Law of Pure Types and Some Exotic Spectra of Fractal Spectral Measures

Xin-Rong Dai, Xing-Gang He and Chun-Kit Lai

Abstract Let μ be a Borel probability measure with compact support in $\mathbb{R}^d$ and let $E(\Lambda) = \{e^{-2\pi\lambda\cdot x} : \lambda \in \Lambda\}$. We make a review on some recent progress about spectral measures. We first show that the law of pure types holds for spectral measures, i.e. if $E(\Lambda)$ is a frame for $L^2(\mu)$, then μ is discrete or absolutely continuous or singular continuous with respect to Lebesgue measure (see [HLL13]). And we discuss the spectral properties of Cantor measures (see [DaHL13]), where we focus on some exotic properties of the spectra of some Cantor measures.

Keywords Cantor measures fourier frames · Law of pure types · Spectral measures

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1 Introduction to the General Spectral Measures

Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. We call a family $E(\Lambda) = \{e_\lambda := e^{-2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ (Λ is a countable set) a *Fourier frame* for the Hilbert space $L^2(\mu)$ if there exist $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle_\mu|^2 \le B\|f\|^2, \quad \forall\, f \in L^2(\mu). \tag{1.1}$$

Here the inner product is defined as usual, $x \cdot y = \sum_{i=1}^{d} x_i y_i$ for $x, y \in \mathbb{R}^d$ and

$$\langle f, e_\lambda\rangle_\mu = \int_{\mathbb{R}^d} f(x) e_{-\lambda} d\mu(x).$$

$E(\Lambda)$ is called an *(exponential) Riesz basis* if it is both a basis and a frame for $L^2(\mu)$. Fourier frames and exponential Riesz bases are natural generalizations of exponential orthonromal bases in $L^2(\mu)$. They have fundamental importance in non-harmonic Fourier analysis and wavelet. When (1.1) is satisfied, $f \in L^2(\mu)$ can be expressed as $f(x) = \sum_{\lambda\in\Lambda} c_\lambda e^{2\pi i\lambda x}$, and the expression is unique if it is a Riesz basis.

When $E(\Lambda)$ is an orthonormal basis (Riesz basis, or frame) for $L^2(\mu)$, we say that μ is a *spectral measure* (*R-spectral measure, or F-spectral measure respectively*) and Λ is called a *spectrum* (*R-spectrum, or F-spectrum respectively*) of μ. We will also use the term *orthonormal spectrum* instead of spectrum when we need to emphasize the orthonormal property. If $E(\Lambda)$ only satisfies the upper bound condition in (1.1), then it is called a *Bessel sequence*; for convenience, we also call Λ a Bessel sequence of $L^2(\mu)$.

Since Fuglede proposed the spectral set conjecture [Fug74] and Jorgensen and Pedersen [JP98] discovered the first singular fractal spectral measure, there has been a lot of interest in understanding which kind of measures are spectral and its delicate connection with translational tiling. In this short note, we aim at giving a systematic survey on the recent progress in this line of research and some more detailed explanations about our discovery in [HLL13, DaHL13] will be given.

The first fundamental result is about the law of pure type. It was proved by He et al. [HLL13], which generalized the early investigation of spectral measures by Łaba and Wang [LaW06]. Recall that a σ-finite Borel measure μ on $\mathbb{R}^d$ can be decomposed uniquely as *discrete*, *singularly continuous* and *absolutely continuous* measures, i.e., $\mu = \mu_d + \mu_s + \mu_a$. The measure μ is said to be of *pure types* if μ equals only one of the three components.

Theorem 1.1 *Let μ be an F-spectral measure on $\mathbb{R}^d$. Then it must be one of the three pure types: discrete* (*and finite*)*, singularly continuous or absolutely continuous.*

By the law of pure types, we can study spectral measures according to its type. When μ is a discrete counting measure with finite support, it is an R-spectral measure

[HLL13]; When μ is absolutely continuous, Lai proved that μ is an F-spectral measure if and only if its density function is bounded above and bounded away from 0 almost everywhere on its support [Lai11]. Furthermore, Dutkay and Lai proved that if μ is a spectral measure, then its density function is a constant on its support, that is, μ is essentially the Lebesgue measure restricted on its support [DL00]. However, classification of spectral measures is far from complete. For the study of R-spectral absolutely continuous measures, one can refer to the recent work of Lev et al. with their emphasis on the use of quasicrystals [KN00, Lev12, GL14].

From now on, we concentrate on orthogonally spectral measures. We call Λ an *orthogonal set* if $E(\Lambda)$ is a mutually orthogonal sequence for $L^2(\mu)$. Define

$$Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\widehat{\mu}(\xi + \lambda)|^2,$$

where the Fourier transform of μ is define as usually by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi \xi \cdot x} d\mu(x).$$

Q_Λ is crucial in determining whether $E(\Lambda)$ is complete. It is well-known that an orthogonal sequence $E(\Lambda)$ is complete in $L^2(\mu)$ if and only if $Q_\Lambda \equiv 1$ [JP98]. Here, we give a slight generalization of this result and also exploit the analytic property of Q_Λ.

Theorem 1.2 *Let μ be a compactly supported Borel probability measure with compact support in $\mathbb{R}^d$.*

(i) Suppose that $\overline{\text{span}} E(\Gamma) = L^2(\mu)$ and $E(\Lambda)$ is an orthogonal set for $L^2(\mu)$. Then Λ is a spectrum of μ if and only if

$$Q_\Lambda(\gamma) = 1, \quad \text{for } \gamma \in \Gamma.$$

(ii) Suppose that $E(\Lambda)$ is a Bessel sequence for $L^2(\mu)$. Then $Q_\Lambda(\cdot)$ is an entire function in $\mathbb{C}^d$.

The entire property is a simple extension of [JP98, Lemma 4.3]. In our proofs, this property helps us establish the completeness by allowing us to focus on small values of ξ.

Our main interest on the spectral measures is when μ is singularly continuous. The one-fourth contraction Cantor measure was the first example of such spectral measures, which was found by Jorgensen and Pedersen [JP98] in 1998. From that time on, various properties of singular spectral measures are studied extensively [Dai12, DHJ09, DHS09, DHSW11, LaW02, LaW06]. In particular, many exotic spectra were discovered and they do not appear in their absolutely continuous counterpart. Here, we list some of the interesting ones.

(1) There exists a spectrum Λ of a singularly continuous measure μ such that $k\Lambda$ is also a spectrum of μ for some $k \neq 1$;
(2) There exists a Λ so that $E(\Lambda)$ is a maximal orthogonal collection of exponentials for $L^2(\mu)$, but not a basis;
(3) There exists a spectrum Λ of a singularly continuous measure so that its Beurling dimension is zero.

Property (1) means that we can sort of dilate a spectrum but preserve its completeness. It was first given by Łaba and Wang [LaW02] and some studies are given in [DJ12].

Property (2) has two types of variants. First, some measures have maximal orthogonal collections of infinite cardinality without being spectral [HuL08, Dai12]. Second, even though the measure is spectral, there still exists some incomplete maximal orthogonal collections. In [DHS09], Dutkay et al. tried to give a classification on maximal orthogonal collection for one-fourth Cantor measures and tried to study which of them are complete. This investigation was generalized and improved in [DaHL13]. Furthermore, we can demonstrate the existence of spectrum satisfying property (3). Beurling dimension is a concept defined in [DHSW11], who tried to generalize Beurling density and the elegant result of Landau [Lan67] on Fourier frame spectra to fractal setting. Their work gave some partial positive results, letting alone a technical assumption on the spectra. In person communication with Wang in 2011, we were told that he can construct an example such that a spectral measure can have a spectrum with zero Beurling dimension. However, he cannot explain why there can be such phenomenon. Our construction gave a better picture of it.

For the rest of our paper, we will prove Theorem 1.1 and 1.2 in Sect. 2. In Sect. 3, we will present a simplified content of [DaHL13] and the examples of zero Beurling dimension spectra will be given. For more results on this issue, reader may refer to [DaHL13, DHS09].

2 Law of Pure Types

In this section, we will present a self-contained proof for the law of pure types of F-spectral measures. First, we need the following proposition, which was proved in [DHSW11]. This can be viewed as the stability of Bessel sequence under a constant perturbation of a Bessel sequence. It has its origin in the paper of Duffin and Schaeffer [DS52].

Proposition 2.1 *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a Bessel sequence of μ. If there exists C such that $|\lambda_n - \gamma_n| \leq C$ for $n \geq 0$, then $\{\gamma_n\}_{n=0}^{\infty}$ is also a Bessel sequence of μ.*

Proof It is sufficient to show that all $\gamma_n = (\gamma_1^{(n)}, \ldots, \gamma_d^{(n)})$ differs $\lambda_n = (\lambda_1^{(n)}, \ldots, \lambda_d^{(n)})$ only on the first component, and the statement follows by induction on the number of components.

Let $\text{supp}\mu \subseteq [-P, P]^d$ for some $P > 0$. We have that

$$\begin{aligned}
\sum_{n=0}^{\infty} \left| < f(x), e^{-2\pi i \gamma_n \cdot x} > \right|^2 &= \sum_{n=0}^{\infty} \left| < f(x) e^{2\pi i (\gamma_n - \lambda_n)\cdot x}, e^{-2\pi i \lambda_n \cdot x} > \right|^2 \\
&= \sum_{n=0}^{\infty} \left| < f(x) e^{2\pi i (\gamma_1^{(n)} - \lambda_1^{(n)}) x_1}, e^{-2\pi i \lambda_n \cdot x} > \right|^2 \\
&= \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \frac{(2\pi i (\gamma_1^{(n)} - \lambda_1^{(n)}))^k}{k!} < f(x) x_1^k, e^{-2\pi i \lambda_n \cdot x} > \right|^2 \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2\pi C)^{2k}}{k!} \sum_{k=0}^{\infty} \frac{| < f(x) x_1^k, e^{-2\pi i \lambda_n \cdot x} > |^2}{k!} \\
&\leq e^{(2\pi C)^2} \sum_{k=0}^{\infty} \frac{B \| f(x) x_1^k \|^2}{k!} \\
&\leq B e^{(2\pi C)^2 + P^2} \| f \|^2.
\end{aligned}$$

Note that the fourth line above uses Cauchy-Schwarz inequality. Hence, the assertion follows. □

In the proof of the pure type property of the F-spectral measures, we need to use the *lower Beurling density* of an infinite discrete set $\Lambda \subset \mathbb{R}^d$:

$$D^{-}\Lambda := \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d},$$

where $Q_h(x)$ is the standard cube of side length h centered at x. Intuitively Λ is distributed like a lattice if $D^{-}\Lambda$ is positive. In the seminal paper [Lan67], Landau gave an elegant and useful necessary condition for Λ to be an F-spectrum on $L^2(\Omega)$: $D^{-}\Lambda \geq \mathcal{L}(\Omega)$, where $\mathcal{L}$ is the Lebesgue measure. The following proposition provides some relationships between the lower Beurling density and the types of the measures.

Proposition 2.2 *Let μ be a compactly supported probability measure on $\mathbb{R}^d$ and let Λ be an F-spectrum of μ. We have*

(i) *If $\mu = \sum_{c\in\mathcal{C}} p_c \delta_c$ is discrete, then $\#\Lambda < \infty$ and $\#\mathcal{C} < \infty$;*
(ii) *If μ is singularly continuous, then $D^{-}\Lambda = 0$;*
(iii) *If μ is absolutely continuous, then $D^{-}\Lambda > 0$.*

Proof (i) By the definition of Fourier frame, we have for all $f \in L^2(\mu)$,

$$\sum_{\lambda\in\Lambda} \left| \sum_{c\in\mathcal{C}} f(c) e^{2\pi i \langle \lambda, c\rangle} p_c \right|^2 \leq B \sum_{c\in\mathcal{C}} |f(c)|^2 p_c.$$

Taking $f = \chi_{c_0}$, where χ_{c_0} is the indictor function of the set $\{c_0\}$ and $p_{c_0} > 0$, we have $(\#\Lambda) \cdot p_{c_0}^2 \le Bp_{c_0}$. Hence $\#\Lambda \le B/p_{c_0} < \infty$. This implies $\#\mathcal{C} < \infty$ by the completeness of Fourier frame.

(ii) Suppose on the contrary that $D^-\Lambda \ge c > 0$. We claim that $\mathbb{Z}^d$ is a Bessel sequence of $L^2(\mu)$. By the definition of $D^-\Lambda$, we can choose a large $h \in \mathbb{N}$ such that

$$\inf_{x\in\mathbb{R}^d} (\#(\Lambda \cap Q_h(x))) \ge ch^d > 1.$$

Taking $x = h\mathbf{n}$, where $\mathbf{n} \in \mathbb{Z}^d$, we see that all cubes of the form $h\mathbf{n} + [-h/2, h/2)^d$ contains at least one points of Λ, say $\lambda_{\mathbf{n}}$. Since Λ is an F-spectrum, $\{\lambda_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ is a Bessel sequence. Observing that

$$|\lambda_{\mathbf{n}} - h\mathbf{n}| \le \operatorname{diam}([-h/2, h/2)^d) = \sqrt{d}\, h,$$

then $h\mathbb{Z}^d$ is also a Bessel sequence of $L^2(\mu)$ by Proposition 2.1. As a Bessel sequence is invariant under translation, we see that the finite union $\mathbb{Z}^d = \bigcup_{\mathbf{k}\in\{0,\ldots,h-1\}^d} (h\mathbb{Z}^d + \mathbf{k})$ is again a Bessel sequence of $L^2(\mu)$, which proves the claim.

Now consider

$$G(x) := \sum_{\mathbf{n}\in\mathbb{Z}^d} |\widehat{\mu}(x+\mathbf{n})|^2.$$

G is a periodic function (mod $\mathbb{Z}^d$). As $\mathbb{Z}^d$ is a Bessel sequence, applying the definition to e_{-x}, we see that $G(x) \le B < \infty$. Hence $G \in L^1([0,1)^d)$ and

$$\int_{\mathbb{R}^d} |\widehat{\mu}(x)|^2\, dx = \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{[0,1)^d} |\widehat{\mu}(x+\mathbf{n})|^2 dx = \int_{[0,1)^d} |G(x)| dx < \infty.$$

This means that $\widehat{\mu} \in L^2(\mathbb{R}^d)$, which implies that μ must be absolutely continuous. This is a contradiction.

(iii) If μ is absolutely continuous, we write $d\mu(x) = \varphi(x)dx$, for some L^1 function φ and denote by Ω the support of μ. Let

$$E_N = \left\{x \in \Omega : \frac{1}{N} \le \varphi(x) \le N\right\}.$$

Since μ is absolutely continuous, the support Ω must have positive Lebesgue measure and E_N also has positive Lebesgue measure for N large, which we may assume it holds for all E_N. Now, we claim that Λ is an F-spectrum of $L^2(E_N)$. To see this, let $f \in L^2(E_N)$, then we have $\int_{E_N} |\frac{f(x)}{\varphi(x)}|^2\varphi(x)dx \le N \int_{E_N} |f|^2 < \infty$. Hence,

$$\sum_{\lambda\in\Lambda}|\int_{E_N} f(x)e^{2\pi i\lambda x}dx|^2 = \sum_{\lambda\in\Lambda}|\int_{E_N} \frac{f(x)}{\varphi(x)}e^{2\pi i\lambda x}\varphi(x)dx|^2$$
$$\leq B\int_{E_N}|\frac{f(x)}{\varphi(x)}|^2\varphi(x)dx \leq BN\int_{E_N}|f(x)|^2dx.$$

This establishes the upper frame bound. The lower bound can also be established analogously. This justify the claim. By the Landau's density theorem, we have $D^-\Lambda \geq \mathcal{L}(E_N)$. As E_N are increasing sequence of sets and $\bigcup_N E_N = \Omega$ up to a Lebesgue measure zero set, we have

$$D^-\Lambda \geq \mathcal{L}(\Omega) > 0. \qquad \square$$

Now it is easy to conclude that an F-spectral measure is of pure type.

Proof of Theorem 1.1 First let us assume that if μ is decomposed into non-trivial discrete and continuous parts, $\mu = \mu_d + \mu_c$. Let Λ be an F-spectrum of μ. As $L^2(\mu_d)$ and $L^2(\mu_s)$ are non-trivial subspaces of $L^2(\mu)$, it is easy to see that Λ is also an F-spectrum of both $L^2(\mu_d)$ and $L^2(\mu_c)$. Then $\#\Lambda < \infty$ by Proposition 2.2(i); but $\#\Lambda = \infty$ since $L^2(\mu_c)$ is an infinite dimensional Hilbert space. This contradiction shows that μ is either discrete or purely continuous.

Suppose μ is continuous and has non-trivial singular part μ_s and absolutely continuous part μ_a. By applying the same argument as the above, Λ is an F-spectrum of $L^2(\mu_s)$ and $L^2(\mu_a)$. This is impossible in view of the Beurling density of Λ in Proposition 2.2(ii) and (iii). $\square$

The following corollary is immediate from Theorem 1.1.

Corollary 2.3 *A spectral measure or an R-spectral measure must be of pure type.*

In the rest of this section, we will prove Theorem 1.2 and it will be needed in the next section.

Proof of Theorem 1.2 (i) It is easy to see that the necessity follows by applying Parseval's identity to e_γ for $\gamma \in \Gamma$. Now we show the sufficiency. By the hypotheses, it is sufficient to show that $e_\gamma \in \overline{\text{span}}E(\Lambda)$ for each $\gamma \in \Gamma$. Let Π be the projection from $L^2(\mu)$ to $\overline{\text{span}}E(\Lambda)$. Then $e_\gamma = \Pi(e_\gamma) + (Id - \Pi)(e_\gamma)$ and thus $1 = \|\Pi(e_\gamma)\|^2 + \|(Id-\Pi)(e_\gamma)\|^2$. Note that

$$\|\Pi(e_\gamma)\|^2 = \sum_{\lambda\in\Lambda}|\langle\Pi(e_\gamma), e_\lambda\rangle|^2 = \sum_{\lambda\in\Lambda}|\langle e_\gamma, e_\lambda\rangle|^2 = \|e_\gamma\|^2 = 1.$$

Then $(Id - \Pi)(e_\gamma) = 0$ and thus $e_\gamma \in \overline{\text{span}}E(\Lambda)$.

(ii) Let $M > 0$ so that $\text{supp}\mu \subseteq B(0, M)$, where $B(0, M)$ is the ball with center at 0 and radius M. Denote $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ and

$$Q_N(w) = \sum_{n=0}^{N}|\widehat{\mu}(w + \lambda_n)|^2, \quad \forall w \in \mathbb{C}^d.$$

Let B be the upper bound of $E(\Lambda)$. Note that

$$Q_N(w) = \sum_{n=0}^{N} \left| < e_{-w}, e_{\lambda_n} > \right|^2 \leq B\|e_{-w}\|^2 \leq Be^{4\pi M|\mathcal{I}m(w)|},$$

where $\mathcal{I}m(w)$ is the imaginary part of w. This implies that the sequence $\{Q_N(w)\}_{N=1}^{\infty}$ is uniformly bounded on each compact set of $\mathbb{C}^d$. By Montel theorem (see, e.g., [Gun90] p. 54), we have Q_Λ is an entire function on $\mathbb{C}^d$ and

$$|Q_\Lambda(w)| \leq Be^{4\pi M|\mathcal{I}m(w)|}, \quad \forall w \in \mathbb{C}^d.$$

□

Now the standard Jorgensen-Pedersen Lemma follows as a corollary.

Corollary 2.4 *An orthogonal sequence $E(\Lambda)$ is complete in $L^2(\mu)$ if and only if $Q_\Lambda \equiv 1$.*

Proof We only need to show that $\operatorname{span} E(\mathbb{R}^d)$ is dense in $L^2(\mu)$ by Theorem 1.2. Let $K = \operatorname{supp} \mu$. Since $\operatorname{span} E(\mathbb{R}^d)$ is a subalgebra of Banach algebra $C(K)$, the space of all continuous function on K, and it separates points K. By Stone-Weierstrass theorem, we have that $\operatorname{span} E(\mathbb{R}^d)$ is dense in the space $C(K)$. According to Lusin theorem, $C(K)$ is dense in $L^2(\mu)$. This implies the assertion. □

3 Spectral Properties of Cantor Measures on $\mathbb{R}$

This section is devoted to a simplified content of [DaHL13]. Our aim is to show the existence of spectra with zero Beurling dimension (Theorem 3.5) when the measures are the Cantor measure with consecutive digits. Let b, q be two integers > 1 with $b > q$ and $q \mid b$. Then there exists unique Borel probability measure, denoted by $\mu_{b,q}$, satisfying

$$\mu_{b,q}(\cdot) = \frac{1}{q} \sum_{i=0}^{q-1} \mu_{b,q}(q \cdot -i). \tag{3.1}$$

$\mu_{b,q}$ is called a *Cantor measure* (with consecutive digit). It is well-known that the Hausdorff dimension of the support of $\mu_{b,q}$ is $\ln q / \ln b < 1$ and thus $\mu_{b,q}$ is singularly continuous with respect to Lebesgue measure. We will construct a class of orthogonal set of $\mu_{b,q}$.

Denote $\Sigma_q = \{0, \cdots, q-1\}$, $\Sigma_q^0 = \{\vartheta\}$ and $\Sigma_q^n = \underbrace{\Sigma_q \times \cdots \times \Sigma_q}_{n}$. Let $\Sigma_q^* = \bigcup_{n=0}^{\infty} \Sigma_q^n$ be the set of all finite words. Given $\sigma = \sigma_1\sigma_2 \cdots \in \Sigma^*$, we define $\vartheta\sigma = \sigma$, $\sigma|_k = \sigma_1 \cdots \sigma_k$ for $k \geq 0$ where $\sigma|_0 = \vartheta$ for any σ and adopt the notation $0^k = \underbrace{0\cdots 0}_{k}$ and $\sigma\sigma'$ is the concatenation of σ and σ'. We start with two definitions.

Definition 3.1 Let Σ_q^* be all the finite words defined as above. We say it is a *$q-$adic tree* if we set naturally the root is ϑ, all the k-th level nodes are Σ_q^k for $k \geq 1$ and all the offsprings of $\sigma \in \Sigma_q^*$ are σi for $i = 0, 1, \ldots, q-1$.

Definition 3.2 Let Σ_q^* be a $q-$adic tree, τ is called a *regular mapping* from Σ_q^* to $\{-1, 0, ..., b-2\}$ if it satisfies

(i) $\tau(\vartheta) = \tau(0^n) = 0$ for all $n \geq 1$.
(ii) For $\sigma_1\cdots\sigma_k \in \Sigma_q^k$, $\tau(\sigma_1\cdots\sigma_k) \in (\sigma_k + q\mathbb{Z}) \cap \{-1, 0, ..., b-2\}$.
(iii) For any $\sigma \in \Sigma_q^*$, $\tau(\sigma 0^\ell) = 0$ for ℓ large enough.

Let τ be a regular mapping from Σ_q^* to $\{-1, 0, ..., b-2\}$. For any $n \in \mathbb{N}$ with $q^{N-1} \leq n < q^N$, there exists unique $\sigma = \sigma_1\cdots\sigma_N \in \Sigma_q^N$ such that $\sigma_N \neq 0$ and

$$n = \sigma_1 + \sigma_2 q + \cdots + \sigma_N q^{N-1}.$$

Associated to τ, we define a sequence of integers by $\lambda_0 = 0$ and

$$\lambda_n = \tau(\sigma|_1) + \tau(\sigma|_2)b + \cdots + \tau(\sigma|_N)b^{N-1} + \sum_{k=N}^{\infty} \tau(\sigma 0^{k-N+1})b^k.$$

Note that λ_n is uniquely determined by $\tau(\sigma|_1), \tau(\sigma|_2), \ldots, \tau(\sigma|_N) = \tau(\sigma)$. We call $\Lambda = \{\lambda_n\}_{n=0}^\infty$ a *τ-sequence*. Let ℓ_n be the number of nonzero terms in the sum $\sum_{k=N}^\infty \tau(\sigma 0^{k-N+1})b^k$, that is

$$\ell_n = \#\{k : \tau(\sigma 0^k) \neq 0 \text{ for } k \geq 1\}. \tag{3.2}$$

We assume that $b, q, r = b/q$ are integers with $b > q$. The following are our main theorems.

Theorem 3.3 *Let τ be a regular mapping from Σ_q^* to $\{-1, 0, ..., b-2\}$ and let $\Lambda = \{\lambda_n\}_{n=0}^\infty$ be the τ-sequence. Then $E(r\Lambda)$ is a maximal orthogonal collection of exponentials for $L^2(\mu_{b,q})$.*

Theorem 3.4 *Let τ be a regular mapping and let $\Lambda = \{\lambda_n\}_{n=0}^\infty$ be the τ-sequence. We have the following:*

(i) *If $\max_{n\geq 1}\{\ell_n\} < \infty$, then $r\Lambda$ is a spectrum of $\mu_{b,q}$;*
(ii) *If $\ell_n \geq \log_q n$ for sufficient large n, then $r\Lambda$ is not a spectrum of $\mu_{b,q}$.*

Theorem 3.5 *Let $g(x)$ be an increasing non-negative function on $[0, \infty)$. Then there exists a spectrum Λ of $L^2(\mu_{b,q})$ such that*

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}} \frac{\#(\Lambda \cap (x-R, x+R))}{g(R)} = 0. \tag{3.3}$$

Let $\mu_{b,q}$ be the Cantor measure given by (3.1) and let

$$M(\xi) = \frac{1}{q}(1 + e^{2\pi i\xi} + \cdots + e^{2\pi i(q-1)\xi}).$$

Then it is easy to obtain that

$$\widehat{\mu}_{b,q}(\xi) = M(b^{-1}\xi)\widehat{\mu}_{b,q}(b^{-1}\xi) = \prod_{k=1}^{\infty} M(b^{-k}\xi). \tag{3.4}$$

Note that $|M(\xi)| = |\sin q\pi\xi|/q|\sin \pi\xi|$. Then

$$\mathcal{Z}_M := \{\xi : M(\xi) = 0\} = \frac{1}{q}(\mathbb{Z} \setminus q\mathbb{Z})$$

and

$$\mathcal{Z}_\mu := \{\xi : \widehat{\mu}_{b,q}(\xi) = 0\} = r\{b^k a : k \geq 0, a \in \mathbb{Z} \setminus q\mathbb{Z}\}.$$

Clearly, Θ is an orthogonal set of $\mu_{b,q}$ if and only if

$$\Theta - \Theta \subseteq \mathcal{Z}_\mu \cup \{0\}. \tag{3.5}$$

Proof of Theorem 3.3 We first prove the orthogonal property of $E(r\Lambda)$. Denote $\lambda_{n'} = \tau(\sigma'|_1) + \tau(\sigma'|_2)b + \cdots + \tau(\sigma'|_{N'})b^{N'-1} + \sum_{k=N'}^{\infty} \tau(\sigma 0^{k-N'+1})b^k$ and $n \neq n'$. Let s be the smallest index such that $\tau(\sigma' 0^{|\sigma|}|_s) \neq \tau(\sigma 0^{|\sigma'|}|_s)$, where $|\sigma|$ is the length of σ. Then

$$\lambda_{n'} - \lambda_n = (\tau(\sigma'|_s) - \tau(\sigma|_s))b^s + b^{s+1}M$$

for some $M \in \mathbb{Z}$. Then $r(\lambda_{n'} - \lambda_n)$ is the zero point of $M(b^{-s+1}\xi)$ by the definition of τ and thus is a zero point of $\widehat{\mu}_{b,q}$ by (3.4). This implies that $r\Lambda$ is an orthogonal set of $\mu_{b,q}$.

Now we show the maximal property of $E(r\Lambda)$. Suppose that $r\Lambda \cup \{\gamma\}$ is an orthogonal set of $\mu_{b,q}$ with $\gamma \notin r\Lambda$. By (3.5) and $0 \in r\Lambda$, we have $\gamma = rb^k a$ for some $k \geq 0$ and $a \in \mathbb{Z} \setminus q\mathbb{Z}$. Since a can be expressed uniquely as

$$a = a_0 + a_1 b + \cdots + a_m b^m,$$

where all $a_i \in \{-1, 0, 1, \dots, b-1\}$, $a_m \neq 0$ and $a_0 \in \mathbb{Z} \setminus q\mathbb{Z}$, there exists unique $i_0 \in \{1, 2, \dots, q-1\}$ such that $a_0 - \tau(0^k i_0) \in q\mathbb{Z}$. By the assumption we have $ab^k - \lambda_{i_0 q^k} \in \mathcal{Z}_\mu$, that is,

$$\frac{ab^k - \lambda_{i_0 q^k}}{b^k} = a_0 - \tau(0^k i_0) + \sum_{s=1}^{m} (a_i - \tau(0^k i_0 0^s))b^s - \sum_{s=m+1}^{\infty} \tau(0^k i_0 0^s) b^s \in \mathcal{Z}_\mu.$$

Since $q \mid (a_0 - \tau(0^k i_0))$, but $b \nmid (a_0 - \tau(0^k i_0))$ if $a_0 \neq \tau(0^k i_0)$, one has $a_0 = \tau(0^k i_0)$. Similarly, there exists unique $i_1 \in \{0, 1, \dots, q-1\}$ such that $a_1 - \tau(0^k i_0 i_1) \in q\mathbb{Z}$. From $ab^k - \lambda_{i_0 q^k + i_1 q^{k+1}} \in \mathcal{Z}_\mu$, one has $a_1 = \tau(0^k i_0 i_1)$. By m-steps one has $a_s = \tau(0^k i_0 \cdots i_s)$ for $0 \le s \le m$.

Let $p = \sum_{s=0}^{m} i_s q^{k+s}$. We claim that $\gamma = r\lambda_p$ and the result follows if the claim holds. In fact,

$$\begin{aligned}\frac{ab^k - \lambda_p}{b^k} &= \sum_{s=0}^{m} (a_s - \tau(0^k i_0 \cdots i_s))b^s - \sum_{s=m+1}^{\infty} \tau(0^k i_0 \cdots i_m 0^{s-m}) b^s \\ &= -\sum_{s=m+1}^{\infty} \tau(0^k i_0 \cdots i_m 0^{s-m}) b^s.\end{aligned}$$

If $ab^k \neq \lambda_p$, the above implies that $ab^k - \lambda_p \notin \mathcal{Z}_\mu$, which contradicts to the assumption. Hence the claim follows. □

Let δ_a be the Dirac measure with center a. We define

$$\delta_{\mathcal{E}} = \frac{1}{\#\mathcal{E}} \sum_{e \in \mathcal{E}} \delta_e$$

for any finite set $\mathcal{E}$, where $\#\mathcal{E}$ is the cardinality of $\mathcal{E}$. Write $\mathcal{D} = \{0, 1, \dots, q-1\}$ and $D_N = \frac{1}{b}\mathcal{D} + \cdots + \frac{1}{b^N}\mathcal{D}$ for $N \ge 1$. Let $\mu_N = \delta_{\mathcal{D}_N}$. Then

$$\widehat{\mu_N}(\xi) = \prod_{j=1}^{N} M(b^{-j}\xi).$$

By (3.4) we have

$$\widehat{\mu}_{b,q}(\xi) = \widehat{\mu_N}(\xi)\widehat{\mu}_{b,q}(\frac{\xi}{b^N}). \tag{3.6}$$

Lemma 3.6 *Let τ be a regular mapping and let $\{\lambda_n\}_{n=0}^{\infty}$ be the τ-sequence. Then for all $N \ge 1$,*

$$\sum_{n=0}^{q^N - 1} |\widehat{\mu_N}(\xi + r\lambda_n)|^2 \equiv 1. \tag{3.7}$$

Proof Since the dimension of $L^2(\mu_N)$ is q^N, the assertion follows by Corollary 2.4 if $\{r\lambda_n\}_{n=0}^{q^N-1}$ is an orthogonal set of μ_N, which can be proved by the same proof of Theorem 3.3. $\square$

For $m \geq 1$, let

$$Q_m(\xi) = \sum_{n=0}^{q^m-1} \left|\widehat{\mu}_{b,\,q}(\xi + r\lambda_n)\right|^2 \text{ and } Q(\xi) = \sum_{n=0}^{\infty} \left|\widehat{\mu}_{b,\,q}(\xi + r\lambda_n)\right|^2.$$

Let $\mu = \mu_{b,q}$. For any $m, p > 0$, we have the following identity:

$$\begin{aligned} Q_{m+p}(\xi) &= Q_m(\xi) + \sum_{n=q^m}^{q^{m+p}-1} |\widehat{\mu}(\xi + r\lambda_n)|^2 \\ &= Q_m(\xi) + \sum_{n=q^m}^{q^{m+p}-1} \left|\widehat{\mu_{m+p}}(\xi + r\lambda_n)\right|^2 \left|\widehat{\mu}(\frac{\xi + r\lambda_n}{b^{m+p}})\right|^2. \end{aligned} \tag{3.8}$$

Our goal is see whether $Q(\xi) \equiv 1$. Then by invoking Corollary 2.4, we can determine whether we have a spectrum. As Q is an entire function by Theorem 1.2(ii), we just need to see the value of $Q(\xi)$ for some small values of ξ. To do this, we need to make a fine estimation of the terms $\left|\widehat{\mu}(\frac{\xi+r\lambda_n}{b^{m+p}})\right|^2$ in the above. Write

$$\alpha = \min\left\{|M(\xi)\widehat{\mu}(\xi)|^2 : |\xi| \leq \frac{b-1}{qb}\right\} > 0$$

and

$$\beta = \max\left\{|M(\xi)|^2 : \frac{1}{b^2} \leq |\xi| \leq \frac{b-1}{qb}\right\} < 1.$$

where $|M(\xi)| = \frac{|\sin \pi q\xi|}{q|\sin \pi\xi|}$.

Proposition 3.7 *Let* $|\xi| \leq \frac{r(b-2)}{b-1}$ *and let* $t = \xi + \sum_{k=1}^{N} d_i b^{n_k}$, *where* $d_i \in \{1, 2, \cdots r-1\}$ *and* $1 \leq n_1 < \cdots < n_N$. *Then*

$$\alpha^{N+1} \leq |\widehat{\mu}(t)|^2 \leq \beta^N. \tag{3.9}$$

Proof First it is easy to check that, for $|\xi| \leq \frac{r(b-2)}{b-1}$ and all $d_k \in \{0, 1, 2, \ldots, r-1\}$, we have

$$\left|\frac{\xi+\sum_{k=1}^{n} d_k b^k}{b^{n+1}}\right| \le \frac{1}{b^{n+1}}\left(\frac{r(b-2)}{b-1}+(r-1)(b+b^2+\cdots+b^n)\right)$$
$$= \frac{r(b-2)+(r-1)(b^{n+1}-b)}{b^{n+1}(b-1)}$$
$$\le \frac{b-1}{qb} \tag{3.10}$$

for $n \ge 1$. The inequality in the last line follows from a direct comparison of the difference and $q \ge 2$. To simplify notations, we let $n_0 = 0$ and $n_{N+1} = \infty$. Then $|\widehat{\mu}(t)|^2$ equals

$$\prod_{j=1}^{\infty}\left|M\left(b^{-j}t\right)\right|^2 = \prod_{i=0}^{N}\prod_{j=n_i+1}^{n_{i+1}}\left|M\left(b^{-j}t\right)\right|^2. \tag{3.11}$$

We now estimate the products one by one. By (3.10), we have

$$\left|\frac{\xi+\sum_{k=1}^{i} d_k b^{n_k}}{b^{n_i+1}}\right| \le \frac{b-1}{qb}.$$

Hence, together with the integral periodicity of $M(\xi)$ and the definition of α, we have for all $i > 0$,

$$\prod_{j=n_i+1}^{n_{i+1}}\left|M(b^{-j}t)\right|^2 = \prod_{j=n_i+1}^{n_{i+1}}\left|M\left(b^{-j}(\xi+\sum_{k=1}^{i} d_k b^{n_k})\right)\right|^2$$
$$\ge \prod_{j=0}^{\infty}\left|M\left(b^{-j}\left(\frac{\xi+\sum_{k=1}^{i} d_k b^{n_k}}{b^{n_i+1}}\right)\right)\right|^2 \ge \alpha. \tag{3.12}$$

For the case $i = 0$, it is easy to see that $\left|\frac{\xi}{b}\right| \le \frac{b-2}{q(b-1)} < \frac{b-1}{qb}$. Hence, $\prod_{j=n_0+1}^{n_1}\left|M(b^{-j}t)\right|^2 \ge \prod_{j=0}^{\infty}\left|M\left(b^{-j}(\xi/b)\right)\right|^2 \ge \alpha$. Putting this fact and (3.12) into (3.11), we have $|\widehat{\mu}_{b,q}(t)|^2 \ge \alpha^{N+1}$.

We next prove the upper bound. From $|M(\xi)| \le 1$, (3.11) and the integral periodicity of $M(\xi)$,

$$|\widehat{\mu}(t)|^2 \le \prod_{i=1}^{N}\left|M\left(b^{-(n_i+1)}t\right)\right|^2 = \prod_{i=1}^{N}\left|M\left(b^{-(n_i+1)}(\xi+\sum_{k=1}^{i} d_k b^{n_k})\right)\right|^2. \tag{3.13}$$

By (3.10) we have

$$|\xi+\sum_{k=1}^{i} d_k b^{n_k}| \geq b^{n_i} - |\xi+\sum_{k=1}^{i-1} d_k b^{n_k}| \geq b^{n_i} - \frac{b^{n_{i-1}}(b-1)}{q} \geq b^{n_i-1}.$$

By (3.10), (3.13), the above and the definition of β, we obtain that $|\widehat{\mu}(t)|^2 \leq \beta^N$. □

Proof of Theorem 3.4 (i) Without loss generality we assume that $|\xi| \leq \frac{r(b-2)}{b-1}$. Recall that

$$Q_{m+p}(\xi) = Q_m(\xi) + \sum_{n=q^m}^{q^{m+p}-1} \left|\widehat{\mu}_{m+p}(\xi+r\lambda_n)\right|^2 \left|\widehat{\mu}\left(\frac{\xi+r\lambda_n}{q^{m+p}}\right)\right|^2. \tag{3.14}$$

Let also $L = \max_{n\geq 1} \ell_n$ ($<\infty$ by assumption). For $q^m \leq n < q^{m+p}$, there exists unique N, $m < N \leq m+p$, such that $q^{N-1} \leq n < q^N$. By the definition of τ, we have $\tau(\sigma 0^k) \in \{0, q, 2q, \ldots, (r-1)q\}$ for $k \geq 1$. We therefore have

$$\begin{aligned}
\xi + r\lambda_n &= \xi + r\tau(\sigma|_1) + r\tau(\sigma|_2)b + \cdots + r\tau(\sigma|_N)b^{N-1} + \sum_{s=N}^{\infty} r\tau(\sigma 0^{s-N+1})b^s \\
&= \xi + r\sum_{i=1}^{N} \tau(\sigma|_i)b^{i-1} + r\tau(\sigma 0)b^N + \cdots + r\tau(\sigma 0^{m+p-N})b^{m+p-1} \\
&\quad + \sum_{s=m+p}^{\infty} d_s b^{s+1}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\xi+r\lambda_n}{b^{m+p}} &= \frac{1}{b^{m+p}}\left(\xi + r\sum_{i=1}^{N} \tau(\sigma|_i)b^{i-1} + r\tau(\sigma 0)b^N + \cdots + r\tau(\sigma 0^{m+p-N})b^{m+p-1}\right) \\
&\quad + \sum_{s=m+p}^{\infty} d_s b^{s+1-(m+p)} := t + \sum_{s=m+p}^{\infty} d_s b^{s+1-(m+p)}.
\end{aligned}$$

Note that, from $|\tau(\sigma)| \leq b-2$ for any multi-indices σ,

$$|t| \leq \frac{1}{b^{m+p}}\left(|\xi| + r(b-2)(1+b+b^2+\cdots+b^{m+p-1})\right) \leq \frac{r(b-2)}{b-1}.$$

Also, $d_s \in \{0, 1, \ldots, r-1\}$ and there are at most L non-zero terms. By Proposition 3.7, we conclude that $\left|\widehat{\mu}\left(\frac{\xi+r\lambda_n}{q^{m+p}}\right)\right|^2 \geq \alpha^{L+1}$. Using (3.14) and Lemma 3.6. we obtain

$$
\begin{aligned}
Q_{m+p}(\xi) &\geq Q_m(\xi) + \alpha^{L+1} \sum_{n=q^m}^{q^{m+p}-1} \left|\widehat{\mu}_{m+p}(\xi + r\lambda_n)\right|^2 \\
&= Q_m(\xi) + \alpha^{L+1} \left(1 - \sum_{n=0}^{q^m-1} \left|\widehat{\mu}_{m+p}(\xi + r\lambda_n)\right|^2\right).
\end{aligned}
$$

Fixing m, we first let p approaches infinity and obtain

$$
Q(\xi) \geq Q_m(\xi) + \alpha^{L+1} \left(1 - \sum_{n=0}^{q^m-1} |\widehat{\mu}(\xi + r\lambda_n)|^2\right).
$$

We then finally let m goes to infinity.

$$
\alpha^{L+1} \left(1 - \sum_{n=0}^{\infty} |\widehat{\mu}(\xi + r\lambda_n)|^2\right) \leq 0.
$$

This means that $Q(\xi) \geq 1$ for $|\xi| \leq r(b-2)/(b-1)$. As $Q(\xi) \leq 1$ for mutually orthogonal sets and by the entire function property of Q on $\mathbb{C}$, we must have $Q(\xi) \equiv 1$ and hence Λ is a spectrum for μ.

(ii) With loss of generality we assume that $\ell_n \geq \log_q n$ for $n \geq 1$. Again we begin with

$$
Q_m(\xi) = Q_{m-1}(\xi) + \sum_{n=q^{m-1}}^{q^m-1} |\widehat{\mu}_m(\xi + r\lambda_n)|^2 \left|\widehat{\mu}(\frac{\xi + r\lambda_n}{q^m})\right|^2.
$$

Note that for $q^{m-1} \leq n < q^m$, $\ell_n \geq \log_q n \geq m-1$. Using it and the same estimate as in (i) so as to apply Proposition 3.7, we have

$$
\begin{aligned}
Q_m(\xi) &\leq Q_{m-1}(\xi) + \sum_{n=q^{m-1}}^{q^m-1} |\widehat{\mu}_m(\xi + r\lambda_n)|^2 \beta^{\ell_n} \\
&\leq Q_{m-1}(\xi) + \beta^{m-1} \sum_{n=q^{m-1}}^{q^m-1} |\widehat{\mu}_m(\xi + r\lambda_n)|^2 \\
&= Q_{m-1}(\xi) + \beta^{m-1}(1 - \sum_{n=0}^{q^{m-1}-1} |\widehat{\mu}_m(\xi + r\lambda_n)|^2) \\
&\leq Q_{m-1}(\xi) + \beta^{m-1}(1 - Q_{m-1}(\xi)).
\end{aligned}
$$

Consequently,

$$1 - Q_m(\xi) \geq (1 - Q_{m-1}(\xi))(1 - \beta^{m-1}) \geq (1 - Q_1(\xi)) \prod_{k=1}^{m-1} (1 - \beta^k).$$

By letting m to infinity, we have

$$1 - Q(\xi) \geq (1 - Q_1(\xi)) \prod_{k=1}^{\infty} (1 - \beta^k).$$

Since $Q_1(\xi) < 1$ for almost all $\xi \in \mathbb{R}$, the second assertion follows by Corollary 2.4. □

Proof of Theorem 3.5 Let $\{m_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers with $m_1 \geq 2$. Then $m_k > k$ for $k \geq 1$. We now define a regular mapping in terms of this sequence by induction. Let $\tau(\vartheta) = \tau(0^k) = 0$ for $k \geq 1$. For $\sigma \in \{1, 2, \dots, q-1\} \subset \Sigma_q^1$, we define $\tau(\sigma) = \sigma$ and $\tau(\sigma 0^l) = 0$ or q according to $l \neq m_\sigma$ or $l = m_\sigma$, respectively. Suppose we have defined all $\tau(\sigma)$, $\sigma = \sigma_1 \cdots \sigma_s$ with $s \leq k$ and $\sigma_s \neq 0$, and $\tau(\sigma 0^l)$ for $l \geq 1$. For $\sigma = \sigma_1 \cdots \sigma_{k+1} \in \Sigma_q^{k+1}$ with $\sigma_{k+1} \neq 0$, we define $\tau(\sigma) = \sigma_{k+1}$ and $\tau(\sigma 0^l) = 0$ or q according to $l \neq m_{p_\sigma}$ or $l = m_{p_\sigma}$, respectively, where $p_\sigma = \sum_{i=1}^{k+1} \sigma_i q^{i-1}$. By induction we have well-defined a regular mapping from the q-adic tree to $\{-1, 0, 1, \dots, b-1\}$.

For any $n \in \mathbb{N}$, there exists unique $k \geq 1$ such that $q^{k-1} \leq n < q^k$. Then n can be expressed by

$$n = \sum_{j=1}^{k} \sigma_j q^{j-1}, \tag{3.15}$$

where all $\sigma_j \in \{0, 1, \dots, q-1\}$ and $\sigma_k \neq 0$. By the definition of τ-sequence, we have $\lambda_0 = 0$ and

$$\lambda_n = \sum_{j=1}^{k} \tau(\sigma_1 \cdots \sigma_j) b^{j-1} + q b^{m_n},$$

consequently, $\ell_n = 1$ and by Theorem 3.4(i), $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ is a spectrum of $\mu_{b,q}$.

We now find Λ satisfying (3.9) by choosing m_n. To do this, we first note that there exists a strictly increasing continuous function $h(t)$ from $[0, \infty)$ onto itself such that $h(t) \leq g(t)$ for $t \geq 0$ and it is sufficient to replace $g(t)$ by $h(t)$ in the proof. In this way, the inverse of $h(t)$ exists, and we denote it by $h^{-1}(t)$.

Now, note that

$$\lambda_n \leq q \frac{b^k - 1}{b - 1} + q b^{m_n} \leq (q+1) b^{m_n}.$$

Hence,

$$\lambda_{n+1} - \lambda_n \geq q b^{m_{n+1}} - (q+1) b^{m_n} \geq b^{m_n + 1}. \tag{3.16}$$

Therefore, we choose m_n so that $b^{m_n} \geq 2h^{-1}(b^{n+1})$ for all $n \geq 1$. For any $h(R) \geq 1$, there exists unique $s \in \mathbb{N}$ such that $b^{s-1} \leq h(R) < b^s$. Then

$$\frac{\sup_{x\in\mathbb{R}} \#(\Lambda \cap (x-R, x+R))}{h(R)} \leq \frac{\sup_{x\in\mathbb{R}} \#(\Lambda \cap (x-h^{-1}(b^s), x+h^{-1}(b^s)))}{b^{s-1}}. \tag{3.17}$$

Note from (3.16) that the length of the open intervals $(x-h^{-1}(b^s), x+h^{-1}(b^s))$ is less than $\lambda_{n+1}-\lambda_n$ whenever $n \geq s$. This implies that the set $\Lambda \cap (x-h^{-1}(b^s), x+h^{-1}(b^s))$ contains at most one λ_n where $n \geq s$. We therefore have

$$\sup_{x\in\mathbb{R}} \#(\Lambda \cap (x-h^{-1}(b^s), x+h^{-1}(b^s))) \leq s+1.$$

Thus the result follows by taking limit in (3.17). □

We conclude the paper with some remarks.

Remark (1) When observing the proofs of theorems, the main crux of the proof to spectra of zero Beurling dimension is in Proposition 3.7. The uniform control on the Fourier transform depends only on the number of non-zero digits in the b-adic expansion rather than the size of the frequencies.

(2) Indeed, all maximal orthogonal exponentials for $\mu_{b,q}$ can be classified through either *regular* or *irregular* mappings. This note discusses only the regular mappings. For irregular mappings, we can discuss its spectral properties if the number of irregular paths is finite. One can refer the details to [DaHL13].

(3) Much less is known about dilating a spectrum of a spectral measure. A standard example is that if $\Lambda = \{0, 1\} \oplus 4\{0, 1\} \oplus ....$, then 5Λ is also a spectrum for the standard one-fourth Cantor measure (i.e. $q = 2, b = 4$) [DHSW11]. However, one can prove that the tree mapping corresponding to 5Λ is irregular with infinitely many irregular paths. To see this, we re-write the following elements 5Λ into our standard 4-adic expansions.

$$5 \cdot 4^n + 5 \cdot 4^{n+1} + \cdots + 5 \cdot 4^m = 4^n + 2 \cdot 4^{n+1} + 2 \cdot 4^{n+2} + ... + 2 \cdot 4^m + 4^{m+1}.$$

This means the paths $0^{n-1}10^{\infty}$ are irregular paths. Hence, there are infinitely many such paths. This example of spectra cannot be covered by our theory and is also the first example of spectra with infinitely many irregular paths .

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