ \alpha \geq 0 \implies \alpha u \in M.$$

*Then the following statements are equivalent:*

*(i) There exists a $u_0 \in M$ and a constant $C_0 > 0$ such that for all $\xi \in \mathbb{R}^m$*

$$u_0 + \xi \in M \implies |\xi| \leq C_0.$$

*(ii) There exists a constant $C > 0$ such that for all $u \in M$*

$$\|u\|_{L^p(\Omega;\mathbb{R}^m)} \leq C\|\nabla u\|_{L^p(\Omega;\mathbb{R}^{m\times n})}.$$

**Proposition 2.3.14** *Let $p > n$ be a constant. Then*

$$L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \subseteq \mathcal{C}(\overline{\Omega_T}).$$

*Proof.* Let $z \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$. We can choose a representative such that $z \in \mathcal{C}([0,T];L^2(\Omega))$ and $z(t) \in W^{1,p}(\Omega)$ for all $t \in [0,T]$. By employing the embedding $W^{1,p}(\Omega) \subseteq \mathcal{C}(\overline{\Omega})$ (note that $p > n$), we obtain a representant $z : \overline{\Omega_T} \to \mathbb{R}$ such that

$$z \in \mathcal{C}([0,T];L^2(\Omega)) \text{ and } z(t) \in \mathcal{C}(\overline{\Omega}) \text{ for all } t \in [0,T]. \tag{2.2}$$

Let $(x_m,t_m) \in \overline{\Omega_T}$ be arbitrary with $(x_m,t_m) \to (x,t)$ in $\overline{\Omega_T}$ as $m \to \infty$. We have

$$|z(x,t) - z(x_m,t_m)| \leq \underbrace{|z(x,t) - z(x,t_m)|}_{A_m} + \underbrace{|z(x,t_m) - z(x_m,t_m)|}_{B_m}.$$

Assume that $A_m \not\to 0$ as $m \to \infty$. Then, there exists a subsequence of $\{A_m\}$ (also denoted by $\{A_m\}$) such that $\lim_{m\to\infty} A_m > 0$. Using this subsequence, it holds $z(\cdot,t_m) \to z(\cdot,t)$ in $L^2(\Omega)$ due to (2.2). We obtain again a subsequence (we omit the additional subscript) such that $z(y, t_m) \to z(y, t)$ as $m \to \infty$ for a.e. $y \in \Omega$. Therefore, we can choose $y_m \to x$ in $\overline{\Omega}$ such that $|z(y_m, t) - z(y_m, t_m)| \to 0$ as $m \to \infty$. It follows

$$A_m \leq \underbrace{|z(x,t) - z(y_m,t)|}_{A_m^1} + \underbrace{|z(y_m,t) - z(y_m,t_m)|}_{A_m^2} + \underbrace{|z(y_m,t_m) - z(x,t_m)|}_{A_m^3}$$

The continuity of $z(\cdot, t)$ due to (2.2) implies $A_m^1 \to 0$ as $m \to \infty$. $A_m^2$ converges to 0 by the construction of $\{y_m\}$. To treat the term $A_m^3$, we apply the Poincaré inequality from Theorem 2.3.13 with $M := \{u \in W^{1,p}(B_1(q_0)) \,|\, u(q_0) = 0\}$ and obtain

$$\|g\|_{L^p(B_1(q_0))} \leq C\|\nabla g\|_{L^p(B_1(q_0))} \tag{2.3}$$

for all $g \in M$, where $q_0 \in \mathbb{R}^n$, and $C > 0$ is independent of $g$ and $q_0$. Note that, due to $g \in W^{1,p}(B_1(q_0)) \subseteq \mathcal{C}(\overline{B_1(q_0)})$, $g$ is pointwise defined. By utilizing (2.3) and using a scaling argument, we gain a $C > 0$ such that for all $\varepsilon > 0$ and all $g \in W^{1,p}(B_\varepsilon(q_0))$ with $g(q_0) = 0$ it follows

$$\begin{aligned} \|g\|_{\mathcal{C}(\overline{B_\varepsilon(q_0)})} = \|g(\varepsilon\cdot)\|_{\mathcal{C}(\overline{B_1(q_0)})} &\leq C\|g(\varepsilon\cdot)\|_{W^{1,p}(B_1(q_0))} \\ &\leq C\|\varepsilon\nabla g(\varepsilon\cdot)\|_{L^p(B_1(q_0))} \\ &= C\varepsilon^{\frac{p-n}{p}}\|\nabla g\|_{L^p(B_\varepsilon(q_0))}. \end{aligned}$$

By setting $g_m(\cdot) := z(y_m, t_m) - z(\cdot, t_m)$ and $\varepsilon_m := 2|y_m - x|$, we can estimate $A_m^3$ in the following way (note that $g_m(y_m) = 0$):

$$A_m^3 \leq \|g_m\|_{\mathcal{C}(\overline{B_{\varepsilon_m}(y_m)})} \leq C\varepsilon_m^{\frac{p-n}{p}}\|\nabla g_m\|_{L^p(B_{\varepsilon_m}(y_m))}.$$

Since $z \in L^\infty(0, T; W^{1,p}(\Omega)) \cap H^1(0, T; L^2(\Omega))$, $\|\nabla g_m\|_{L^p(B_{\varepsilon_m}(y_m))}$ is bounded with respect to $m$. In conclusion, $A_m^3 \to 0$ as $m \to \infty$. Hence, we end up with a contradiction. Therefore, $A_m \to 0$ as $m \to \infty$.

The convergence $B_m \to 0$ as $m \to \infty$ can be shown as for $A_m^3 \to 0$. $\square$

**Remark 2.3.15** *The inclusion in Proposition 2.3.14 is a special version of a more general compactness result in [Sim86, Corollary 5].*

### 2.3.3 Approximations of $W^{1,p}(\Omega)$ and $L^q(0, T; W^{1,p}(\Omega))$-functions

This subsection presents an approximation method that can be used for passaging to the limit of certain variational problems. To the author's best knowledge, the main result in this subsection (Lemma 2.3.18) has not been investigated in the literature. For more details concerning the application, see step (iii) from the proof of Theorem 4.2.5.

**Lemma 2.3.16 (Approximation of test functions)** *Let $p > n$ and $f, \zeta \in W_+^{1,p}(\Omega)$ with $\{\zeta = 0\} \supseteq \{f = 0\}$. Furthermore, let $\{f_M\}_{M\in\mathbb{N}} \subseteq W_+^{1,p}(\Omega)$ be a sequence with $f_M \rightharpoonup f$ in $W^{1,p}(\Omega)$ as $M \to \infty$. Then, there exist a sequence $\{\zeta_M\}_{M\in\mathbb{N}} \subseteq W_+^{1,p}(\Omega)$ and constants $\nu_M > 0$, $M \in \mathbb{N}$, such that* *(i)* $\zeta_M \to \zeta$ *in* $W^{1,p}(\Omega)$ *as* $M \to \infty$,

*(ii)* $\zeta_M \leq \zeta$ *in* $\Omega$ *for all* $M \in \mathbb{N}$,

*(iii)* $\nu_M \zeta_M \leq f_M$ *in* $\Omega$ *for all* $M \in \mathbb{N}$.

*Proof.* We give the following proof.

- Without loss of generality, we may assume $\zeta \not\equiv 0$ on $\overline{\Omega}$. Otherwise, the statement follows directly.

- Let $\{\delta_k\}$ be a sequence with $\delta_k \to 0^+$ as $k \to \infty$ and $\delta_k > 0$. Define for every $k \in \mathbb{N}$ the approximation function $\widetilde{\zeta}_k \in W^{1,p}_+(\Omega)$ as

$$\widetilde{\zeta}_k := [\zeta - \delta_k]^+,$$

  where $[\cdot]^+$ stands for $\max\{0, \cdot\}$.

- Let $0 < \alpha < 1 - \frac{n}{p}$ be a fixed constant. In the following, we use the compact embedding

$$W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$$

  due to Theorem 2.3.4 (ii). In particular, we obtain $f, f_M, \zeta, \widetilde{\zeta}_k \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ and $f_M \to f$ in $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ as $M \to \infty$.

- Set the constant $R_k$, $k \in \mathbb{N}$, to

$$R_k := \Big(\delta_k / \|\zeta\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}\Big)^{1/\alpha} > 0.$$

- We obtain the inclusion

$$\{\widetilde{\zeta}_k = 0\} \supseteq \overline{\Omega} \cap B_{R_k}(\{\zeta = 0\}). \tag{2.4}$$

  Indeed, let $x \in \overline{\Omega} \cap B_{R_k}(\{\zeta = 0\})$. Then, it follows $dist(x, \{\zeta = 0\}) < R_k$. This implies the existence of a $y \in \{\zeta = 0\}$ with $|x - y| < R_k$. Now, we can estimate as follows:

$$|\zeta(x)| = |\zeta(x) - \underbrace{\zeta(y)}_{=0}| \leq \|\zeta\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} |x - y|^\alpha < \|\zeta\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} (R_k)^\alpha = \delta_k.$$

  We end up with $x \in \{\widetilde{\zeta}_k = 0\}$.

- Taking also the assumption $\{\zeta = 0\} \supseteq \{f = 0\}$ into account, we obtain

$$\{\widetilde{\zeta}_k = 0\} \supseteq \overline{\Omega} \cap B_{R_k}(\{\zeta = 0\}) \supseteq \overline{\Omega} \cap B_{R_k}(\{f = 0\}). \tag{2.5}$$

- Since $\zeta \not\equiv 0$, it follows from $\{\zeta = 0\} \supseteq \{f = 0\}$ that $f \not\equiv 0$. Thus, if $k$ is large enough, we obtain $\overline{\Omega} \setminus B_{R_k}(\{f = 0\}) \neq \emptyset$. By possibly modifying the sequence $\{\delta_k\}$, we may assume, without loss of generality, $\overline{\Omega} \setminus B_{R_k}(\{f = 0\}) \neq \emptyset$ for all $k \in \mathbb{N}$.

- Let $k \in \mathbb{N}$ be arbitrary and fixed for the moment. Since $\overline{\Omega} \setminus B_{R_k}(\{f = 0\})$ is a compact set contained in $\{f > 0\}$ (note that $\{f > 0\} := \{x \in \overline{\Omega} \,|\, f(x) > 0\}$), it follows
$$\eta_k := \inf\{f(x) \,|\, x \in \overline{\Omega} \setminus B_{R_k}(\{f = 0\})\} > 0.$$
Due to $f_M \to f$ in $\mathcal{C}(\overline{\Omega})$, there exists an $\widetilde{M} \in \mathbb{N}$ such that for all $M \geq \widetilde{M}$:
$$f_M \geq \eta_k/2 \text{ in } \overline{\Omega} \setminus B_{R_k}(\{f = 0\}).$$

- Therefore, we find a strictly increasing sequence $\{M_k\} \subseteq \mathbb{N}$ such that for all $k \in \mathbb{N}$:
$$f_M \geq \eta_k/2 \text{ in } \overline{\Omega} \setminus B_{R_k}(\{f = 0\}) \text{ for all } M \geq M_k. \tag{2.6}$$

- For all $M \geq M_k$, we obtain $\widetilde{\nu}_k \widetilde{\zeta}_k \leq f_M$ in $\overline{\Omega}$ with the constant
$$\widetilde{\nu}_k := \eta_k/(2\|\zeta\|_{L^\infty(\Omega)}) > 0. \tag{2.7}$$
Indeed, for $x \in \overline{\Omega} \cap B_{R_k}(\{f = 0\})$, we get $x \in \{\widetilde{\zeta}_k = 0\}$ by (2.5) and, therefore,
$$\widetilde{\nu}_k \widetilde{\zeta}_k(x) = 0 \leq f_M(x).$$
In the case, $x \in \overline{\Omega} \setminus B_{R_k}(\{f = 0\})$, we can use (2.7) and (2.6), and estimate as follows:
$$\widetilde{\nu}_k \widetilde{\zeta}_k(x) = \frac{\eta_k}{2} \frac{\widetilde{\zeta}_k(x)}{\|\zeta\|_{L^\infty(\Omega)}} \leq f_M(x).$$

- The claim follows with $\zeta_M := 0$ and $\nu_k := 1$ for $M \in \{1, \ldots, M_1 - 1\}$ and $\zeta_M := \widetilde{\zeta}_{\delta_k}$ and $\nu_M := \widetilde{\nu}_k$ for each $M \in \{M_k, \ldots, M_{k+1} - 1\}$, $k \in \mathbb{N}$. □

**Remark 2.3.17** *We remark that $\{\zeta = 0\} \supseteq \{f = 0\}$ means in the context of Lemma 2.3.16*
$$\{x \in \overline{\Omega} \,|\, \zeta(x) = 0\} \supseteq \{x \in \overline{\Omega} \,|\, f(x) = 0\}$$
*by using the embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ $(p > n)$.*

**Lemma 2.3.18 (Approximation of time-dependent test functions)** *Let $p > n$, $q \geq 1$ and $f, \zeta \in L^q(0, T; W^{1,p}_+(\Omega))$ with $\{\zeta(t) = 0\} \supseteq \{f(t) = 0\}$ for a.e. $t \in (0, T)$. Furthermore, let $\{f_M\}_{M \in \mathbb{N}} \subseteq L^q(0, T; W^{1,p}_+(\Omega))$ be a sequence with $f_M(t) \rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ as $M \to \infty$ for a.e. $t \in (0, T)$. Then, there exist a sequence $\{\zeta_M\}_{M \in \mathbb{N}} \subseteq L^q(0, T; W^{1,p}_+(\Omega))$ and constants $\nu_{M,t} > 0$ such that* *(i)* $\zeta_M \to \zeta$ *in* $L^q(0,T;W^{1,p}(\Omega))$ *as* $M \to \infty$,

*(ii)* $\zeta_M \le \zeta$ *a.e. in* $\Omega_T$ *for all* $M \in \mathbb{N}$ *(in particular* $\{\zeta_M = 0\} \supseteq \{\zeta = 0\}$*),*

*(iii)* $\nu_{M,t}\zeta_M(t) \le f_M(t)$ *in* $\Omega$ *for a.e.* $t \in (0,T)$ *and for all* $M \in \mathbb{N}$.

*If, in addition,* $\zeta \le f$ *a.e. in* $\Omega_T$ *then condition (iii) can be refined to*

*(iii)'* $\zeta_M \le f_M$ *a.e. in* $\Omega_T$ *for all* $M \in \mathbb{N}$.

*Proof.* Let $\{\delta_k\}$ with $\delta_k \to 0^+$ as $k \to \infty$ and $\delta_k > 0$ be a sequence and $0 < \alpha < 1 - \frac{n}{p}$ be a fixed constant. As in the proof of the previous lemma, we use the compact embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$ in the following.

We construct the approximations $\zeta_M \in L^q(0,T;W^{1,p}_+(\Omega))$, $M \in \mathbb{N}$, as follows:

$$\zeta_M(t) := \sum_{k=1}^{M} \chi_{A_M^k}(t)[\zeta(t) - \delta_k]^+, \tag{2.8}$$

where $\chi_{A_M^k} : [0,T] \to \{0,1\}$ is defined as the characteristic function of the measurable set $A_M^k$ given by

$$A_M^k := \begin{cases} P_M^k \setminus \left(\bigcup_{i=k+1}^M P_M^i\right) & \text{if } k < M, \\ P_M^M & \text{if } k = M, \end{cases} \tag{2.9}$$

with

$$\begin{aligned} P_M^k := \Big\{ t \in [0,T] \,\big|\, &\overline{\Omega} \setminus B_{R_k(t)}(\{f(t) = 0\}) \neq \emptyset \\ &\text{and } f_M(t) \ge \eta_k(t)/2 \text{ in } \overline{\Omega} \setminus B_{R_k(t)}(\{f(t) = 0\}) \Big\}, \end{aligned} \tag{2.10}$$

where the functions $R_k : [0,T] \to [0,\infty]$ and $\eta_k : [0,T] \to [0,\infty)$ are defined by

$$\begin{aligned} R_k(t) &= \begin{cases} \left(\delta_k / \|\zeta(t)\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}\right)^{1/\alpha}, & \|\zeta(t)\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} > 0, \\ \infty, & \text{otherwise}, \end{cases} \\ \eta_k(t) &= \inf\{ f(t,x) \,|\, x \in \overline{\Omega} \setminus B_{R_k(t)}(\{f(t) = 0\})\}. \end{aligned}$$

Note that $A_M^k$, $1 \le k \le M$, are pairwise disjoint by construction.

We are going to prove that the construction of $\zeta_M$ satisfies (i)-(iii).

(i) By the assumptions, it holds for a.e. $t \in (0,T)$:

- $f_M(t) \rightharpoonup f(t)$ in $W^{1,p}(\Omega)$ as $M \to \infty$
- $\{\zeta(t) = 0\} \supseteq \{f(t) = 0\}$

Take such a $t$ and consider the case $\zeta(t) \not\equiv 0$. Then, $f(t) \not\equiv 0$.

Let $K \in \mathbb{N}$ be arbitrary but large enough such that $R_K(t) > 0$ is so small that we have $\overline{\Omega} \setminus B_{R_K(t)}(\{f(t) = 0\}) \neq \emptyset$.

Since $\overline{\Omega} \setminus B_{R_K(t)}(\{f(t) = 0\})$ is a compact set contained in $\{f(t) > 0\}$ and $f_M(t) \to f(t)$ in $\mathcal{C}(\overline{\Omega})$ as $M \to \infty$, we find an $\widetilde{M} \in \mathbb{N}$ such that

$$f_M(t) \geq \eta_K(t)/2 \text{ in } \overline{\Omega} \setminus B_{R_K(t)}(\{f(t) = 0\}) \text{ for all } M \geq \widetilde{M}.$$

In other words,

$$\forall K \in \mathbb{N} \textit{ large enough}, \exists \widetilde{M} \geq K, \forall M \geq \widetilde{M} : \; t \in P_M^K. \tag{2.11}$$

A visualization of this statement is shown Figure 2.1 and, in particular, it implies (see the definition of $A_M^k$ in (2.9)):

$$\begin{aligned} &\forall M \in \mathbb{N} \textit{ large}, \exists K \in \{1, \dots, M\} : \; t \in P_M^K, \\ &\forall M \in \mathbb{N} \textit{ large: } t \in A_M^{k_M} \textit{ with } k_M := \max\{k \in \{1, \dots, M\} \,|\, t \in P_M^k\}. \end{aligned} \tag{2.12}$$

Next, we will prove that the properties (2.11) and (2.12) imply $k_M \to \infty$ as $M \to \infty$.

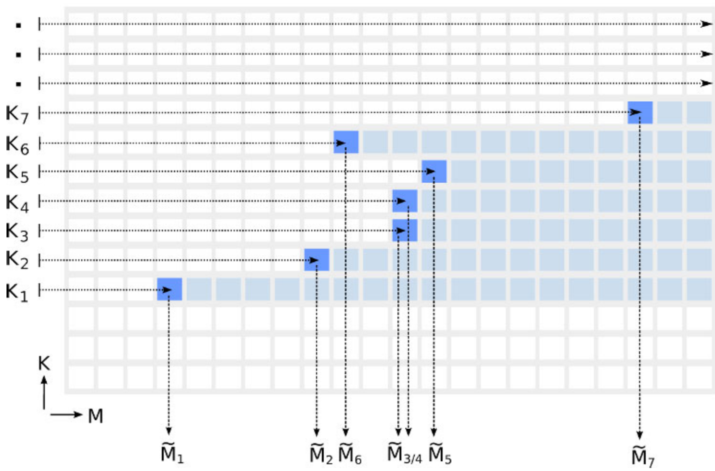


**Figure 2.1:** *An example to illustrate statement* (2.11)*: a filled box in this matrix at position* $(M, K) \in \mathbb{N} \times \mathbb{N}$ *indicates* $t \in P_M^K$*, otherwise* $t \notin P_M^K$.

- In fact, for every large $K \in \mathbb{N}$, we find an $\widetilde{M} \geq K$ such that $t \in P_M^K$ for all $M \geq \widetilde{M}$. Then, by using (2.12), we obtain for every $M \geq \widetilde{M}$:

$$k_M = \max\{k \in \{1, \ldots, M\} \,|\, t \in P_M^k\} \geq K.$$

  Consequently, $k_M \to \infty$ as $M \to \infty$.

Thus $\delta_{k_M} \to 0^+$ as $M \to \infty$.

Furthermore, $t \in A_M^{k_M}$ shows $\zeta_M(t) = [\zeta(t) - \delta_{k_M}]^+$ for every large $M \in \mathbb{N}$ by (2.8). We end up with $\zeta_M(t) \to \zeta(t)$ in $W^{1,p}(\Omega)$ as $K \to \infty$.

Taking also the estimate $\|\zeta_M(t)\|_{W^{1,p}(\Omega)} \leq \|\zeta(t)\|_{W^{1,p}(\Omega)}$ (follows from (2.8)) for all $M$ and a.e. $t$ into account, Lebesgue's convergence theorem shows

$$\int_0^T \|\zeta_M(t) - \zeta(t)\|^p_{W^{1,p}(\Omega)} \,\mathrm{d}t \to 0 \text{ as } M \to \infty.$$

(ii) Property (ii) follows from (2.8) and that $A_M^k$, $k = 1, \ldots, M$, are pairwise disjoint.

(iii) Let $t$ be as in (i). Since $A_M^k$, $k = 1, \ldots, M$, are pairwise disjoint, the definition in (2.8) implies for every $M \in \mathbb{N}$ one of the two alternatives:

- $\zeta_M(t) = 0$,
- there exists an $k \in \{1, \ldots, M\}$ such that $\zeta_M(t) = [\zeta(t) - \delta_k]^+$ and $t \in A_M^k$.

In the first case, the estimate in (iii) is fulfilled for any value $\nu_{M,t} > 0$.

In the second case, we can argue as in the proof of Lemma 2.3.16. More precisely, we obtain with the same argumentation

$$\{\zeta_M(t) = 0\} = \{[\zeta(t) - \delta_k]^+ = 0\} \supseteq \overline{\Omega} \cap B_{R_k(t)}(\{f(t) = 0\})$$

and, since $t \in P_M^k$, we obtain

$$f_M(t) \geq \eta_k(t)/2 \text{ in } \overline{\Omega} \setminus B_{R_k(t)}(\{f(t) = 0\}).$$

Now, (iii) is fulfilled with

$$\nu_{M,t} := \eta_k(t)/(2\|\zeta(t)\|_{L^\infty(\Omega)}) > 0.$$

In the case $\zeta \leq f$, we use instead of (2.10) the set

$$P_M^k := \Big\{t \in [0, T] \,\Big|\, \|f_M(t) - f(t)\|_{\mathcal{C}(\overline{\Omega})} \leq \delta_k\Big\}.$$

With a similar argumentation, $\{\zeta_M\}$ fulfills (i), (ii) and (iii)'. □

### 2.3.4 Characterization of certain variational inequalities

Here, we present a technique of how one can drop a certain restriction on the space of test functions for some types of variational inequalities. To the author's best knowledge, the result in this subsection has not been investigated in the literature. In combination with the approximation technique in Subsection 2.3.3, we obtain a tool for establishing coupled variational inequalities arising from (1.1c). For more details, see step (iv) from the proof of Theorem 4.2.5.

In the following, the spelling "$\{f = 0\} \supseteq \{z = 0\}$ in an a.e. sense" means that there exists a subset $N \subseteq \Omega$ with $\mathcal{L}^n(N) = 0$ such that $\big(\{f = 0\} \setminus N\big) \supseteq \big(\{z = 0\} \setminus N\big)$.

**Lemma 2.3.19** *Let $p > n$ and let $f \in L^{p/(p-1)}(\Omega;\mathbb{R}^n)$, $g \in L^1(\Omega)$ and $z \in W^{1,p}_+(\Omega)$ with $f \cdot \nabla z \geq 0$ a.e. in $\Omega$ and $\{f = 0\} \supseteq \{z = 0\}$ in an a.e. sense. Furthermore, we assume that*

$$\int_\Omega \big(f \cdot \nabla\zeta + g\zeta\big)\,\mathrm{d}x \geq 0 \quad \textit{for all } \zeta \in W^{1,p}_-(\Omega) \textit{ with } \{\zeta = 0\} \supseteq \{z = 0\}. \tag{2.13}$$

*Then*

$$\int_\Omega \big(f \cdot \nabla\zeta + g\zeta\big)\,\mathrm{d}x \geq \int_{\{z=0\}} [g]^+\zeta\,\mathrm{d}x \quad \textit{for all } \zeta \in W^{1,p}_-(\Omega). \tag{2.14}$$

*Proof.* We assume $z \not\equiv 0$ in $\Omega$. Let $\zeta \in W^{1,p}_-(\Omega)$ be a test function. If $\delta > 0$ is small enough we obtain

$$\overline{\Omega} \setminus B_\delta(\{z = 0\}) \neq \emptyset, \tag{2.15}$$

since $z \in \mathcal{C}(\overline{\Omega})$ due to the embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$. Then, we can define the approximation $\zeta_\delta \in W^{1,p}_-(\Omega)$ by

$$\zeta_\delta := \max\big\{\zeta, -z\|\zeta\|_{L^\infty} C_\delta^{-1}\big\} \tag{2.16}$$

with the constant

$$C_\delta := \inf\big\{z(x)\,|\,x \in \overline{\Omega} \setminus B_\delta(\{z = 0\})\big\}\,. \tag{2.17}$$

It holds $C_\delta > 0$ due to $z \geq 0$, property (2.15) and the continuity of $z$. We consider the following partition of $\overline{\Omega}$:

$$\overline{\Omega} = \Sigma_1^\delta \cup \Sigma_2^\delta \cup \Sigma_3^\delta$$

with

$$\begin{aligned}
&\Sigma_1^\delta := \overline{\Omega} \setminus B_\delta(\{z = 0\}),\\
&\Sigma_2^\delta := \overline{\Omega} \cap B_\delta(\{z = 0\}) \cap \{\zeta \leq -z\|\zeta\|_{L^\infty} C_\delta^{-1}\},\\
&\Sigma_3^\delta := \overline{\Omega} \cap B_\delta(\{z = 0\}) \cap \{\zeta > -z\|\zeta\|_{L^\infty} C_\delta^{-1}\}.
\end{aligned}$$

By the definition (2.16) with the constant (2.17), the sequence $\{\zeta_\delta\}_{\delta\in(0,1)}$ satisfies

$$\zeta_\delta(x) = \begin{cases} \zeta(x), & \text{if } x \in \Sigma_1^\delta \cup \Sigma_3^\delta, \\ -z(x)\|\zeta\|_{L^\infty} C_\delta^{-1}, & \text{if } x \in \Sigma_2^\delta, \end{cases} \tag{2.18}$$

as well as (in an a.e. sense)

$$\nabla\zeta_\delta(x) = \begin{cases} \nabla\zeta(x), & \text{if } x \in \Sigma_1^\delta \cup \Sigma_3^\delta, \\ -\nabla z(x)\|\zeta\|_{L^\infty} C_\delta^{-1}, & \text{if } x \in \Sigma_2^\delta. \end{cases} \tag{2.19}$$

From (2.16) and $\zeta \le 0$ in $\Omega$, we infer that $\{\zeta_\delta = 0\} \supseteq \{z = 0\}$ and

$$-\int_{\{z=0\}} [g]^+\zeta \,\mathrm{d}x \ge -\int_{\{z=0\}} g\zeta \,\mathrm{d}x = -\int_{\{z=0\}} g(\zeta - \zeta_\delta)\,\mathrm{d}x. \tag{2.20}$$

We calculate

$$\begin{aligned}
&\int_\Omega \big(f\cdot\nabla\zeta + g\zeta\big)\,\mathrm{d}x - \int_{\{z=0\}} [g]^+\zeta\,\mathrm{d}x \\
&\quad= \int_\Omega \big(f\cdot\nabla(\zeta-\zeta_\delta) + g(\zeta-\zeta_\delta)\big)\,\mathrm{d}x + \underbrace{\int_\Omega \big(f\cdot\nabla\zeta_\delta + g\zeta_\delta\big)\,\mathrm{d}x}_{\ge 0 \text{ by } (2.13)} \underbrace{- \int_{\{z=0\}} [g]^+\zeta\,\mathrm{d}x}_{\text{apply } (2.20)} \\
&\quad\ge \int_\Omega \big(f\cdot\nabla(\zeta-\zeta_\delta) + g(\zeta-\zeta_\delta)\big)\,\mathrm{d}x - \int_{\{z=0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x \\
&\quad= \int_\Omega f\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x + \int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x \\
&\quad= \underbrace{\int_{\Sigma_1^\delta\cup\Sigma_3^\delta} f\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x}_{=0 \text{ by } (2.19)} + \int_{\Sigma_2^\delta} f\cdot\nabla(\zeta-\zeta_\delta)\,\mathrm{d}x + \int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x \\
&\quad= \int_{\Sigma_2^\delta} f\cdot\nabla\zeta\,\mathrm{d}x - \underbrace{\int_{\Sigma_2^\delta} f\cdot\nabla\zeta_\delta\,\mathrm{d}x}_{\text{using } (2.19)} + \int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x \\
&\quad= \int_{\Sigma_2^\delta} f\cdot\nabla\zeta\,\mathrm{d}x + \|\zeta\|_{L^\infty} C_\delta^{-1} \underbrace{\int_{\Sigma_2^\delta} f\cdot\nabla z\,\mathrm{d}x}_{\ge 0 \text{ by assumption}} + \int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x \\
&\quad= \underbrace{\int_{\Sigma_2^\delta} f\cdot\nabla\zeta\,\mathrm{d}x}_{\text{using } \{f=0\}\supseteq\{z=0\} \text{ a.e.}} + \underbrace{\int_{\{z>0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x}_{\text{using } (2.18)} \\
&\quad= \int_{\Sigma_2^\delta\setminus\{z=0\}} f\cdot\nabla\zeta\,\mathrm{d}x + \int_{\Sigma_2^\delta\setminus\{z=0\}} g(\zeta-\zeta_\delta)\,\mathrm{d}x
\end{aligned} \tag{2.21}$$

The two terms on the right hand side can be treated as follows:

- The set $\{z=0\}$ is closed because $z$ is continuous. Therefore, we obtain

$$\bigcap_{\delta>0} B_\delta(\{z=0\}) = \{z=0\}$$

and, consequently,

$$\bigcap_{\delta>0} \big(B_\delta(\{z=0\}) \setminus \{z=0\}\big) = \emptyset.$$

The monotonicity of the measure $\mathcal{L}^n$ yields $\mathcal{L}^n\big(B_\delta(\{z=0\}) \setminus \{z=0\}\big) \to 0$ as $\delta \to 0^+$. This implies

$$\mathcal{L}^n\big(\Sigma_2^\delta \setminus \{z=0\}\big) \le \mathcal{L}^n\big(B_\delta(\{z=0\}) \setminus \{z=0\}\big) \to 0 \tag{2.22}$$

and we end up with

$$\int_{\Sigma_2^\delta \setminus \{z=0\}} f \cdot \nabla \zeta \,\mathrm{d}x \to 0 \tag{2.23}$$

as $\delta \to 0^+$.

- Since $\zeta \le 0$ and $z \ge 0$, it follows from the definition of $\zeta_\delta$ in (2.16) that $\zeta \le \zeta_\delta \le 0$. Therefore, $\zeta_\delta$ is uniformly bounded in the $L^\infty(\Omega)$ norm and we can argue as follows by taking (2.22) into account:

$$\left| \int_{\Sigma_2^\delta \setminus \{z=0\}} g(\zeta - \zeta_\delta) \,\mathrm{d}x \right| \le C \int_{\Sigma_2^\delta \setminus \{z=0\}} |g| \,\mathrm{d}x \to 0 \tag{2.24}$$

as $\delta \to 0^+$.

From (2.21) and (2.23) and (2.24), we infer the claim (2.14). □

**Remark 2.3.20** *Let us remark that we suppose $\{f=0\} \supseteq \{z=0\}$ only in an a.e. sense in Lemma 2.3.19.*

## 2.4 Shrinking sets and admissible subsets

The aim of this section is to prove covering and representation results for shrinking sets as well as to introduce local Sobolev spaces on shrinking sets. We call a space-time subset $G \subseteq \overline{\Omega_T}$ shrinking if $G$ is relatively open in $\overline{\Omega_T}$ and $G(s) \subseteq G(t)$ holds whenever $0 \le t \le s \le T$. Here, the $t$-cut of $G$ at time $t \in [0,T]$, i.e., $G \cap (\overline{\Omega} \times \{t\})$, is denoted by $G(t) := \{x \in \overline{\Omega} \,|\, (x,t) \in G\}$.

### 2.4.1 Covering and representation results

In the following, we are going to study shrinking sets $G$. They will appear in the analysis of the complete damage systems in Chapter 6 and Chapter 7 as subsets of $\overline{\Omega_T}$ where the damage is not complete. Due to the possibly bad smoothness property of $G$, we will need to represent certain parts of $G$ as a countable union of Lipschitz domains. In this context, it is convenient to introduce the notion of *fine representation* and of *admissible subset.*

**Definition 2.4.1 (Fine representation)** *Let $H \subseteq \overline{\Omega}$ be a relatively open subset. We call a countable family $\{U_k\}$ of open sets $U_k \subset\subset H$ a fine representation for $H$ if for every $x \in H$ there exist an open set $U \subseteq \mathbb{R}^n$ with $x \in U$ and an $k \in \mathbb{N}$ such that $\big(U \cap \Omega\big) \subseteq U_k$.*

**Remark 2.4.2** *See Figure 2.2 for an example. Note that $H \cap \partial\Omega$ is not covered by $\{U_k\}$.*

For a given relatively open subset of $\overline{\Omega}$, we will be interested in the subsets where every path-connected component is connected to the Dirichlet boundary $\Gamma_{\mathrm{D}}$.

**Definition 2.4.3 (Admissible subsets of $\overline{\Omega}$ with respect to $\Gamma_{\mathrm{D}}$)**

*(i) Let $F \subseteq \overline{\Omega}$ be a relatively open subset and*

$$P_F(x) := \big\{ y \in F \,|\, x \text{ and } y \text{ are connected by a path in } F \,\big\}$$

*for $x \in F$. We say that $F$ is admissible with respect to the Dirichlet boundary $\Gamma_{\mathrm{D}}$ if for every $x \in F$ the condition*

$$\mathcal{H}^{n-1}(P_F(x) \cap \Gamma_{\mathrm{D}}) > 0$$

*is fulfilled. Furthermore, $\mathfrak{A}_{\Gamma_{\mathrm{D}}}(F)$ denotes the maximal admissible subset of $F$ with respect to $\Gamma_{\mathrm{D}}$, i.e.,*

$$\mathfrak{A}_{\Gamma_{\mathrm{D}}}(F) := \bigcup \{ G \subseteq F \,|\, G \text{ is admissible with respect to } \Gamma_{\mathrm{D}} \} \,.$$

*(ii) For a relatively open subset $F \subseteq \overline{\Omega_T}$, the set $\mathfrak{A}_{\Gamma_{\mathrm{D}}}(F)$ is given by $(\mathfrak{A}_{\Gamma_{\mathrm{D}}}(F))(t) := \mathfrak{A}_{\Gamma_{\mathrm{D}}}(F(t))$.*

In the remaining part of this subsection, we are going to prove certain covering and representation results.

**Lemma 2.4.4 (Finite covering)** *Let $G \subseteq \overline{\Omega_T}$ be a relatively open subset and the sequence $\{t_m\}$ containing $T$ be dense in $[0,T]$. Furthermore, let $\{U_k^m\}_{k\in\mathbb{N}}$ be a fine representation for $G(t_m)$ for every $m \in \mathbb{N}$. Then, for every compact set $K \subseteq G$ there exist a finite set $I \subseteq \mathbb{N}$ and values $m_k \in \mathbb{N}$, $k \in I$, such that $K \cap \Omega_T \subseteq \bigcup_{k\in I} \big(U_k^{m_k} \times (0, t_{m_k})\big)$.*

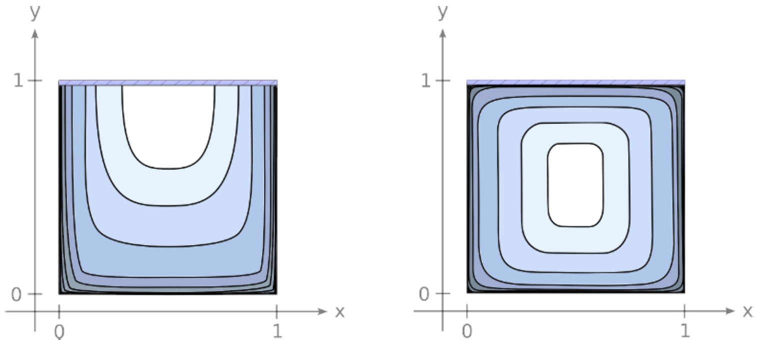


**Figure 2.2:** *Left: a fine representation for the relatively open subset* $H = (0,1) \times (0,1]$ *of* $\overline{\Omega} = [0,1] \times [0,1]$*; Right: not a fine representation for* $H$*.*

*Proof.* To every element $p = (x,t) \in K$, we will construct a neighborhood $\Theta_p \subseteq \overline{\Omega_T}$ of $p$ in the subspace topology of $\overline{\Omega_T}$ such that there exists $k, m \in \mathbb{N}$ with $\Theta_p \cap \Omega_T \subseteq U_k^m \times (0, t_m)$. Then the claim follows by the Heine-Borel theorem.

Indeed, to every $p = (x,t) \in K$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(p) \cap \overline{\Omega_T} \subseteq G$ since $G \subseteq \overline{\Omega_T}$ is relatively open. Therefore, if $t < T$, $(x, t_m) \in G$ for all $m \in \mathbb{N}$ such that $t < t_m < t + \varepsilon$. This implies $(x, t_m) \in G \cap (\overline{\Omega} \times \{t_m\}) = G(t_m) \times \{t_m\}$. Then, we find $p \in G(t_m) \times J$ with $J = [0, t_m)$. In the case $t = T$, it holds $p \in G(T) \times J$ with $J = [0, T]$. Since $\{U_k^m\}_{k \in \mathbb{N}}$ is a fine representation of $G(t_m)$, let $\delta > 0$ such that $B_\delta(x) \cap \Omega \subseteq U_k^m$ for some $k \in \mathbb{N}$. Finally, $\Theta_p := (B_\delta(x) \cap \overline{\Omega}) \times J$ is the required neighborhood of $p$. □

**Lemma 2.4.5 (Partition of unity property)** *Let* $G$, $\{t_m\}$ *and* $\{U_k^m\}$ *be as in Lemma 2.4.4. Then, for every compact subset* $K \subseteq G$ *there exist a finite set* $I \subseteq \mathbb{N}$*, values* $m_k \in \mathbb{N}$*,* $k \in I$ *and functions* $\psi_k \in \mathcal{C}^\infty(\overline{\Omega_T})$*,* $k \in I$*, such that*

*(i)* $K \cap \Omega_T \subseteq \bigcup_{k \in I} \left( U_k^{m_k} \times (0, t_{m_k}) \right)$,

*(ii)* $\operatorname{supp}(\psi_k) \subseteq \overline{U_k^{m_k}} \times [0, t_{m_k}]$,

*(iii)* $\sum_{k \in I} \psi_k \equiv 1$ *on* $K$.

*Proof.* We extend the family of open sets $\{V_k^m\}$ given by $V_k^m := U_k^m \times (0, t_{m_k})$ in the following way. Define

$$\mathcal{P} := \Big\{ \{W_k^m\} \,\Big|\, W_k^m \subseteq \mathbb{R}^{n+1} \text{ is open with } W_k^m \cap \Omega_T = U_k^m \times (0, t_{m_k}) \Big\}.$$

We see that $\mathcal{P}$ is non-empty and every totally ordered subset of $\mathcal{P}$ has an upper bound with respect to the "$\leq$" ordering defined by

$$\{W_k^m\} \leq \{\widetilde{W}_k^m\} \Leftrightarrow W_k^m \subseteq \widetilde{W}_k^m \text{ for all } k, m \in \mathbb{N}.$$

By Zorn's lemma, we find a maximal element $\{\widetilde{V}_k^m\}$. It holds

$$G \subseteq \bigcup_{k,m\in\mathbb{N}} \widetilde{V}_k^m. \tag{2.25}$$

Assume that this condition fails. Because of $G \cap \Omega_T = \bigcup_{k,m\in\mathbb{N}} V_k^m$, there exists a $p = (x,t) \in G \cap \partial(\Omega_T)$ with $p \notin \bigcup_{k,m\in\mathbb{N}} \widetilde{V}_k^m$.

Let us consider the case $t < T$. Since $F \subseteq \overline{\Omega_T}$ is relatively open, we find an $m_0 \in \mathbb{N}$ with $x \in G(t_{m_0})$ and $t_{m_0} > t$. By the fine representation property of $\{U_k^{m_0}\}_{k\in\mathbb{N}}$ for $G(t_{m_0})$, we find an open set $U \subseteq \mathbb{R}^n$ with $x \in U$ and $k_0 \in \mathbb{N}$ such that $U \cap \Omega \subseteq U_{k_0}^{m_0}$.

The family $\{\widetilde{W}_k^m\}$ given by

$$\widetilde{W}_k^m := \begin{cases} \widetilde{V}_k^m \cup U \times (-\infty, t_{m_0}) & \text{if } k = k_0 \text{ and } m = m_0, \\ \widetilde{V}_k^m & \text{else,} \end{cases}$$

satisfies $\{\widetilde{W}_k^m\} \in \mathcal{P}$ and $p \in \bigcup_{k,m\in\mathbb{N}} \widetilde{W}_k^m$ which contradicts the maximality property of $\{\widetilde{V}_k^m\}$.

In the case $t = T$, we also find $k_0, m_0 \in \mathbb{N}$ and an open set $U \subseteq \mathbb{R}^n$ with $x \in U$ such that $U \cap \Omega \subseteq U_{k_0}^{m_0}$ and $t_{m_0} = T$. The family $\{\widetilde{W}_k^m\}$ given by

$$\widetilde{W}_k^m := \begin{cases} \widetilde{V}_k^m \cup U \times \mathbb{R} & \text{if } k = k_0 \text{ and } m = m_0, \\ \widetilde{V}_k^m & \text{else,} \end{cases}$$

also contradicts the maximality of $\{\widetilde{V}_k^m\}$. Therefore, (2.25) is proven.

Heine-Borel theorem yields

$$K \subseteq \bigcup_{k\in I} \widetilde{V}_k^{m_k}$$

for a finite set $I \subseteq \mathbb{N}$ and values $m_k \in \mathbb{N}$, $k \in I$. Together with a partition of unity argument, we get functions $\psi_k \in \mathcal{C}^\infty(\overline{\Omega_T})$ such that (i)-(iii) hold. $\square$

If a relatively open set $H \subseteq \overline{\Omega}$ is admissible with respect to $\Gamma_{\mathrm{D}}$ we can construct a fine representation for $H$ with Lipschitz domains in the following sense.

**Lemma 2.4.6 (Lipschitz representation of admissible sets)** *Let $H \subseteq \overline{\Omega}$ be relatively open and admissible with respect to $\Gamma_{\mathrm{D}}$. Then, there exists a fine representation $\{U_m\}$ for $H$ such that*

*(i) $U_m$ is a Lipschitz domain for all $m \in N$,*

*(ii) $\mathcal{H}^{n-1}(\partial U_m \cap \Gamma_{\mathrm{D}}) > 0$ for all $m \in N$.*

*Proof.* We will sketch a possible construction for reader's convenience.

We assume WLOG that $H$ is path-connected because $H$ can only have at most countably many path-connected components and for each component we can apply the construction below.

Let us choose a reference point $x_0 \in \Gamma_\mathrm{D} \cap H$ with the property

$$\mathcal{H}^{n-1}\Big(\partial\big(B_\varepsilon(x_0) \cap \Omega\big) \cap \Gamma_\mathrm{D}\Big) > 0 \text{ for all } \varepsilon > 0, \tag{2.26}$$

which is possible since $\mathcal{H}^{n-1}(\Gamma_\mathrm{D} \cap H) > 0$. The relatively open subset $D_m \subseteq \overline{\Omega}$ for $m \in \mathbb{N}$ is defined as

$$D_m := H \setminus \overline{B_{1/m}(\overline{\Omega} \setminus H)}.$$

If $m$ is large enough we have $x_0 \in D_m$ since $H \subseteq \overline{\Omega}$ is relatively open. We define

$$D'_m := \{x \in D_m \,|\, x \text{ is path-connected to } x_0 \text{ in } D_m\}.$$

Hence, we obtain an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \cap \overline{\Omega} \subseteq D'_m$ since $D'_m$ is relatively open in $\overline{\Omega}$. In combination with (2.26), this yields $\mathcal{H}^{n-1}(\partial D'_m \cap \Gamma_\mathrm{D}) > 0$. Because of $D'_m \subset\subset H$, there exists a Lipschitz domain $U_m \subseteq \Omega$ with $D'_m \subseteq \overline{U_m} \subseteq H$ (e.g. the part of the boundary $\partial U_m \setminus \partial\Omega$ of $U_m$ can be constructed by polygons such that $\partial U_m$ fulfills the Lipschitz boundary condition, see Figure 2.3). The family $\{U_m\}$ satisfies all the desired properties.

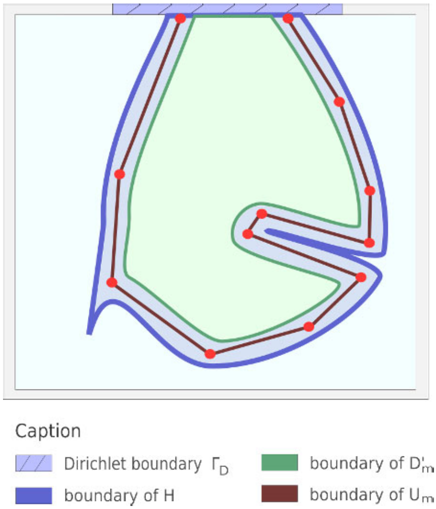


**Figure 2.3:** *Visualization of the construction of $U_m$ in 2D.*

□

**Corollary 2.4.7** *Let $G \subseteq \overline{\Omega_T}$ be a shrinking set where $G(t)$ is admissible with respect to $\Gamma_\mathrm{D}$ for all $t \in [0, T]$. Furthermore, let $\{t_m\} \subseteq [0, T]$ be a dense sequence containing $T$.*

*Then, there exists a countable family $\{U_k^m\}_{k\in\mathbb{N}}$ of Lipschitz domains $U_k^m \subset\subset G(t_m)$ for each $m \in \mathbb{N}$ such that*

*(i)* $\mathcal{H}^{n-1}(\partial U_k^m \cap \Gamma_\mathrm{D}) > 0$ *for all* $m \in \mathbb{N}$,

*(ii)* $\{U_k^m\}_{k\in\mathbb{N}}$ *is a fine representation for* $G(t_m)$ *for all* $m \in \mathbb{N}$,

*(iii)* $G = \bigcup_{m=1}^\infty G(t_m) \times [0, t_m]$.

### 2.4.2 Local Sobolev spaces on shrinking sets

Given a shrinking set $G$, the space of local Sobolev functions on $G$ which are of $L^2(H^q)$-type will be introduce. This space will appear in weak formulations of the complete damage approach in Chapter 6 and Chapter 7.

**Definition 2.4.8 (Space-time local Sobolev functions)** *Let $N \in \mathbb{N}$, $q \geq 1$ and $G \subseteq \overline{\Omega_T}$ be a shrinking set. Define*

$$L_t^2 H_{x,\mathrm{loc}}^q(G;\mathbb{R}^N) := \Big\{ v : G \to \mathbb{R}^N \;\Big|\; \forall t \in (0,T],\ \forall U \subset\subset G(t) \text{ open}: \\ v|_{U\times(0,t)} \in L^2(0,t;H^q(U;\mathbb{R}^N)) \Big\}. \tag{2.27}$$

*As usual, we set $L_t^2 H_{x,\mathrm{loc}}^q(G) := L_t^2 H_{x,\mathrm{loc}}^q(G;\mathbb{R})$.*

**Remark 2.4.9** *(i) Note that we do not demand that $G$ is an open set.*

*(ii) Applying Lemma 2.4.5 (partition of unity) shows that $L_t^2 H_{x,\mathrm{loc}}^0(G;\mathbb{R}^N)$ coincides with $L_{\mathrm{loc}}^2(G;\mathbb{R}^N)$, where $L_{\mathrm{loc}}^2(G;\mathbb{R}^N)$ denotes the classical local $L^2$-Lebesgue space on $G$ given by*

$$L_{\mathrm{loc}}^2(G;\mathbb{R}^N) := \Big\{ v : G \to \mathbb{R}^N \;\Big|\; v|_V \in L^2(V;\mathbb{R}^N) \text{ for all open } V \subset\subset G \Big\}.$$

*This can be seen as follows. The inclusion*

$$L_{\mathrm{loc}}^2(G;\mathbb{R}^N) \subseteq L_t^2 H_{x,\mathrm{loc}}^0(G;\mathbb{R}^N)$$

*follows from Definition 2.4.8 since $\big(U \times (0,t)\big) \subset\subset G$ for every $U \subset\subset G(t)$. Now, let $v \in L_t^2 H_{x,\mathrm{loc}}^0(G;\mathbb{R}^N)$ and $V \subset\subset G$ be arbitrary. Furthermore, let $\{U_k^m\}$ be as in Corollary 2.4.7.*

*By Lemma 2.4.5 applied to $K = \overline{V}$, we find a finite covering $\{U_k^{m_k} \times (0,t_{m_k})\}_{k\in I}$ of $K \cap \Omega_T$. For each $k \in I$, it holds $v|_{U_k^{m_k}\times(0,t_{m_k})} \in L^2(0,t_{m_k};H^0(U_k^{m_k};\mathbb{R}^N))$ by (2.27). Using the partition of unity yields $v|_V \in L^2(V;\mathbb{R}^N)$. Thus $v \in L_{\mathrm{loc}}^2(G;\mathbb{R}^N)$.*

*(iii) At fixed time points $t \in (0,T)$, we find $v(t) \in H_{\mathrm{loc}}^q(G(t);\mathbb{R}^N)$.*

Given $v \in L_t^2 H_{x,\mathrm{loc}}^q(G;\mathbb{R}^N)$, we say that $v = b$ on $(\Gamma_{\mathrm{D}})_T \cap G$ if for every $t \in (0,T)$ and every open set $U \subset\subset G(t)$ with Lipschitz boundary

$$\widetilde{v}(s) = b(s) \text{ on } \partial U \cap \Gamma_{\mathrm{D}} \text{ in the sense of traces for a.e. } s \in (0,t), \tag{2.28}$$

is fulfilled with $\widetilde{v} := v|_{U\times(0,t)} \in L^2(0,t;H^1(U;\mathbb{R}^N))$.

We write $\nabla v$ for the weak derivative with respect to the spatial variable as well as $\epsilon(v) := \frac{1}{2}(\nabla v + (\nabla v)^{\mathrm{t}})$ for its symmetric part. The precise definition and characterization of $\nabla v$ can be found in the following proposition.

**Proposition 2.4.10** *Let $G \subseteq \overline{\Omega_T}$ be a shrinking subset and let $\{t_m\}$ and $\{U_k^m\}$ be as in Lemma 2.4.4. Furthermore, let $v : G \to \mathbb{R}^N$ be a function.*

*(a) The following statements are equivalent:*

*(i) $v \in L_t^2 H_{x,\mathrm{loc}}^1(G;\mathbb{R}^N)$*

*(ii) $v|_{U_k^m \times (0,t_m)} \in L^2(0,t_m; H^1(U_k^m;\mathbb{R}^N))$ for all $k, m \in \mathbb{N}$*

*(iii) $v \in L^2_{\mathrm{loc}}(G;\mathbb{R}^N)$ and there exists a function $g \in L^2_{\mathrm{loc}}(G;\mathbb{R}^{N\times n})$ such that*

$$\int_G v \cdot \operatorname{div}(\zeta)\,\mathrm{d}x\,\mathrm{d}t = -\int_G g : \zeta\,\mathrm{d}x\,\mathrm{d}t \tag{2.29}$$

*for all $\zeta \in \mathcal{C}_c^\infty(\operatorname{int}(G);\mathbb{R}^{N\times n})$*

*If one of these conditions is satisfied we write $\nabla v := g$ and $\epsilon(v) := \frac{1}{2}(\nabla v + (\nabla v)^{\mathrm{t}})$.*

*(b) Assume that each $U_k^m$ has a Lipschitz boundary. Then the following statements are equivalent:*

*(i) $v = b$ on the boundary $D_T \cap G$*

*(ii) for every $k, m \in \mathbb{N}$, condition (2.28) is satisfied for $U = U_k^m$ and $t = t_m$*

*Proof.*

(a) (i)$\Longrightarrow$(ii) and (iii)$\Longrightarrow$(i) are trivial.

(ii)$\Longrightarrow$(iii): Let the function $\widehat{g} : G \to \mathbb{R}^{N\times n}$ be $\mathcal{L}^{n+1}$-a.e. defined as follows. For each $k, m \in \mathbb{N}$, we set $\widehat{g}|_{U_k^m} := \widehat{g}_k^m$ where $\widehat{g}_k^m \in L^2(U_k^m \times (0,t_m);\mathbb{R}^{N\times n})$ is the weak derivative of $v|_{U_k^m\times(0,t_m)}$. The function $\widehat{g}$ is well-defined on $G \cap \Omega_T$ since

$$G \cap \Omega_T = \bigcup_{k,m\in\mathbb{N}} U_k^m \times (0,t_m)$$

and $\widehat{g}_{k_1}^{m_1} = \widehat{g}_{k_2}^{m_2}$ on $U_{k_1}^{m_1} \times (0,t_{m_1}) \cap U_{k_2}^{m_2} \times (0,t_{m_2})$ for all $k_1, k_2, m_1, m_2 \in \mathbb{N}$ in an $\mathcal{L}^{n+1}$-a.e. sense. Let $t \in (0,T]$ and $U \subset\subset G(t)$ be open. By Lemma 2.4.4, $U \times (0,t)$ can be covered by finitely many sets $U_k^m \times (0,t_m)$. In particular, $\widehat{g}|_{U\times(0,t)} \in L^2(0,t;L^2(U;\mathbb{R}^{N\times n}))$. Thus $\widehat{g} \in L^2_{\mathrm{loc}}(G;\mathbb{R}^{N\times n}))$.

Let $\zeta \in \mathcal{C}_c^\infty(\operatorname{int}(G);\mathbb{R}^{N\times n})$. Applying Lemma 2.4.4 again, there exists a finite set $I \subseteq \mathbb{N}$ such that $\operatorname{supp}(\zeta) \subseteq \bigcup_{k\in I} U_k^{m_k} \times (0,t_{m_k}) =: U$. By a partition of unity argument over $U$, (2.29) holds for $g = \widehat{g}$.

(b) (ii)$\Longrightarrow$(i): Let $t \in (0,T)$ and $U \subset\subset G(t)$ be an arbitrary open subset. By Lemma 2.4.4, we find a finite set $I \subseteq \mathbb{N}$ such that $U \subseteq \bigcup_{k\in I} U_k^{m_k}$ and $t_{m_k} \geq t$. The claim follows. $\square$

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