Chapter 2
Historical Overview of the Mittag-Leffler Functions

2.1 A Few Biographical Notes on Gösta Magnus Mittag-Leffler

Gösta Magnus Mittag-Leffler was born on March 16, 1846, in Stockholm, Sweden. His father, John Olof Leffler, was a school teacher, and was also elected as a member of the Swedish Parliament. His mother, Gustava Vilhelmina Mittag, was a daughter of a pastor, who was a person of great scientific abilities. At his birth Gösta was given the name Leffler and later (when he was a student) he added his mother's name "Mittag" as a tribute to this family, which was very important in Sweden in the nineteenth century. Both sides of his family were of German origin.

At the Gymnasium in Stockholm Gösta was training as an actuary but later changed to mathematics. He studied at the University of Uppsala, entering it in 1865. In 1872 he defended his thesis on applications of the argument principle and in the same year was appointed as a Docent (Associate Professor) at the University of Uppsala.

In the following year he was awarded a scholarship to study and work abroad as a researcher for 3 years. In October 1873 he left for Paris.

In Paris Mittag-Leffler met many mathematicians, such as Bouquet, Briot, Chasles, Darboux, and Liouville, but his main goal was to learn from Hermite. However, he found the lectures by Hermite on elliptic functions difficult to understand.

In spring 1875 he moved to Berlin to attend the lectures by Weierstrass whose research and teaching style was very close to his own. From Weierstrass' lectures Mittag-Leffler learned many ideas and concepts which would later become the core of his scientific interests.

In Berlin Mittag-Leffler received news that professor Lorenz Lindelöf (Ernst Lindelöf's father) had decided to leave a chair at the University of Helsingfors (now Helsinki). At the same time Weierstrass requested from the ministry of education the installation of a new position at his institute and suggested Mittag-Leffler for the

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position. In spite of this, Mittag-Leffler applied for the chair at Helsingfors. He got the chair in 1876 and remained at the University of Helsingfors for the next 5 years.

In 1881 the new University of Stockholm was founded, and Gösta Mittag-Leffler was the first to hold a chair in Mathematics there. Soon afterwards he began to organize the setting up of the new international journal *Acta Mathematica*. In 1882 Mittag-Leffler founded *Acta Mathematica* and served as the Editor-in-Chief of the journal for 45 years. The original idea for such a journal came from Sophus Lie in 1881, but it was Mittag-Leffler's understanding of the European scene, together with his political skills, that ensured the success of the journal. Later he invited many well-known mathematicians (Cantor, Poincaré and many others) to submit papers to this journal. Mittag-Leffler was always a good judge of the quality of the work submitted to him for publication.

In 1882 Gösta Mittag-Leffler married Signe af Linfors and they lived together until the end of his life.

Mittag-Leffler made numerous contributions to mathematical analysis, particularly in the areas concerned with limits, including calculus, analytic geometry and probability theory. He worked on the general theory of functions, studying relationships between independent and dependent variables.

His best known work deals with the analytic representation of a single-valued complex function, culminating in the Mittag-Leffler theorem. This study began as an attempt to generalize results from Weierstrass's lectures, where Weierstrass had described his theorem on the existence of an entire function with prescribed zeros each with a specified multiplicity. Mittag-Leffler tried to generalize this result to meromorphic functions while he was studying in Berlin. He eventually assembled his findings on generalizing Weierstrass' theorem to meromorphic functions in a paper which he published (in French) in 1884 in *Acta Mathematica*. In this paper Mittag-Leffler proposed a series of general topological notions on infinite point sets based on Cantor's new set theory.

With this paper Mittag-Leffler became the sole proprietor of a theorem that later became widely known and so he took his place in the circle of internationally known mathematicians. Mittag-Leffler was one of the first mathematicians to support Cantor's theory of sets but, one has to remark, a consequence of this was that Kronecker refused to publish in *Acta Mathematica*. Between 1899 and 1905 Mittag-Leffler published a series of papers which he called "Notes" on the summation of divergent series. The aim of these notes was to construct the analytical continuation of a power series outside its circle of convergence. The region in which he was able to do this is now called Mittag-Leffler's star. Andre Weyl in his memorial [Weil82] says: "*A well-known anecdote has Oscar Wilde saying that he had put his genius into his life; into his writings he had put merely his talent. With at least equal justice it may be said of Mittag-Leffier that the Acta Mathematica were the product of his genius, while nothing more than talent went into his mathematical contributions. Genius transcends and defies analysis; but this may be a fitting occasion for examining some of the qualities involved in the creating and in the editing of a great mathematical journal.*"

In the same period Mittag-Leffler introduced and investigated in five subsequent papers a new special function, which is now very popular and useful for many applications. This function, as well as many of its generalizations, is now called the "Mittag-Leffler" function.[1]

His contribution is nicely summed up by Hardy [Har28a]: "*Mittag-Leffler was a remarkable man in many ways. He was a mathematician of the front rank, whose contributions to analysis had become classical, and had played a great part in the inspiration of later research; he was a man of strong personality, fired by an intense devotion to his chosen study; and he had the persistence, the position, and the means to make his enthusiasm count.*"

Gösta Mittag-Leffler passed away on July 7, 1927. During his life he received many honours. He was an honorary member or corresponding member of almost every mathematical society in the world including the Accademia Reale dei Lincei, the Cambridge Philosophical Society, the Finnish Academy of Sciences, the London Mathematical Society, the Moscow Mathematical Society, the Netherlands Academy of Sciences, the St.-Petersburg Imperial Academy, the Royal Institution, the Royal Belgium Academy of Sciences and Arts, the Royal Irish Academy, the Swedish Academy of Sciences, and the Institute of France. He was elected a Fellow of the Royal Society of London in 1896. He was awarded honorary degrees from the Universities of Oxford, Cambridge, Aberdeen, St. Andrews, Bologna and Christiania (now Oslo).

2.2 The Contents of the Five Papers by Mittag-Leffler on New Functions

Let us begin with a description of the ideas which led to the introduction by Mittag-Leffler of a new transcendental function.

In 1899 Mittag-Leffler began the publication of a series of articles under the common title "*Sur la représentation analytique d'une branche uniforme d'une fonction monogène*" ("On the analytic representation of a single-valued branch of a monogenic function") published mainly in *Acta Mathematica* ([ML5-1, ML5-2, ML5-3, ML5-4, ML5-5, ML5-6]). The first articles of this series were based on three reports presented by him in 1898 at the Swedish Academy of Sciences in Stockholm.

His research was connected with the following question:

Let $k_0, k_1, \ldots$ be a sequence of complex numbers for which

$$\lim_{\nu\to\infty} |k_\nu|^{1/\nu} = \frac{1}{r} \in \mathbb{R}_+$$

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[1] Since it is the subject of this book we will give below a wider discussion of these five papers and of the role of the Mittag-Leffler functions.

is finite. Then the series

$$FC(z) := k_0 + k_1 z + k_2 z^2 + \dots$$

is convergent in the disk $D_r = \{z \in \mathbb{C} : |z| < r\}$ and divergent at any point with $|z| > r$. It determines a single-valued analytic function in the disk D_r.[2]

The questions discussed were:

1. To determine the maximal domain on which the function $FC(z)$ possesses a single-valued analytic continuation;
2. To find an analytic representation of the corresponding single-valued branch.

Abel [Abe26a] had proposed (see also [Lev56]) to associate with the function $FC(z)$ the entire function

$$F_1(z) := k_0 + \frac{k_1 z}{1!} + \frac{k_2 z^2}{2!} + \dots + \frac{k_\nu z^\nu}{\nu!} + \dots = \sum_{\nu=0}^{\infty} \frac{k_\nu z^\nu}{\nu!}.$$

This function was used by Borel (see, e.g., [Bor01]) to discover that the answer to the above question is closely related to the properties of the following integral (now called the *Laplace–Abel integral*):

$$\int_0^\infty \mathrm{e}^{-\omega} F_1(\omega z)\mathrm{d}\omega. \tag{2.2.1}$$

An intensive study of these properties was carried out at the beginning of twentieth century by many mathematicians (see, e.g., [ML5-3,ML5-5] and references therein).

Mittag-Leffler introduced instead of $F_1(z)$ a one-parametric family of (entire) functions

$$F_\alpha(z) := k_0 + \frac{k_1 z}{\Gamma(1 \cdot \alpha + 1)} + \frac{k_2 z^2}{\Gamma(2 \cdot \alpha + 1)} + \dots = \sum_{\nu=0}^{\infty} \frac{k_\nu z^\nu}{\Gamma(\nu \cdot \alpha + 1)}, \ (\alpha > 0),$$

and studied its properties as well as the properties of the generalized Laplace–Abel integral

$$\int_0^\infty \mathrm{e}^{-\omega^{1/\alpha}} F_\alpha(\omega z)\mathrm{d}\omega^{1/\alpha} = \int_0^\infty \mathrm{e}^{-\omega} F_\alpha(\omega^\alpha z)\mathrm{d}\omega. \tag{2.2.2}$$

[2]The notation $FC(z)$ is not described in Mittag-Leffler's paper. The letter "C" probably indicates the word 'convergent' in order to distinguish this function from its analytic continuation $FA(z)$ (see discussion below).

The main result of his study was: in a maximal domain A (star-like with respect to origin) the analytic representation of the single-valued analytic continuation $FA(z)$ of the function $FC(z)$ can be represented in the following form

$$FA(z) = \lim_{\alpha \to 1} \int_0^\infty e^{-\omega} F_\alpha(\omega^\alpha z) d\omega. \tag{2.2.3}$$

For this reason analytic properties of the functions $F_\alpha(z)$ become highly important.

Due to this construction Mittag-Leffler decided to study the most simple function of the type $F_\alpha(z)$, namely, the function corresponding to the unit sequence k_ν. This function

$$E_\alpha(z) := 1 + \frac{z}{\Gamma(1 \cdot \alpha + 1)} + \frac{z^2}{\Gamma(2 \cdot \alpha + 1)} + \ldots = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\Gamma(\nu \cdot \alpha + 1)}, \tag{2.2.4}$$

was introduced and investigated by G. Mittag-Leffler in five subsequent papers [ML1, ML2, ML3, ML4, ML5-5] (in particular, in connection with the above formulated questions). This function is known now as the *Mittag-Leffler function*.

In the *first paper* [ML1], devoted to his new function, Mittag-Leffler discussed the relation of the function $F_\alpha(z)$ with the above problem on analytic continuation. In particular, he posed the question of whether the domains of analyticity of the function

$$\lim_{\alpha \downarrow 1} \int_0^\infty e^{-\omega} F_\alpha(\omega^\alpha z) d\omega$$

and the function (introduced and studied by Le Roy [LeR00])

$$\lim_{\alpha \downarrow 1} \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu\alpha + 1)}{\Gamma(\nu + 1)} k_\nu z^\nu$$

coincide.

In the *second paper* [ML2] the new function (i.e., the Mittag-Leffler function) appeared. Its asymptotic properties were formulated. In particular, Mittag-Leffler showed that $E_\alpha(z)$ behaves as $e^{z^{1/\alpha}}$ in the angle $-\frac{\pi\alpha}{2} < \arg z < \frac{\pi\alpha}{2}$ and is bounded for values of z with $\frac{\pi\alpha}{2} < |\arg z| \leq \pi$.[3]

In the *third paper* [ML3] the asymptotic properties of $E_\alpha(z)$ were discussed more carefully. Mittag-Leffler compared his results with those of Malmquist [Mal03], Phragmén [Phr04] and Lindelöf [Lin03] which they obtained for similar functions (the results form the background of the classical Phragmén–Lindelöf theorem [PhrLin08]).

[3] The behaviour of $E_\alpha(z)$ on critical rays $|\arg z| = \pm\frac{\pi\alpha}{2}$ was not described.

The *fourth paper* [ML4] was completely devoted to the extension of the function $E_\alpha(z)$ (as well as the function $F_\alpha(z)$) to complex values of the parameter α.

Mittag-Leffler's most creative paper on the new function $E_\alpha(z)$ is his *fifth paper* [ML5-5]. There he:

(a) Found an integral representation for the function $E_\alpha(z)$;
(b) Described the asymptotic behaviour of $E_\alpha(z)$ in different angle domains;
(c) Gave the formulas connecting $E_\alpha(z)$ with known elementary functions;
(d) Provided the asymptotic formulas for

$$E_\alpha(z) = \frac{1}{2\pi i}\int_L \frac{1}{\alpha} e^{\omega^{1/\alpha}} \frac{d\omega}{\omega - z}$$

by using the so-called *Hankel integration path*;
(e) Obtained detailed asymptotics of $E_\alpha(z)$ for negative values of the variable, i.e. for $z = -r$;
(f) Compared in detail his asymptotic results for $E_\alpha(z)$ with the results obtained by Malmquist;
(g) Found domains which are free of zeros of $E_\alpha(z)$ in the case of "small" positive values of parameter, i.e. for $0 < \alpha < 2,\ \alpha \neq 1$;
(h) Applied his results on $E_\alpha(z)$ to answer the question of the domain of analyticity of the function $FA(z)$ and its analytic representation (see formula (2.2.3)).

2.3 Further History of Mittag-Leffler Functions

The importance of the new function was understood as soon as the first analytic results for it appeared. First of all, it is a very simple function playing the key role in the solution of a general problem of the theory of analytic functions. Secondly, the Mittag-Leffler function can be considered as a direct generalization of the exponential function, preserving some of its properties. Furthermore, $E_\alpha(z)$ has some interesting properties which later became essential for the description of many problems arising in applications.

After Mittag-Leffler's introduction of the new function, one of the first results on it was obtained by Wiman [Wim05a]. He used Borel's method of summation of divergent series (which Borel applied to the special case of the Mittag-Leffler function, namely, for $\alpha = 1$, see [Bor01]). Using this method, Wiman gave a new proof of the asymptotic representation of $E_\alpha(z)$ in different angle domains. This representation was obtained for positive rational values of the parameter α. He also noted[4] that analogous asymptotic results hold for the two-parametric generalization $E_{\alpha,\beta}(z)$ of the Mittag-Leffler function (see (1.0.3)). Applying the

[4]But did not discuss in detail.

obtained representation Wiman described in [Wim05b] the distribution of zeros of the Mittag-Leffler function $E_\alpha(z)$. The main focus was on two cases – to the case of real values of the parameter $\alpha \in (0, 2]$, $\alpha \neq 1$, and to the case of complex values of α, $\mathrm{Re}\,\alpha > 0$.

In [Phr04] Phragmén proved the generalization of the Maximum Modulus Principle for the case of functions analytic in an angle. For this general theorem the Mittag-Leffler function plays the role of the key example. It satisfies the inequality $|E_\alpha(z)| < C_1 \mathrm{e}^{|z|^\rho}$, $\rho = 1/\{\mathrm{Re}\,\alpha\}$, in an angular domain z, $|\arg z| \leq \frac{\pi}{2\rho}$, but although it is bounded on the boundary rays it is not constant in the whole angular domain. This means that the Mittag-Leffler function possesses a maximal angular domain (in the sense of the *Phragmén* or *Phragmén–Lindelöf theorem*, see [PhrLin08]) in which the above stated property holds.

One more paper devoted to the development of the asymptotic method of Mittag-Leffler appeared in 1905. Malmquist (a student of G. Mittag-Leffler) applied this method to obtain the asymptotics of a function similar to $E_\alpha(z)$, namely

$$\sum_\nu \frac{z^\nu}{\Gamma(1+\nu a_\nu)},$$

where the sequence a_ν tends to zero as $\nu \to \infty$. The particular goal was to construct a simple example of an entire function which tends to zero along almost all rays when $|z| \to \infty$. Such an example

$$G(z) = \sum_\nu \frac{z^\nu}{\Gamma(1 + \frac{\nu}{(\log \nu)^\alpha})}, \quad 0 < \alpha < 1, \tag{2.3.1}$$

was constructed [Mal05] and carefully examined by using the calculus of residues for the integral representation of $G(z)$ (which is also analogous to that for E_α).

At the beginning of the twentieth century many mathematicians paid great attention to obtaining asymptotic expansions of special functions, in particular, those of hypergeometric type. The main reason for this was that these functions play an important role in the study of differential equations, which describe different phenomena. In the fundamental paper [Barn06] Barnes proposed a unified approach to the investigation of asymptotic expansions of entire functions defined by Taylor series. This approach was based on the previous results of Barnes [Barn02] and Mellin [Mel02]. The essence of this approach is to use the representation of the quotient of the products of Gamma functions in the form of a contour integral which is handled by using the method of residues. This representation is now known as the *Mellin–Barnes integral formula* (see Appendix D). Among the functions which were treated in [Barn06] was the Mittag-Leffler function. The results of Barnes were further developed in his articles, including applications to the theory of differential equations, as well as in the articles by Mellin (see, e.g., [Mel10]). In fact, the idea of employing contour integrals involving Gamma functions of the variable in the subject of integration is due to Pincherle, whose suggestive paper [Pin88] (see also

[MaiPag03]) was the starting point of Mellin's investigations (1895), although the type of contour and its use can be traced back to Riemann, as Barnes wrote in [Barn07b, p. 63].

Generalizations of the Mittag-Leffler function are proposed among other generalizations of the hypergeometric functions. For them similar approaches were used. Among these generalizations we should point out the collection of *Wright functions*, first introduced in 1934, see [Wri34],

$$\phi(z;\rho,\beta) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\Gamma(\rho n+\beta)} = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\rho n+\beta)}; \tag{2.3.2}$$

the collection of generalized hypergeometric functions, first introduced in 1928, see [Fox28],

$${}_pF_q(z) = {}_pF_q\left(\alpha_1,\alpha_2,\ldots,\alpha_p;\beta_1,\beta_2,\ldots,\beta_q;z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdot(\alpha_2)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdot(\beta_2)_k\cdots(\beta_q)_k}\frac{z^k}{k!}{}^{5}; \tag{2.3.3}$$

the collection of *Meijer G-functions* introduced in 1936, see [Mei36], and intensively treated in 1946, see [Mei46],

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_T \frac{\prod\limits_{i=1}^{m}\Gamma(b_i+s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-s)}{\prod\limits_{i=m+1}^{q}\Gamma(1-b_i-s)\prod\limits_{i=n+1}^{p}\Gamma(a_i+s)} z^{-s}\mathrm{d}s, \tag{2.3.4}$$

and the collection of more general *Fox H-functions* introduced in 1961 [Fox61]

$$H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\ (b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_T \frac{\prod\limits_{i=1}^{m}\Gamma(b_i+\beta_i s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod\limits_{i=m+1}^{q}\Gamma(1-b_i-\beta_i s)\prod\limits_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)} z^{-s}\mathrm{d}s. \tag{2.3.5}$$

Some generalizations of the Mittag-Leffler function appeared as a result of developments in integral transform theory. In this connection in 1953 Agarval and

[5] Here $(\cdot)_k$ is the Pochhammer symbol, see (A.1.17).

Humbert (see [Hum53, Aga53, HumAga53]) and independently in 1954 Djrbashyan (see [Dzh54a, Dzh54b, Dzh54c]) introduced and studied the *two-parametric Mittag-Leffler function* (or *Mittag-Leffler type function*)

$$E_{\alpha,\beta}(z) := \sum_{\nu=0}^{\infty} \frac{z^{\nu}}{\Gamma(\nu \cdot \alpha + \beta)}. \tag{2.3.6}$$

We note once more that, formally, the function (2.3.6) first appeared in the paper of Wiman [Wim05a], who did not pay much attention to its extended study.

In 1971 Prabhakar [Pra71] introduced the *three-parametric Mittag-Leffler function* (or *generalized Mittag-Leffler function*, or *Prabhakar function*)

$$E_{\alpha,\beta}^{\rho}(z) := \sum_{\nu=0}^{\infty} \frac{(\rho)_{\nu} z^{\nu}}{\Gamma(\nu \cdot \alpha + \beta)}. \tag{2.3.7}$$

This function appeared in the kernel of a first order integral equation which Prabhakar treated by using fractional calculus.

Recently, other *three-parametric Mittag-Leffler functions* (also called *generalized Mittag-Leffler functions* or *Mittag-Leffler type functions*, or *Kilbas–Saigo functions*) were introduced by Kilbas and Saigo (see, e.g., [KilSai95a])

$$E_{\alpha,m,l}(z) := \sum_{n=0}^{\infty} c_n z^n, \tag{2.3.8}$$

where

$$c_0 = 1, \ c_n = \prod_{i=0}^{n-1} \frac{\Gamma[\alpha(im + l) + 1]}{\Gamma[\alpha(im + l + 1) + 1]}.$$

These functions appeared in connection with the solution of new types of integral and differential equations and with the development of the fractional calculus.

For real $\alpha_1, \alpha_2 \in \mathbb{R}$ $(\alpha_1^2 + \alpha_2^2 \neq 0)$ and complex $\beta_1, \beta_2 \in \mathbb{C}$ the following function was introduced by Dzherbashian (=Djrbashian) [Dzh60] in the form of the series (in fact only for $\alpha_1, \ \alpha_2 > 0$)

$$E_{\alpha_1,\beta_1;\alpha_2,\beta_2}(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1)\Gamma(\alpha_2 k + \beta_2)} \quad (z \in \mathbb{C}). \tag{2.3.9}$$

Generalizing the four-parametric Mittag-Leffler function (2.3.9) Al-Bassam and Luchko [Al-BLuc95] introduced the following Mittag-Leffler type function

$$E((\alpha, \beta)_m; z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j k + \beta_j)} \quad (m \in \mathbb{N}) \tag{2.3.10}$$

with $2m$ real parameters $\alpha_j > 0; \beta_j \in \mathbb{R}$ $(j = 1, \dots, m)$ and with complex $z \in \mathbb{C}$. In [Al-BLuc95] an explicit solution to a Cauchy type problem for a fractional differential equation is given in terms of (2.3.10). The theory of this class of functions was developed in the series of articles by Kiryakova et al. [Kir99, Kir00, Kir08, Kir10a, Kir10b].

In the last several decades the study of the Mittag-Leffler function has become a very important branch of Special Function Theory. Many important results have been obtained by applying integral transforms to different types of functions from the Mittag-Leffler collection. Conversely, Mittag-Leffler functions generate new kinds of integral transforms with properties making them applicable to various mathematical models.

A number of more general functions related to the Mittag-Leffler function will be discussed in Chap. 6 below.

Nowadays the Mittag-Leffler function and its numerous generalizations have acquired a new life. The recent notable increased interest in the study of their relevant properties is due to the close connection of the Mittag-Leffler function to the Fractional Calculus and its application to the study of Differential and Integral Equations (in particular, of fractional order). Many modern models of fractional type have recently been proposed in Probability Theory, Mechanics, Mathematical Physics, Chemistry, Biology, Mathematical Economics etc. Historical remarks concerning these subjects will be presented at the end of the corresponding chapters of this book.

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