Factorization of Motions

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1 Extended Abstract

We define *motion polynomials* as polynomials with coefficients in the dual quaternions and study their factorizations. The motion polynomials correspond to motions in 3D space, and factoring into linear factors means to compose the motion into translations and rotations. This allows to realize the motion by a linkage with revolute or prismatic joints. This is joint work with G. Hegedüs (Univ. Oboda), Z. Li (RICAM), and H.-P. Schröcker (Univ. Innsbruck). The results are published in [1]. This research has been supported by the Austrian Science Fund (FWF): DK W 1214-N15.

Let $\mathbb{H} = \langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle_{\mathbb{R}}$ be the skew field of quaternions. It is well known [2] that $\mathbb{H}$ is algebraically closed, in the sense that every univariate left polynomial $P \in \mathbb{H}[t]$ can be written as a product of linear polynomials. Here, the variable t is supposed to commute with the coefficients. To decompose P, one looks for right zeroes in $\mathbb{H}$: if $P(q) = 0$, then $(t - q)$ is a right factor of P, and the polynomial quotient has degree one less.

In order to find right zeroes, we compute the norm polynomial $N(t) = P(t)\overline{P}(t)$, where $\overline{P}$ is obtained by conjugating all coefficients. It is a real polynomial that does not assume negative values when evaluated at real numbers. Generically, it has no real zeroes, so that it can be written as a product of irreducible quadratic factors. For any such factor Q, there is a unique common right zero of P and Q in $\mathbb{H}$, and this common right zero can be computed by polynomial division: the polynomial remainder of P mod Q is linear.

The factorization algorithm can be extended to skew ring $\mathbb{DH} = \langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle_{\mathbb{D}}$ of dual quaternions, where $\mathbb{D} = \mathbb{R} \oplus \mathbb{R}\epsilon$ is the two-dimensional $\mathbb{R}$-algebra generated by $\mathbb{R}$ and ϵ with $\epsilon^2 = 0$. We are especially interested in polynomials with real

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norm polynomials; these polynomials are called *motion polynomials*. Since $\mathbb{D}$ is not a field, the algorithm may sometimes fail, so that there exist polynomials in $\mathbb{DH}[t]$ without factorization into linear ones. For generic motion polynomials of degree d, the algorithm works, and one gets $d!$ different factorizations into linear motion polynomials. (In contrast to the commutative case, it is not allowed to permute the factors.)

The special interest in motion polynomials comes from a well-known isomorphism of the six-dimensional Lie group SE_3 of Euclidean displacements and the multiplicative group of dual quaternions with nonzero real norm modulo multiplication by nonzero real scalars. Motions are curves in SE_3, and in this sense motion polynomials parameterize motions. Conversely, every motion that has a parameterization by rational functions can also be parameterized by a motion polynomial.

Linear motion polynomials parameterize revolutions around a fixed axis or translational pushes in fixed directions. Hence the factorization into linear motion polynomials decomposes the parameterized motion into revolutions or pushes, and the motion can be realized by a chain of revolute or prismatic joints.

A generic quadratic motion has two factorizations into two revolutions. The two chains of revolute joints can be combined to a movable closed chain with four links and four revolute joints. This linkage is called the Bennett linkage after its discoverer Bennett [3].

For $d > 2$, a generic motion of degree d can be decomposed into d revolutions in $d!$ different ways. Again, it is possible to combine the corresponding chains into one linkage. For instance, for $d = 3$ we obtain a movable linkage with 8 links connected by 12 revolute joints. Since the 6 decompositions are in relation to the permutations of the 3 irreducible factors of the norm polynomial of the parameterizing cubic motion polynomial, and the group of permutations is generated by transpositions, one can construct all decompositions by composing two neighboring revolutions and decomposing in the second way, as above. We call this operation *Bennett flip*.

By multiplying linear motion polynomials and applying Bennett flips, one can construct various families of closed overconstrained linkages. For instance, let us multiply two linear motion polynomials parameterizing revolutions around the same axes and a third linear motion polynomial; then we do two Bennett flips and construct a closed 5R linkage. This linkage is called the Goldberg 5R linkage after its discoverer [4]. It was shown in [5] that the Goldberg 5R linkage is the only movable 5R linkage with all 5 joints actually moving that is neither planar nor spherical.

Similar constructions lead to various families of 6R linkages, some well known, and some new. It should be mentioned that the classification of closed 6R linkages is a famous open problem in kinematics. Not all known families have a motion that can be rationally parameterized. An upper bound for the genus of the motion of a 6R linkage is given in [6].

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