Characters of Modules of Irrational Vertex Algebras

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Abstract We review several properties of characters of vertex algebra modules in connection to q-series and modular-like objects. Four representatives of conformal vertex algebras: regular, C_2-cofinite, tamely irrational and wild, are discussed from various points of view.

1 Introduction

Unlike many algebraic structures, vertex algebras have for long time enjoyed natural and fruitful connection with modular forms. This connection came first to light through the monstrous moonshine, a fascinating conjecture connecting modular forms (or more precisely the Hauptmodulns) and representations of the Monster, the largest finite sporadic simple group. This mysterious connection was partially explained first in the work of Frenkel et al. [37] who constructed a vertex operator algebra $V^{\natural}$, called the moonshine module, whose graded dimension is $j(q) - 744$ and whose automorphism group is the Monster. The connection with McKay-Thompson series was later proved by Borcherds [21] thus proving the full Conway-Norton conjecture. What is amazing about the vertex algebra $V^{\natural}$ is that on one hand it is arguably one of the most complicated objects constructed in algebra, yet it has an extremely simple representations theory (that of a field!).

Another important closely related concept in vertex algebra theory (and two-dimensional conformal field theory) is that of modular invariance of characters. This property, proposed by physicists as a consequence of the axioms of rational conformal field theory, was put on firm ground first in the seminal work of Zhu [62]. Among many applications of Zhu's result we point out its power to "explain"

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modular invariance of characters of integrable highest weight modules for affine Kac-Moody Lie algebras, discovered previously by Kac and Peterson [48]. As rational vertex algebras (with some extra properties) give rise to Modular Tensor Categories [45], this rich underlying structure can be used for to prove the general Verlinde formula [45] (see [20] for definition). This formula, discovered first by Verlinde [61], gives an important connection between the fusion coefficients in the tenor product coefficients of the S-matrix coming from both categorical and analytic $SL(2,\mathbb{Z})$-action on the "space" of modules. In addition, it also gives a fascinating link between analytic q-dimensions and the coefficients of the S-matrix.

There are other important connections between two subjects such as ADE classification of modular invariant partition functions, vertex superalgebras and mock modular forms, orbifold theory, elliptic genus, generalized moonshine, etc.

Everything that we mentioned so far comes from a very special class of vertex algebras called C_2-cofinite rational vertex algebras [62], the moonshine module being a prominent example. In this note we do not try to say much about rational vertex algebras (although we do give some definition and list known results) and almost nothing about the moonshine. Our modest goal is simply to argue that even non-rational (and sometimes even non C_2-cofinite) vertex algebra seem to enjoy properties analogous to properties of rational VOA, but much more complicated, yet reach enough that exploring them leads to some interesting mathematics related to modular forms and other modular-like objects. Another pedagogical aspect of these notes is to convey some ideas and aspects of the theory rarely considered in the literature on vertex algebras. We focus on four different types of vertex algebras:

- rational C_2-cofinite or *regular* (the category of modules has modular tensor category structure, q-dimensions are closely related to categorical dimensions).
- irrational C_2-cofinite (tensor product theory and a version modular invariance are available, a Verlinde-type formula is still to be formulated and proved)
- non C_2-cofinite, mildly irrational (there is evidence of braided tensor category structure on the category of module, or suitable sub-category. A version of modular invariance holds with continuous part added. Usually involve atypical and typical modules, the latter parametrized by continuous parameters. q-dimensions of irreps are finite and nonzero).
- non C_2-cofinite, badly irrational (not likely to have good categorical structure. For example two modules under fusion can give infinitely many modules. Consequently, q-dimensions may be infinite).

As a working example of rational C_2-cofinite vertex algebra we shall use the lattice vertex algebra V_L, where L is an even positive definite lattice. This is, from many different points of view, the most important source of vertex algebras, and in particular leads to the moonshine module via the Leech lattice and orbifolding.

When we move beyond rational vertex algebras, many difficulties arise, and this transition really has to be done in two steps. The nicest examples worth exploring are of course C_2-cofinite vertex algebras. These vertex algebras admit finitely-many inequivalent irreducible modules. Here the most prominent example is triplet vertex algebra [4, 33, 39, 50] being a conformal vertex subalgebra of the rank one lattice vertex algebra of certain rational central charge. Another prominent example is the symplectic fermion vertex superalgebra [1, 50, 59]. As we shall see the triplet vertex algebra enjoys many interesting properties including a version of modular invariance, even a conjectural version of the Verlinde formula.

If we move one step lower in the hierarchy this leads us to non C_2-cofinite vertex algebras. There are at least several candidates here. One is of course the vertex algebra associated to free bosons, called the Heisenberg vertex algebra [37, 49, 51]. Because this algebra has a fairly simple representation theory [37] we decided to consider another family of irrational vertex algebras—certain subalgebras of the Heisenberg vertex algebra. As we shall see this so-called "singlet" vertex algebras involve two types of irreducible representations: typical and atypical, something that persists for many $\mathscr{W}$-algebras. Quite surprisingly, there is a version of modular invariance for the singlet family, including a Verlinde-type formula inferred from the characters [24].

Finally, at the bottom of the barrel sort of speaking, we are left with badly behaved irrational conformal vertex algebra, namely those that are vacuum modules for the Virasoro algebra (or more general affine $\mathscr{W}$-algebras [18]) or for affine Lie algebras [49, 51]. One reason for this type of vertex algebra not being very interesting is due to lack of modular-like properties. Also, their fusion product is somewhat ill behaved. For example, two irreducible modules can produce infinitely many non-isomorphic modules under the fusion.

Four examples representing four types entering our discussion are connected with a chain of VOA embeddings:

$$\boxed{L(c_{p,1},0) \hookrightarrow W(2,2p-1) \hookrightarrow W(p) \hookrightarrow V_{\sqrt{2p}}\,.}$$

At the end of the paper we show that this diagram can be extended to an arbitrary ADE type simple Lie algebra, the above diagram being the simplest instance coming from $\mathfrak{sl}_2$.

2 Vertex Algebras and Their Characters

We begin by recalling the definition of a vertex operator algebra following primarily [51] (cf. [37, 49]).

Definition 1. A vertex operator algebra is a quadruple $(V, Y, \mathbf{1}, \omega)$ where V is a $\mathbb{Z}$-graded vector space

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}$$

together with a linear map $Y(\cdot, x) : V \to (\mathrm{End}\, V)[[x, x^{-1}]]$ and two distinguished elements $\mathbf{1}$ and $\omega \in V$, such that for $u, v \in V$ we have

$$Y(u, x)v \in V((x)),$$

$$Y(\mathbf{1}, x) = 1,$$

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v,$$

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}c$$

for $m, n \in \mathbb{Z}$, where

$$Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2},$$

and $c \in \mathbb{C}$ (the so-called *central charge*); we also have

$$L(0)w = nw \qquad \text{for } n \in \mathbb{Z} \text{ and } v \in V_{(n)},$$

and the $L(-1)$-axiom

$$Y(L(-1)u, x) = \frac{d}{dx}Y(u, x)$$

and the following Jacobi identity

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1) \tag{1}$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y\,(Y(u, x_0)v, x_2)\,. \tag{2}$$

If we omit the Virasoro axiom and the grading the structure is called vertex algebra, but we have to replace $L(-1)$-axiom with the D-derivative axiom [51]. In some constructions it is useful to have another VOA structure on the same space. This is important when we pass to a different coordinate system on the torus E_τ discussed below. With $Y(u, x)$ as above and u homogeneous, we let

$$Y[u, x] = Y(e^{x\deg(u)}u, e^x - 1),$$

which is well-defined if we expand $1/(e^x - 1)^m$, for $m \geq 0$, in finitely many negative powers of x. Then it can be shown [62] that $(V, Y[\cdot, x], \mathbf{1}, \omega - \frac{c}{24}\mathbf{1})$ is also a vertex operator algebra isomorphic to the original one. We also define bracket modes of vertex operator

$$Y[u,x] = \sum_{n\in\mathbb{Z}} u[n]x^{-n-1}; \;\; u[n] \in End(V).$$

Definition 2 (Sketch). We say that a vector space W together with a linear map

$$Y_W(\cdot, x) : V \to (\text{End } W)[[x, x^{-1}]]$$

is a weak V-module if Y_W satisfies the Jacobi identity, and "all other defining properties of a vertex algebra that make sense hold". If in addition the space is graded by $L(0)$-eigenvalues such that the grading is compatible with that of V, we say that M is an ordinary module.

Not all vertex algebra modules are of interest to us right now.

Definition 3. An admissible V-module is a weak V-module M which carries a $\mathbb{Z}_{\geq 0}$-grading

$$M = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} M(n)$$

satisfying the following condition: if $r, m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}$ and $a \in V_r$ then

$$a_m M(n) \subseteq M(r+n-m-1). \tag{3}$$

We call an admissible V-module M irreducible in case 0 and M are the only submodules. An ordinary module is an admissible module where the above grading is decomposition into finite-dimensional $L(0)$-eigenspaces.

A vertex algebra V is called *rational* if every admissible V-module is a direct sum of simple admissible V-modules. That is, we have complete reducibility of admissible V-modules. Observe that the definition of rationality does not seem to involve any internal characterization or property of vertex algebras. The next definition is analogous to "finite-dimensionality" for associative algebras.

Definition 4 (C_2-cofiniteness). A vertex algebra V is said to be C_2-cofinite if the space generated by vectors $\{a_{-2}b, \;\; a, b \in V\}$ is of finite codimension (in V).

An important consequence of this definition is that a C_2-cofinite vertex algebra has finitely many irreducible modules up to equivalence, which explains "finite-dimensionality" hinted earlier. It is a conjecture that every rational vertex algebra is C_2-cofinite, but the converse is known not to be true (see below).

2.1 One-Point Functions on Torus

To an admissible V-module M with finite dimensional graded subspaces we can associate its modified *graded dimension* or simply *character* [62]:

$$\text{ch}_M(q) := \text{tr}_M q^{L(0)-c/24}, \tau \in \mathbb{H},$$

where c is the central charge. Strictly speaking this function does not necessarily converge so it should be viewed only formally, but in almost all known examples it is holomorphic in the whole upper half-plane.

We are also interested in related graded traces that can be computed on M:

$$\mathrm{tr}_M o(a) q^{L(0)-c/24},$$

where $o(a) = a(\deg(a) - 1)$ is the zero weight operator and a is homogeneous, in the sense that it preserved graded components.

As usual we denote by

$$G_{2k}(\tau) = \frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n \geq 1} \frac{q^n n^{2k-1}}{1-q^n}$$

($k \geq 1$) slightly normalized Eisenstein series as in [62] given by their q-expansions.

Denote by $O_q(V)$ the $\mathbb{C}[G_4, G_6]$-submodule of $V \otimes \mathbb{C}[G_4, G_6]$ generate by

$$a[0]b$$

$$a[-2]b + \sum_{k=2}^{\infty} (2k-1) a[2k-2]b \otimes G_{2k}(\tau)$$

Definition 5. Let V be a VOA. A map $S(-,-) : V[G_2, G_4] \times \mathbb{H} \to \mathbb{C}$ satisfying the following conditions is called a one-point function on the torus $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$:

(1) For any $a \in V \otimes \mathbb{C}[G_4, G_6]$ the functions $S(a, \tau)$ is holomorphic in $\tau \in \mathbb{H}$.
(2) $S(\sum_i v_i \otimes f_i(\tau), \tau) = \sum_i f_i(\tau) S(a_i, \tau)$ for all $a_i \in V$ and $f_i \in \mathbb{C}[G_4, G_6]$.
(3) $S(a, \tau) = 0$ for all $a \in O_q(V)$,
(4) $S(L[-2]a, \tau) = (q\frac{d}{dq}) S(a, \tau) + \sum_{k=1}^{\infty} G_{2k}(\tau) S(L[2k-2]a, \tau)$.

We denote the space of one-point functions by $\mathscr{C}(V)$. Then any element of the form $S(\mathbf{1}, \tau)$, where $S \in V$, is called a (virtual) *generalized character*. It is possible to show [62], that graded traces $\mathrm{tr}_M o(a) q^{L(0)-c/24}$ give a one-point function on the torus. So in particular an (ordinary) character can be viewed as a generalized character.

Let us explain results pertaining rational vertex algebras first. We denote by M_i, $i \in I$ irreducible V-modules (so I is finite). Later we shall also assume that $i = 0 \in I$ is reserved for the VOA itself, which is also assumed to be simple. We shall also use $\mathrm{Irrep}(V)$ to denote the set of equivalence classes of irreducible V-module.

Theorem 1 (Zhu). *Let V be a rational C_2-cofinite vertex algebra. Then for every homogeneous $a \in V$ with respect to $L[0]$, the expressions $\{\mathrm{tr}_{M_i} o(a) q^{L(0)-c/24}\}$, $i \in I$ defines a vector valued modular form of weight $\deg(a)$. In particular, for $a = \mathbf{1}$ this weight is zero. Moreover, the space of one point functions on torus is $|Irrep(V)|$-dimensional and $a \mapsto \mathrm{tr}_{M_i} o(a) q^{L(0)-c/24}$, $i \in I$, is a basis of $\mathscr{C}(V)$.*

Observe that another consequence of this result is that for rational C_2-cofinite vertex algebras every generalized character is an ordinary character.

Because the category of modules of rational vertex algebras has a semisimple braided tensor category structure [47], we have the fusion product:

$$M_i \boxtimes M_j = \sum_{k \in I} N_{ij}^k M_k,$$

where $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ are the fusion coefficients and $\boxtimes$ is Huang, Lepowsky and Zhang's tensor product [47]. On the other hand, the previous theorem furnishes us with a $|Irrep(V)|$-dimensional representations of $SL(2,\mathbb{Z})$ acting on the space of ordinary characters. In particular, we have the special matrix $S \in SL(2,\mathbb{Z})$, called the S-matrix, corresponding to $\tau \to -\frac{1}{\tau}$. If in addition, the vertex algebra is C_2-cofinite the category of V-Mod is a modular tensor category (an important result of Huang [45]), so it also admits a categorical action of $SL(2,\mathbb{Z})$ on the space generated by the equivalence classes of irreducible modules $M_i, i \in I$. In particular $\tau \to -\frac{1}{\tau}$ induces a matrix called the s-matrix. In turns out that $S = s$ [29], after suitable rescaling of s. One important property of MTCs is the Verlinde formula [20] (first formulated in [61]) that allows us to express fusion coefficients simply from the coefficients of the s (and hence S) matrix. The precise statement is: Denote by N_{ij}^k the fusion coefficients, then we have

$$N_{ij}^k = \sum_r \frac{S_{ir} S_{jr} S_{k^* r}}{S_{0r}}, \tag{4}$$

where $r \mapsto r^*$ is the map on indices induced by taking dual of irreducible modules $M_i \mapsto M_i^*$.

Another related important notion in two-dimensional conformal field theory is that of (analytic) q-dimension. For a V-module M we let

$$q\dim(M) = \lim_{y \to 0+} \frac{\mathrm{ch}_M(iy)}{\mathrm{ch}_V(iy)} \tag{5}$$

Of course, such a quantity may not need even exist. But again, for V rational and C_2-cofinite, it is known to be closely related to categorical q-dimension $\dim_q(M)$, computed as the trace of the identity endofunctor, which also equals $\frac{s_{i0}}{s_{00}}$ [20]. Under some favorable conditions on the vertex algebra, this categorical version of the q-dimension coincide with the analytic (see [29], conditions (V1) and (V2) and formula (3.1)):

Proposition 1. *Let V be a rational C_2-cofinite VOA with lowest conformal weights of irreducible modules positive expect for $i = 0$, then*

$$dim_q(M_i) = \frac{S_{i0}}{S_{00}} = qdim(M_i).$$

This proposition is known to hold

Categorical q-dimensions are known to have good properties with respect to tensor products and direct sums:

$$dim_q(M \boxtimes N) = dim_q(M) \cdot dim_q(N),$$
$$dim_q(M \oplus N) = dim_q(M) + dim_q(N).$$

If V is only C_2-cofinite, we shall see in the next sections that Zhu's modular invariance theorem fails and not every one point function on the torus is an ordinary trace. This is closely related to non-semisimplicity of Zhu's algebra $A(V) = V/O(V)$, where $O(V)$ is spanned by $\mathrm{Res}_x \frac{(1+x)^{\deg(a)}}{x^2} Y(a,x)b$. Rationality implies that the space of one-point functions on the torus is isomorphic to the vector space of symmetric functions on $A(V)$:

$$S^V = (A(V)/[A(V), A(V)])^*.$$

But in general this space does not carry a precise description of one-point functions. Still, there is a satisfactory result essentially due to Miyamoto [56]. Assume for completeness that the central charge of the vertex algebra is non-zero (so finite-dimensional V-modules are excluded—these only appear for $c = 0$). Then there is a connection between one-point functions and the Zhu algebra (Miyamoto).

Theorem 2. *The vector space $\mathscr{C}(V)$ admits a finite basis $\mathscr{B}$ such that each $S \in \mathscr{B}$ admits an expansion*

$$S(a,\tau) = \sum_{j=0}^{d} \sum_{k=0}^{\infty} S_{jk}(a) q^{r-c_c/24+k} (2\pi i \tau)^j$$

for all $a \in V$, where $r \in \mathbb{C}$ and $S_{00} \in S^V$, a symmetric linear functional on $A(V)$. Moreover, $S \mapsto S_{00}$ is an embedding. In particular, the dimension of $\mathscr{C}(V)$ is bounded by the dimension of S^V.

This version of the theorem is proven in [19], but something similar is implicitly used in [56] (see also [6]). One striking feature of the theorem is the appearance of τ-powers, so no q-expansion of one point functions exists in general. This is closely tied to existence of $L(0)$ non-diagonalizable modules, called logarithmic modules [40,53]. For more about this subject and connection to Logarithmic Conformal Field Theory we refer the reader to another review paper [13,40,44], as we do not discuss this subject here. In the aforementioned paper of Miyamoto, he constructs $S(a,\tau)$ via certain *pseudotraces* maps ϕ expressed as $\mathrm{tr}^{\phi,M} o(a) q^{L(0)-c/24}$ where M is a particular module "interlocked" with ϕ. We should point out that in many examples of interest this object is hard to construct explicitly. A slightly more efficient way of constructing one-point functions was obtained by Arike-Nagatomo's paper [19], although it is not clear whether their construction works in general.

3 Rational VOA: Lattice Vertex Algebras

We review the construction of a vertex operator algebra coming from an even lattice following [51] (see also [37, 49]). Let L be a rank $d \in \mathbb{N}$ even positive definite lattice of rank $d \in \mathbb{N}$ with an integer valued nondegenerate symmetric $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle : L \times L \to \mathbb{Z}$.

Form the vector space

$$\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C} \tag{6}$$

so that $\dim(\mathfrak{h}) = d$ and extend the bilinear form from L to $\mathfrak{h}$. Now we shall consider the affinization of $\mathfrak{h}$ viewed as an abelian Lie algebra

$$\hat{\mathfrak{h}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}\mathbf{k}, \tag{7}$$

with bracket relations

$$\begin{aligned}[][\alpha \otimes t^m, \beta \otimes t^n] &= \langle\alpha,\beta\rangle m\delta_{m+n,0}\mathbf{k} \\ [\mathbf{k}, \hat{\mathfrak{h}}] &= 0\end{aligned} \tag{8}$$

for $\alpha, \beta \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. Consider

$$\hat{\mathfrak{h}}_+ = \mathfrak{h} \otimes t\mathbb{C}[t] \ \text{ and } \ \hat{\mathfrak{h}}_- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]. \tag{9}$$

We now form a vertex operator algebra associated to $\hat{\mathfrak{h}}$ with central charge 1, $M(1)$, by adding structure to the symmetric algebra of $\hat{\mathfrak{h}}_-$. As vector spaces we have

$$M(1) = U(\hat{\mathfrak{h}}_-) = S(\hat{\mathfrak{h}}_-). \tag{10}$$

If we let $\{u^{(1)}, \ldots, u^{(d)}\}$ be an orthonormal basis of $\mathfrak{h}$ we define the conformal vector

$$\omega = \frac{1}{2}\sum_{i=1}^{d} u^{(i)}(-1)u^{(i)}(-1)\mathbf{1}. \tag{11}$$

So we have the Virasoro algebra operators

$$L(n) = \mathrm{Res}_x x^{n+1} Y(\omega, x) = \frac{1}{2}\sum_{i=1}^{d}\sum_{m\in\mathbb{Z}} {}^{\circ}_{\circ} u^{(i)}(m)u^{(i)}(n-m) {}^{\circ}_{\circ}. \tag{12}$$

It is easy to construct irreducible $M(1)$-modules. Those are simply Fock spaces F_λ where $\lambda \in \mathfrak{h}^*$. This is again just an induced module such that $h \in \mathfrak{h}$ acts on the highest weight vector as multiplication by $\lambda(h)$, so as a vector space $F_\lambda \cong M(1)$ and $F_0 = M(1)$. This space will again become relevant in later sections.

The space $S(\hat{\mathfrak{h}}_-)$ makes up one part of the vertex operator algebra associated with L. The other portion is related to the group algebra $\mathbb{C}[L]$. In order to ensure the Jacobi identity we need to modify the product associated to $\mathbb{C}[L]$ so that

$$e^\alpha e^\beta = (-1)^{\langle\alpha,\beta\rangle} e^\beta e^\alpha, \tag{13}$$

for $\alpha, \beta \in L$. To accomplish this we use a central extension, $(\hat{L}, \bar{\ })$, of L by the cyclic group $\langle\kappa|\kappa^2 = 1\rangle$. For $\alpha, \beta \in L$ define the map

$$c_0 : L \times L \to \mathbb{Z}/2\mathbb{Z} \tag{14}$$

as follows:

$$\begin{aligned} c_0(\alpha,\alpha) &= 0 + 2\mathbb{Z}, \\ c_0(\alpha,\beta) &= \langle\alpha,\beta\rangle + 2\mathbb{Z} \text{ and,} \\ c_0(\beta,\alpha) &= -c_0(\alpha,\beta). \end{aligned} \tag{15}$$

This is indeed the commutator map associated to the central extension of the lattice. It may also be uniquely defined by the condition $ab = \kappa^{c_0(\bar{a},\bar{b})} ba$ for $a, b \in \hat{L}$. Define a section of $\hat{L}$, $e : L \to \hat{L}$, so that $\alpha \mapsto e_\alpha$. So e is such that $\bar{\ } \circ e = id_L$. Let

$$\epsilon_0 : L \times L \to \mathbb{Z}/2\mathbb{Z} \tag{16}$$

be the *corresponding* 2-*cocyle*, defined by

$$e_\alpha e_\beta = \kappa^{\epsilon_0(\alpha,\beta)} e_{\alpha+\beta} \tag{17}$$

Let $\chi : \langle\kappa\rangle \to \mathbb{C}^\times$ be defined by $\chi(\kappa) = -1$. View $\mathbb{C}$ as a $\langle\kappa\rangle$-module where κ acts as -1 and denoted this module as $\mathbb{C}_\chi$. Define

$$\mathbb{C}\{L\} = \mathrm{Ind}_{\langle\kappa\rangle}^{\hat{L}} \mathbb{C}_\chi = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\langle\kappa\rangle]} \mathbb{C}_\chi = \mathbb{C}[\hat{L}]/(\kappa - (-1))\mathbb{C}[\hat{L}]. \tag{18}$$

Let ι be the inclusion $\hat{L} \hookrightarrow \mathbb{C}\{L\}$ such that $\iota(a) = a \otimes 1$. Notice our section e allows us to view $\mathbb{C}\{L\}$ and $\mathbb{C}[L]$ as isomorphic vector spaces with $\iota(e_\alpha) \mapsto e^\alpha$ for $\alpha \in L$.

Now define maps $c, \epsilon : L \times L \to \mathbb{C}^\times$ by $c(\alpha,\beta) = (-1)^{c_0(\alpha,\beta)}$ and $\epsilon(\alpha,\beta) = (-1)^{\epsilon_0(\alpha,\beta)}$. Now we can see the action of $\hat{L}$ on $\mathbb{C}[L]$, for $\alpha, \beta \in L$

$$\begin{aligned} e_\alpha \cdot e^\beta &= \epsilon(\alpha,\beta) e^{\alpha+\beta} \\ \kappa \cdot e^\beta &= -e^\beta \\ e_\alpha \cdot 1 &= e^\alpha \end{aligned} \tag{19}$$

Now set

$$V_L = M(1) \otimes \mathbb{C}\{L\} \tag{20}$$

and

$$\mathbf{1} = 1 \otimes \iota(1) \in V_L. \tag{21}$$

We now add more structure to the space V_L. First we will view $M(1)$ as a trivial $\hat{L}$-module, so that for $\alpha \in L$, e_α acts as $1 \otimes e_\alpha \in End(V_L)$. Also view $\mathbb{C}\{L\}$ as a trivial $\hat{\mathfrak{h}}_*$-module and for $h \in \mathfrak{h}$, define

$$h(0) : \mathbb{C}\{L\} \to \mathbb{C}\{L\} \ \text{ so that } \ \iota(a) \mapsto \langle h, \bar{a} \rangle\, \iota(a) \tag{22}$$

for $a \in \hat{L}$. By making the identification $M(1) \cong S(\hat{\mathfrak{h}}) \otimes e^0$ we can transport the structure of a Virasoro algebra module to V_L with the grading given by the action of $L(0)$

$$L(0) \cdot \iota(e_\alpha) = (\mathrm{wt}\, \iota(e_\alpha))\, \iota(e_\alpha) = \frac{1}{2}\, \langle \alpha, \alpha \rangle\, \iota(e_\alpha). \tag{23}$$

We keep the same conformal vector so the central charge of V_L is $\mathrm{rank}(L)$.

In order to define the vertex operator $Y(\iota(e_\alpha), x)$ we need the following operator for $h \in \mathfrak{h}$,

$$E^{\pm}(-h, x) = \exp\left(\sum_{n \in \pm\mathbb{Z}} \frac{-h(n)}{n} x^{-n} \right) \in (\mathrm{End}\ V_L)[[x, x^{-1}]]. \tag{24}$$

and define

$$Y(\iota(e_\alpha), x) = E^{-}(-\alpha, x) E^{+}(-\alpha, x) e_\alpha x^{\alpha} \in (\mathrm{End}\ V_L)[[x, x^{-1}]]. \tag{25}$$

where x^α acts on V_L as

$$x^\alpha (v \otimes \iota(a)) = x^{\langle \alpha, \bar{a} \rangle} (v \otimes \iota(a)). \tag{26}$$

This explains how to construct lattice vertex algebra structure on V_L. If the lattice is of rank one, no central extension is needed. Thus $\mathbb{C}[L] = \mathbb{C}\{L\}$. Also, to simplify the notation we shall write e^α instead of $\iota(e_\alpha)$, where no confusion arise. Everything about representation theory of lattice vertex algebras can be summarized in the following elegant result by Dong (see [27] and [51] for instance):

Theorem 3. *The vertex algebra V_L is rational. Moreover, the set*

$$\{V_{L+\lambda}; \lambda + L \in L^\circ / L\}$$

(where L^0 is the dual lattice) is a complete set of inequivalent irreducible V_L-modules (strictly speaking, we never defined $V_{L+\lambda}$ but this is easily done by replacing $\mathbb{C}[L]$ in the definition with $\mathbb{C}[L+\lambda]$. For a full account on this see [51]).

Characters of V_L-modules are easily determined (keep in mind $c = \mathrm{rank}(L)$). We have

$$\mathrm{ch}_{V_{L+\lambda}}(q) = \frac{\sum_{\alpha \in L+\lambda} q^{\langle \alpha,\alpha \rangle/2}}{\eta(\tau)^c}.$$

By using a well-known formula for the modular transformation formula for the higher rank theta function, we infer

$$\mathrm{ch}_{V_{L+\lambda}}(-\frac{1}{\tau}) = \sum_{\bar{\nu} \in L^\circ/L} S_{\lambda\nu} \mathrm{ch}_{V_{L+\nu}}(\tau),$$

where $S_{\lambda\nu}$ denote the S-matrix of the transformation. Observe that $S_{0\nu} = \frac{1}{\sqrt{det(S)}}$, where S is the Gram matrix of L. This modular invariance part also follows from Zhu's theorem (the vertex algebra V_L is C_2-cofinite). The fusion product for the lattice vertex algebras is simply

$$V_{L+\lambda} \boxtimes V_{L+\nu} = V_{L+\lambda+\nu}.$$

The q-dimensions are also easy to compute and $dim_q(V_{L+\lambda}) = 1$ for all λ.

4 C_2-Cofinite Irrational Case: The Triplet VOA

In this section we examine properties of a specific irrational C_2-cofinite vertex algebra.

4.1 *The Triplet*

Let V_L be as in the previous section, where L is of rank one. First we construct a subalgebra of V_L called the triplet algebra. We should point out that lattice vertex algebra are rarely mentioned in the physics literature, where triplet is usually treated as an extended conformal algebra with $SO(3)$ symmetry [41, 42, 50] , or as a part of an extended Felder's complex in which we extract the kernel instead of cohomology. Our approach here is slightly different and it follows [4, 33, 39], where the triplet algebra is constructed as kernel of a screening operator acting (as we shall see) among two V_L-modules.

Let $p \in \mathbb{Z}$, $p \geq 2$, and

$$L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 2p,$$

or simply $L = \sqrt{2p}\mathbb{Z}$, with the usual multiplication. We are interested in the central charge $c_{p,1} = 1 - \frac{6(p-1)^2}{p}$, so we choose

$$\omega = \frac{1}{4p}\alpha(-1)^2\mathbf{1} + \frac{p-1}{2p}\alpha(-2)\mathbf{1}.$$

We also define conformal weights

$$h_{r,s}^{p,q} = \frac{(ps - rq)^2 - (p-q)^2}{4pq}.$$

With this central charge, the generalized vertex algebra V_{L° [28] admits two screenings:

$$\tilde{Q} = e_0^{-\alpha/p} \text{ and } Q = e_0^{\alpha}.$$

Then we let

$$\mathscr{W}(p) = \mathrm{Ker}_{V_L} e_0^{-\alpha/p} \subset V_L, \tag{27}$$

a subalgebra of V_L called the triplet algebra.

The above construction can be recast in terms of automorphisms of infinite order and *generalized twisted* modules introduced by Huang [46]. Consider $\nu = \exp(e_0^{-\alpha/p})$. This operator does not preserve V_L but it can be viewed as an automorphism of V_{L° of infinite order. Then the triplet is $V_{L^\circ}^{\nu} \cap V_L$, where V_L^{ν} denote the ν-fixed vertex subalgebra. In fact, V_{L° can be also replaced by $V_L \oplus V_{L-\alpha/p}$ (see [7]).

As shown in [4], $\mathscr{W}(p)$ is strongly generated by the conformal vector ω and three *primary* vectors

$$F = e^{-\alpha},\ H = QF,\ E = Q^2 e^{-\alpha}.$$

There is another useful description of $\mathscr{W}(p)$ [32, 39]. As a module for the Virasoro algebra, V_L is not completely reducible but it has a semisimple filtration whose maximal semisimple part is $\mathscr{W}(p)$. More precisely,

$$\begin{aligned} \mathscr{W}(p) &= \mathrm{soc}_{Vir}(V_L) \\ &= \bigoplus_{n=0}^{\infty} \bigoplus_{j=0}^{2n} U(Vir).Q^j e^{-n\alpha} \\ &\cong \bigoplus_{n=0}^{\infty} (2n+1) L(c_{p,1}, h_{1,2n+1}^{p,1}),, \end{aligned} \tag{28}$$

where $L(c,h)$ denote the highest weight Virasoro module of central charge c and lowest conformal weight h. For other examples of irrational C_2-cofinite vertex (super)algebras see [5, 11, 12, 14–16] .

4.2 Irreducible Modules and Characters

The triplet $\mathscr{W}(p)$ is known to be C_2-cofinite but irrational [4] (see also [23]). It also admits precisely $2p$ inequivalent irreducible modules [4] which are usually denoted by:

$$\Lambda(1),\ldots,\Lambda(p),\Pi(1),\ldots,\Pi(p).$$

These modules were previously studied in [33, 34, 39] was proposed as a complete list of irreducibles. Since irreps are admissible, for $1 \leq i \leq p$, the top component of $\Lambda(i)$ is one-dimensional and has lowest conformal weight $h_{i,1}^{p,1}$, and the top component of $\Pi(i)$ is two-dimensional with conformal weight $h_{3p-i,1}^{p,1}$.

The characters of irreducible $\mathscr{W}(p)$-modules are well-known and computed in many papers on logarithmic conformal field theories starting with [39]. For $1 \leq i \leq p$, the formulas are

$$\begin{aligned}\mathrm{ch}_{\Lambda(i)}(\tau) &= \frac{i\Theta_{p,p-i}(\tau)+2\partial\Theta_{p,p-i}(\tau)}{p\eta(\tau)},\\ \mathrm{ch}_{\Pi(i)}(\tau) &= \frac{i\Theta_{p,i}(\tau)-2\partial\Theta_{p,i}(\tau)}{p\eta(\tau)},\end{aligned} \tag{29}$$

where

$$\Theta_{i,p}(\tau)=\sum_{n\in\mathbb{Z}}q^{(2np+i)^2/4p},\quad \partial\Theta_{i,p}(\tau)=\sum_{n\in\mathbb{Z}}(n+\frac{i}{2p})q^{(2np+i)^2/4p}.$$

From here we infer that the space spanned by characters of irreps is not modular invariant! To understand this better observe that in addition to $\Theta_{p,i}$ and $\partial\Theta_{p,i}$ series we also need $\tau\partial\Theta_{p,i}$ series to preserve modularity. This gives indication that one-point functions on the torus might be bigger than the number of irreps.

The next theorem (essentially taken from [6]) settles the problem of finding the space of one-point functions on the torus for the triplet algebra.

Theorem 4. *The space of one-point functions for the triplet vertex algebra is* $3p-1$*-dimensional.*

The proof breaks down on studying generalized characters. By using general properties of one-point functions and the triplet vertex algebra we first prove that every generalized character $S(\mathbf{1},\tau)$ satisfies

$$D^{3p-1}S(\mathbf{1},\tau) + \sum_{i=0}^{3p-2} H_i(q)D^i S(\mathbf{1},\tau) = 0, \tag{30}$$

where

$$H_i(q) \in \mathbb{C}[G_4, G_6]_{2h-2i}$$

is a modular form of weight $2h - 2i$. and

$$D_h = (q\frac{d}{dq}) + hG_2(q)$$

where $h \in \mathbb{Z}_{\geq 0}$ and

$$D^n := D_{2n-2} \cdots D_2 D_0.$$

This fact immediately implies several things. First, because the space of solutions of the differential equations is modular invariant, the space of generalized characters is at most $3p-1$-dimensional. But at the same time each ordinary trace associated to an irrep must be a solution to thus equation. So for modular invariance to be preserved the space is at least $3p-1$-dimensional. Therefore there must be contribution coming from $p-1$ generalized characters. Once we observe that $\dim(\mathscr{C}(V)) \leq \dim(S^{\mathscr{W}(p)})$, where the right hand side is known to be $3p-1$-dimensional by [9], we have the proof and observation that $\mathscr{C}(V)$ is as large as it can be. By using a method from [19] we can construct all the missing one-point functions explicitly.

4.3 Verlinde-Type Formula for $\mathscr{W}(p)$-Mod

As there is no general Verlinde formula for C_2-cofinite vertex algebras, in what follows "Verlinde-type formula" refers to the following concepts extracted from the (generalized) characters:

1. A way of constructing a genuine finite-dimensional $SL(2,\mathbb{Z})$ representation on the space of irreducible and possible larger generalized characters.
2. By using the S-matrix from (1), for a fixed triple i, j, k, the standard Verlinde sum,that is, the right hand-side of (4), recovers non-negative integers that agree with the known (or at least conjectural) fusion coefficients N_{ij}^k. Because the category of representation is semisimple these fusion coefficients should be understood as multiplicities in the Grothendieck ring.

We do not claim that there is a unique procedure for extracting the S-matrix here, so there might be more than one Verlinde-type formula giving the same answer.

Next, we outline a Verlinde-type formula for the triplet algebra obtained in [39], with some crucial modifications in [43]. We already listed all irreps earlier with their explicit characters. Form a $2n \times 1$ character vector

$$\chi_p(\tau) := (\mathrm{ch}_{\Lambda(p)}, \mathrm{ch}_{\Pi(p)}, \mathrm{ch}_{\Lambda(1)}, \mathrm{ch}_{\Pi(p-1)}, \cdots, \mathrm{ch}_{\Lambda(p-1)}, \mathrm{ch}_{\Pi(1)})^T,$$

where $(\cdot)^T$ stands for the transpose. Easy computation—by using modular transformation formulas for $\Theta_{p,i}$ and $\partial\Theta_{p,i}$—shows that

$$\chi_p(-\frac{1}{\tau}) = S_p(\tau) \cdot \chi_p(\tau), \quad \chi_p(\tau+1) = T_p(\tau) \cdot \chi_p(\tau),$$

where the entries of the matrix are computed by using the formula

$$\begin{aligned}\mathrm{ch}_{\Lambda(s)}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2p}}\Bigg\{ \frac{s}{p}\Bigg(&\mathrm{ch}_{\Lambda(p)}(\tau) + (-1)^{p-s}\chi_{\Pi(p)}(\tau) \\ &+ \sum_{s'=1}^{p-1} 2cos(\frac{2(p-s)s'}{p})(\mathrm{ch}_{\Lambda(p-s')}(\tau) + \mathrm{ch}_{\Pi(s')}(\tau))\Bigg) \\ - \sum_{s'=1}^{p-1}(-1)^{p+s+s'} 2sin(\frac{2ss'}{p}) i\tau(&\frac{p-s'}{p}\mathrm{ch}_{\Lambda(s')}(\tau) - \frac{s'}{p}\mathrm{ch}_{\Pi(p-s')}(\tau))\Bigg\}\end{aligned}$$

and a similar formula for $\chi_{\Pi(s)}(-\frac{1}{\tau})$. The matrix $T_p(\tau)$ is clearly independent of τ and diagonal (we omit its explicit form here). This way we do not obtain a $2p$-dimensional representation of the modular group due to τ-dependence. To fix this problem it is convenient to introduce a suitable automorphy factor $j(\gamma, \tau)$, $\gamma \in SL(2, \mathbb{Z})$, satisfying the cocycle condition

$$j(\gamma\gamma', \tau) = j(\gamma', \tau) j(\gamma, \gamma'\tau).$$

In addition, we can define $j(\gamma, \tau)$ such that the modified S,T-matrices

$$\mathbf{S}_p := j(S, \tau) S_p(\tau), \;\; \mathbf{T}_p := j(T, \tau) T_p(\tau),$$

do not depend on τ, so $\mathbf{S}_p$ and $\mathbf{T}_p$ define a genuine representation of $SL(2, \mathbb{Z})$. This was achieved explicitly in [39, Sect. 3]. Again we omit explicit formulas for $\mathbf{S}_p$ for brevity. Equipped with a right candidate for the S-matrix we are ready to compute the Verlinde sum

$$N_{ij}^k := \sum_{r \in \{0,\ldots,2p-1\}} \frac{\mathbf{S}_p(ir)\mathbf{S}_p(jr)\mathbf{S}_p(k^*r)}{\mathbf{S}_p(0r)}.$$

These numbers turn out to be non-negative integers, so we can form a free $\mathbb{Z}$-module generated by the equivalence classes of irreps, and on it we let

$$X_I \times X_J := \sum_K N_{IJ}^K X_K. \tag{31}$$

Theorem 5. *The previous product defines an associative ring structure. Moreover,*

$$\begin{aligned}
\Lambda(s) \times \Lambda(t) &= \sum_{r=|s-t|+1, by\ 2}^{min(s+t-1, 2p-s-t-1)} \Lambda(r) \oplus \bigoplus_{r=2p-s-t+1; step=2}^{p,p-1} P_r^+ \\
\Lambda(s) \times \Pi(t) &= \sum_{r=|s-t|+1, by\ 2}^{min(s+t-1, 2p-s-t-1)} \Pi(r) \oplus \bigoplus_{r=2p-s-t+1; step=2}^{p,p-1} P_r^- \\
\Pi(s) \times \Pi(t) &= \sum_{r=|s-t|+1, by\ 2}^{min(s+t-1, 2p-s-t-1)} \Lambda(r) \oplus \bigoplus_{r=2p-s-t+1; step=2}^{p,p-1} P_r^+,
\end{aligned} \tag{32}$$

where $P_r^{\pm}$ are given by

$$P_r^+ = 2\Lambda(r) + 2\Pi(p-r), \quad P_r^- = 2\Pi(r) + 2\Lambda(p-r) \tag{33}$$

and where the summation is up to $p-1$ or p depending on whether $r+s+t$ is even or odd, respectively.

Tsuchiya and Wood in [60] (see also [58]) proved that the above product recovers correct multiplication in the Grothendieck ring of the category $\mathscr{W}(p) - \mathrm{Mod}$ (this one exists thanks to [47]). Moreover, the $P_r^{\pm}$ summands in the formulas should be viewed as projective modules. The approach in [60] is based on the notion of fusion expressed as a certain space of coinvariants. Some special cases of the fusion rules are computed in [7] by using intertwining operators.

Observe also that for $X = \Pi$ or Λ and $1 \leq s \leq p$ we have

$$q\dim(X(s)) = s,$$

which can be easily verified by considering asymptotic properties of the given q-series [22]. It is a priori not clear if this agrees with the categorical q-dimension.

We conclude this section with a comment that we believe this pattern persists for other C_2-cofinite vertex algebras or at least those that are of CFT type and where the vertex algebra is simple. Moreover, we conjecture that in a favorable situation when $V - \mathrm{Mod}$ is rigid [57] the analytic q-dimension agrees with the categorical one. Rigidity in general seems to fail for $\mathscr{W}_{p,q}$ triplet vertex algebras studied in [7, 8, 10, 31, 32].

5 Beyond C_2-Cofinite Vertex Algebras

Very little is known about general categories of representations of irrational non C_2-cofinite vertex algebras (let alone any modularity-type properties!). We only focus on those vertex algebras with good categorical properties in the sense that they admit a subcategory where irreducibles and perhaps projective modules can be classified. An obvious candidate here is the Heisenberg vertex algebra $M(1)$ already discussed in the setup of lattice vertex algebras. The category of $\mathfrak{h}$-diagonalizable $M(1)$-modules is known to be semisimple and the irreps are F_λ, $\lambda \in \mathfrak{h}^*$ [37]. In other words, all irreducible modules are "generic". In addition, the formal fusion product is given by $F_\lambda \times F_\nu = F_{\lambda+\nu}$. A better candidate (in terms of richness of representations) for discuss here is the *singlet* vertex algebra [2, 3, 50], a proper subalgebra of the full rank one Fock space $M(1)$, so all Heisenberg algebra modules are already included. In addition the singlet is included inside the triplet algebra $\mathcal{W}(p)$. So in addition to Fock space modules, the singlet admits a special infinite family of representations that do not look like F_λ and come from decomposition of irreducible $\mathcal{W}(p)$-modules.

The setup is as in the previous section. We fix the central charge to be $c_{p,1}$ and choose the same conformal vector in $M(1)$. Following the notation from [2] (see also [3]), we define

$$\mathcal{W}(2, 2p-1) = \mathrm{Ker}_{M(1)} \tilde{Q}.$$

called the *singlet vertex algebra* of central charge $c_{p,1}$. Since $\tilde{Q}$ commutes with the action of the Virasoro algebra, we have

$$L(c_{p,1}, 0) \subset \mathcal{W}(2, 2p-1).$$

The vertex operator algebra $\mathcal{W}(2, 2p-1)$ is completely reducible as a Virasoro algebra module and the following decomposition holds:

$$\mathcal{W}(2, 2p-1) = \bigoplus_{n=0}^{\infty} U(Vir).\, u^{(n)}; \quad u^{(n)} = Q^n e^{-\alpha n} \cong \bigoplus_{n=0}^{\infty} L(c_{p,1}, n^2 p + np - n),$$

As shown in [2] (see also [3]) all irreducible $\mathcal{W}(2, 2p-1)$-modules are constructed as subquotients of the Fock spaces F_λ. What is peculiar about these irreps is that they come in two groups with very distinct features:

- (Typical or generic) Those isomorphic to irreducible Virasoro Fock spaces denoted by F_λ (it simply means that λ does not satisfy a certain integrability condition).
- (Atypical or generic) A certain family $M_{r,s}$ of subquotients of Fock spaces $F_{\frac{r-1}{2}\sqrt{2p}+\frac{s-1}{\sqrt{2p}}}$, $r \in \mathbb{Z}$, and $1 \leq s \leq p$. Each $M_{r,s}$ is isomorphic to an infinite direct sum of Virasoro irreps.

Each irrep $M_{r,s}$ decomposes as an infinite direct sum of irreducible Virasoro algebra (for explicit decomposition formulas see [4, 24]). This is then used to show:

$$\mathrm{ch}[M_{r,s}](\tau) = \frac{P_{pr-s,p}(0,\tau) - P_{pr+s,p}(0,\tau)}{\eta(\tau)},$$

where

$$P_{a,b}(u,\tau) = \sum_{n=0}^{\infty} z^{n+\frac{a}{2b}} q^{b(n+\frac{a}{2b})^2}, \quad z = e(u). \tag{34}$$

The last expression is what is usually called *partial theta function* of and its properties are well-recorded in the literature [17]. In particular, for $M_{1,1} = \mathscr{W}(2, 2p-1)$, we get

$$\mathrm{ch}[\mathscr{W}(2,2p-1)](\tau) = \frac{\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{p(n+\frac{p-1}{2p})^2}}{\eta(\tau)},$$

which is precisely false theta function of Rogers.

If we try to naively compute

$$P_{a,b}(\frac{u}{\tau}, -\frac{1}{\tau})$$

some divergent integrals appear, so instead we introduce a regularization, a method used in physics to handle divergent quantities.

Now, we define the regularized characters by introducing a parameter ϵ to achieve better modular properties. We let

$$\begin{aligned} \mathrm{ch}[F_\lambda^\epsilon](\tau) &= e^{2\pi\epsilon(\lambda-\alpha_0/2)} \frac{q^{(\lambda-\alpha_0/2)^2/2}}{\eta(\tau)} \\ \mathrm{ch}[M_{r,s}^\epsilon](u;\tau) &= \frac{1}{\eta(\tau)} \sum_{n=0}^{\infty} \mathrm{ch}[F^\epsilon_{\alpha_{r-2n-1,p-s}}](\tau) - \mathrm{ch}[F^\epsilon_{\alpha_{r-2n-2,s}}](\tau), \end{aligned} \tag{35}$$

where $\alpha_0 = \alpha_+ + \alpha_-$, $\alpha_+ = \sqrt{2p}$ and $\alpha_- = -\sqrt{2/p}$.

Observe that typical ϵ-regularized characters are simply $\mathrm{tr}_{F_\lambda} e^{2\pi\epsilon(\frac{\alpha(0)}{\sqrt{2p}}-\alpha_0/2)} q^{L(0)-c/24}$. But atypical regularization is more subtle and it has no obvious interpretation as graded trace. Let $\beta^\pm_{r,s} = ((r-1)\alpha_+ \pm s\alpha_-)/2$, then the atypical characters are

$$\begin{aligned} &\mathrm{ch}[M^\epsilon_{r,s}](\tau) \\ &\quad = \mathrm{ch}[F^\epsilon_{\alpha_0/2-\beta^-_{r,s}}](\tau) P_{\alpha_+\epsilon}(-\alpha_+\beta^-_{r,s}\tau; \alpha_+^2\tau) \\ &\qquad - \mathrm{ch}[F^\epsilon_{\alpha_0/2-\beta^+_{r,s}}](\tau) P_{\alpha_+\epsilon}(-\alpha_+\beta^+_{r,s}\tau; \alpha_+^2\tau). \end{aligned}$$

We can easily show that

$$\mathrm{ch}[F^{\epsilon}_{\lambda+\alpha_0/2}](\frac{-1}{\tau}) = \int_{\mathbb{R}} S^{\epsilon}_{\lambda+\alpha_0/2,\mu+\alpha_0/2}\mathrm{ch}[F^{\epsilon}_{\mu+\alpha_0/2}](\tau)d\mu,$$

with $S^{\epsilon}_{\lambda+\alpha_0/2,\mu+\alpha_0/2} = e^{2\pi\epsilon(\lambda-\mu)}e^{-2\pi i\lambda\mu}$.

The next result taken from [24] gives S-"matrix" expressed as a kernel.

Theorem 6.

$$ch[M^{\epsilon}_{r,s}]\Big(-\frac{1}{\tau}\Big) = \int_{\mathbb{R}} S^{\epsilon}_{(r,s),\mu+\alpha_0/2}\mathrm{ch}[F^{\epsilon}_{\mu+\alpha_0/2}](\tau)d\mu + X^{\epsilon}_{r,s}(\tau)$$

with

$$S^{\epsilon}_{(r,s),\mu+\alpha_0/2} = -e^{-2\pi\epsilon((r-1)\alpha_+/2+\mu)}e^{\pi i(r-1)\alpha_+\mu}\frac{\sin\big(\pi s\alpha_-(\mu+i\epsilon)\big)}{\sin\big(\pi\alpha_+(\mu+i\epsilon)\big)}$$

and

$$X^{\epsilon}_{r,s}(\tau) = \frac{1}{4i\eta(\tau)}(\mathrm{sgn}(\mathrm{Re}(\epsilon))+1)\sum_{n\in\mathbb{Z}}(-1)^{rn}e^{\pi i\frac{s}{p}n}q^{\frac{1}{2}(\frac{n^2}{\alpha_+}-\epsilon^2)}\big(q^{-i\epsilon n/\alpha_+^2}-q^{i\epsilon n/\alpha_+^2}\big).$$

5.1 Brewing a Verlinde-Type Formula

If we have a continuous type S-matrix as the one above, the right approach for defining fusion coefficients seems to be [25, 26]

$$\int_{\mathbb{R}}\frac{S^{\epsilon}_{a\rho}S^{\epsilon}_{b\rho}\overline{S^{-\bar{\epsilon}}_{\rho\mu}}}{S^{\epsilon}_{(1,1)\rho}}d\rho.$$

But this integral badly diverges, so we either have to pass to heuristic approach as in [25] and [26] where the integrals are interpreted as a sum of the Dirac delta functions, or we can simply change the order of integration so that the fusion coefficients are genuine distributions. Thus, we redefine the product in the Verlinde algebra of characters as

$$\mathrm{ch}[X_a]\times\mathrm{ch}[X_b] := \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{S^{\epsilon}_{a\rho}S^{\epsilon}_{b\rho}\overline{S^{-\bar{\epsilon}}_{\rho\mu}}}{S^{\epsilon}_{(1,1)\rho}}\mathrm{ch}[F^{\epsilon}_{\mu}]d\mu\right)d\rho \qquad (36)$$

It is worth to point out that the map $X_a \mapsto \mathrm{ch}[X_a^\epsilon]$ is injective on irreducible modules, so we don't loose any information by working with the characters, and we can even take the approach as in (31) with integrals added.

It can be shown that this product converges for all irreps, and it give rise to a commutative associative algebra. Finally, we have this remarkable formula [24]

Theorem 7. *With $Re(\epsilon) < 0$, the Verlinde-type algebra of regularized characters is given by*

$$\mathrm{ch}[F_\lambda^\epsilon] \times \mathrm{ch}[F_\mu^\epsilon] = \sum_{\ell=0}^{p-1} \mathrm{ch}[F^\epsilon_{\lambda+\mu+\ell\alpha_-}]$$

$$\mathrm{ch}[M_{r,s}^\epsilon] \times \mathrm{ch}[F_\mu^\epsilon] = \sum_{\substack{\ell=-s+2 \\ \ell+s=0 \bmod 2}}^{s} \mathrm{ch}[F^\epsilon_{\mu+\alpha_{r,\ell}}]$$

$$\begin{aligned}\mathrm{ch}[M_{r,s}^\epsilon] \times \mathrm{ch}[M_{r',s'}^\epsilon] = & \sum_{\substack{\ell=|s-s'|+1 \\ \ell+s+s'=1 \bmod 2}}^{min\{s+s'-1,p\}} \mathrm{ch}[M^\epsilon_{r+r'-1,\ell}] \\ & + \sum_{\substack{\ell=p+1 \\ \ell+s+s'=1 \bmod 2}}^{s+s'-1} \Big(\mathrm{ch}[M^\epsilon_{r+r'-2,\ell-p}] + \mathrm{ch}[M^\epsilon_{r+r'-1,2p-\ell}] \\ & \qquad + \mathrm{ch}[M^\epsilon_{r+r',\ell-p}]\Big).\end{aligned}$$

Remark 1. The previous result is expected to give relations in the Grothendieck ring of a suitable (sub)category of $\mathcal{W}(2, 2p-1)$-modules. It is not clear to us whether any of the current results in VOA theory (including [47]) gives braided category structure on this category. Also, we conjecture equivalence of categories $\mathcal{W}(2, 2p-1) - Mod \cong U_q(?) - Mod$ where $U_q(?)$ is yet-to-be defined quantum group at $2p$-th root of unity.

6 "Bare" Virasoro Vertex Algebra

We are finally left with the lonely Virasoro vertex operator algebra $L(c_{p,1}, 0)$ sitting inside the singlet $\mathcal{W}(2, 2p-1)$. In this section we also allow $p = 1$. Irreducible admissible $L(c_{p,1}, 0)$-modules are simple module of the form $L(c_{p,1}, h)$, where $h \in \mathbb{C}$. There is a distinguished family of highest weight modules which are not Verma modules, that is $V(c_{p,1}, 0) \neq L(c_{p,1}, 0)$. This is if and only if $h = h_{i,s} = \frac{(ip-s)^2-(p-1)^2}{4p}$, $i > 0$, $0 < s \leq p$. We call them atypical modules.

6.1 Modular-Like Transformation Properties?

It is a well-known fact (due to Feigin and Fuchs) that

$$\mathrm{ch}_{L(c_{p,1},h_{i,s})}(\tau) = \frac{(1-q^{is})q^{\frac{(ip-s)^2}{4p}}}{\eta(\tau)}.$$

$$\mathrm{ch}_{L(c_{p,1},h)}(\tau) = \frac{q^{h+(p-1)^2/4p}}{\eta(\tau)}; \quad h \neq h_{i,s}$$

Evaluating $\tau \mapsto \tau + 1$ transformation on the character is trivial as usual. If we consider $\tau \mapsto -\frac{1}{\tau}$, as in the singlet case of Fock modules, we obtain one or two Gauss' integrals. But this answer will lead to new problems when we start computing a Verlinde-type formula. It turns out that two irreducible modules for this vertex algebra can produce infinitely many (more precisely, uncountably many) irreducible modules after the fusion, so we conclude that there cannot be a reasonable fusion algebra for $L(c_{p,1}, 0)$-modules unless of course we allow some kind of completions that we do not dwell into. Similar problem is already evident at the level of q-dimensions. Observe that for $h \neq h_{i,s}$

$$q\dim(L(c_{p,1}, h)) = \lim_{\tau \to 0} \frac{q^h}{1-q} = \infty.$$

Yet, in sharp contrast, we have

$$q\dim(L(c_{p,1}, h_{i,s})) = \lim_{\tau \to 0} \frac{q^{h_{i,s}}(1-q^{is})}{1-q} = is,$$

indicating that these atypical modules ought to behave much better under the fusion. This is also clear because of the following results (cf. [35, 52–54]):

$$L(c_{p,1}, h_{r,s}) \times L(c_{p,1}, h_{r',s'}) = \sum_{r'' \in A(r,r'), s'' \in A(s,s')} L(c_{p,1}, h_{r'',s''}),$$

where we assume that all indices are positive and $A_{i,j} = \{i+j-1, i+j-3, \cdots, |i-j|+1\}$. We should say that this formula only indicates triples of atypical modules whose fusion rules are 1 and not a relation in a hypothetical Grothendieck ring. We also have

$$qdim(L(c_{p,1}, h_{r,s}) \times L(c_{p,1}, h_{r',s'})) = qdim(L(c_{p,1}, h_{r,s})) \cdot qdim L(c_{p,1}, h_{r',s'}))$$

Because of the infinities involved we do not expect the irreducible modules can be organized in a way that the Verlinde formula holds. This is why to vertex algebras with similar properties we refer to as "wild".

7 Generalization and Higher Rank False Theta Functions

The story told in the previous sections can be generalized by considering the sequence of embeddings of vertex algebras:

$$\boxed{W_p(\mathfrak{g}) \hookrightarrow \mathscr{W}_Q^0(p) \hookrightarrow \mathscr{W}_Q(p) \hookrightarrow V_{\sqrt{p}Q}\,,}$$

where Q is a root lattice of ADE type, $\mathfrak{g}$ is the corresponding simple Lie algebra, $W_p(\mathfrak{g})$ is the affine W-algebra of central charge $c_p(\mathfrak{g})$ associated to $\mathfrak{g}$, and where $\mathscr{W}_Q^0(p)$ and $\mathscr{W}(p)_Q$ are vertex algebras defined below. In the special case of $\mathfrak{g} = sl_2$ and $p \geq 2$ we recover the embedding of vertex algebras given in the introduction.

The affine $\mathscr{W}$-algebra associated to $\hat{\mathfrak{g}}$ at level $k \neq -h^\vee$, denoted by $\mathscr{W}_k(\mathfrak{g})$ is usually defined as the cohomology group obtained via a quantized BRST complex for the Drinfeld-Sokolov hamiltonian reduction [38]. As shown by Feigin and Frenkel (cf. [38] and [36] and citations therein) this cohomology is nontrivial only in the degree zero. Moreover, it is known that $\mathscr{W}_k(\mathfrak{g})$ is a quantum $\mathscr{W}$-(vertex) algebra, in the sense that is freely generated by rank$(\mathfrak{g})$ primary fields, not counting the conformal vector.

We will be following the notation from Sect. 3. As before denote by L° the dual lattice of L. Now, we specialize $L = \sqrt{p}Q$, where $p \geq 2$ and Q is root lattice of ADE type. We equip V_L with a vertex algebra structure as earlier in Sect. 3 (by choosing an appropriate 2-cocycle). Let α_i denote the simple roots of Q. For the conformal vector we conveniently choose

$$\omega = \omega_{st} + \frac{p-1}{2\sqrt{p}} \sum_{\alpha \in \Delta_+} \alpha(-2)\mathbf{1},$$

where ω_{st} is the standard (quadratic) Virasoro generator [37, 51] . Then V_L is a conformal vertex algebra of central charge[1]

$$\mathrm{rank}(L) + 12(\rho,\rho)(2 - p - \frac{1}{p}),$$

where ρ is the half-sum of positive roots. Consider the operators

$$e_0^{\sqrt{p}\alpha_i}, \quad e_0^{-\alpha_j/\sqrt{p}}, \;\; 1 \leq i, j \leq rank(L) \tag{37}$$

acting between V_L and V_L-modules. These are the so-called *screening operators*. More precisely [55]

Lemma 1. *For every i and j the operators $e_0^{\sqrt{p}\alpha_i}$ and $e_0^{-\alpha_j/\sqrt{p}}$ commute with each other, and they both commute with the Virasoro algebra.*

[1]Without the linear term the central charge is $rank(L)$.

We shall refer to $e_0^{\sqrt{p}\alpha_i}$ and $e_0^{-\alpha_j/\sqrt{p}}$, as the *long* and *short* screening, respectively. It is well-known that the intersection of the kernels of residues of vertex operators is a vertex subalgebra (cf. [36]), so the next construction seems very natural.

An important theorem of Feigin and Frenkel [36] says that for k generic and $\mathfrak{g}$ is simply-laced, there is an alternative description of $\mathscr{W}_k(\mathfrak{g})$ in terms of free fields. For this purpose, we let $\nu = k + h^\vee$, where k is generic. Then there are (as above) appropriately defined screenings[2]

$$e_0^{-\alpha_i/\sqrt{\nu}} : M(1) \longrightarrow M(1, -\alpha_i/\sqrt{\nu}),$$

such that

$$\mathscr{W}_\nu(\mathfrak{g}) = \bigcap_{i=1}^{l} \mathrm{Ker}_{M(1)}(e_0^{-\alpha_i/\sqrt{\nu}}),$$

where $l = \mathrm{rank}(L)$. If we assume in addition that $\mathfrak{g}$ is simply laced (ADE type) we also have the following important duality [36]

$$\mathscr{W}_\nu(\mathfrak{g}) = \bigcap_{i=1}^{l} \mathrm{Ker}_{M(1)}(e_0^{\sqrt{\nu}\alpha_i}).$$

Generic values of ν do not have integrality property so in particular the screening operators $e_0^{-\alpha_i/\sqrt{\nu}}$ cannot be extended to a lattice vertex algebra. Still this idea can be used to define much larger vertex algebras which we now describe.

Theorem 8. *Let $\mathfrak{g}$ be simply laced. Then $p = k + h^\vee \in \mathbb{N}_{\geq 2}$ is non-generic. More precisely,*

$$\mathscr{W}^0(p)_Q := \bigcap_{i=1}^{l} \mathrm{Ker}_{M(1)} e_0^{-\alpha_i/\sqrt{p}}$$

is a vertex algebra containing $\mathscr{W}_p(\mathfrak{g})$ as a proper subalgebra. In particular, for $Q = A_1$ this algebra is simply the singlet $\mathscr{W}(2, 2p-1)$ discussed earlier.

The previous algebra can be maximally extended leading to

$$\mathscr{W}(p)_Q := \bigcap_{i=1}^{l} \mathrm{Ker}_{V_L} e_0^{-\alpha_j/\sqrt{p}}. \tag{38}$$

Again, if we let $Q = A_1$ this is just the triplet vertex algebra $\mathscr{W}(p)$.

[2]These screenings to do not extend to a lattice vertex algebra in general.

The following conjecture was mentioned in [13].

Conjecture 1. The vertex algebra $\mathscr{W}(p)_Q$ is C_2-cofinite.

Although there are not many rigorous results on the representation theory of $\mathscr{W}^0(p)_Q$ and of $\mathscr{W}(p)_Q$, we again expect that irreps of $\mathscr{W}(p)_Q$ can be understood as subquotients of V_L-modules, while all atypical irreps of $\mathscr{W}^0(p)_Q$ all appear in the decomposition of irreducible $\mathscr{W}(p)_Q$-modules and typical representations. As the structure of Fock spaces in the higher rank is not well-understood well, one can take a different geometric approach to guess the characters of relevant modules (see [30]).

7.1 Characters of $\mathscr{W}(p)_Q$-Modules

Let ρ denote the half-sum of positive roots, by W we denote the Weyl group and by $(\cdot,\cdot)$ the usual inner product in $L \otimes_{\mathbb{Z}} \mathbb{Q}$ normalized such that $(\alpha,\alpha) = 2$ for each root α. We also let $(\beta,\beta) = ||\beta||^2$. Let

$$\prod_{\alpha \in \Delta_-} (1 - \mathbf{z}^{\alpha})$$

denote the Weyl denominator (here Δ_- is the set of negative roots) and

$$\mathbf{z}^{\alpha} = z^{(\alpha_1,\alpha)} \cdots z^{(\alpha_n,\alpha)}.$$

There are two expression that we are concerned about here. The first does not give a proper character but only auxiliary expression to compute the proper (conjectural) characters. Assume $\lambda \in L^{\circ}$ and let [30]

$$\mathrm{ch}_{\mathscr{W}(p,\lambda)_Q}(\tau,\mathbf{z}) = \frac{\eta(\tau)^{-rank(Q)}}{e^{\rho}\prod_{\alpha\in\Delta_-}(1-\mathbf{z}^{\alpha})} \sum_{w\in W}\sum_{\beta\in Q} (-1)^{l(w)} q^{\frac{||p\beta+\lambda+(p-1)(\rho)||^2}{2p}} \mathbf{z}^{w(\beta+\hat{\lambda}+\rho)}.$$

This expression cannot be evaluated at $z_i = 1$, but the limit

$$\mathrm{ch}_{\mathscr{W}(p,\lambda)_Q}(\tau) = \lim_{\mathbf{z}\to 1} \mathrm{ch}_{W_Q(p,\lambda)}(\tau)$$

is conjecturally expected to give the character of $\mathscr{W}(p,\lambda)_Q$, an irreducible $\mathscr{W}(p)_Q$-module. It is not hard to see by using L'hopital rule that the resulting expression is a linear combination of quasi-modular forms of different weight generalizing the formula in (29). A much harder question to ask is to determine its modular closure [30].

7.2 Characters of $\mathscr{W}^0(p)_Q$-Modules

The previous computation can be motivated to compute (conjectural) expressions for the characters of atypical irreducible $\mathscr{W}^0(p)_Q$-modules. Of course, for typical modules we have $\mathrm{ch}_{F_\lambda}(\tau)$ is just a pure power of q divided with the rank(Q)-th power of the Dedekind η-function. Characters of atypical $\mathscr{W}^0(p)_Q$-modules should be parameterized by $\lambda \in L^0$ (cf. with the singlet algebra)

$$\begin{aligned}
&\mathrm{ch}_{\mathscr{W}^0(p,\lambda)_Q}(\tau) \\
&= \mathrm{CT}_{\mathbf{z}} \left\{ \frac{\eta(\tau)^{-rank(Q)}}{e^{\rho} \prod_{\alpha \in \Delta_-}(1-\mathbf{z}^{\alpha})} \sum_{w \in W} \sum_{\beta \in Q} (-1)^{l(w)} q^{\frac{||p\beta+\lambda+(p-1)(\rho)||^2}{2p}} \mathbf{z}^{w(\beta+\hat{\lambda}+\rho)}. \right\},
\end{aligned}$$

where $\mathrm{CT}_{\mathbf{z}}$ denote the constant term w.r.t. $\mathbf{z}$. Observe that this is precisely in the analogy with the singlet modules. The previous definition motivates our proposal of higher rank false theta functions, generalizing the $\mathfrak{sl}_2$ case, of "weight" $|\Delta_+| - \frac{rank(L)}{2}$:

$$F_{p,\lambda}(\tau) = \mathrm{CT}_{\mathbf{z}} \left\{ \frac{\sum_{w \in W} \sum_{\beta \in Q} (-1)^{l(w)} q^{\frac{||p\beta+\lambda+(p-1)(\rho)||^2}{2p}} \mathbf{z}^{w(\beta+\hat{\lambda}+\rho)}.}{e^{\rho} \prod_{\alpha \in \Delta_-}(1-\mathbf{z}^{\alpha})} \right\},$$

Remark 2. We expect many properties of generalized false theta functions to follow the pattern observed in the rank one case, including modularity-like properties of regularized false thetas, etc. This will be the subject of our forthcoming joint work with Bringmann [22].

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