

Chapter 2

SPH Models for Geo-disasters

The Smoothed Particle Hydrodynamics (SPH) method is one of the earliest mesh-free methods of pure Lagrangian description, and has been widely used in many fields of engineering. In this chapter, the development history, basic concept, and essential formulations of SPH are introduced. Governing equations and constitutive laws are incorporated into the SPH framework, and SPH models for geo-disaster modeling and analysis are established based on existing research (Liu and Liu 2003; Moriguchi 2005; Nonoyama 2011).

2.1 Basic Concept of SPH

SPH is a true mesh-free particle method based on a pure Lagrange description, which was first developed to solve astrophysical problems in three-dimensional open space, particularly polytopes (Lucy 1977; Gingold and Monaghan 1977). Later, Monaghan and Lattanzio (1985) summarized basic concepts of discretization for the governing equations, including continuity, momentum, and energy. Selection of the smoothing kernel function and techniques used in deriving SPH formulations for complex partial differential equations (PDEs) have been discussed. Considering the instability of the SPH numerical solution, Swegle et al. (1995), Dyka (1994), and Chen et al. (1999) proposed stabilization schemes. Johnson and Beissel (1996) put forth a calculation method for stress.

The core of this method is fully implied in the three words Smoothed Particle Hydrodynamics. “Smoothed” represents the smoothed approximation nature of using weighted averages of neighboring particles for stability. “Particle” indicates that the method is based on mesh-free particle theory. The computing domain is treated using a discrete particle instead of continuous entities. “Hydrodynamics” points to the fact that the SPH method was first applied to hydrodynamics problems.

The basic concept of SPH is that a continuous fluid is represented by a set of arbitrarily distributed particles. The moving particles possess material properties. By providing accurate and stable numerical solutions for hydrodynamic equations

and tracking movements of each particle, the method can describe the mechanical behavior of an entire system. Therefore, the key facet of SPH is how to solve the PDEs using a series of arbitrarily distributed particles carrying field variables, such as mass, density, energy, and stress tensors. In an actual situation, it is usually difficult to obtain an analytical solution of these PDEs, which gives rise to the need for numerical methods for them. The first step is to discretize the problem domain of the PDEs. Then, there is a need to approximate the variable function and its derivative for the arbitrary particles. Finally, the approximate functions are applied to the PDEs to obtain a series of discretization Ordinary Differential Equations (ODEs), which are only related to time.

The core concept of the SPH method can be summarized as follows.

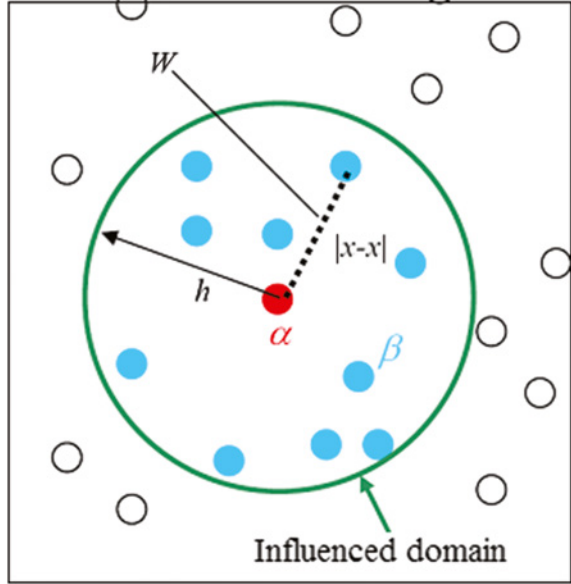
1. In the SPH model, the problem domain is replaced by a series of arbitrarily distributed particles. There is no connectivity between these particles, which reflects the mesh-free nature of this method. The major concern of this method is how to ensure the stability of numerical solutions, especially in applying the arbitrarily distributed particles to solve problems with derivative boundary conditions.
2. One of the most important steps is to represent a function in continuous form as an integral representation using an interpolation function. This step is usually called kernel approximation. The integral has a smoothing effect, similar to the weak form equations. In reality, the kernel approximation stabilizes numerical calculation of the SPH.
3. Another important step is that the value of a function at computing particle a is approximated using the averages of function values at all neighboring particles within the horizon of particle a . This step is termed particle approximation. The role of this approximation is to generate banded or sparse discretized system matrices, which are extremely important for calculation efficiency.
4. Using an explicit integration algorithm to solve differential equations can achieve fast time stepping. The time history of all field variables for all the particles can also be obtained. An appropriate method to determine the time step must be selected in the SPH method.

In summary, the mesh-free, adaptive, stable, and Lagrangian-description SPH method can be used as a dynamics problem solver.

2.2 SPH Approximation

The SPH method is built on interpolation theory, with two essential aspects. The first is smoothed (or kernel) approximation, which represents a function in continuous form as an integral representation. The other is particle approximation, which represents the problem domain using a set of discrete particles within the influence domain to estimate field variables for those computing particles. The value of a function at computing particle a is approximated using the average of those values

Fig. 2.1 SPH concept figure (reprinted from Huang et al. (2011) with permission of Springer)



of the function at all particles in the influence domain of particle a , weighted by the smoothing function. The radius of influence domain is defined as h , depending on the precision of specific problems. As shown in Fig. 2.1, W is the smoothing kernel function, α is the computing particle, and β is the neighboring particle.

2.2.1 Kernel Approximation

The integral representation of a function $f(x)$ used in the SPH method can be rewritten in a continuous form:

$$f(x) = \int_D f(x') \delta(x - x') dx', \quad (2.1)$$

where $f(x)$ is a function of the three-dimensional position vector x , D is the volume of the integral that contains x , and $\delta(x - x')$ is the Dirac delta function, given by

$$\delta(x - x') = \begin{cases} 1 & x = x' \\ 0 & x \neq x' \end{cases}. \quad (2.2)$$

If $f(x)$ is defined as a continuous function in D , Eq. (2.1) is exact or rigorous. If the Delta function kernel $\delta(x - x')$ is replaced by a smoothing kernel function $W(x - x', h)$, the integral representation of $f(x)$ is approximated by

$$f(x) \approx \int_D f(x') W(x - x', h) dx'. \quad (2.3)$$

The smoothing kernel function is also known as the interpolation kernel function, and has the following two characteristics:

$$\int_D W(x - x', h) dx' = 1, \quad (2.4)$$

$$\lim_{h \rightarrow 0} W(x - x', h) = \delta(x - x'). \quad (2.5)$$

W is usually chosen according to the above requirements, which should also be a differentiable even function. The Gaussian kernel, index kernel, cubic spline functions, B-spline function, and quartic spline functions are common smoothing kernel functions.

The above describes the kernel approximation, which is denoted by angle brackets $\langle \rangle$. Hence, Eq. (2.3) can be written as

$$\begin{aligned} \langle f(x) \rangle &= \int_D f(x') W(x - x', h) dx' \\ &= \int_D f(x') W(r, h) dx'. \end{aligned} \quad (2.6)$$

It is obvious that Eq. (2.4) is a unity condition. Equation (2.5) considers the value of W at the point $x' = x$, and is a strongly peaked function. If $|x - x'| > h$, the value of W is zero. If h tends to zero, W is the delta function kernel $\delta(x - x')$ and it can therefore be given as $\lim_{h \rightarrow 0} \langle f(x) \rangle = f(x)$. The $\langle f(x) \rangle$ is a kernel approximation of $f(x)$, which can be considered as a smoothing or filter for $W(x - x', h)$, filtering local statistical fluctuation of $W(x - x', h)$.

Equation (2.6) is a standard expression of the kernel approximation of a function. The approximation of the spatial derivative $\nabla f(x)$ is obtained simply by replacing $f(x)$ with $\nabla f(x)$ in Eq. (2.6), which gives

$$\begin{aligned} \langle \nabla f(x) \rangle &= \int_D \nabla f(x') W(x - x', h) dx' \\ &= \int_D \nabla f(x') W(r, h) dx'. \end{aligned} \quad (2.7)$$

Because the above equation cannot be calculated directly, the following deformation is conducted:

$$[\nabla f(x')] W(x - x', h) = \nabla[f(x') W(x - x', h)] - f(x') \nabla W(x - x', h). \quad (2.8)$$

$$\begin{aligned} \langle \nabla f(x) \rangle &= \int_D \nabla [f(x') W(x - x', h)] dx' - \int_D f(x') \nabla W(x - x', h) dx' \\ &= \int_S f(x') W(x - x', h) \mathbf{n} dS - \int_D f(x') \nabla W(x - x', h) dx', \end{aligned} \quad (2.9)$$

where S is the boundary surface of the integral volume and $\mathbf{n}$ is the unit vector normal to surface S . Because of the compact support condition of W , the surface integral of Eq. (2.5) is zero in the influence domain D . Then Eq. (2.9) can be rewritten as

$$\langle \nabla f(x) \rangle \approx - \int_D f(x') \nabla W(x - x', h) dx'. \quad (2.10)$$

Equation (2.10) is the most commonly used form of function approximation for $\nabla f(x)$.

2.2.2 Particle Approximation

If kernel function W is an n -time differentiable function, it can be deduced that $\langle f(x) \rangle$ is also such a function.

Assuming that the fluid is flowing with density $\rho(x)$, it is divided into N volume elements. Masses of the volume elements are $m_1, m_2, m_3, \dots, m_N$, respectively, and positions of corresponding centers of mass are $x_1, x_2, x_3, \dots, x_N$, respectively. The continuous SPH integral representation for $f(x)$ can be written in the following form of discretized particle approximation:

$$\langle f(x) \rangle = \sum_{j=1}^N m_j \frac{f_j(x')}{\rho_j} W(x - x', h). \quad (2.11)$$

Based on Eq. (2.11), Eqs. (2.6) and (2.10) can be written as

$$\langle f_i(x) \rangle = \sum_{j=1}^N \frac{m_j}{\rho_j} f_j(x') W(x - x', h), \quad (2.12)$$

$$\langle \nabla f_i(x) \rangle = - \sum_{j=1}^N \frac{m_j}{\rho_j} f_j(x') \nabla W(x - x', h). \quad (2.13)$$

Following the same argument, the particle approximation for the spatial derivative of the function is

$$\begin{aligned}
 \frac{\partial \langle f(x) \rangle}{\partial x_i} &= \frac{\partial}{\partial x_i} \int f_j(x') W(x - x', h) dx' \\
 &= \int f_j(x') \frac{\partial W(x - x', h)}{\partial x_i} dx' \\
 &= \sum_{j=1}^N \frac{m_j}{\rho_j} f_j(x') \frac{\partial W(x - x', h)}{\partial x_i}.
 \end{aligned} \tag{2.14}$$

Here, we use $\rho(x)$ instead of $f(x)$. From Eq. (2.11), $\rho(x)$ can be obtained as

$$\langle \rho_i(x) \rangle = \sum_{j=1}^N m_j W(x - x', h). \tag{2.15}$$

This states that the density of particles can be approximated by the weighted average of particle densities within the influence domain of a particle. This is the meaning of the “smoothed particle.”

2.3 SPH Model for Hydrodynamics

2.3.1 Governing Equations

Navier–Stokes (N–S) equations are introduced in this section as the governing equations to describe the incompressible Newtonian fluid.

2.3.1.1 Equation of Continuity

The continuity equation is based on the conservation of mass, which is obtained as

$$\frac{d\rho}{dt} = -\rho \frac{\partial u^\beta}{\partial x^\beta}, \tag{2.16}$$

where ρ is density, t is time, u represents the velocity vector, and the superscript β denotes the coordinate directions (as does the parameter α in subsequent equations). The above equation can be transformed as follows:

$$\frac{d\rho}{dt} = -\frac{\partial (\rho u^\beta)}{\partial x^\beta} + u^\beta \frac{\partial \rho}{\partial x^\beta}. \tag{2.17}$$

2.3.1.2 Equation of Motion

In general, the equation of motion is given by

$$\frac{du^\alpha}{dt} = \frac{1}{\rho} \frac{\partial \sigma^{\alpha\beta}}{\partial x^\beta} + F, \quad (2.18)$$

where σ represents the stress tensor and F the external force vector. The first term on the right side of Eq. (2.18) can be expanded as follows:

$$\frac{1}{\rho} \frac{\partial \sigma^{\alpha\beta}}{\partial x^\beta} = \frac{\partial}{\partial x^\beta} \left(\frac{\sigma^{\alpha\beta}}{\rho} \right) + \frac{\sigma^{\alpha\beta}}{\rho^2} \frac{\partial \rho}{\partial x^\beta}. \quad (2.19)$$

2.3.1.3 Equation of Energy

The equation of energy is based on the conservation of energy, which is a representation of the first law of thermodynamics.

$$\frac{\partial e}{\partial t} = \frac{\sigma^{\alpha\beta}}{\rho} \frac{\partial v^\alpha}{\partial x^\beta}. \quad (2.20)$$

2.3.2 SPH Formulation for Governing Equations

Using the smoothing approximation and particle approximation mentioned in the last section, the SPH version of the governing equations can be expressed as follows:

$$\text{Equation of continuity : } \frac{d\rho_i}{dt} = \sum_{j=1}^N m_j \left(u_i^\beta - u_j^\beta \right) \frac{\partial W_{ij}}{\partial x_i^\beta}, \quad (2.21)$$

$$\text{Equation of motion : } \frac{du_i^\alpha}{dt} = \sum_{j=1}^N m_j \left[\frac{\sigma_i^{\alpha\beta}}{(\rho_i)^2} + \frac{\sigma_j^{\alpha\beta}}{(\rho_j)^2} \right] \frac{\partial W_{ij}}{\partial x_j^\beta} + F_i, \quad (2.22)$$

$$\text{Equation of energy : } \frac{De_i}{Dt} = \frac{1}{2} \sum_{j=1}^N m_j \left(\frac{P_i + P_j}{\rho_i \rho_j} \right) v_{ij}^\beta \frac{\partial W_{ij}}{\partial x_i^\beta} + \frac{\mu_i}{2\rho_i} \varepsilon_i^{\alpha\beta} \varepsilon_j^{\alpha\beta}, \quad (2.23)$$

where W_{ij} is the smoothing function of particle i evaluated at particle j , and is closely related to h .

$$W_{ij} = W(x_i - x_j, h) = W(R_{ij}, h), \quad (2.24)$$

where $R_{ij} = r_{ij}/h = |x_i - x_j|/h$ is the relative distance of particles i and j , and r_{ij} is the distance between these two particles.

In this monograph, we consider the temperature of soil material as a constant. Therefore, the energy in Eq. (2.23) is not considered.

2.3.3 Newton Fluid Constitutive Model

The Newton fluid stress tensor is

$$\sigma_{ij} = -P\delta_{ij} + \xi D_{kk}\delta_{ij} + 2\eta D_{ij}, \quad (2.25)$$

where

$$D_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2.26)$$

where σ_{ij} is a stress tensor, P is pressure, δ_{ij} is the Kronecker delta function, ξ is the second viscosity coefficient, η is the viscosity coefficient, and D_{ij} is a shear strain tensor.

Introducing the Stokes hypothesis

$$\xi + \frac{2}{3}\eta = 0. \quad (2.27)$$

Then Eq. (2.25) can be rewritten as

$$\sigma_{ij} = -P\delta_{ij} - \frac{2}{3}\eta D_{ij}\delta_{ij} + 2\eta D_{ij}. \quad (2.28)$$

2.3.4 SPH Formulation for Poisson's Equation

To solve the N-S equations used in this monograph, we should first obtain the pressure by solving Poisson's equation. Together with velocity u_k , pressure p_k , and external force F_k , we introduce a virtual velocity u^* to construct the equation. Then the N-S equations may be obtained.

From Eqs. (2.18), (2.26) and (2.28), the incompressible N-S governing equation for the momentum equation becomes

$$\frac{Du_i}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x_j} + \frac{\eta}{\rho} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} \right) + F_i. \quad (2.29)$$

That is,

$$\frac{Du}{Dt} = -\frac{1}{\rho} \text{grad}P + \nu \nabla^2 u + F. \quad (2.30)$$

Here, we introduce u^* , which is defined as

$$u^* = u^k + \Delta t \left(-\frac{1}{\rho} \text{grad}P^k + \nu \nabla^2 u^k + F^k \right). \quad (2.31)$$

Simultaneously, we assume u^k and P^k as

$$u^{k+1} = u^k + u', \quad (2.32)$$

$$P^{k+1} = P^k + P'. \quad (2.33)$$

Adding Eq. (2.31) to (2.32), we obtain the following:

$$\begin{aligned}
 (u_i^\alpha)^{k+1} &= (u_i^\alpha)^k + \Delta t \left(-\frac{1}{\rho} \text{grad}(P^\alpha)^{k+1} + \nu \nabla^2 (u_i^\alpha)^k + F^k \right) \\
 &= (u_i^\alpha)^* - \Delta t \left(-\frac{1}{\rho} \text{grad}(P^\alpha)^k + \nu \nabla^2 (u_i^\alpha)^k + F^k \right) \\
 &\quad + \Delta t \left(-\frac{1}{\rho} \text{grad}(P^\alpha)^{k+1} + \nu \nabla^2 (u_i^\alpha)^k + F^k \right) \\
 &= (u_i^\alpha)^* + \Delta t \frac{1}{\rho} \left(\text{grad}(P^\alpha)^k - \text{grad}(P^\alpha)^{k+1} \right),
 \end{aligned} \tag{2.34}$$

$$\begin{aligned}
 u_i^{\alpha k+1} &= u_i^{\alpha k} + \Delta t \left(-\frac{1}{\rho} \frac{\partial P^{k+1}}{\partial x_i} + \eta \frac{\partial^2 u_i^{\alpha k}}{\partial x_j^2} + F^k \right) \\
 &= u_i^{\alpha*} - \Delta t \frac{1}{\rho} \frac{\partial P'}{\partial x_i},
 \end{aligned} \tag{2.35}$$

where α is the computing particle in the center and β represents neighboring particles within the influence domain.

The derivative of Eq. (2.35) is

$$\frac{\partial (u_i^\alpha)^{k+1}}{\partial x_i} = \frac{\partial (u_i^\alpha)^*}{\partial x_i} - \Delta t \frac{\partial}{\partial x_i} \left(\frac{1}{\rho^\alpha} \nabla(P'^\alpha) \right). \tag{2.36}$$

We note that

$$\frac{\partial (u_i^\alpha)^{k+1}}{\partial x_i} = \text{div}(u_i^\alpha)^{k+1} = 0. \tag{2.37}$$

Consequently, Eq. (2.36) can be rewritten as

$$\frac{\partial (u_i^\alpha)^*}{\partial x_i} = \Delta t \frac{\partial}{\partial x_i} \left(\frac{1}{\rho^\alpha} \nabla(P'^\alpha) \right). \tag{2.38}$$

The continuity equation gives

$$\frac{D\rho}{Dt} + \rho_0 \nabla u = 0. \tag{2.39}$$

Because fluid density ρ is proportional to the number of particles n , we can obtain

$$\frac{1}{n^0} \frac{Dn}{Dt} + \nabla \cdot u = 0. \tag{2.40}$$

Here, we introduce n^* . From the definition of the smoothed function, we can obtain

$$n^* = \sum_{\alpha \neq \beta}^N W^{\alpha\beta}, \tag{2.41}$$

and

$$\frac{1}{n^0} \frac{n^0 - n^*}{\Delta t} + \nabla \cdot u = 0. \quad (2.42)$$

The n^0 is obtained by

$$n^0 = n^{k+1} = n^* + n'. \quad (2.43)$$

From simultaneous Eqs. (2.38), (2.42), and (2.43), we can obtain

$$\nabla u = -\frac{n^* - n^0}{n^0 \Delta t} = \Delta t \frac{\partial}{\partial x_i} \left(\frac{1}{\rho^\alpha} \nabla (P'^\alpha) \right). \quad (2.44)$$

Then the Poisson's equation is obtained by:

$$\nabla \left(\frac{1}{\rho^\alpha} \nabla (P'^\alpha)^{k+1} \right) = -\frac{n^* - n^0}{n^0 \Delta t^2}. \quad (2.45)$$

Referring to the SPH formulation for the energy equation proposed by Cleary and Monaghan (1999),

$$\frac{1}{\rho} \nabla \cdot (k \nabla T) = \frac{\partial U^\alpha}{\partial t} = \sum_{\beta}^N \frac{4m^\beta}{\rho^\alpha \rho^\beta} \frac{k^\alpha k^\beta}{k^\alpha + k^\beta} \frac{(T^\alpha - T^\beta)(x^\alpha - x^\beta) \cdot \frac{\partial W^{\alpha\beta}}{\partial x_i}}{(x^\alpha - x^\beta)^2 + (y^\alpha - y^\beta)^2}. \quad (2.46)$$

Applying the SPH discrete equations to the left side of Poisson's equation, we can obtain the following:

$$\begin{aligned} \nabla \cdot \left(\frac{1}{\rho^\alpha} \nabla (P'^\alpha) \right) &= \sum_{\beta}^N \frac{4m^\beta}{\rho^\beta} \frac{\frac{1}{\rho^\alpha} \frac{1}{\rho^\beta}}{\frac{1}{\rho^\alpha} + \frac{1}{\rho^\beta}} \frac{((P'^\alpha) - (P'^\beta))(x_i^\alpha - x_i^\beta) \cdot \frac{\partial W^{\alpha\beta}}{\partial x_i}}{(x^\alpha - x^\beta)^2 + (y^\alpha - y^\beta)^2} \\ &= \sum_{\beta}^N \frac{4m^\beta}{\rho^\beta} \frac{1}{\rho^\alpha + \rho^\beta} \frac{((P'^\alpha) - (P'^\beta))(x_i^\alpha - x_i^\beta) \cdot \frac{\partial W^{\alpha\beta}}{\partial x_i}}{(x^\alpha - x^\beta)^2 + (y^\alpha - y^\beta)^2}. \end{aligned} \quad (2.47)$$

The two-dimensional Poisson equation is represented as follows:

$$\begin{aligned} \nabla \cdot \left(\frac{1}{\rho^\alpha} \nabla (P'^\alpha) \right) &= \sum_{\beta}^N \frac{4m^\beta}{\rho^\beta} \frac{1}{\rho^\alpha + \rho^\beta} \frac{((P'^\alpha) - (P'^\beta))(x^\alpha - x^\beta) \cdot \frac{\partial W^{\alpha\beta}}{\partial x}}{(x^\alpha - x^\beta)^2 + (y^\alpha - y^\beta)^2} \\ &\quad + \sum_{\beta}^N \frac{4m^\beta}{\rho^\beta} \frac{1}{\rho^\alpha + \rho^\beta} \frac{((P'^\alpha) - (P'^\beta))(y^\alpha - y^\beta) \cdot \frac{\partial W^{\alpha\beta}}{\partial y}}{(x^\alpha - x^\beta)^2 + (y^\alpha - y^\beta)^2} \\ &= \sum_{\beta}^N \frac{4m^\beta}{\rho^\beta} \frac{1}{\rho^\alpha + \rho^\beta} \left(\frac{(x^\alpha - x^\beta) \cdot \frac{\partial W^{\alpha\beta}}{\partial x} + (y^\alpha - y^\beta) \cdot \frac{\partial W^{\alpha\beta}}{\partial y}}{(x^\alpha - x^\beta)^2 + (y^\alpha - y^\beta)^2} \right) [(P'^\alpha) - (P'^\beta)]. \end{aligned} \quad (2.48)$$

$$-\frac{n^* - n^0}{n^0 \Delta t^2} = \sum_{\beta}^N \frac{4m^{\beta}}{\rho^{\beta}} \frac{1}{\rho^{\alpha} + \rho^{\beta}} \left(\frac{(x^{\alpha} - x^{\beta}) \cdot \frac{\partial W^{\alpha\beta}}{\partial x} + (y^{\alpha} - y^{\beta}) \cdot \frac{\partial W^{\alpha\beta}}{\partial y}}{(x^{\alpha} - x^{\beta})^2 + (y^{\alpha} - y^{\beta})^2} \right) ((P'^{\alpha}) - (P'^{\beta})). \quad (2.49)$$

That is,

$$\begin{aligned} -\frac{n^* - n^0}{n^0 \Delta t^2} &= \sum_{\beta}^N \frac{4m^{\beta}}{\rho^{\beta}} \frac{1}{\rho^{\alpha} + \rho^{\beta}} \left(\frac{(x^{\alpha} - x^{\beta}) \cdot \frac{\partial W^{\alpha\beta}}{\partial x} + (y^{\alpha} - y^{\beta}) \cdot \frac{\partial W^{\alpha\beta}}{\partial y}}{(x^{\alpha} - x^{\beta})^2 + (y^{\alpha} - y^{\beta})^2} \right) (P'^{\alpha}) \\ &\quad - \sum_{\beta}^N \frac{4m^{\beta}}{\rho^{\beta}} \frac{1}{\rho^{\alpha} + \rho^{\beta}} \left(\frac{(x^{\alpha} - x^{\beta}) \cdot \frac{\partial W^{\alpha\beta}}{\partial x} + (y^{\alpha} - y^{\beta}) \cdot \frac{\partial W^{\alpha\beta}}{\partial y}}{(x^{\alpha} - x^{\beta})^2 + (y^{\alpha} - y^{\beta})^2} \right) (P'^{\beta}). \end{aligned} \quad (2.50)$$

The above equation can be solved by the incomplete Cholesky decomposition conjugate gradient (ICCG) method.

2.4 SPH Model for Solid Mechanics

Compared with other numerical methods, SPH has shown a distinct advantage in dealing with large deformation problems. The method has been widely applied in many disciplines, and has evolved from a hydrodynamics technique to a mechanics one. In this section, the Drucker–Prager material law has been incorporated into the SPH model to promote the application of that model in solid mechanics.

2.4.1 Governing Equations

2.4.1.1 SPH Approximation of Equation of Continuity

When SPH is used to calculate the elasto-plastic mechanic problems, there is still a need to rewrite the equation of continuity in the SPH formulations.

$$\rho_i = \sum_{j=1}^N m_j W_{ij}. \quad (2.51)$$

Similarly, Eq. (2.52) is used to treat the free-surface and material interface.

$$\rho_i = \frac{\sum_{j=1}^N m_j W_{ij}}{\sum_{j=1}^N \frac{m_j}{\rho_j} W_{ij}}. \quad (2.52)$$

Equation (2.53) is the mass conservation equation in a Euler description. In a Lagrangian description, it can be expressed as Eq. (2.54). Therefore, another SPH formulation for the equation of continuity is obtained as Eq. (2.55).

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0, \quad (2.53)$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_i)}{\partial x_i} + v_i \frac{\partial \rho}{\partial x_i}, \quad (2.54)$$

$$\frac{d\rho}{dt} = -\sum_{\beta}^N \frac{m^{\beta}}{\rho^{\beta}} \rho^{\beta} v_i^{\beta} \frac{\partial W^{\alpha\beta}}{\partial x_i} + v_i \sum_{\beta}^N \frac{m^{\beta}}{\rho^{\beta}} \rho^{\beta} \frac{\partial W^{\alpha\beta}}{\partial x_i} = \sum_{\beta}^N m^{\beta} (v_i^{\alpha} - v_i^{\beta}) \frac{\partial W^{\alpha\beta}}{\partial x_i^{\beta}}, \quad (2.55)$$

2.4.1.2 SPH Approximation of Equation of Momentum

The equation of momentum in the elasto-plastic SPH model is similar to that in fluid dynamics.

$$\frac{dv_i}{dt} = \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} + F_i, \quad (2.56)$$

where σ_{ij} is the stress tensor and F_i is body force.

Evaluating the first item on the right side of Eq. (2.56), then

$$\frac{\partial}{\partial x_i} \left(\frac{\sigma_{ij}}{\rho} \right) = \frac{\rho \frac{\partial \sigma_{ij}}{\partial x_i} - \sigma_{ij} \frac{\partial \rho}{\partial x_i}}{\rho^2} = \frac{\partial}{\partial x_i} \left(\frac{\sigma_{ij}}{\rho} \right) + \frac{\sigma_{ij} \frac{\partial \rho}{\partial x_i}}{\rho^2}. \quad (2.57)$$

Substituting Eq. (2.57) into (2.56), we have

$$\frac{dv_i}{dt} = \frac{\partial}{\partial x_i} \left(\frac{\sigma_{ij}}{\rho} \right) + \frac{\sigma_{ij} \frac{\partial \rho}{\partial x_i}}{\rho^2} + F_i. \quad (2.58)$$

Using the smoothing and particle approximations, the SPH formulations for the equation of momentum in the elasto-plastic SPH model can be described by

$$\begin{aligned} \frac{Dv_i^{\alpha}}{Dt} &= \frac{\sigma_i^{\alpha\beta}}{\rho_i^2} \sum_{j=1}^N \frac{m_j}{\rho_j} \rho_j \frac{\partial W_{ij}}{\partial x_i^{\beta}} + \sum_{j=1}^N \frac{m_j}{\rho_j} \frac{\sigma_j^{\alpha\beta}}{\rho_j} \frac{\partial W_{ij}}{\partial x_i^{\beta}} + F_i \\ &= \sum_{j=1}^N m_j \left(\frac{\sigma_i^{\alpha\beta}}{\rho_i^2} + \frac{\sigma_j^{\alpha\beta}}{\rho_j^2} \right) \frac{\partial W_{ij}}{\partial x_i^{\beta}} + F_i. \end{aligned} \quad (2.59)$$

An additional artificial viscosity is usually incorporated into the pressure terms of the equation of motion to convert kinetic energy to heat (Monaghan and Gingold 1983; see Eq. (2.60)). This avoids numerical oscillation and resists penetration between particles.

$$\frac{Dv_i^\alpha}{Dt} = \sum_{j=1}^N m_j \left(\frac{\sigma_i^{\alpha\beta}}{\rho_i^2} + \frac{\sigma_j^{\alpha\beta}}{\rho_j^2} - \delta^{\alpha\beta} \Pi_{ij} \right) \frac{\partial W_{ij}}{\partial x_i^\beta} + F_i. \quad (2.60)$$

This viscosity is defined by the following formulas:

$$\Pi_{ij} = \frac{-a\bar{c}\mu_{ij} + b(\mu_{ij})^2}{\rho^{\alpha\beta}}, \quad (2.61)$$

$$\mu_{ij} = \frac{h(v_i - v_j) \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}}{(x_i - x_j)^2 + (y_i - y_j)^2 + kh^2}, \quad (2.62)$$

$$\bar{c} = \frac{1}{2}(c_i + c_j), \quad (2.63)$$

$$\bar{\rho} = \frac{1}{2}(\rho_i + \rho_j), \quad (2.64)$$

where a and b are constant parameters (often chosen to be 1.0) that improve numerical stability of the code (Monaghan 1988; Evard 1988). k is a free parameter used to avoid numerical divergence when particles approach each other, and can be taken as 0.01 (Liu and Liu 2003). c_i and c_j are the velocity of sound at points i and j , respectively.

2.4.2 Elasto-Plastic Constitutive Model

2.4.2.1 Stress–Strain Relation

For elasto-plastic materials, the constitutive model in the form of (Jaumann) stress rate can be used to establish a relationship between stress states and particle motion. To deal with the large deformation problem, total deformation can be separated into elastic and plastic deformations, but the former is relatively small.

From elastic mechanics, the stress-strain relationship of elastic deformation can be represented as

$$\sigma_{ij} = D_{ijkl}^e \varepsilon_{kl}^e, \quad (2.65)$$

where σ_{ij} is the elastic stress tensor, D_{ijkl}^e is a fourth-order elastic coefficient matrix, and ε_{kl}^e is elastic strain. The incremental form of the elastic stress–strain relationship can be represented as

$$d\sigma_{ij} = D_{ijkl}^e d\varepsilon_{kl}^e, \quad (2.66)$$

$$[D^e] = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}, \quad (2.67)$$

where parameters λ and μ are

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad (2.68)$$

$$\mu = \frac{E}{2(1 + \nu)}, \quad (2.69)$$

where E is Young's modulus and ν is Poisson's ratio.

Combining Eqs. (2.66) and (2.67), the stress-strain matrix can be obtained as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & \lambda + 2\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^e \\ \varepsilon_{yy}^e \\ \varepsilon_{zz}^e \\ \varepsilon_{xy}^e \\ \varepsilon_{yz}^e \\ \varepsilon_{zx}^e \end{bmatrix}. \quad (2.70)$$

In the plain strain problem, σ_{yz} , σ_{zx} , ε_{zz} , ε_{yz} , and ε_{zx} are all zero. Therefore, D_{ijkl}^e can be obtained as follows:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \mu & 0 \\ \mu & \lambda + 2\mu & 0 \\ \mu & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^e \\ \varepsilon_{yy}^e \\ \gamma_{xy}^e \end{bmatrix}. \quad (2.71)$$

It is well known that three problems must be considered to establish an elasto-plastic constitutive model. The first is the plastic potential function, which determines the direction of plastic strain. The second is the yield function, which determines whether plastic strain emerges. The last is the compatibility equation, which determines the magnitude of plastic strain.

Simple tensile test results show that the full yield strain ε_{ij} is composed of two parts, elastic strain ε^e and plastic strain ε^p :

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p. \quad (2.72)$$

The differential of Eq. (2.72) is

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p. \quad (2.73)$$

It can be rewritten as

$$d\varepsilon_{ij}^e = d\varepsilon_{ij} - d\varepsilon_{ij}^p. \quad (2.74)$$

Substituting the above equation into Eq. (2.66), we can obtain

$$d\sigma_{ij} = D_{ijkl}^e (d\varepsilon_{ij} - d\varepsilon_{ij}^p). \quad (2.75)$$

The flow rule of plastic strain is represented as

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial g}{\partial \sigma_{ij}} \quad (d\lambda \geq 0), \quad (2.76)$$

where $d\lambda$ depends on particle position and the load level. This represents the property of the plastic material and can determine the magnitude of plastic strain. g is the plastic potential function. When the yield surface coincides with the plastic potential surface, the yield function can act as the plastic potential function, which can be expressed as

$$\begin{cases} d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}} & (d\lambda \geq 0). \\ f = g \end{cases} \quad (2.77)$$

Because of the plastic deformation, g is related not only to the stress state but also to the loading history. Here, a hardening parameter is introduced to represent this history. Then, the yield function f can be written as

$$f(\sigma_{ij}, L) = 0. \quad (2.78)$$

The total derivative of the above equation may be written as

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial L} \frac{\partial L}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p = 0. \quad (2.79)$$

Substituting Eq. (2.75) into the above equation, we can obtain

$$\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e (d\varepsilon_{kl} - d\varepsilon_{kl}^p) + \frac{\partial f}{\partial L} \frac{\partial L}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p = 0. \quad (2.80)$$

This can be rewritten as

$$\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e d\varepsilon_{kl} = \left(\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e - \frac{\partial f}{\partial L} \frac{\partial L}{\partial \varepsilon_{kl}^p} \right) d\varepsilon_{kl}^p. \quad (2.81)$$

Substituting Eq. (2.77) into the above equation, we can obtain

$$\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e d\varepsilon_{kl} = \left(\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e - \frac{\partial f}{\partial L} \frac{\partial L}{\partial \varepsilon_{kl}^p} \right) d\lambda \frac{\partial f}{\partial \sigma_{kl}}, \quad (2.82)$$

where $d\lambda$ can be solved as

$$d\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e d\varepsilon_{kl}}{\left(\frac{\partial f}{\partial \sigma_{ij}} D_{ijkl}^e - \frac{\partial f}{\partial L} \frac{\partial L}{\partial \varepsilon_{kl}^p} \right) \frac{\partial f}{\partial \sigma_{kl}}}. \quad (2.83)$$

Substituting Eq. (2.77) into (2.75), it can be rewritten as

$$d\sigma_{ij} = D_{ijkl}^e \left(d\varepsilon_{kl} - d\lambda \frac{\partial f}{\partial \sigma_{kl}} \right). \quad (2.84)$$

From the above equation, the strain–stress relationship of elasto-plastic material can be expressed as

$$d\sigma_{ij} = \left(D_{ijkl}^e - \left(d\varepsilon_{kl} - d\lambda \frac{\partial f}{\partial \sigma_{kl}} \right) \frac{D_{ijmnl}^e \frac{\partial f}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{pq}} D_{ijkl}^e}{\frac{\partial f}{\partial \sigma_{mn}} D_{mnpq}^e \frac{\partial f}{\partial \sigma_{pq}} - \frac{\partial f}{\partial L} \frac{\partial L}{\partial \varepsilon_{mn}^p} \frac{\partial f}{\partial \sigma_{mn}}} \right) d\varepsilon_{kl}. \quad (2.85)$$

We introduce the Drucker–Prager condition of the yield function, which is

$$f = q + \alpha p - \kappa, \quad (2.86)$$

where p and q are defined as follows:

$$p = \frac{1}{3} \sigma_{kk}, \quad (2.87)$$

$$q = \sqrt{\frac{3}{2} s_{ij} s_{ij}}, \quad (2.88)$$

where s_{ij} is deviatoric stress, defined as

$$s_{ij} = \sigma_{ij} - p \delta_{ij}. \quad (2.89)$$

Parameters α and k used in the yield function are dimensionless constants. Here, we choose α and k as follows:

$$\alpha = \frac{\tan \phi}{\sqrt{9 + 12 \cdot \tan^2 \phi}}, \quad (2.90)$$

$$\kappa = \frac{3 \cdot c}{\sqrt{9 + 12 \cdot \tan^2 \phi}}, \quad (2.91)$$

where c is cohesion and ϕ is an internal friction angle.

2.4.2.2 Jaumann Stress Rate

In the SPH method, deformation and motion of the spatial continuum are represented by motion of the discrete particles. A distinct feature of the Jaumann stress rate is that it can form a relationship between the stress rate and strain rate tensors.

$$\hat{\sigma}_{ij} = \dot{\sigma}_{ij} + \sigma_{ik}\omega_{kj} - \omega_{ik}\sigma_{kj}, \quad (2.92)$$

where w_{ij} is the rotation tensor and $\dot{\sigma}_{ij}$ is the Cauchy stress rate tensor. They are defined in the following:

$$\omega_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right), \quad (2.93)$$

$$\dot{\sigma}_{ij} = \frac{d\sigma_{ij}}{dt}. \quad (2.94)$$

The Jaumann stress rate $\hat{\sigma}_{ij}$ can be written as

$$\hat{\sigma}_{ij} = E_{ijkl}D_{kl}, \quad (2.95)$$

where E_{ijkl} is a fourth-order matrix that represents an elastic stiffness matrix. D_{ij} is a stretch tensor.

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \quad (2.96)$$

After the SPH approximations, D_{ij} can be rewritten in a discrete form:

$$D_{ij} = \frac{1}{2} \sum_j^N \frac{m_j}{\rho_j} \left[\left(v_x^j - v_x^i \right) \frac{\partial W_{ij}}{\partial x} + \left(v_y^j - v_y^i \right) \frac{\partial W_{ij}}{\partial y} \right]. \quad (2.97)$$

Similarly, we can obtain the discrete SPH form for Eq. (2.93) as

$$\omega_{ij} = \frac{1}{2} \sum_j^N \frac{m_j}{\rho_j} \left[\left(v_x^j - v_x^i \right) \frac{\partial W_{ij}}{\partial x} - \left(v_y^j - v_y^i \right) \frac{\partial W_{ij}}{\partial y} \right]. \quad (2.98)$$

The two-dimensional Jaumann stress rate, stretch tensor, and rotation tensor can be expressed as follows:

$$\dot{\sigma}_{xx} = \hat{\sigma}_{xx} + 2\omega_{xy}\sigma_{yx}, \quad (2.99)$$

$$\dot{\sigma}_{yy} = \hat{\sigma}_{yy} - 2\sigma_{xy}\omega_{xy}, \quad (2.100)$$

$$\dot{\sigma}_{xy} = \hat{\sigma}_{xy} - \sigma_{xx}\omega_{xy} + \sigma_{yy}\omega_{xy}, \quad (2.101)$$

$$D_{xx} = \frac{1}{2} \sum_j^N \frac{m_j}{\rho_j} \left[(v_x^j - v_x^i) \frac{\partial W_{ij}}{\partial x} \right], \quad (2.102)$$

$$D_{yy} = \frac{1}{2} \sum_j^N \frac{m_j}{\rho_j} \left[(v_y^j - v_y^i) \frac{\partial W_{ij}}{\partial y} \right], \quad (2.103)$$

$$D_{xy} = D_{yx} = \frac{1}{2} \sum_j^N \frac{m_j}{\rho_j} \left[(v_x^j - v_x^i) \frac{\partial W_{ij}}{\partial x} + (v_y^j - v_y^i) \frac{\partial W_{ij}}{\partial y} \right], \quad (2.104)$$

$$\omega_{xx} = \frac{1}{2} \sum_j^N \frac{m_j}{\rho_j} \left[(v_x^j - v_x^i) \frac{\partial W_{ij}}{\partial x} - (v_y^j - v_y^i) \frac{\partial W_{ij}}{\partial x} \right], \quad (2.105)$$

$$\omega_{yy} = \frac{1}{2} \sum_j^N \frac{m_j}{\rho_j} \left[(v_x^j - v_x^i) \frac{\partial W_{ij}}{\partial y} - (v_y^j - v_y^i) \frac{\partial W_{ij}}{\partial y} \right], \quad (2.106)$$

$$\omega_{xy} = -\omega_{yx} = \frac{1}{2} \sum_j^N \frac{m_j}{\rho_j} \left[(v_x^j - v_x^i) \frac{\partial W_{ij}}{\partial y} - (v_y^j - v_y^i) \frac{\partial W_{ij}}{\partial x} \right]. \quad (2.107)$$

2.5 Numerical Aspects of the SPH Model

2.5.1 Initial Settings

Appropriate initial settings should be made for different computational problems. For specific requirements of varying computing cases, the initial state including the kernel function, type of particles and their coordinates, initial particle spacing, density, pressure, and velocity should be determined at the beginning of the calculation. h was chosen to be twice the particle spacing. The unit time step can be determined through

$$\Delta t = \min(0.4 \frac{h}{c}, 0.25 \sqrt{\frac{h}{f}}, 0.125 \frac{h^2}{\nu}), \quad (2.108)$$

where c is the speed of the sound, f is the external force, ν is the viscosity coefficient, and 0.4, 0.25, and 0.125 are safety factors from experience.

2.5.2 Neighboring Particle Searching Algorithm

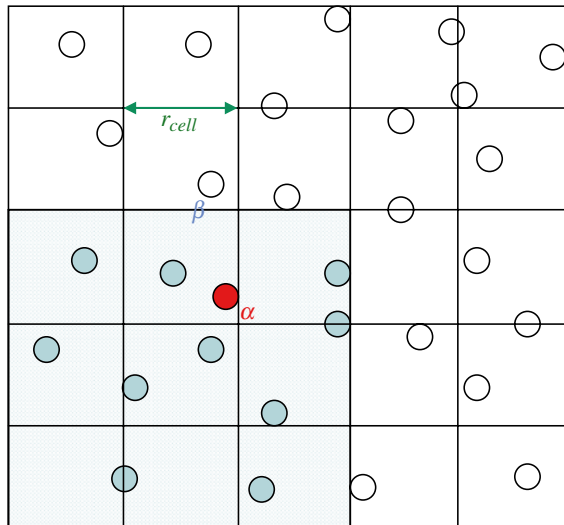
As mentioned above, SPH is a mesh-free method based on interactions with the closest neighboring particles. During the movement of each particle, the spatial location is constantly changing. Therefore, the neighbor list of each particle should be updated every time step. Therefore, creation of the neighbor list is important for high performance of the code. The efficiency of the SPH model strongly depends on the construction and use of this list.

There are currently two major approaches for determining the search area in the mesh-free methods: the Verlet neighbor list and linked-cell neighbor list methods. Put simply, the difference between these methods is that the Verlet neighbor list first defines a limit distance and then calculates the relative distance of each pair of particles. If this distance is less than the limit value, the particle can be considered a neighbor particle. The two-dimensional computational domain of each particle has a circular appearance. The linked-cell neighbor list generates a virtual mesh and determines the amount of computation by limiting the number of close grids. In this approach, the two-dimensional computational domain of each particle is a rectangle.

2.5.2.1 Linked-Cell Neighbor List Method

The linked-cell neighbor list method (Allen and Tildesley 1990; see Fig. 2.2) is a particle search method that takes searching scope determination as its principle. The greatest advantage is that its complexity is of the order $O(N)$ rather than $O(N^2)$. The entire calculation area is divided into several rectangular grids. Particles are allocated to different grids according to their coordinates. Side length r_{cell} is the

Fig. 2.2 Linked-cell neighbor list algorithm (based on Dominguez et al. 2011)



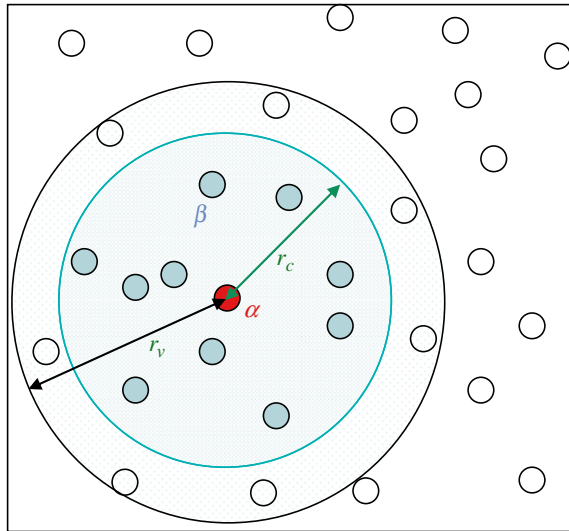
determinant of the mesh dimension, and its value directly impacts the effectiveness and accuracy of the method. After all the particles are included in the various grids, their motion information is stored in different lists and associated with each other. For example, in the calculation, if the pointer points to information of a particle in the list, the information of all the other particles in that list is obtained. For any problem, the complexity of this neighboring particle search algorithm is always of the order $O(N)$. During the particle search, the calculation amount accumulates linearly with the change of particle number. However, large memory is required by this algorithm, and some cached data are lost.

2.5.2.2 Verlet Neighbor List Method

The Verlet neighbor list was named after Loup Verlet, who proposed this method in 1967 (Verlet 1967). The sketch of this algorithm is shown in Fig. 2.3. In its first use, the method greatly improved the computing speed for particle interactions in a small area. The core concept of the method is as follows. First, the distance between each particle pair is calculated, then a potential neighbor list is constructed in which particle pair distances are all within a “skin” layer radius r_v . In this list, only pairs of particles within cutoff radius r_c interact, resulting in another neighbor list. The neighbor list is updated every time step, whereas the potential neighbor list is updated every N time steps. The value of N was suggested to be 7 (Dominguez et al. 2011). According to Verlet’s original paper (Verlet 1967), r_c and r_v can be taken as $2.5r$ and $3.2r$, respectively, where r is the radius of the interacting particles.

Figures 2.2 and 2.3 show the difference between these two methods when dividing the calculation area for the same problem. Each method has its strong points. The Verlet neighbor list method was selected for the numerical models in this monograph.

Fig. 2.3 Verlet neighbor list algorithm (reprinted from Dai et al. (2014), with permission of Elsevier)



2.5.3 Kernel Function

As mentioned above, the kernel function is important in establishing the governing equations in the SPH method. The function determines the accuracy and validity of the approximation for the field function. The order, symmetry, and stability of the interpolation function should be considered when selecting the kernel function. The following lists some of the most frequently used kernel functions in the SPH literature.

A. Gauss kernel

$$W(x - x', h) = \left[\frac{1}{\sqrt{\pi}h} \right]^d \exp \left[-\frac{(x - x')^2}{h^2} \right], \quad (2.109)$$

where d represents the dimension of space.

B. Super Gauss kernel

$$W(x - x', h) = \left[\frac{1}{\sqrt{\pi}h} \right]^d \left[\frac{5}{2} - \frac{(x - x')^2}{h^2} \right] \exp \left[-\frac{(x - x')^2}{h^2} \right]. \quad (2.110)$$

C. Exponential kernel

$$W(x - x', h) = \frac{1}{8\pi h^3} \exp \left[-\frac{(x - x')}{h} \right]. \quad (2.111)$$

D. Cubic spline function

$$W(s, h) = \frac{C}{h^d} \begin{cases} 1 - \frac{3}{2}s^2 + \frac{3}{4}s^3 & 0 \leq s \leq 1 \\ \frac{1}{4}(2 - s)^2 & 1 \leq s \leq 2 \\ 0 & s \geq 2. \end{cases} \quad (2.112)$$

Here, $s = |x - x'|/h$ and C is $2/3$, $10/7\pi$, and $1/\pi$, respectively, in one-, two-, and three- dimensional space, for the unity requirement.

E. B-spline function

Monaghan and Lattanzio (1985) devised the following smoothing function based on the cubic spline functions known as the B-spline function (Fig. 2.4):

$$W(s, h) = a_d \begin{cases} \frac{2}{3} - s^2 + \frac{1}{2}s^3 & 0 \leq s \leq 1 \\ \frac{1}{6}(2 - s)^3 & 1 \leq s \leq 2 \\ 0 & s \geq 2. \end{cases} \quad (2.113)$$

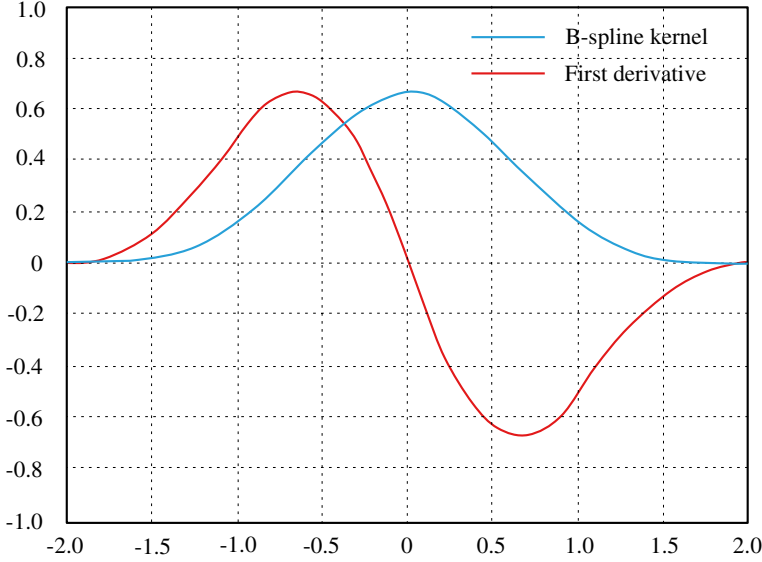


Fig. 2.4 B-spline kernel and its first derivative

Here, s is the same as above. In one-, two-, and three-dimensional space, $a_d = 1/h$, $15/7 \pi h^2$, and $3/2 \pi h^2$, respectively. The cubic spline function has been the most widely used smoothing function in the SPH literature, since it resembles a Gaussian function while having a narrower compact support.

In this monograph, the B-spline function was selected as the kernel function. Taking the derivative of the above equation,

$$\nabla_x \omega = \frac{\partial \omega}{\partial s} \cdot \frac{\partial s}{\partial r} \cdot \frac{\partial r}{\partial x} = \alpha_d \times \begin{cases} \left(-2s + \frac{3}{2}s^2\right) \left(\frac{1}{h}\right) \left(\frac{x-x'}{r}\right) & 0 \leq s < 1 \\ \left(-\frac{1}{2}(2-s)^2\right) \left(\frac{1}{h}\right) \left(\frac{x-x'}{r}\right) & 1 \leq s < 2 \\ 0 & s \geq 2. \end{cases} \quad (2.114)$$

2.5.4 Free-Surface Boundary Treatment

Boundary conditions vary with the research subject in this monograph. In the SPH fluid dynamics mode, it is important to distinguish the free-surface boundaries. If the particle density meets the following requirement, then the particle is considered a free-surface particle:

$$\rho < \beta \rho_0. \quad (2.115)$$

where ρ_0 is the real density of the fluid and β is a parameter less than 1, with a value between 0.8 and 0.98. The specific value can be determined by experiment.

In the SPH solid mechanics mode, various distinguishing marks of the particles in different collections are tagged to determine whether a particle is involved in the calculation and, if so, its degrees of freedom. For example, N is set as the distinguishing mark. If $N = 1$, the particle is fixed in the X direction; if $N = 2$, it is fixed in the Y direction; if $N = 3$, it is fixed in both the X and Y directions.

2.5.5 Averaging the Velocity Field

In a Lagrangian method, the new position of a particle is derived by time integration of the velocity at every moment. A variant called XSPH was proposed in the SPH literature, with the goal of modifying and smoothing SPH particle movement. However, the acceleration equation remained unchanged. The smoothed velocity was defined by an average over the velocities of neighboring particles, according to Monaghan (1992):

$$v_i = v_i + \chi \sum_{j=1}^N \frac{m_j}{\rho_{ij}} (v_j - v_i) W_{ij}, \quad (2.116)$$

where χ is a constant, the value of which is between 0 and 1. Numerical experiments show that $\chi = 0.5$ is effective at smoothing local fluctuations of velocity (Monaghan 2002). ρ_{ij} denotes an average density of particles i and j .

2.5.6 Solid Boundary Treatment

Near the boundary, only neighboring particles inside the boundary contribute to the SPH summation of particle interaction; no contribution comes from outside the boundary because there are no particles there (Fig. 2.5). Therefore, SPH simulations may have a problem of particle deficiency near the boundary, which leads to inaccurate solutions.

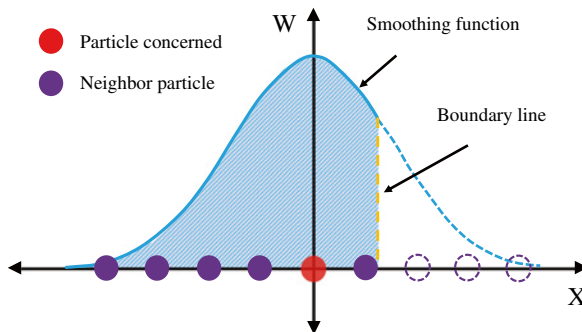


Fig. 2.5 SPH particle approximations for the one-dimensional case

To address this problem, some improvements have been proposed to deal with the boundary condition. Among such methods, free-slip boundaries have been widely used in SPH simulation for free-surface flows. This type of simulation uses boundary particles that exert strong repulsive forces that prevent SPH particles from penetrating the solid surface (Monaghan 1994). These boundary particles do not contribute to the density of the free SPH particles. Libersky et al. (1993) introduced ghost particles of opposite velocity to reflect a symmetrical surface boundary condition. Later, a more general treatment was proposed (Randles and Libersky 1996), in which all the ghost particles were assigned the same boundary field variable to calculate the values of the interior particles. The present work incorporates a no-slip boundary condition first proposed by Morris et al. (1997), which is similar to that used by Takeda et al. (1994). Figure 2.5 illustrates the principal concept for a curved boundary. The yellow dashed line represents the boundary line. The red particle is the flow particle concerned. Pseudo-boundary particles (purple) are set on and outside the boundary line. They contribute to the density of the flow particles, thus avoiding the problem of particle deficiency near the solid boundary. A tangent plane to the boundary surface is defined and normal distances of the fluid particle and boundary particle to this tangent plane, d_A and d_B , are calculated. The velocity of particle A is extrapolated across the tangent plane, assuming zero velocity on the plane itself (no-slip). Then, the velocity of each image boundary particle can be calculated using

$$V_B = -(d_B/d_A) \cdot V_A, \quad (2.117)$$

The artificial velocities are used here to calculate the viscous force rather than to update boundary particle positions. The relative velocities between fluid and boundary particles V_{AB} can then be defined by

$$V_{AB} = V_A - V_B = (1 + d_B/d_A) \cdot V_A. \quad (2.118)$$

According to Morris et al. (1997), viscous forces from the solid boundary can be calculated using relative velocities. To avoid exerting very large viscous forces when the fluid particles approach the boundary surface too closely, the magnitude of V_{AB} must be restricted through the following equations:

$$V_{AB} = \beta \cdot V_A, \quad (2.119)$$

$$\beta = \min(\beta_{\max}, 1 + d_B/d_A), \quad (2.120)$$

where β is a safety parameter to prevent numerical singularities and $\beta_{\max}$ is an empirical parameter (often chosen as 1.5).

2.5.7 Acceleration Calculation

Acceleration of a particle can be generated by internal and external forces. In the fluid model, particle density should initially be calculated using the continuity equation. Second, free-surface particles should be caught and their densities modified. Then, the pressure is calculated via the equation of state, and acceleration is

caused by the external force. The final acceleration can then be obtained through the momentum equation. For the solid model, the Jaumann stress rate is used to calculate the change of stress and strain of the particles. Then, the final acceleration is determined by accumulating those from internal and external forces.

2.5.8 Integration Schemes

To calculate forces acting on the SPH particles after discretization, a neighbor list is created for each particle in the current configuration. In the subsequent calculations, the neighbor list is referred to, so the total force acting on a particle can be determined. Since the SPH method reduces the original PDEs to sets of ODEs, any stable time-step algorithm for ODEs can be introduced. The Velocity Verlet algorithm (Ercolessi 1997), similar to the leapfrog method, is introduced here for the time integration to update positions, velocities, and accelerations:

$$x(t + \Delta t) = x(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2, \quad (2.121)$$

$$v(t + \Delta t/2) = v(t) + \frac{1}{2}a(t)\Delta t, \quad (2.122)$$

$$v(t + \Delta t) = v(t + \Delta t/2) + \frac{1}{2}a(t + \Delta t)\Delta t, \quad (2.123)$$

$$a(t) = \frac{1}{\rho}f(t). \quad (2.124)$$

2.5.9 Outputs of Calculation

Given the Lagrangian property of the SPH method, information on each particle, such as density, position, velocity, and pressure, can be captured at every time step. According to the actual needs of a study, these variable values can be selectively output, easily visualized, and rapidly analyzed, and time history curves can be readily obtained.

2.6 Summary

SPH is a novel mesh-free particle method based on a pure Lagrangian description. The basic idea of the method is that a continuous fluid is represented by a set of arbitrarily distributed particles. By providing accurate and stable numerical solutions for hydrodynamic equations and tracking the movements of each particle, the mechanical behavior of the full system can be determined. SPH is characterized

by a mesh-free, adaptive, and Lagrangian description, which makes it suitable for handling the problems of large deformation and free surfaces.

On a theoretical level, SPH can accurately describe the mechanical process as long as there are sufficient numbers of particles. Although the precision of results depends on particle arrangement, the requirement for this arrangement is much less stringent than the demands of a grid. Without the grid, there is no connectivity between the particles. Therefore, severe mesh distortions caused by large deformation are avoided, thereby improving computational accuracy. In addition, SPH can conveniently handle the interface of different materials. Another advantage of the method is the Lagrangian description, which avoids the difficult interface of the grid and material. Hence, the SPH method is especially suitable for solving complex flow problems. In particular, the method has the following advantages: (1) there is no migration term in the PDEs, so the program design is simple and efficient; (2) it is easy to track the time history of all field variables for all particles; (3) it can automatically exert the boundary condition to track the free surfaces, material interfaces, and moving boundaries; (4) it is easy to handle irregular and complex geometry shapes. SPH is therefore a novel and promising method for computational fluid mechanics.

In summary, we recap the following major points:

- (1) We briefly summarized the origin of the SPH method and its main concepts. Two core approximations, those of the kernel and particle, were described in detail.
- (2) Based on SPH basic theories, SPH formulations for the N-S equations in a Lagrangian description were established. It was pointed out that accuracy near both free boundaries and material interfaces is the criterion for elevating the approximate formats.
- (3) The key issue for establishing the SPH formulations for elasto-plastic mechanics is analysis of the constitutive relationship between material stress and strain, and internal particle motion. Based on the constitutive model in the form of the Jaumann stress rate, the SPH model can be applied to many types of materials.
- (4) The implicit algorithm of Poisson's equation for calculating the pressure in SPH can improve the stability and accuracy of the numerical solution in the fluid dynamics model.
- (5) Computational efficiency and accuracy are key elements for evaluating a numerical simulation method. Relative to the mesh-based method, the unique techniques in the SPH method, including the special treatment in the solution method, efficient particle searching, and selection of the smoothing kernel function, help SPH analyze physical problems efficiently and accurately.

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