

Some Recent Developments in Systems and Control Theory on Infinite Dimensional Banach Space: Part 2

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Abstract The objective of this chapter is to present some recent developments in systems and control theory on infinite dimensional spaces. In part 1, we considered evolution equations with mild solutions and presented some results on optimal control. This is part 2 and here we consider evolution equations which do not admit mild solutions but have measure solutions. We introduce and use the notion of measure valued solutions for both deterministic and stochastic systems. Following this we consider the question of existence of optimal feedback controls and formulate several interesting control problems on the space of measures. This is then applied to stochastic Navier–Stokes equations and its optimal feedback control and nonlinear filtering problem aimed at finding from an admissible class the best operator modeling the observation equation. In the final section we consider hybrid systems driven by vector measures and operator valued measures which include impulsive systems as special cases and present some results asserting existence of optimal measure valued controls.

Keywords Optimal feedback control · Measure solutions and optimal control · Stochastic Navier–Stokes equation · Hybrid systems · Structural control

1 Measure Solutions and Optimal Control

The notion of measure solution is relatively new. See the work of [Fat97], and the author [Ahm97, Ahm00, Ahm05, Ahm06, Ahm04, Ahm07, Ahm98]. The overriding reason for its introduction will be clear from the following discussion.

If f is **merely continuous** and even uniformly bounded on X , the Cauchy problem given by the following evolution equation

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$$\dot{x} = Ax + f(x), x(0) = x_0 \quad (1)$$

may have **no** solution. On the other hand if X is finite dimensional, then just continuity is sufficient to prove the existence of a local solution which may blow up in finite time. But if $\dim(X) = \infty$ continuity is not good enough. To support this statement we present the following simple example due to Dieudonne [Die50]. Let $X = c_0$ the sequence space $x = (x_n, n \geq 1) = (x_1, x_2, \dots)$ with norm $\|x\| \equiv \sup_{n \in N} |x_n|$ satisfying $\lim_{n \rightarrow \infty} x_n = 0$. Clearly with respect to this norm it is a Banach space. Define the map

$$f(x) \equiv \left\{ f_n(x_n) = \sqrt{|x_n|} + (1/1 + n), x_n \in R, n = 1, 2, \dots \right\}.$$

It is clear that f is a continuous map from X to X . Now consider the evolution equation on X

$$\dot{x} = f(x), x(0) = 0, t \geq 0.$$

For each $n \in N$, it is easy to verify that the following scalar equation

$$\dot{x}_n = f_n(x_n), x_n(0) = 0$$

has a continuous solution and that $|x_n(t)| \geq t^2/4$ for all $n \in N$ and therefore $x(t) = (x_n(t), n \geq 1) \notin X$ for any $t > 0$. Another example due to Godunov [God74] on a Hilbert space also demonstrates the nonexistence of a solution even though f is continuous. This led to the concept of measure valued solutions [Fat97, Ahm97, Ahm00, Ahm99, Ahm06, Ahm05, Ahm06, Ahm04, Ahm00, Ahm98].

1.1 Deterministic Systems and Measure Solutions

Measure Solutions: Let X be a B-space and $BC(X)$ the class of continuous and bounded real valued functions on X and define

$$\Phi \equiv \{ \varphi \in BC(X) : \varphi(\xi) \text{ and } D\varphi(\xi) (\in X^*) \text{ have bounded supports} \}.$$

Let $M_{rba}(X)$ denote the class of regular bounded finitely additive (f.a) measures on $\mathcal{B}(X)$, the Borel sets of X , and $\Pi_{rba}(X) \subset M_{rba}(X)$ denote the class of f.a. probability measures. Since X is a Banach space, it is a normal topological space. It is known that for such topological spaces (see Dunford–Schwartz [DS96], p257) the topological dual of $BC(X)$ is given by $M_{rba}(X)$, that is,

$$(BC(X))^* \cong M_{rba}(X).$$

Thus for $\nu \in M_{rba}(X)$ and $h \in BC(X)$, $\langle \nu, h \rangle \equiv \nu(h) \equiv \int_X h(\xi) \nu(d\xi)$ is well defined duality product. Define the operator F on $BC(X)$ by

$$F(\varphi)(\xi) \equiv (A^*D\varphi(\xi), \xi)_{X^*,X} + (D\varphi(\xi), f(\xi))_{X^*,X}$$

and introduce the operator $\mathcal{A}$ as follows:

$$\mathcal{D}(\mathcal{A}) \equiv \{\varphi \in \Phi : F(\varphi)(\xi) \in BC(X)\}$$

and set $\mathcal{A}\varphi = F(\varphi)$ for $\varphi \in \mathcal{D}(\mathcal{A})$. Let $\Pi_{rba}(X) \subset M_{rba}(X)$ denote the set of regular finitely additive probability measures.

Definition 1.1 A function $\mu : I \longrightarrow \Pi_{rba}(X)$ is said to be a measure solution of equation (1) if it satisfies the following identity

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(\mathcal{A}\varphi) ds, \quad \forall t \in I, \quad (2)$$

for every $\varphi \in \mathcal{D}(\mathcal{A}) \subset \Phi$, where $\mu_0 = \delta_{x_0}$ is the Dirac measure.

Theorem 1.2 Consider the system (1) with $A \in \mathcal{G}_0(M, \omega)$ and $f : X \longrightarrow X$ continuous and bounded on bounded sets. Then for each $x_0 \in X$ (or generally, $\mu_0 = \nu \in \Pi_{rba}(X)$) the system (1) has at least one weak* continuous measure solution μ with values $\mu_t \in \Pi_{rba}(X)$.

Proof For details see [7, Theorem 3.2] and its generalized version in [32, Theorem 3.2]. Here we present a very brief outline. In the first step the operator A is approximated by its Yosida regularization $A_n = nAR(n, A)$ for $n \in \rho(A) \cap N$ where $\rho(A)$ denotes the resolvent set of A and the function $f : X \longrightarrow X$ is approximated by a sequence $\{f_n\}$ such that $f_n(x) \in D(A)$ for $x \in X$ satisfying uniform Lipschitz and growth condition on X with the Lipschitz and growth coefficients $\{\alpha_n\} \rightarrow \infty$ as $n \rightarrow \infty$. Similarly the initial condition is approximated by $x_{0,n} = nR(n, A)x_0$. Clearly, the resulting evolution equation $\dot{x} = A_n x + f_n(x)$, $x(0) = x_{0,n}$ has a unique strong solution $x_n \in C(I, X)$ for each $n \in N \cap \rho(A)$. Then for any once continuously Fréchet differentiable function $\varphi \in C_b(X)$, one can easily verify that

$$\mu_t^n(\varphi) = \mu_0^n(\varphi) + \int_0^t \mu_s^n(\mathcal{A}_n \varphi) ds, \quad t \in I,$$

where $\mathcal{A}_n \varphi = (A_n^* D\varphi(\xi), \xi)_{X^*,X} + (D\varphi(\xi), f_n(\xi))_{X^*,X}$ and $\mu_t^n = \delta_{x_n(t)}(d\xi)$ is the Dirac measure evaluated at $x_n(t)$ and $\mu_0^n = \delta_{x_{0,n}}$. Clearly the sequence $\{\mu^n\} \in L_\infty^w(I, \Pi_{rba}(X))$. It follows from Alaoglu's Theorem that there exists a $\mu \in L_\infty^w(I, \Pi_{rba}(X))$ such that along a generalized subsequence (sub net), if necessary, relabeled as the original sequence, $\mu^n \xrightarrow{w^*} \mu$ in $L_\infty^w(I, \Pi_{rba}(X))$. Then, for

any test function $\varphi \in D(\mathcal{A})$, with an elementary algebraic manipulation and then taking the limit in the above expression, we arrive at the following expression

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(\mathcal{A}\varphi)ds, t \in I.$$

It is clear from the identity that $t \rightarrow \mu_t$ is weak star continuous. This completes the outline of our proof. For more general results on the question of existence of measure valued solutions see [Ahm97, Ahm00, Ahm99, Ahm06, Ahm00, Ahm98].

Remarks 1.4 (A) This result implies the existence of a w^* continuous semigroup $\{U(t), t \geq 0\}$ on $M_{rba}(X)$ such that $\mu_t = U(t)\mu_0, t \geq 0$; and $\Pi_{rba}(X) \subset M_{rba}(X)$ is an invariant set under $U(t), t \geq 0$. (B) In fact $U(t)$ is a w^* continuous contraction s-g (semigroup) on $M_{rba}(X)$.

Remarks 1.5 Theorem 1.3 also holds for Borel measurable vector fields f bounded on bounded sets with $\Pi_{rba}(X)$ replaced by $\Pi_{ba}(X) \subset M_{ba}(X) \cong B^*(X)$. For details see [Ahm06, Ahm04].

Theorem 1.5 (simplest of its kind) *is an earlier result and it has been extended in several directions such as (E1): Quasilinear systems, (E2): Systems driven by vector measures, (E3): Stochastic systems driven by Martingale measures, (E4): Differential Inclusions, (E5): Optimal Feedback controls. For details see [Ahm97, Ahm00, Ahm99, Ahm06, Ahm05, Ahm06, Ahm04, Ahm00, Ahm98].*

Systems driven by Vector measures (Impulsive Systems) : Systems driven by vector measures are generalizations of classical impulsive systems. This is discussed more in Sect. 3. We end this section with a result for the system

$$dx = Axdt + f(t, x)dt + g(t, x(t-))\nu(dt), x(0) = \xi, t \in I, \quad (3)$$

where $\nu \in M_{cabv}(\Sigma_I)$, the space of signed measures. It is known that, equipped with the total variation norm, it is a Banach space. Here we need two operators as presented below.

$$\begin{aligned} (\mathcal{A}\varphi)(t, \xi) &\equiv \langle A^*D\varphi, \xi \rangle_{X^*, X} + \langle D\varphi(\xi), f(t, \xi) \rangle_{X^*, X} \\ (\mathcal{C}\varphi)(t, \xi) &\equiv \int_0^1 d\theta \langle D\varphi(\xi + \theta g(t, \xi)\nu(\{t\})), g(t, \xi) \rangle_{X^*, X}. \end{aligned}$$

In case ν is non atomic the operator $\mathcal{C}$ reduces to $\mathcal{C}\varphi = \langle D\varphi, g \rangle_{X^*, X}$.

Theorem 1.6 *Suppose X is a separable B -space, $A \in \mathcal{G}_0(M, \omega)$, $\nu \in M_{cabv}(\Sigma_I)$, f, g Borel measurable on $I \times X$ to X bounded on bounded sets, and $f(\cdot, \xi) \in L_1(I, X)$ and $g(\cdot, \xi) \in L_1(|\nu|, X)$ for each $\xi \in X$. Then, for each $x_0 \in X$, system*

(3) has a measure valued solution $\mu \in L_\infty^w(\ell, \Pi_{ba}(X)) \cap L_\infty^w(|\nu|, \Pi_{ba}(X))$ meaning that, for each $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{C})$ having bounded supports,

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(\mathcal{A}\varphi)ds + \int_0^t \mu_{s-}(\mathcal{C}\varphi)ds, t \in I. \quad (4)$$

Proof See [Ahm06, Theorem 3.2, p. 83]. For more on this topic see [Ahm04, Ahm07].

1.2 Stochastic Systems and Measure Solutions

Let H, E be two separable Hilbert spaces. Consider on H the stochastic system of the form

$$dx = Axdt + f(x)dt + G(x(t-))M(dt), t \in I, x(0) = x_0 \quad (5)$$

where M is a martingale measure on a complete filtered probability space $(\Omega, \mathcal{F} \supset \mathcal{F}_t \uparrow, P)$ with values in E satisfying the following properties:

(1): $\{M(J), M(K)\}$ are stochastically independent for $J \cap K = \emptyset$.

(2): $\mathcal{E}\{(M(J), \zeta)_E(M(K), \eta)_E\} = \pi(J \cap K)(\zeta, \eta)_E$, where π is a countably additive bounded positive measure.

The functions f, G are continuous maps bounded on bounded sets. Clearly, under these assumptions, there may not exist any mild solution. To consider the question of measure valued solutions for this system we must introduce the following operators:

$$\begin{aligned} \mathcal{A}\varphi &\equiv (1/2)\text{Tr}(D^2\varphi GG^*), \quad \mathcal{B}\varphi \equiv \langle A^*D\varphi, \xi \rangle_H + \langle D\varphi, f \rangle_H, \\ \mathcal{C}\varphi &\equiv G^*D\varphi \in C(H, E), \end{aligned}$$

with $D(\mathcal{A}) \cap D(\mathcal{B})$ being those $\varphi \in BC(H)$ for which $\mathcal{A}(\varphi), \mathcal{B}(\varphi) \in BC(H)$. It is well known that the B-spaces $BC(H), \mathcal{M}_{rba}(H)$ do not satisfy the Radon-Nikodym-Property (RNP). Hence one invokes the theory of lifting to define the duals of Banach spaces like $L_1(I, L_2(\Omega, BC(H)))$ giving $L_\infty^w(I, L_2(\Omega, \mathcal{M}_{rba}(H)))$. Let $M_{1,2}(I \times \Omega, BC(H))$ denote the closed linear subspace of the Banach space $L_1(I, L_2(\Omega, BC(H)))$ consisting of elements which are also progressively measurable with respect to the filtration $\mathcal{F}_t, t \in I$. The topological dual of this space is denoted by $M_{\infty,2}^w(I \times \Omega, \mathcal{M}_{rba}(H))$. Let $\Pi_{rba}(H)$ denote the space of regular bounded finitely additive probability measures contained in $\mathcal{M}_{rba}(H)$.

Definition 1.7 A measure valued random process

$$\mu \in M_{\infty,2}^w(I \times \Omega, \Pi_{rba}(H)) \subset M_{\infty,2}^w(I \times \Omega, \mathcal{M}_{rba}(H))$$

is a measure solution of system (5) if for every $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ and $t \in I$ the following identity holds P-a.s

$$\mu_t(\varphi) = \phi(x_0) + \int_0^t \mu_s(\mathcal{A}\varphi)\pi(ds) + \int_0^t \mu_s(\mathcal{B}\varphi)ds + \int_0^t \langle \mu_s - (\mathcal{C}\varphi), M(ds) \rangle_E. \quad (6)$$

Theorem 1.8 Suppose $A \in \mathcal{G}_0(M, \omega)$, $f : H \rightarrow H$, $G : H \rightarrow \mathcal{L}(E, H)$ are continuous and bounded on bounded sets satisfying similar approximation properties as in 1.3. The martingale measure M is nonatomic satisfying the properties (1)–(2). Then, for every $\mathcal{F}_0$ measurable random element x_0 satisfying $P(|x_0|_H < \infty) = 1$, the evolution Eq. (5) has at least one measure solution $\mu \in M_{\infty,2}^w(I \times \Omega, \Pi_{rba}(H))$.

Proof For detailed proof see [Ahm05, Theorem 3.3, p136] for continuous vector fields and [Ahm05, 4.1, p145] for measurable vector fields. See also [Ahm06] for impulsive systems with measurable vector fields, and detailed discussion on uniqueness of measure solutions.

Remarks 1.6 Generally measures on infinite dimensional spaces are only finitely additive as seen above. To obtain countable additivity and also to prevent escape of mass from the original space, some time one uses Stone-Čech compactification of the original space X giving $X^+ = \beta X$ (see [Ahm99, Ahm98]). This turns out to be a compact Hausdorff space and the original space is homeomorphic to a subspace of X^+ which is dense in X^+ . Thus all the measures discussed in this chapter do have countably additive extension provided one uses the Stone-Čech compactification.

Optimal Feedback Control : Now we can consider the controlled system with direct state feedback control as follows:

$$dx = Axdx + f(x)dt + F(x)u(t, x)dt + G(x(t-))M(dt). \quad (7)$$

Let $\mathcal{E}$ denote another separable Hilbert space where the controls take their values from, $F : H \rightarrow \mathcal{L}(\mathcal{E}, H)$ and U is a nonempty closed bounded convex subset of $\mathcal{E}$ and $u : I \times H \rightarrow U \subset \mathcal{E}$. As usual A is the generator of a C_0 -semigroup and f, F, G are Borel measurable maps bounded on bounded sets. The set of admissible controls is given by the family of Borel measurable functions denoted by

$$\mathcal{U}_{ad} \equiv \{u \in BM(I \times H, \mathcal{E}) : u(t, x) \in U \text{ for } (t, x) \in I \times X\}.$$

This is endowed with the topology τ_{wu} of weak convergence in $\mathcal{E}$ uniformly on compact subsets of $I \times H$. Since $\{f, F, u, G\}$ are merely Borel measurable maps, one cannot expect to prove the existence of mild solutions. So we must consider measure valued solution and therefore we must formulate the Eq. (7) as an evolution equation on the space of measures $\mathcal{M}_{rba}(H)$ written in the weak form as follows:

$$\begin{aligned} d\mu_t(\varphi) &= \mu_t(\mathcal{A}\varphi)\pi(dt) + \mu_t(\mathcal{B}\varphi)dt + \mu_t(\mathcal{G}_u\varphi)dt + \langle \mu_t - (\mathcal{C}\varphi), M(dt) \rangle_E, \\ \mu_0(\varphi) &= \nu(\varphi), \varphi \in D(\mathcal{A}) \cap D(\mathcal{B}), \end{aligned} \quad (8)$$

where the operators $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ are as defined above and

$$\mathcal{G}_u(\varphi)(t, \xi) \equiv \langle F(\xi)u(t, \xi), D\varphi(\xi) \rangle_H.$$

Equation (8), in its integral form, is given by

$$\begin{aligned} \mu_t(\varphi) &= \phi(x_0) + \int_0^t \mu_s(\mathcal{A}\varphi)\pi(ds) + \int_0^t \mu_s(\mathcal{B}\varphi)ds \\ &\quad + \int_0^t \mu_s(\mathcal{G}_u\varphi)ds + \int_0^t \langle \mu_s - (\mathcal{C}\varphi), M(ds) \rangle_E \end{aligned}$$

which holds for all $t \in I$ and $\varphi \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$.

Existence of solution for the controlled system (8) essentially follows from 1.8. For details see [Ahm05, Theorem 4.1, Lemma 5.1, p145].

Here we mention the following three control problems. For other interesting control problems involving measure solutions see [Ahm12, Ahm12, Ahm97, Ahm00, Ahm99, Ahm06, Ahm05, Ahm06].

(PA) A standard cost functional is given by

$$J(u) \equiv \mathcal{E} \left\{ \int_{I \times H} \ell(t, x) \mu_t^u(dx) dt + \int_H \psi(x) \mu_T^u(dx) \right\}. \quad (9)$$

The problem is to find $u \in \mathcal{U}_{ad}$ that minimizes this functional. Many more interesting and general control problems of this nature and existence of optimal controls for such problems can be found in [11, s 5.2, 5.3, 5.4, 5.5, 5.6, 5.7] and [Ahm06].

(PB) (Target seeking) Let $C \subset H$ be a closed set. This is a friendly zone and our objective is to find a control law that drives the system to this set and possibly keep it there. In other words the objective is to maximize the probability of residence in C during the period of interest I . We may consider a slightly more general problem. For any countably additive bounded positive measure λ , find a control law u such that

$$J(u) \equiv \mathcal{E} \left\{ \int_I \mu_t^u(C) \lambda(dt) \right\} \longrightarrow \sup. \quad (10)$$

For example, if λ is the Dirac measure concentrated at T , we have $J(u) = \mathcal{E} \{ \mu_T^u(C) \}$. Maximizing this is equivalent to maximizing the probability of reaching or hitting the target C at time T .

Theorem 1.10 Consider the system (8) with the objective functional (10) and suppose $\mathcal{U}_{ad}$ is τ_{wu} compact. Then there exists an optimal control law $u^o \in \mathcal{U}_{ad}$ maximizing the functional (10).

Proof For detailed proof see [Ahm05]. We present only an outline. Let $\{u^n, u^o\}$ be any generalized sequence (net) from $\mathcal{U}_{ad}$ and $\{\mu^n, \mu^o\}$ the corresponding generalized sequence of (measure valued) solutions of the system (8). We can show that, along a subsequence if necessary, whenever $u^n \xrightarrow{\tau_{wu}} u^o$ we have $\mu_t^n \xrightarrow{w} \mu_t^o$ for almost every $t \in I$ with probability one. Since C is a closed set it follows from a well known property of weakly convergent measures that

$$\limsup_n \mu_t^n(C) \leq \mu_t^o(C), \text{ for a.e. } t \in I, P - a.s.$$

From this inequality we obtain

$$\limsup_n J(u^n) = \limsup \mathcal{E} \int_I \mu_t^n(C) \lambda(dt) \leq \mathcal{E} \int_I \mu_t^o(C) \lambda(dt) = J(u^o).$$

This means that $u \rightarrow J(u)$ is weakly upper semicontinuous on $\mathcal{U}_{ad}$ which is τ_{wu} compact. Hence J attains its supremum on $\mathcal{U}_{ad}$ and therefore an optimal control law exists.

(PC) (Obstacle avoidance) Let $D \subset H$ be an open set, a danger zone. Find a control u that tries to keep the system away from this zone during the period I or equivalently minimizes the probability of contact with this set during the time interval I . Since the solution $\{\mu_t^u, t \geq 0\}$ is a measure valued stochastic process we may consider the expected value of the contact probability during the time period I . Thus the correct problem is given by

$$J(u) \equiv \mathcal{E} \left\{ \int_I \mu_t^u(D) \lambda(dt) \right\} \rightarrow \inf. \quad (11)$$

Using the same technique as in 1.9, we can prove the existence of an optimal control law for this problem. For details see [Ahm05, Ahm12, Ahm06].

2 Some Applications of Measure Solutions

Here we consider two important applications involving stochastic Navier–Stokes equation and nonlinear filtering. Our objectives are (A1): Applications to Stochastic Navier–Stokes equation (SNSE): problem: $\rightarrow$ minimize the intensity of turbulence. (A2): Zakai equations for nonlinear filtering with measure valued solutions, problem: $\rightarrow$ optimum selection of observation operator.

2.1 Stochastic NSE

The classical Navier–Stokes equation determines the dynamics of incompressible fluid flow, and this is given by the following nonlinear PDE,

$$\begin{aligned} (\partial/\partial t)\hat{v} - \nu\Delta\hat{v} + (\hat{v} \cdot \nabla)\hat{v} + \nabla p &= g \text{ on } I \times \Sigma \\ \operatorname{div} \hat{v} &= 0, \quad v|_{\partial\Sigma} = 0, \quad \hat{v}(0, x) = \hat{v}_0(x), x \in \Sigma, \end{aligned}$$

where $\hat{v}$ denotes the velocity vector of the fluid confined in an open bounded domain $\Sigma \subset R^n$, $n \geq 2$. The top equation represents the conservation of momentum. Its first term denotes the inertial force, the second the viscous force (dissipative), the third term denotes the convective force, the fourth term denotes the hydrodynamic pressure force and g the volume force giving the momentum balance. For incompressible fluid the density ρ is constant and hence it follows from the continuity equation (equivalently the mass conservation equation) $(\partial/\partial t)\rho + \operatorname{div}(\rho\hat{v}) = 0$, that $\operatorname{div} \hat{v} = 0$. The homogeneous Dirichlet boundary condition represents physically the nonslip boundary condition. The last equation gives the initial state of the fluid.

It is well known [Ahm91, p182, p210] that by projecting to the solenoidal (divergence free) vector field the classical Navier–Stokes equation can be transformed into an abstract differential equation on the Hilbert space $H \equiv L_2^\sigma(\Sigma, R^3)$ which consists of divergence free R^3 valued (norm) square integrable functions defined on the set $\Sigma \subset R^n$. In other words, $L_2(\Sigma) = H \oplus (1 - P)H$ where P is the projection of $L_2(\Sigma)$ onto H . By applying this projection operator P on either side of the above equation we arrive at the following abstract differential equation

$$(d/dt)\hat{v} + \nu A(\hat{v}) + B(\hat{v}) = f_0, \quad \hat{v}(0) = \hat{v}_0,$$

on the Hilbert space H where for any test function $\varphi \in C^2(\Sigma, R^3)$ we have $A\varphi = -P(\Delta\varphi)$, $B(\varphi) = P((\varphi \cdot \nabla)\varphi)$ and $f_0 = Pg$. Since $P(\nabla p) = 0$, the pressure term disappears. This is the deterministic Navier–Stokes equation. In practice there are always unaccounted forces in nature that affects the dynamics. Thus a stochastic term may be added to include the impact of such forces. Including the stochastic term and the control, the simplest such model is given by the equation

$$d\hat{v} + (\nu A\hat{v} + B(\hat{v}))dt = F_0(t, u)dt + F(t)dw, \quad \hat{v}(0) = \hat{v}_0, t \geq 0;$$

where $F_0(t, u) \equiv f_0(t) + Cu$ with f_0 denoting the natural (uncontrolled) volume force and Cu the volume force induced by the control u all projected to the divergence free vector field. Considering the stochastic term, let E (possibly a closed linear subspace of H) denote a suitable Hilbert space and suppose that w is an E -valued Brownian motion (all supported in a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, P)$) and F is a linear bounded operator valued function with values in

$\mathcal{L}(E, H)$.

For analysis surrounding this system we need the pivot space $H \equiv L_2^\sigma(\Sigma, R^3)$, and the NSE-related Gelfand triple: $V \hookrightarrow H \hookrightarrow V^*$ where

$$\begin{aligned} V &\equiv c\ell^{H^1} \{\psi \in C^\infty(\Sigma, R^3) : \operatorname{div} \psi = 0\} \\ H &\equiv c\ell^{L^2} \{\psi \in C^\infty(\Sigma, R^3) : \operatorname{div} \psi = 0\} \end{aligned}$$

with V^* being the topological dual of V . We are interested in measure valued solutions of the NSE as introduced for the first time in [Ahm98] and later in [Ahm12]. In [Ahm98] we considered measure valued solutions for the deterministic system (NSE) proving existence of solutions based on our previous chapter [Ahm98] and also considered optimal control. In [Ahm12] we considered stochastic system (SNSE). To construct the dynamic model for measure valued solutions we need to introduce the following generating operators (related to the original system)

$$\begin{aligned} \mathcal{A}(t)\varphi(\xi) &\equiv (1/2)Tr[D^2\varphi(\xi)(F(t)F^*(t))], \\ \mathcal{B}(\varphi)(\xi) &\equiv \langle -\nu A\xi - B(\xi), D\varphi(\xi) \rangle_{V^*, V}, \\ \mathcal{C}_1(u)(t)\varphi(\xi) &\equiv \langle F_0(t, u), D\varphi(\xi) \rangle_H, \quad \mathcal{C}_2(t)\varphi(\xi) \equiv F^*(t)D\varphi(\xi) \end{aligned}$$

where $\varphi \in C_b^2(H)$ so that the above is well defined taking values in $BC(H)$. Let $\mathcal{M}_b(H)$ denote the space of bounded (signed) Borel measures defined on the Borel field $\mathcal{B}(H)$. Furnished with the standard total variation norm, it is a Banach space. Then define $M_2(H)$ as

$$M_2(H) \equiv \{\mu \in \mathcal{M}_b(H) : \int_H |\xi|_H^2 |\mu|(d\xi) < \infty\}$$

where $|\mu|$ denotes the positive measure induced by its variation. This is a closed linear subspace of $\mathcal{M}_b(H)$ and so a Banach space. Now we can write the controlled stochastic NSE in its weak form on the B-space of measures $M_2(H)$, as follows:

$$\begin{aligned} d\mu_t(\varphi) &= \mu_t(\mathcal{A}\varphi)dt + \mu_t(\mathcal{B}\varphi)dt + \mu_t(\mathcal{C}_1(u)\varphi)dt + \langle \mu_t(\mathcal{C}_2\varphi), dw \rangle, \\ \mu_0(\varphi) &= \nu_0(\varphi), \text{ for } \varphi \in D(\mathcal{A}) \cap D(\mathcal{B}), t \in I = [0, T], \end{aligned} \quad (12)$$

where ν_0 is the measure induced by the initial state $\hat{v}_0$. We use a sensor (observation) and a linear output feedback control law based on the sensor output (observation) given by the composition $u \equiv \Gamma L(\cdot)$ where $L : H \rightarrow Y$, the output space (a Hilbert space), is the sensor not necessarily linear, and $\Gamma \in \mathcal{FC}_{ad} = B_1(\mathcal{L}(Y, U))$ (closed unit ball in $\mathcal{L}(Y, U)$). It is not necessary to restrict to the unit ball; any closed ball of finite radius will do. Existence of a measure solution follows from the general result as stated in theorem 1.8. Further, for each $\Gamma \in \mathcal{FC}_{ad}$, the solution $\mu^\Gamma \in \mathcal{M}_{Na}$ where

$$\mathcal{M}_{Na} \equiv L_\infty^{wa}(I, L_1(\Omega, M_2(H))) \cap L_1^{wa}(I, L_1(\Omega, M_2(V))).$$

This is the space of measure valued random processes weakly adapted (indicated by the superscript *wa*) to the filtration $\mathcal{F}_t, t \geq 0$, in the sense that for any test function φ and any $\mu \in \mathcal{M}_{Na}$, $t \longrightarrow \mu_t(\varphi)$ is an $\mathcal{F}_t$ adapted random process. Further

$$\sup\{\mathcal{E} \|\mu_t\|_{M_2(H)}, t \in I\} < \infty, \quad \mathcal{E} \int_I \|\mu_t\|_{M_2(V)} dt < \infty$$

for any $\mu \in \mathcal{M}_{Na}$. The reachable set (of measure valued processes) is given by

$$\mathcal{R} \equiv \left\{ \mu \in \mathcal{M}_{Na} : \mu = \mu^\Gamma \text{ solution of (12), for some } \Gamma \in \mathcal{F}C_{ad} \right\}.$$

For proof of existence of measure solution of the system (12), interested reader may see [Ahm12, Theorem 7.8.1, p189], [Ahm99, Theorem 3.2, p80] and [Ahm05, Theorem 3.3, p136]. Here we concentrate on applications.

Problem (A) (Turbulence Control) For any control $\Gamma \in \mathcal{F}C_{ad}$, let μ^Γ denote the corresponding solution of the evolution Eq. (12). Find $\Gamma \in \mathcal{F}C_{ad}$ that minimizes the functional,

$$J(\Gamma) \equiv \mathcal{E} \int_I \mu_t^\Gamma(\psi) \lambda(dt) = \mathcal{E} \int_I \int_H \psi(\xi) \mu_t^\Gamma(d\xi) \lambda(dt)$$

where the function ψ is given by $\psi(\xi) = \beta d^2(\xi, B_r(H))$ with $d(\xi, B_r(H))$ denoting the distance of ξ from the closed ball $B_r(H)$ of radius $r > 0$ (centered at the origin) and any $\beta \geq 1$. Clearly, for any given $r > 0$, the larger this quantity is the more energetic the fluid is and if the energy lies in $B_r(H)$ it is zero. Thus minimizing this functional amounts to minimizing the turbulence.

We have proved existence of optimal $\Gamma \in \mathcal{F}C_{ad}$ for the problem (A) [Ahm12, Theorem 7.8.4]. The proof is based on continuity of the functional $\Gamma \longrightarrow J(\Gamma)$ in the weak operator topology and the well known fact that the closed unit ball $B_1(\mathcal{L}(Y, U))$ is compact in the weak operator topology iff U is reflexive. In our case U is Hilbert so reflexive. See Dunford [DS96, p. 512].

Problem B (Control of Blow up Time) Suppose initially the fluid is calm, that is, there exists a closed ball $V_r \equiv B_r(V)$ of V of radius $r > 0$ and centered at the origin such that $\mu_0(V_r) = 1$. Let $\overline{V}_r$ denote the closure of V_r in H . Define the Markov time

$$\tau_m(\Gamma) \equiv \inf\{t \geq 0 : \int_0^t \mu_s^\Gamma(H \setminus \overline{V}_r) > 0\}$$

which gives the first time the measure μ starts loosing its support in $\overline{V}_r$ and hence V_r . If the underlying set is empty we set it equal to ∞ . This means that the fluid stays in $\overline{V}_r$ for all time. The objective functional is given by its expected value

$J(\Gamma) \equiv \mathcal{E}\{\tau_m(\Gamma)\}$. The problem is to find a Γ^o that maximizes J on $\mathcal{FC}_{ad}$ thereby maximizing the expected first time to blow up. Clearly, for each $t \geq 0$, the set $\{\tau_m(\Gamma) > t\}$ is $\mathcal{F}_t$ measurable and hence it is a well defined stopping time. In [4, 7.8.10] we have proved existence of optimal feedback operator Γ^o that maximizes this. For more on similar problems on general Banach spaces see [Ahm12].

2.2 Optimal Measurement Strategy in Nonlinear Filtering

Nonlinear filtering problem starts with the following system of stochastic differential equations:

$$\begin{aligned} d\xi &= A\xi dt + B(\xi)dt + R_1^{1/2}dW_1 + CdW_2, \xi(0) = \xi_0, \text{ in } H \\ dy &= h(\xi)dt + R_2^{1/2}dW_2, y(0) = 0, \text{ in } R^d. \end{aligned} \quad (13)$$

The first equation describes the dynamics of the process of interest ξ not accessible to the observer. The second equation describes the dynamics of the observer giving the output process y which carries some noisy information about the process ξ . For details see [Ahm96, AFZ97, AC06]. The filtering problem is concerned with the best estimate of any functional like $\phi(\xi(t))$ given the history $\mathcal{F}_t^y \equiv \sigma\{y(s), s \leq t\}$ of $\{y\}$ up to time $t > 0$. It is well known see [Ahm88, p. 142], [Ahm96, AFZ97, AC06, Ahm98] that the unbiased minimum variance filter is given by the conditional expectation of the process $\{\phi(\xi(t))\}$ given the history of observation up to time t . In other words the best estimate is given by

$$E\{\phi(\xi(t))|\mathcal{F}_t^y\} = \int_H \phi(z)Q_t^y(dz) \quad (14)$$

where $Q_t^y(\sigma) = Prob.\{\xi(t) \in \sigma|\mathcal{F}_t^y\}$, $\sigma \in \mathcal{B}(H)$, is the conditional probability measure on H . The un normalized bounded positive measure valued process $\mu_t^y(\cdot) \equiv \mu_t^y(H)Q_t^y(\cdot)$ defined on $\mathcal{B}(H)$, the Borel sets of H , satisfies in the weak sense the following linear evolution equation on $\mathcal{M}_b(H)$

$$\begin{aligned} d\mu_t(\varphi) &= \mu_t(\mathcal{A}\varphi)dt + \mu_t(\mathcal{B}\varphi) \cdot dv, t \geq 0, \\ \mu_0(\varphi) &= \nu(\varphi), \text{ where } v \text{ is an } R^d\text{-valued innovation process.} \end{aligned} \quad (15)$$

This equation is usually known as Zakai equation (18), (30). Here the operators $\{\mathcal{A}, \mathcal{B}\}$ are given by

$$\begin{aligned} \mathcal{A}\varphi(\xi) &\equiv (1/2)Tr(R_1 D^2\varphi) + (A^*D\varphi, \xi) + (D\varphi, B(\xi)) \\ \mathcal{B}\varphi(\xi) &\equiv (R_2^{-1/2}h + C^*D)\varphi. \end{aligned}$$

For detailed derivation see [AFZ97] and also [Ahm98] where the original Zakai equation is derived in the metric space of bounded positive measures. The best filter given by (14) through (15), however, corresponds to a given observation operator h and it is clear that the best estimate depends on the choice of the operator h . If one has a choice in the design of the observation channel, it is possible to choose one that gives the best performance. Let μ^h denote the measure solution corresponding to the choice of $h \in \mathcal{H}_{ad} \subset BC(H, R^d) \subset BM(H, R^d)$. Suppose $BC(H, R^d)$ is endowed with the compact open topology. Considering any finite interval $[0, T]$, the reader can easily verify that the filtering error at the final time T is given by the expression

$$J_T(h)(v) \equiv (1/\mu_T^h(H))^2 \left(\mu_T^h(H) \int_H \phi^2(x) \mu_T^h(dx) - \left(\int_H \phi(x) \mu_T^h(dx) \right)^2 \right). \quad (16)$$

This is a random variable dependent on the innovation process $\{v\}$ driving the system (15). Let $(\Omega, \mathcal{B}_\Omega, \mu_W) \equiv (C(I, R^d), \mathcal{B}_C, \mu^v)$ denote the classical wiener measure space induced by the innovation process v with μ_W denoting the Wiener measure. Our problem is to find $h \in \mathcal{H}_{ad}$ that minimizes the functional

$$\rho(h) = \int_{\Omega} J_T(h)(v) \mu_W(dv). \quad (17)$$

Theorem 2.1 *Consider the filtering problem (17) subject to the dynamics (15) with the observation operator h as defined above and suppose the admissible set of observers $\mathcal{H}_{ad} \subset BC(H, R^d)$, furnished with the compact-open topology, satisfies the conditions of abstract Ascoli characterizing compact sets. Then there exists an optimal observer $h^o \in \mathcal{H}_{ad}$ at which ρ attains its minimum.*

Proof See [Ahm12, 7.7.2, p182]. To determine the optimal observer, one can use the necessary conditions of optimality as given in [AC06].

3 Hybrid Systems and Control

Hybrid systems (classical): Classical models for hybrid systems on a Banach space X are given by a system of differential and difference equations as follows:

$$\begin{aligned} \dot{x} &= Ax + f(t, x), t \in I \setminus D \\ x(t_i+) &= x(t_i) = x(t_i-) + g(t_i, x(t_i-)), t_i \in D \end{aligned} \quad (18)$$

where $D \subset I$ consists of a countable (possibly finite) set of points in $I \equiv (0, \infty)$. The top equation evolves continuously on $I \setminus D$ and the second equation evolves by pure jumps on D with jump size determined by the vector field g . Similarly, systems governed by differential inclusions are given by

$$\begin{aligned}\dot{x} &\in Ax + F(t, x), t \in I \setminus D \\ x(t_i +) &\in x(t_i -) + G(t_i, x(t_i -)), t_i \in D\end{aligned}\quad (19)$$

where F and G are suitable multi functions from $I \times X$ to $2^X \setminus \emptyset$, for more see [Ahm03, Ahm07, Ahm98, Ahm01, Ahm00, Ahm01] and the references therein.

Hybrid Systems(Modern): The classical models described above are special cases of more general systems driven by vector measures. For example, by use of signed measures, these equations can be described more generally by evolution equations or inclusions of the form:

$$dx = Axd t + f(t, x)dt + g(t, x)\nu(dt), t \in I \quad (20)$$

$$dx \in Axd t + F(t, x)dt + G(t, x)\nu(dt), t \in I, \quad (21)$$

where $f, g : I \times X \longrightarrow X$, $F, G : I \times X \longrightarrow cc(X)$ (the class of nonempty closed convex subsets of X) and ν is a signed measure. For extensive study of such systems and their generalized versions see [Ahm05, PWX12, Ahm03, Ahm07, Ahm09, Ahm09, Ahm07, Ahm01, Ahm00, Ahm01].

Control problem : Let X be a Banach space considered as the state space and E another Banach space where the vector measures considered as controls take their values from. Consider the controlled system

$$dx = Axd t + f(t, x)dt + g(t, x)\nu(dt) + B(t, x)u(dt), x(0+) = x_0, \quad (22)$$

where $f, g : I \times X \longrightarrow X$, ν is any fixed signed measure (system naturally hybrid); and $B : I \times X \longrightarrow \mathcal{L}(E, X)$, and the control $u \in M_{cav}(\Sigma_I, E)$ (the space of E -valued countably additive vector measures having bounded total variations). Let $\mathcal{U}_{ad} \subset M_{cav}(\Sigma_I, E)$ denote the class of admissible controls. The problem is to find $u \in \mathcal{U}_{ad}$ such that

$$J(u) \equiv \int_I \ell(t, x)dt + \Psi(x(T)) + \Phi(|u|_\nu) \longrightarrow \inf_{\mathcal{U}_{ad}}. \quad (23)$$

The reader will find in [Ahm03] some interesting examples involving structural vibration of beams subject to impulsive loads; pulsed radar modeled by Maxwell's equations subject to impulsive inputs. We wish to present here a result on existence of optimal control. This leads us to the question of compactness in the space of vector measures. The necessary and sufficient conditions for weak compactness of $\mathcal{U}_{ad}$ is given by the celebrated due to Bartle-Dunford-Schwartz (BDS) [Ahm12, Theorem 4.1] and [DU77, Theorem 5, p. 105].

Lemma 3.1 (BDS) *Suppose the dual pair $\{E, E^*\}$ satisfies the Radon-Nikodym property (RNP). Then a set $\Gamma \subset M_{cav}(\Sigma_I, E)$ is relatively weakly compact iff (i) Γ is bounded, (ii) Γ is uniformly countably additive and (iii) for each $\sigma \in \Sigma_I$ the set $\{u(\sigma), u \in \Gamma\}$ is a relatively weakly compact subset of E .*

Using the above result we can prove the existence of optimal control. We present one typical result here.

Theorem 3.2 *Consider the system (22) with the cost functional (23). Suppose $A \in \mathcal{G}_0(M, \omega)$, $f : I \times X \rightarrow X$, $g : I \times X \rightarrow X$ are Borel measurable in the first argument and Lipschitz in the second with Lipschitz coefficients $K \in L_1^+(I)$, $L \in L_1^+(I, |\nu|)$ respectively and $B : I \times X \rightarrow \mathcal{L}(E, X)$ is uniformly Lipschitz and $\mathcal{U}_{ad}$ is a weakly compact subset of $M_{cav}(\Sigma_I, E)$. The function ℓ is Borel measurable and real valued satisfying $\ell(t, x) \geq h(t)$ for some $h \in L_1(I)$ and $\Psi(x) \geq c$ for some $c \in \mathbb{R}$ and both are lower semicontinuous on X , and Φ is a nonnegative nondecreasing real valued lower semicontinuous function on $[0, \infty)$ satisfying $\Phi(0) = 0$. Then there exists an optimal control $u^* \in \mathcal{U}_{ad}$.*

Proof For detailed proof see [Ahm12, Theorem 4.2]. Here we present an outline. Under the given assumptions we can prove that $u \rightarrow J(u)$ is weakly lower semicontinuous on $\mathcal{U}_{ad}$. This is the most difficult part. Then the result follows from weak compactness of the set $\mathcal{U}_{ad}$.

For more on existence results and necessary conditions of optimality see [Ahm07, Theorem 3.1, 3.2] and [Ahm09, Theorem 3.6]. For necessary conditions with state constraints see [Ahm09, Theorem 5.1]. For optimal control of systems determined by nonlinear operator valued measures and controlled by vector measures see [Ahm08, Theorem 5.1, 5.2].

Structural Control (Operator Valued measures): In mechanics and possibly social systems, structural control is natural. Proper maneuver and orientation of flexible parts of a moving body can change its course of motion (example: Aircraft). Similarly, by shooting atoms with laser beams one can change the electronic structure and hence the potential perturbing the ground state Hamiltonian. This helps produce materials with desirable properties. Similarly, structural change of the network of communication can change the dynamics of social and economic activities. Here we present a result on partially observed stochastic control problem with structural controls which are operator valued measures.

Without further notice, it is always assumed that all the random processes are adapted to the filtration $\mathcal{F}_t, t \geq 0$, of the complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, P)$. Let $\{X, Y, H\}$ be separable Hilbert spaces, X representing the state space, Y the output space and H the space where the Brownian motion W takes values from. The system is described by a stochastic differential equation on X coupled with an algebraic equation representing noisy measurement process in Y as follows:

$$dx = Axdt + B(dt)y(t-) + \sigma(t)dW(t), t \in I \equiv [0, T], x(0) = x_0, \quad (24)$$

$$y(t) = L(t)x(t) + \xi(t), t \in I. \quad (25)$$

The process x is the state and y is the observation. The system is called partially observed since the state x is not directly accessible; only the noisy measured output

y is available for control. The operator A is the infinitesimal generator of a C_0 -semigroup $S(t)$, $t \geq 0$, on X and $B \in M_{cabv}(\Sigma_I, \mathcal{L}(Y, X))$ where Σ_I is the sigma algebra of subsets of the set I , $\sigma \in B_\infty(I, \mathcal{L}(H, X))$, W is an H -valued Brownian motion with covariance $Q \in \mathcal{L}_1^+(H)$, $L \in B_\infty(I, \mathcal{L}(X, Y)) \cap C(I, \mathcal{L}(X, Y))$ and ξ is a measurable random process taking values from Y . No other probabilistic structure for ξ is assumed.

Any change of the operator valued measure B means a structural change of the system. This operator valued measure is considered here as the decision or control variable which interacts with the state in a bilinear fashion. The problem is to find a control policy so as to minimize fluctuation of the trajectory x around its mean path $\bar{x}$ and also guide this mean trajectory along a desirable deterministic path $x_d \in B_\infty(I, X)$ as closely as possible while minimizing the cost of control efforts for doing so. A functional that incorporates all these attributes is given by the following expression

$$J(B) \equiv E \int_I |x(t) - \bar{x}(t)|_X^2 \lambda(dt) + \int_I |\bar{x}(t) - x_d(t)|_X^2 \nu(dt) + |B|_v. \quad (26)$$

The first term is the measure of fluctuation weighted by the countably additive bounded positive measure λ , the second term measures the deviation of the mean trajectory from the desired path weighted by ν , also a positive measure, and the last term is a measure of control cost given by the total variation of the operator valued measure (frequent structural change is costly). Given that the desired path x_d is deterministic, an equivalent expression for the cost functional is given by

$$J(B) \equiv \int_I \text{Tr}(P(t)) \lambda(dt) + \int_I |\bar{x}(t) - x_d(t)|_X^2 \nu(dt) + |B|_v, \quad (27)$$

where $P(t)$ is the covariance operator of the process x given by

$$E\{(x(t) - \bar{x}(t), h)^2\} = (P(t)h, h), h \in X.$$

Let $\Gamma \subset M_{cabv}(\Sigma_I, \mathcal{L}(Y, X))$, denote the set of admissible structural controls. Our objective is to find a $B \in \Gamma$ that minimizes the cost functional (27).

We present a result on existence of optimal structural control.

Theorem 3.3 *Consider the partially observed system (24)–(25) with the cost functional (27) and suppose the following assumptions hold.*

(A1): *A generates a C_0 -semigroup $S(t)$, $t \geq 0$, on X , compact for $t > 0$, and $L \in B_\infty(I, \mathcal{L}(X, Y)) \cap C(I, \mathcal{L}(X, Y))$.*

(A2): *The admissible set $\Gamma \subset M_{cabv}(\Sigma_I, \mathcal{L}(Y, X))$ is weakly compact and there exists a $\mu \in M_{cabv}^+(\Sigma_I)$ and $\gamma > 0$ such that $|B|_v(\sigma) \leq \gamma\mu(\sigma)$ for all $\sigma \in \Sigma_I$ and $B \in \Gamma$.*

(A3): The process $\{\xi\}$ is an $\mathcal{F}_t$ -adapted Y -valued second order centered random process satisfying $\sup\{E|\xi(t)|_Y^2, t \in I\} \leq \beta^2$ with $\beta \in R$.

(A4): $P_0 \in \mathcal{L}_1^+(X)$, and $\sigma(\cdot)Q\sigma^*(\cdot) \in L_1(I, \mathcal{L}_1^+(X))$.

(A5): The measures $\lambda, \nu \in M_{cav}^+(\Sigma)$, $x_d \in B_\infty(I, X)$.

Then, there exists an optimal policy $B_0 \in \Gamma$ such that $J(B_0) \leq J(B) \forall B \in \Gamma$.

Proof For detailed proof see [Ahm11, Theorem 4.4]. We present only a brief outline. Since Γ is weakly compact and hence weakly sequentially compact by Eberlein-Smulian Theorem, it suffices to prove that the cost functional $B \rightarrow J(B)$ is sequentially weakly lower semicontinuous. Under the assumptions (A1)–(A4) we show that whenever a generalized sequence $B_n \in \Gamma$ converges weakly to $B_o \in \Gamma$, the corresponding sequence of covariance operators of the process $\{x_n\}$ and $\{x_o\}$ denoted by P_n, P_o respectively, has the property that $Tr P_n(t) \rightarrow Tr P_o(t)$ for each $t \in I$ and that the sequence is dominated by an integrable function. Thus by dominated convergence $Tr(P^n) \rightarrow Tr(P^o)$ in $L_1(\nu, R)$. Further, by use of the assumption (A1), one can verify that $\bar{x}_n \xrightarrow{s} \bar{x}_o$ in $L_2(\nu, X)$ whenever B_n converges weakly to B_o . Since the norm in any Banach space is weakly lower semicontinuous, the last term of (27) is weakly lower semicontinuous on $\mathcal{M}_{cav}(\Sigma, \mathcal{L}(Y, X))$. Thus the functional $B \rightarrow J(B)$ is weakly lower semicontinuous. The conclusion then follows from the fact that Γ is weakly compact.

For lack of space we can not include many other interesting results on this topic. The reader is referred to [Ahm09, Ahm07, Ahm08, Ahm08, Ahm08, Ahm11] where many more interesting results on existence of optimal structural controls including necessary conditions of optimality are given. Several examples from theoretical mechanics are included for illustration. The mathematical models used are given by either differential equations or inclusions. For characterization of weakly compact sets in $M_{cav}(\Sigma, \mathcal{L}(Y, X))$ see [Ahm11], where some other interesting examples are given.

An Example: We conclude this section with an example of structural control. Structural control arises in molecular dynamics governed by linear or nonlinear schrodinger equation

$$id\psi = H_0\psi dt + V_0(dt)\psi + g(\psi)dt \quad (28)$$

where the change of potential perturbing the ground state Hamiltonian acts as the structural control. Considering the order of time measuring the pulse width ($\leq 10^{-15}$ s), the control is close to Dirac measures and so can be represented by measures. Considering the real and the imaginary parts of the wave function as a vector x in $H \equiv L_2(R^{3n}) \times L_2(R^{3n})$, one can rewrite the Schrödinger equation as an abstract differential equation on H as follows:

$$dx = Axdt + f(x)dt + B(dt)x, \quad (29)$$

where

$$A = \begin{pmatrix} 0 & H_0 \\ -H_0 & 0 \end{pmatrix}, f(x) = \begin{pmatrix} g_2(x) \\ -g_1(x) \end{pmatrix}, B(\sigma) = \begin{pmatrix} 0 & V_0(\sigma) \\ -V_0(\sigma) & 0 \end{pmatrix}.$$

Here B is the operator valued measure defined on Σ_R , the sigma algebra of Borel subsets of R . For details see [Ahm08, Ahm08]. It is well known that the operator A , with domain and range $D(A), R(A) \subset H$, generates a unitary group $U(t), t \in R$, in H and the mild solution is given by the solution of the following integral equation on H ,

$$x(t) = U(t)x_0 + \int_0^t U(t-s)f(x(s))ds + \int_0^t U(t-s)B(ds)x(s-), t \in I. \quad (30)$$

Under local Lipschitz and linear growth assumption on f , for every $B \in \Gamma$, the integral Eq.(30) has a unique mild solution $x \in B(I, H)$, the Banach space of bounded measurable functions with values in X . The control problem is to find $B \in \Gamma$ so that

$$J(B) \equiv \int_I (Q(x - x^d), (x - x^d))_H \lambda(dt) + \Phi(|B|_\nu) \longrightarrow \inf_{B \in \Gamma},$$

where $Q \in \mathcal{L}_s^+(H)$ and $\lambda \in M_{cabv}^+(\Sigma_I)$ and Φ is any nonnegative nondecreasing real valued lower semicontinuous function. Since $U(t), t \in R$, is a unitary group and hence not compact, we need a stronger assumption on the set of admissible structural controls $\Gamma \subset M_{casbv}(\Sigma_I, \mathcal{L}(H))$, the space of $\mathcal{L}(H)$ valued measures which are countably additive in the strong operator topology and have bounded total variations. It suffices to assume that Γ is set wise compact in the strong operator topology on $\mathcal{L}(H)$ in the sense that for every generalized sequence $B_n \in \Gamma$ there exists a generalized subsequence, relabeled as the original sequence, and a $B_o \in \Gamma$ such that $(B_n(\sigma) - B_o(\sigma))h \rightarrow 0$ in H for every $\sigma \in \Sigma_I$ and $h \in H$. In other words, this is the topology on $M_{casbv}(\Sigma_I, \mathcal{L}(H))$ induced by the set wise convergence in the strong operator topology. We may denote this topology by τ_{ss} . Furnished with this topology, it is a locally convex sequentially complete topological vector space. Under these assumptions one can prove that $B \rightarrow J(B)$ is lower semicontinuous in the τ_{ss} topology. Thus J attains its minimum on Γ and so an optimal structural control exists.

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