

## Chapter 2

# Continuum Mechanics of Plasticity

**Abstract** This chapter introduces first the basic equations of plasticity theory. The yield condition, the flow rule, and the hardening rule are introduced. After deriving and presenting these equations for the one-dimensional stress and strain state, the equations are generalized for a two-component  $\sigma$ - $\tau$  stress state. Based on these basic equations, the concept of effective stress and strain as well as the elasto-plastic modulus is introduced. The chapter finishes with an introduction to two different damage concepts, i.e. the LEMAITRE and GURSON damage model, which are derived for the one-dimensional case.

### 2.1 General Comments and Observations

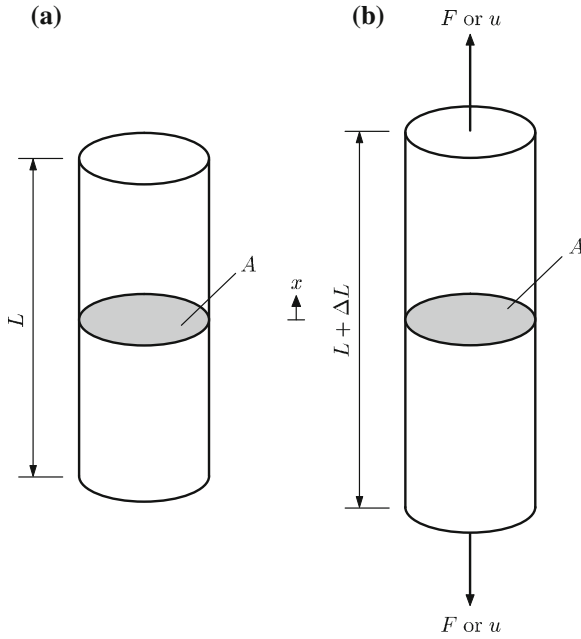
Let us consider in the following a uniaxial tensile test whose idealized specimen is schematically shown in Fig. 2.1. The original dimensions of, for example, the cylindrical specimen are characterized by the cross-sectional area  $A$  and length  $L$ . This specimen is now elongated in a universal testing machine and its length increases to  $L + \Delta L$ . In the case of a real specimen made of a common engineering material, the cross-sectional area would reduce to  $A - \Delta A$ . This phenomenon could be described based on POISSON's ratio.<sup>1</sup> However, if we assume an idealized uniaxial state, i.e. a uniaxial stress *and* strain state, the contraction is disregarded and the cross-sectional area is assumed to remain constant, see Fig. 2.1b. During the tensile test, the force is normally recorded by a load cell attached to the movable or fixed cross-head of the machine. If this force is divided by the (initial) cross-sectional area, the engineering stress is obtained as:

$$\sigma = \frac{F}{A}. \quad (2.1)$$

The deformation or elongation of the specimen can be measured, for example, by an external extensometer which should be directly attached to the specimen.

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<sup>1</sup> Siméon Denis POISSON (1781–1840), French mathematician, geometer, and physicist.



**Fig. 2.1** Schematic representation of **a** an unloaded and **b** an idealized uniaxial tensile specimen loaded by a force  $F$  or a displacement  $u$

These devices are either realized as strain gauge or inductive extensometers.<sup>2</sup> Any measurement based on the movement of the cross-head must be avoided since it does not guarantee an accurate determination of the specimen's behavior. The definition of strain is given in its simplest form as elongation over initial length and the engineering strain can be calculated as:

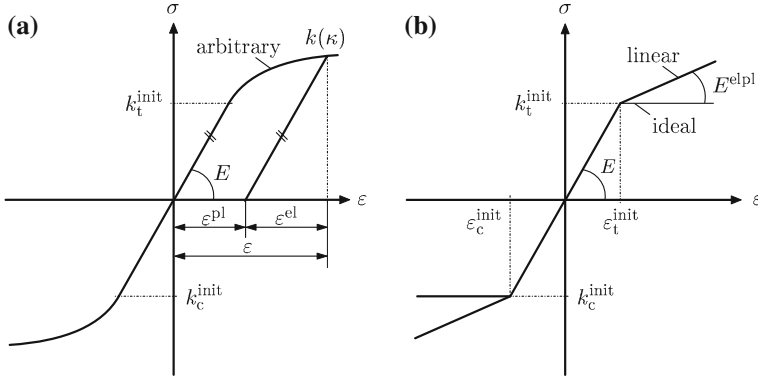
$$\varepsilon = \frac{\Delta L}{L}. \quad (2.2)$$

Relating the stress to its corresponding strain, the engineering stress-strain diagram can be plotted as schematically shown in Fig. 2.2. In the pure elastic range, a linear line is observed and its slope is equal to YOUNG's<sup>3</sup> modulus:

$$E = \frac{\Delta \sigma}{\Delta \varepsilon} \quad \text{or} \quad E = \frac{d\sigma}{d\varepsilon}. \quad (2.3)$$

<sup>2</sup> More modern devices are contactless laser or video based extensometers.

<sup>3</sup> Thomas YOUNG (1773–1829), English polymath.



**Fig. 2.2** Uniaxial stress-strain diagrams for different isotropic hardening laws: **a** arbitrary hardening; **b** linear hardening and ideal plasticity

The last relation is also known as HOOKE's<sup>4</sup> law and often written in the following form for linear elastic behavior:

$$\sigma = E\varepsilon. \quad (2.4)$$

As soon as the initial yield stress  $k_t^{\text{init}}$  is reached, plastic material behavior occurs and the slope of the stress-strain diagram changes.

The characteristic feature of plastic material behavior is that a remaining strain  $\varepsilon^{\text{pl}}$  occurs after complete unloading, see Fig. 2.2a. Only the elastic strains  $\varepsilon^{\text{el}}$  returns to zero at complete unloading. An additive composition of the strains by their elastic and plastic parts

$$\varepsilon = \varepsilon^{\text{el}} + \varepsilon^{\text{pl}} \quad (2.5)$$

is permitted at restrictions to small strains. The elastic strains  $\varepsilon^{\text{el}}$  can hereby be determined via HOOKE's law, whereby  $\varepsilon$  in Eq. (2.3) has to be substituted by  $\varepsilon^{\text{el}}$ .

Furthermore, no explicit correlation is given anymore for plastic material behavior in general between stress and strain, since the strain state is also dependent on the loading history. Due to this, rate equations are necessary and need to be integrated throughout the entire load history. Within the framework of the time-independent plasticity investigated here, the rate equations can be simplified to incremental relations. From Eq. (2.5) the additive composition of the strain increments results in:

$$d\varepsilon = d\varepsilon^{\text{el}} + d\varepsilon^{\text{pl}}. \quad (2.6)$$

<sup>4</sup> Robert HOOKE (1635–1703), English natural philosopher, architect and polymath.

The constitutive description of plastic material behavior includes

- a yield condition,
- a flow rule and
- a hardening law.

In the following, the case of the monotonic loading<sup>5</sup> is considered first, so that isotropic hardening is explained first in the case of material hardening. This important case, for example, occurs in the experimental mechanics at the uniaxial tensile test with monotonic loading. Furthermore, it is assumed that the yield stress is identical in the tensile and compressive regime:  $k_t = k_c = k$ .

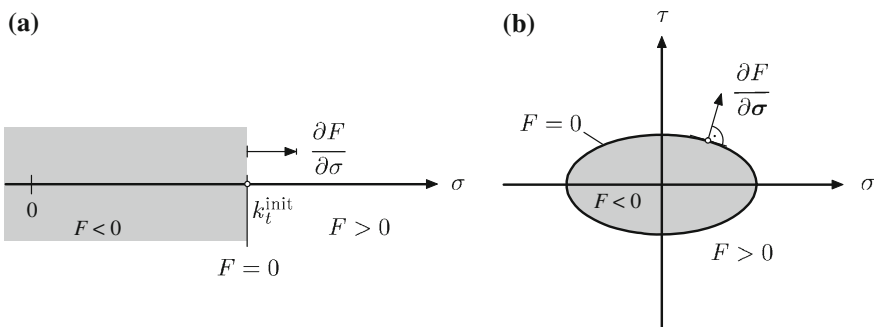
## 2.2 Yield Condition

The yield condition enables one to determine whether the relevant material suffers only elastic or also plastic strains at a certain stress state at a point of the relevant body. In the uniaxial tensile test, plastic flow begins when reaching the initial yield stress  $k^{\text{init}}$ , see Fig. 2.2. The yield condition in its general one-dimensional form can be set as follows ( $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ):

$$F = F(\sigma, \kappa), \quad (2.7)$$

where  $\kappa$  represents the inner variable of isotropic hardening. In the case of ideal plasticity, see Fig. 2.2b, the following is valid:  $F = F(\sigma)$ . The values of  $F$  have the following mechanical meaning, see Fig. 2.3a:

$$F(\sigma, \kappa) < 0 \rightarrow \text{elastic material behavior}, \quad (2.8)$$



**Fig. 2.3** Schematic representation of the values of the yield condition and the direction of the stress gradient in the **a** uniaxial and **b** two-component  $\sigma$ - $\tau$  stress space (in the  $\sigma$ - $\tau$  space, the shown normal vector is the projection of this vector onto this plan)

<sup>5</sup> The case of unloading or alternatively load reversal will be treated separately in Sect. 2.7.

$$F(\sigma, \kappa) = 0 \rightarrow \text{plastic material behavior,} \quad (2.9)$$

$$F(\sigma, \kappa) > 0 \rightarrow \text{invalid.} \quad (2.10)$$

A further simplification results under the assumption that the yield condition can be split into a pure stress fraction  $f(\sigma)$ , the so-called yield criterion,<sup>6</sup> and into an experimental material parameter  $k(\kappa)$ , the so-called flow stress:

$$F(\sigma, \kappa) = f(\sigma) - k(\kappa). \quad (2.11)$$

For a uniaxial tensile test (see Fig. 2.2) the yield condition can be noted in the following form:

$$F(\sigma, \kappa) = |\sigma| - k(\kappa) \leq 0. \quad (2.12)$$

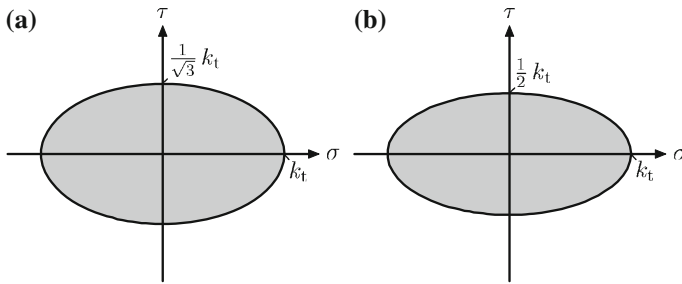
If one considers the idealized case of the linear hardening (see Fig. 2.2b), Eq. (2.12) can be written as (Fig. 2.4)

$$F(\sigma, \kappa) = |\sigma| - (k^{\text{init}} + E^{\text{pl}} \kappa) \leq 0. \quad (2.13)$$

The parameter  $E^{\text{pl}}$  is the plastic modulus (see Fig. 2.5), which becomes zero in the case of ideal plasticity:

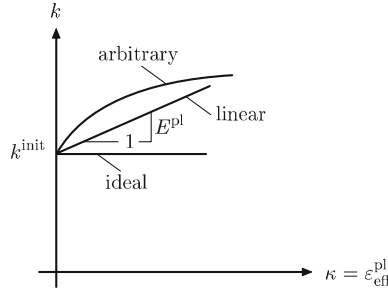
$$F(\sigma, \kappa) = |\sigma| - k^{\text{init}} \leq 0. \quad (2.14)$$

In the following, we are going to extend this concept to a stress state where in addition to a normal stress ( $\sigma$ ) a shear stress component ( $\tau$ ) is acting. This two-component stress state occurs, for example, in the case of general beam formulations, see Sects. 4.3 and 4.4. Two classical representatives of yield conditions for solid dense



**Fig. 2.4** Graphical representation of the yield condition according to **a** VON MISES and **b** TRESCA in the two-component  $\sigma$ - $\tau$  space

<sup>6</sup> If the unit of the yield criterion equals the stress,  $f(\sigma)$  represents the equivalent stress or effective stress. In the general three-dimensional case the following is valid under consideration of the symmetry of the stress tensor  $\sigma_{\text{eff}} : (\mathbb{R}^6 \rightarrow \mathbb{R}_+)$ .



**Fig. 2.5** Flow curve for different isotropic hardening laws

materials (classical metals) will be introduced for this special two-component stress state<sup>7</sup> in the following [53]. The TRESCA<sup>8</sup> yield condition [107] is based on the assumption that yielding begins when the maximum shear stress reaches a certain value and can be written, for example, in its mathematical form based on the two acting stress components as

$$F_{\sigma-\tau} = \sqrt{\sigma^2 + 4\tau^2} - k_t = 0 \quad (\text{TRESCA}), \quad (2.15)$$

where the relationship between shear ( $k_s$ ) and tensile yield stress ( $k_t$ ) is given by  $k_s = \frac{k_t}{2}$ , see [120].

The VON MISES<sup>9</sup> yield condition [69] is based on the assumption that yielding begins when the elastic energy of distortion reaches a critical value [50]. The mathematical formulation of this condition reads in the  $\sigma$ - $\tau$  space as

$$F_{\sigma-\tau} = \sqrt{\sigma^2 + 3\tau^2} - k_t = 0 \quad (\text{VON MISES}), \quad (2.16)$$

where the relationship between shear and tensile yield stress is given by  $k_s = \frac{k_t}{\sqrt{3}}$ , see [20]. Both yield conditions can be written in a more general form as

$$F = F(\boldsymbol{\sigma}) = 0, \quad (2.17)$$

where  $\boldsymbol{\sigma} = [\sigma \ \tau]^T$  is the column matrix of the stress components. The graphical representation of both yield conditions is given in Fig. 2.4. As can be seen, centered ellipses are obtained where the major and minor axes of the ellipses are parallel to the stress axes of the coordinate system. Comparing both shapes, it can be concluded that the TRESCA yield condition gives a more conservative prediction of the yield stress since the minor axis is smaller compared to the VON MISES condition.

<sup>7</sup> A two-dimensional stress state would comprise in its general form, for example, the three components  $\sigma_x$ ,  $\sigma_y$  and  $\tau_{xy}$ .

<sup>8</sup> Henri Édouard TRESCA (1814–1885), French mechanical engineer.

<sup>9</sup> Richard Edler VON MISES (1883–1953), Austrian scientist and mathematician.

**Table 2.1** Different formulations of the VON MISES and TRESCA yield condition in the two-component  $\sigma$ - $\tau$  space

VON MISES	TRESCA
<i>Based on tensile yield stress</i>	
$\sqrt{\sigma^2 + 3\tau^2} - k_t = 0$	$\sqrt{\sigma^2 + 4\tau^2} - k_t = 0$
$\sigma^2 + 3\tau^2 - k_t^2 = 0$	$\sigma^2 + 4\tau^2 - k_t^2 = 0$
<i>Based on shear yield stress</i>	
$\frac{\sqrt{3}}{3} \sqrt{\sigma^2 + 3\tau^2} - k_s = 0$	$\frac{1}{2} \sqrt{\sigma^2 + 4\tau^2} - k_s = 0$
$\frac{1}{3} (\sigma^2 + 3\tau^2) - k_s^2 = 0$	$\frac{1}{4} (\sigma^2 + 4\tau^2) - k_s^2 = 0$

It should be noted here that both conditions given in Eqs. (2.15) and (2.16) reduce for a uniaxial stress state where only a normal stress  $\sigma$  is acting to the formulation presented in Eq. (2.12). Finally, Table 2.1 summarizes different formulations of the considered yield conditions in the two-component  $\sigma$ - $\tau$  space.

## 2.3 Flow Rule

The flow rule serves as a mathematical description of the evolution of the infinitesimal increments of the plastic strain  $d\varepsilon^{\text{pl}}$  in the course of the load history of the body. In its most general one-dimensional form, the flow rule can be set up as follows [96]:

$$d\varepsilon^{\text{pl}} = d\lambda r(\sigma, \kappa), \quad (2.18)$$

whereupon the factor  $d\lambda$  is described as the consistency parameter ( $d\lambda \geq 0$ ) and  $r : (\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R})$  as the function of the flow direction.<sup>10</sup> One considers that solely for  $d\varepsilon^{\text{pl}} = 0$  then  $d\lambda = 0$  results. Based on the stability postulate of DRUCKER<sup>11</sup> [33] the following flow rule can be derived<sup>12</sup>:

$$d\varepsilon^{\text{pl}} = d\lambda \frac{\partial F(\sigma, \kappa)}{\partial \sigma}. \quad (2.19)$$

Such a flow rule is referred to as the normal rule<sup>13</sup> (see Fig. 2.3a) or due to  $r = \partial F(\sigma, \kappa)/\partial \sigma$  as the *associated* flow rule.

<sup>10</sup> In the general three-dimensional case  $r$  hereby defines the direction of the vector  $d\varepsilon^{\text{pl}}$ , while the scalar factor defines the absolute value.

<sup>11</sup> Daniel Charles DRUCKER (1918–2001), US engineer.

<sup>12</sup> A formal alternative derivation of the associated flow rule can occur via the LAGRANGE multiplier method as extreme value with side-conditions from the principle of maximum plastic work [12].

<sup>13</sup> In the general three-dimensional case the image vector of the plastic strain increment has to be positioned upright and outside oriented to the yield surface, see Fig. 2.3b.

Experimental results, among other things from the area of granular materials [13] can however be approximated better if the stress gradient is substituted through a different function, the so-called plastic potential  $Q$ . The resulting flow rule is then referred to as the *non-associated* flow rule:

$$d\varepsilon^{\text{pl}} = d\lambda \frac{\partial Q(\sigma, \kappa)}{\partial \sigma}. \quad (2.20)$$

In quite complicated yield conditions often the case occurs that a more simple yield condition is used for  $Q$  in the first approximation, for which the gradient can easily be determined.

The application of the associated flow rule (2.19) to the yield conditions according to Eqs. (2.12)–(2.14) yields for all three types of yield conditions (meaning arbitrary hardening, linear hardening and ideal plasticity):

$$d\varepsilon^{\text{pl}} = d\lambda \operatorname{sgn}(\sigma), \quad (2.21)$$

where  $\operatorname{sgn}(\sigma)$  represents the so-called sign function,<sup>14</sup> which can adopt the following values:

$$\operatorname{sgn}(\sigma) = \begin{cases} -1 & \text{for } \sigma < 0 \\ 0 & \text{for } \sigma = 0 \\ +1 & \text{for } \sigma > 0 \end{cases}. \quad (2.22)$$

For the two-component  $\sigma$ - $\tau$  stress space, the associated flow rule (2.19) can be written as

$$d\varepsilon^{\text{pl}} = d\lambda \frac{\partial F(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}, \quad (2.23)$$

where  $d\varepsilon^{\text{pl}} = [d\varepsilon^{\text{pl}} \ d\gamma^{\text{pl}}]^T$  is the column matrix of the plastic strain increments. Application of this definition to the yield conditions given in Eqs. (2.15) and (2.16) gives finally:

$$d\varepsilon^{\text{pl}} = \begin{bmatrix} d\varepsilon^{\text{pl}} \\ d\gamma^{\text{pl}} \end{bmatrix} = \frac{d\lambda}{\sqrt{\sigma^2 + 4\tau^2}} \begin{bmatrix} \sigma \\ 4\tau \end{bmatrix} \quad (\text{TRESCA}), \quad (2.24)$$

$$d\varepsilon^{\text{pl}} = \begin{bmatrix} d\varepsilon^{\text{pl}} \\ d\gamma^{\text{pl}} \end{bmatrix} = \frac{d\lambda}{\sqrt{\sigma^2 + 3\tau^2}} \begin{bmatrix} \sigma \\ 3\tau \end{bmatrix} \quad (\text{VON MISES}). \quad (2.25)$$

These two equations can be generally expressed in the manner of Eq. (2.18) as:

$$d\varepsilon^{\text{pl}} = d\lambda \mathbf{r}(\boldsymbol{\sigma}, \kappa). \quad (2.26)$$

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<sup>14</sup> Also signum function; from the Latin ‘signum’ for ‘sign’.



## 2.4 Hardening Rule

The hardening law allows the consideration of the influence of material hardening on the yield condition and the flow rule.

### 2.4.1 Isotropic Hardening

In the case of isotropic hardening, the yield stress is expressed as being dependent on an inner variable  $\kappa$ :

$$k = k(\kappa). \quad (2.27)$$

If the equivalent plastic strain<sup>15</sup> is used for the hardening variable ( $\kappa = |\varepsilon^{\text{pl}}|$ ), then one talks about strain hardening.

Another possibility is to describe the hardening being dependent on the specific<sup>16</sup> plastic work ( $\kappa = w^{\text{pl}} = \int \sigma d\varepsilon^{\text{pl}}$ ). Then one talks about work hardening. If Eq. (2.27) is combined with the flow rule according to (2.21), the evolution equation for the isotropic hardening variable results in:

$$d\kappa = d|\varepsilon^{\text{pl}}| = d\lambda. \quad (2.28)$$

Figure 2.5 shows the flow curve, meaning the graphical illustration of the yield stress being dependent on the inner variable for different hardening approaches.

The yield condition which was expressed in Eq. (2.17) for the case of ideal plasticity can now be expanded to the formulation

$$F = F(\boldsymbol{\sigma}, q) = 0, \quad (2.29)$$

where the internal variable  $q$  considers the influence of the material hardening on the yield condition. The evolution equation for this internal variable can be stated in its most general form based on Eq. (2.29) as

$$dq = d\lambda \times h(\boldsymbol{\sigma}, q), \quad (2.30)$$

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<sup>15</sup> The effective plastic strain is in the general three-dimensional case the function  $\varepsilon_{\text{eff}}^{\text{pl}} : (\mathbb{R}^6 \rightarrow \mathbb{R}_+)$ . In the one-dimensional case, the following is valid:  $\varepsilon_{\text{eff}}^{\text{pl}} = \sqrt{\varepsilon^{\text{pl}} \varepsilon^{\text{pl}}} = |\varepsilon^{\text{pl}}|$ . Attention: Finite element programs optionally use the more general definition for the illustration in the post processor, this means  $\varepsilon_{\text{eff}}^{\text{pl}} = \sqrt{\frac{2}{3} \sum \Delta \varepsilon_{ij}^{\text{pl}} \sum \Delta \varepsilon_{ij}^{\text{pl}}}$ , which considers the lateral contraction at uniaxial stress problems in the plastic area via the factor  $\frac{2}{3}$ . However in pure one-dimensional problems *without* lateral contraction, this formula leads to an illustration of the effective plastic strain, which is reduced by the factor  $\sqrt{\frac{2}{3}} \approx 0.816$ .

<sup>16</sup> This is the volume-specific definition, meaning  $[w^{\text{pl}}] = \frac{\text{N}}{\text{m}^2} \frac{\text{m}}{\text{m}} = \frac{\text{kg m}}{\text{s}^2 \text{m}^2} \frac{\text{m}}{\text{m}} = \frac{\text{kg m}^2}{\text{s}^2 \text{m}^3} = \frac{\text{J}}{\text{m}^3}$ .

where the function  $h$  defines the evolution of the hardening parameter. Assigning for the internal variable  $q = \kappa$  (in the case that  $\kappa$  equals the effective plastic strains, one talks about a strain space formulation) and considering the case of associated plasticity, a more specific rule for the evolution of the internal variable is given as

$$\begin{aligned} d\kappa &= -d\lambda \times \left(D^{\text{pl}}\right)^{-1} \frac{\partial F(\boldsymbol{\sigma}, \kappa)}{\partial \kappa} = -d\lambda \times \frac{\partial \kappa}{\partial k(\kappa)} \frac{\partial F(\boldsymbol{\sigma}, \kappa)}{\partial \kappa} \\ &= -d\lambda \times \frac{1}{E^{\text{pl}}} \frac{\partial F(\boldsymbol{\sigma}, \kappa)}{\partial \kappa}, \end{aligned} \quad (2.31)$$

where  $D^{\text{pl}}$  is the generalized plastic modulus. Considering the yield stress  $k$  as the internal variable, one obtains a stress space formulation as  $F = F(\boldsymbol{\sigma}, k)$  and the corresponding evolution equation for the internal variable is given by:

$$dk = -d\lambda \times D^{\text{pl}} \frac{\partial F(\boldsymbol{\sigma}, k)}{\partial k} = -d\lambda \times E^{\text{pl}} \frac{\partial F(\boldsymbol{\sigma}, k)}{\partial k}, \quad (2.32)$$

where  $dk$  can be written as  $E^{\text{pl}} d\kappa$ . Thus, one may alternatively formulate:

$$d\kappa = -d\lambda \times \frac{\partial F(\boldsymbol{\sigma}, k)}{\partial k}. \quad (2.33)$$

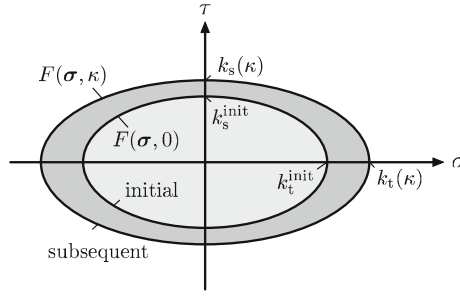
Application of the instruction for the evolution of the internal variable according to Eq. (2.31) or (2.33) to the TRESCA or VON MISES yield condition with  $k_t = k_t(\kappa)$  (cf. Eqs. (2.15) and (2.16)) gives:

$$d\kappa = d\lambda. \quad (2.34)$$

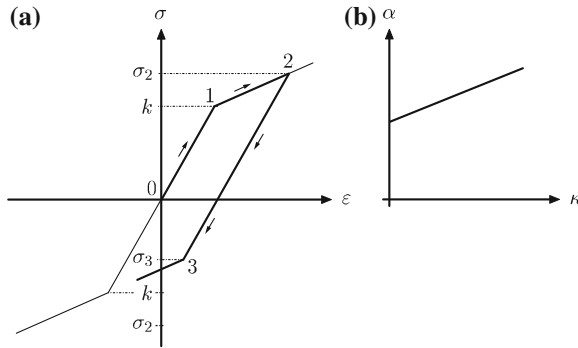
Thus, it turned out that  $h$ , i.e. the evolution equation for the hardening parameter in Eq. (2.30), simplified to  $h = 1$ . However, in the case of more complex yield conditions, the function  $h$  may take a more complex form. This will be shown in Sect. 2.8.2 where the GURSON's damage model is introduced. The graphical representation of the initial and subsequent yield surface for isotropic hardening in the two-component  $\sigma$ - $\tau$  space is shown in Fig. 2.6.

### 2.4.2 Kinematic Hardening

In the case of a pure monotonic loading, i.e. pure tensile or pure compression, it is not possible to distinguish from the stress-strain diagram the cases of isotropic or kinematic hardening. Let us look in the following on a uniaxial test with plastic deformation and stress reversal as schematically shown in Fig. 2.7. The test starts without any initial stress or strain in the origin of the stress-strain diagram (point '0') and a tensile load is continuously increased. The first part of the path, i.e. as long as



**Fig. 2.6** Initial and subsequent yield surface for isotropic hardening in the two-component  $\sigma$ - $\tau$  space



**Fig. 2.7** Uniaxial kinematic hardening: **a** idealized stress-strain curve with BAUSCHINGER effect and **b** kinematic hardening parameter as a function of the internal variable (linear hardening)

the stress is below the yield stress  $k$ , is in the pure elastic range and HOOKE's law describes the stress-strain behavior. Reaching the yield stress  $k$  (point '1'), the slope of the diagram changes and plastic deformation occurs. With ongoing increasing load, the plastic deformation and the plastic strain increases in this part of the diagram. Let us assume now that the load is reversed at point '2'. The unloading is completely elastic and compressive stress develops as soon as the load path passes the strain axis. The interesting question is now when the subsequent plastic deformations starts in the compressive regime. This plastic deformation occurs now in the case of kinematic hardening at a stress level  $\sigma_3$  which is lower than the initial yield stress  $k$  or the subsequent stress  $\sigma_2$ . This behavior is known as the BAUSCHINGER<sup>17</sup> effect [8] and requires plastic pre-straining with subsequent load reversal.

The behavior shown in Fig. 2.7a can be described based on the following yield condition

$$F = |\sigma - \alpha(\kappa)| - k = 0, \quad (2.35)$$

<sup>17</sup> Johann BAUSCHINGER (1834–1893), German mathematician and engineer.

where the initial yield stress  $k$  is constant and the kinematic hardening parameter<sup>18</sup>  $\alpha$  is a function of an internal variable  $\kappa$ . Figure 2.7b shows the case of linear hardening where a linear relationship between kinematic hardening parameter and internal variable is obtained.

The simplest relation between the kinematic hardening parameter and the internal variable was proposed in [68] as

$$\alpha = H\varepsilon^{\text{pl}} \quad \text{or} \quad d\alpha = H d\varepsilon^{\text{pl}}, \quad (2.36)$$

where  $H$  is a constant called the kinematic hardening modulus and the plastic strain is assigned as the internal variable. Thus, Eq. (2.36) describes the case of linear hardening. A more general formulation of Eq. (2.36) is known as PRAGER's<sup>19</sup> hardening rule [80, 81]:

$$d\alpha = H(\sigma, \kappa_i) d\varepsilon^{\text{pl}}, \quad (2.37)$$

where the kinematic hardening modulus is now a scalar function which depends on the state variables  $(\sigma, \kappa_i)$ . One suggestion is to use the effective plastic strain  $\varepsilon_{\text{eff}}^{\text{pl}}$  as internal variable [6]. A further extension is proposed in [60] where the hardening modulus is formulated as a tensor.

Another formulation was proposed by ZIEGLER<sup>20</sup> [94, 118] as

$$d\alpha = d\mu(\sigma - \alpha), \quad (2.38)$$

where the proportionality factor  $d\mu$  can be expressed as:

$$d\mu = a d\varepsilon^{\text{pl}}, \quad (2.39)$$

or in a more general way as  $a = a(\sigma, \kappa_i)$ . The rule given in Eq. (2.38) is known in the literature as ZIEGLER's hardening rule. It should be noted here that the plastic strain increments in Eqs. (2.39) and (2.37) can be calculated based on the flow rules given in Sect. 2.3. Thus, the kinematic hardening rules can be expressed in a more general way as:

$$d\alpha = d\lambda h(\sigma, \alpha). \quad (2.40)$$

For the two-component  $\sigma$ - $\tau/\varepsilon$ - $\gamma$  space, the kinematic hardening rules according to PRAGER and ZIEGLER can be written as

$$d\alpha = H(\sigma, \kappa) d\varepsilon^{\text{pl}} \quad (\text{PRAGER}), \quad (2.41)$$

$$d\alpha = d\mu(\sigma - \alpha) \quad (\text{ZIEGLER}), \quad (2.42)$$

<sup>18</sup> An alternative expression for the kinematic hardening parameter is back-stress.

<sup>19</sup> William PRAGER (1903–1980), German engineer and applied mathematician.

<sup>20</sup> Hans ZIEGLER (1910–1985), Swiss scientist.

or in components as

$$\begin{bmatrix} d\alpha_\sigma \\ d\alpha_\tau \end{bmatrix} = H(\sigma, \kappa) \begin{bmatrix} d\varepsilon^{pl} \\ d\gamma^{pl} \end{bmatrix} \quad (\text{PRAGER}) , \quad (2.43)$$

$$\begin{bmatrix} d\alpha_\sigma \\ d\alpha_\tau \end{bmatrix} = d\mu \begin{bmatrix} \sigma - \alpha_\sigma \\ \tau - \alpha_\tau \end{bmatrix} \quad (\text{ZIEGLER}) . \quad (2.44)$$

Thus, the generalization of the one-dimensional yield condition given in Eq. (2.35) can be written in the case of TRESCA and VON MISES as:

$$F_{\sigma-\tau} = \sqrt{(\sigma - \alpha_\sigma)^2 + 4(\tau - \alpha_\tau)^2} - k_t = 0 \quad (\text{TRESCA}) , \quad (2.45)$$

$$F_{\sigma-\tau} = \sqrt{(\sigma - \alpha_\sigma)^2 + 3(\tau - \alpha_\tau)^2} - k_t = 0 \quad (\text{VON MISES}) . \quad (2.46)$$

The graphical representation of Eqs. (2.45) and (2.46) is schematically shown in Fig. (2.8) where it can be seen that the center of the subsequent yield surface is described by the back-stress vector  $\alpha^T = [\alpha_\sigma \ \alpha_\tau]^T$ .

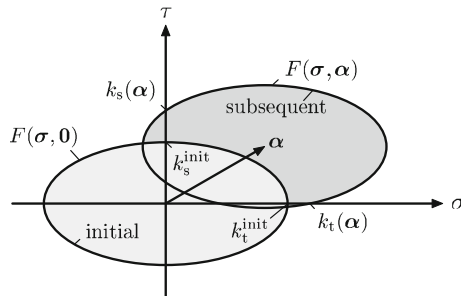
### 2.4.3 Combined Hardening

The isotropic and kinematic hardening rules presented in Sects. 2.4.1 and 2.4.2 can simply be joined together to obtain a combined hardening rule for the one-dimensional yield condition as:

$$F(\sigma, q) = |\sigma - \alpha| - k(\kappa) , \quad (2.47)$$

or for the special case of isotropic linear hardening as

$$F = |\sigma - \alpha| - (k^{\text{init}} + E^{\text{pl}} \varepsilon_{\text{eff}}^{\text{pl}}) , \quad (2.48)$$



**Fig. 2.8** Initial and subsequent yield surface for kinematic hardening in the two-component  $\sigma$ - $\tau$  space

where the back-stress  $\alpha$  can be a function as indicated in Eqs. (2.36)–(2.38). The associated flow rule is then obtained according to Eq. (2.19) as:

$$d\varepsilon^{\text{pl}} = d\lambda \frac{\partial F}{\partial \sigma} = d\lambda \operatorname{sgn}(\sigma - \alpha) \quad (2.49)$$

and the isotropic and kinematic hardening (PRAGER) laws can be written as:

$$d\kappa = d|\varepsilon^{\text{pl}}| = |d\lambda \operatorname{sgn}(\sigma - \alpha)| = d\lambda, \quad (2.50)$$

$$d\alpha = H d\varepsilon^{\text{pl}} = H d\lambda \operatorname{sgn}(\sigma - \alpha). \quad (2.51)$$

The last two equations can be combined and generally expressed as:

$$d\mathbf{q} = d\lambda \mathbf{h}(\sigma, \mathbf{q}), \quad (2.52)$$

or

$$\begin{bmatrix} d\kappa \\ d\alpha \end{bmatrix} = d\lambda \begin{bmatrix} 1 \\ H \operatorname{sgn}(\sigma - \alpha) \end{bmatrix}. \quad (2.53)$$

For the two-component  $\sigma$ - $\tau$ / $\varepsilon$ - $\gamma$  space, Eqs. (2.45), (2.46) and (2.27) can be combined to obtain the yield conditions for combined hardening. In this case, the evolution equations for the isotropic and kinematic hardening parameters can be generally expressed as:

$$d\mathbf{q} = d\lambda \mathbf{h}(\sigma, \mathbf{q}). \quad (2.54)$$

## 2.5 Effective Stress and Effective Plastic Strain

The previous sections introduced a two-component stress state  $\boldsymbol{\sigma} = [\sigma \ \tau]^T$ . If a material is subjected to such a multi-axial stress state, it is difficult to directly compare these different stress components with the experimental uniaxial stress-strain as schematically shown in Fig. 2.2. How to decide if the state is still in the elastic or already in the plastic range? To overcome this problem, one can define the so-called effective stress.<sup>21</sup> This is a mathematical equation which calculates, based on the acting stress components, a single scalar stress value  $\sigma_{\text{eff}}$  which can be directly compared with the experimental stress value  $k$ . If a yield condition is defined, for example, as  $F(\boldsymbol{\sigma}, \kappa) = f(\boldsymbol{\sigma}) - k(\kappa)$ , then  $f(\boldsymbol{\sigma})$  can be considered as the effective stress definition. However, yield conditions may have different formulations

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<sup>21</sup> Sometimes called the equivalent stress.

(see Table 2.1) and we may indicate the yield condition in a more general form as

$$F(\boldsymbol{\sigma}, \kappa) = f(\boldsymbol{\sigma}) - g(k) = 0. \quad (2.55)$$

Thus, the yield condition is split into a fraction  $f(\boldsymbol{\sigma})$  which depends only on the acting stress state and a fraction which involves the experimental stress-strain response of the material. Then, it follows that the function  $f(\boldsymbol{\sigma})$  must be some constant  $c$  times the effective stress to some power  $d$  [22]:

$$f(\boldsymbol{\sigma}) = c \times \sigma_{\text{eff}}^d. \quad (2.56)$$

Let us assume, for example, a yield condition of the form  $\sigma^2 + 3\tau^2 - k_t^2 = 0$ , see Table 2.1. Since the effective stress should reduce to the normal stress component  $\sigma$  in the uniaxial tensile test ( $\sigma \neq 0 \wedge \tau = 0$ ), we can write that

$$\sigma^2 = c \times \sigma_{\text{eff}}^d, \quad (2.57)$$

and comparing coefficients gives  $c = 1$  and  $d = 2$ . Thus, the effective stress in this specific case is given as:

$$\sigma_{\text{eff}} = 1 \times \left( \sigma^2 + 3\tau^2 \right)^{\frac{1}{2}}. \quad (2.58)$$

For the definition of the effective plastic strain  $\varepsilon_{\text{eff}}^{\text{pl}}$ , different approaches can be found in the literature [22]. We will use in the following a definition which is based on the volume-specific plastic work which can be expressed either by the stress and plastic strain components or the corresponding effective values as:

$$dw^{\text{pl}} = \boldsymbol{\sigma}^T d\boldsymbol{\varepsilon}^{\text{pl}} \stackrel{!}{=} \sigma_{\text{eff}} d\varepsilon_{\text{eff}}^{\text{pl}}, \quad (2.59)$$

or rearranged for the effective plastic strain increment as:

$$d\varepsilon_{\text{eff}}^{\text{pl}} = \frac{\boldsymbol{\sigma}^T d\boldsymbol{\varepsilon}^{\text{pl}}}{\sigma_{\text{eff}}}. \quad (2.60)$$

Under the assumption of an associated flow rule, the last equation can be expressed as:

$$d\varepsilon_{\text{eff}}^{\text{pl}} = d\lambda \underbrace{\frac{\boldsymbol{\sigma}^T \frac{\partial F}{\partial \boldsymbol{\sigma}}}{\sigma_{\text{eff}}}}_h. \quad (2.61)$$

In the following let us look at the VON MISES yield condition as given in Eq. (2.16), i.e.

$$F_{\sigma-\tau} = \underbrace{\sqrt{\sigma^2 + 3\tau^2}}_{\sigma_{\text{eff}}} - k_t = 0, \quad (2.62)$$

where the effective stress is easily identified. The derivative of the yield condition with respect to the stress components can be obtained from Eq. (2.25) and the effective plastic strain increment can be written as:

$$d\varepsilon_{\text{eff}}^{\text{pl}} = d\lambda \frac{\begin{bmatrix} \sigma & \tau \end{bmatrix}^T \frac{1}{\sqrt{\sigma^2 + 3\tau^2}} \begin{bmatrix} \sigma \\ 3\tau \end{bmatrix}}{\sqrt{\sigma^2 + 3\tau^2}} = d\lambda \times 1. \quad (2.63)$$

## 2.6 Elasto-Plastic Modulus

The stiffness of a material changes during plastic deformation and the strain state is dependent on the loading history. Therefore, HOOKE's law which is valid for the linear-elastic material behavior according to Eq. (2.3) must be replaced by the following infinitesimal incremental relation:

$$d\sigma = E^{\text{elpl}} d\varepsilon, \quad (2.64)$$

where  $E^{\text{elpl}}$  is the elasto-plastic modulus. The algebraic expression for this modulus can be obtained in the following manner. The total differential of a yield condition  $F = F(\sigma, \mathbf{q})$ , see Eq. (2.47), is given by:

$$dF(\sigma, \mathbf{q}) = \left( \frac{\partial F}{\partial \sigma} \right) d\sigma + \left( \frac{\partial F}{\partial \mathbf{q}} \right)^T d\mathbf{q} = 0. \quad (2.65)$$

If HOOKE's law (2.3) and the flow rule (2.18) are introduced in the relation for the additive composition of the elastic and plastic strain according to Eq. (2.6), one obtains:

$$d\varepsilon = \frac{1}{E} d\sigma + d\lambda r. \quad (2.66)$$

Multiplication of Eq. (2.66) from the left-hand side with  $\left( \frac{\partial F}{\partial \sigma} \right) E$  and inserting in Eq. (2.65) gives, under the consideration of the evolution equation of the hardening variables (2.52), the consistence parameter as:

$$d\lambda = \frac{\left( \frac{\partial F}{\partial \sigma} \right) E}{\left( \frac{\partial F}{\partial \sigma} \right) E r - \left( \frac{\partial F}{\partial \mathbf{q}} \right)^T \mathbf{h}} d\varepsilon. \quad (2.67)$$



This equation for the consistency parameter can be inserted into Eq. (2.66) and solving for  $\frac{d\sigma}{d\varepsilon}$  gives the elasto-plastic modulus as:

$$E^{\text{elpl}} = E - \frac{E \left( \frac{\partial F}{\partial \sigma} \right) E r}{\left( \frac{\partial F}{\partial \sigma} \right) E r - \left( \frac{\partial F}{\partial \mathbf{q}} \right)^T \mathbf{h}}. \quad (2.68)$$

Let us consider now the case of combined linear kinematic and isotropic hardening [see Eq. (2.48)] where the kinematic hardening modulus  $H$  (PRAGER) and the plastic modulus  $E^{\text{pl}}$  are constant. Furthermore, the flow rule is assumed to be associated. We assume in the following that the yield condition is a function of the following internal variables:  $F = F(\sigma, \mathbf{q}) = F(\sigma, \varepsilon^{\text{pl}}, \varepsilon_{\text{eff}}^{\text{pl}})$ . The corresponding terms in the expression for the elasto-plastic modulus are as follows:

$$\left( \frac{\partial F}{\partial \sigma} \right) = \text{sgn} \left| \sigma - H \varepsilon^{\text{pl}} \right|, \quad (2.69)$$

$$r = \frac{\partial F}{\partial \sigma} = \text{sgn} \left| \sigma - H \varepsilon^{\text{pl}} \right|, \quad (2.70)$$

$$\left( \frac{\partial F}{\partial \mathbf{q}} \right) = \begin{bmatrix} -E^{\text{pl}} \\ -\text{sgn}(\sigma - H \varepsilon^{\text{pl}}) H \end{bmatrix}, \quad (2.71)$$

$$\mathbf{h} = \begin{bmatrix} 1 \\ \text{sgn}(\sigma - H \varepsilon^{\text{pl}}) \end{bmatrix}. \quad (2.72)$$

Introducing these four expressions in Eq. (2.68) gives finally:

$$E^{\text{elpl}} = \frac{d\sigma}{d\varepsilon} = \frac{E(H + E^{\text{pl}})}{E + (H + E^{\text{pl}})}. \quad (2.73)$$

A slightly different derivation is obtained by considering the yield condition depending on the following internal variables:  $F = F(\sigma, \mathbf{q}) = F(\sigma, \alpha, \varepsilon_{\text{eff}}^{\text{pl}})$ . Then, the following different expressions are obtained:

$$\left( \frac{\partial F}{\partial \mathbf{q}} \right) = \begin{bmatrix} -E^{\text{pl}} \\ -\text{sgn}(\sigma - H \varepsilon^{\text{pl}}) \end{bmatrix}, \quad (2.74)$$

$$\mathbf{h} = \begin{bmatrix} 1 \\ H \text{sgn}(\sigma - H \varepsilon^{\text{pl}}) \end{bmatrix}, \quad (2.75)$$

which result again in Eq. (2.73). The different general definitions of the moduli used in this derivation are summarized in Table 2.2.

**Table 2.2** Comparison of the different definitions of the stress-strain characteristics (moduli) in the case of the one-dimensional  $\sigma$ - $\varepsilon$  space

Range	Definition	Graphical representation
Elastic	$E = \frac{d\sigma}{d\varepsilon^{\text{el}}}$	Figure 2.2
Plastic	$E^{\text{elpl}} = \frac{d\sigma}{d\varepsilon}$ for $\varepsilon > \varepsilon^{\text{init}}$	Figure 2.2b
	$E^{\text{pl}} = \frac{dk}{d \varepsilon^{\text{pl}} }$	Figure 2.5

For the two-component  $\sigma$ - $\tau/\varepsilon$ - $\gamma$  space, the incremental relation between the stresses and strains reads  $d\sigma = \mathbf{C}^{\text{elpl}} d\varepsilon$  where  $\mathbf{C}^{\text{elpl}}$  is the elasto-plastic modulus matrix. The yield condition can be generally stated as  $F = F(\sigma, q)$  and the total differential of such a yield condition is given as:

$$dF(\sigma, q) = \left( \frac{\partial F}{\partial \sigma} \right)^T d\sigma + \left( \frac{\partial F}{\partial q} \right)^T dq = 0. \quad (2.76)$$

Following the same line of reasoning as in the one-dimensional case, the consistence parameter is obtained as

$$d\lambda = \frac{\left( \frac{\partial F}{\partial \sigma} \right)^T \mathbf{C}}{\left( \frac{\partial F}{\partial \sigma} \right)^T \mathbf{C} \mathbf{r} - \left( \frac{\partial F}{\partial q} \right)^T \mathbf{h}} d\varepsilon, \quad (2.77)$$

and finally the elasto-plastic modulus matrix as:

$$\mathbf{C}^{\text{elpl}} = \mathbf{C} - \frac{\left( \mathbf{C} \frac{\partial F}{\partial \sigma} \right) \times (\mathbf{C} \mathbf{r})^T}{\left( \frac{\partial F}{\partial \sigma} \right)^T \mathbf{C} \mathbf{r} - \left( \frac{\partial F}{\partial q} \right)^T \mathbf{h}}, \quad (2.78)$$

or under consideration of the dyadic product ' $\otimes$ ' (see Sect. A.14.3):

$$\mathbf{C}^{\text{elpl}} = \mathbf{C} - \frac{\left( \mathbf{C} \frac{\partial F}{\partial \sigma} \right) \otimes (\mathbf{C} \mathbf{r})}{\left( \frac{\partial F}{\partial \sigma} \right)^T \mathbf{C} \mathbf{r} - \left( \frac{\partial F}{\partial q} \right)^T \mathbf{h}}. \quad (2.79)$$

Let us consider now the case of combined linear kinematic and isotropic hardening (see Sect. 2.4.3) where the kinematic hardening modulus  $H$  (PRAGER) and the plastic modulus  $E^{\text{pl}}$  are constant. Furthermore, the flow rule is assumed to be associated. We assume in the following that the yield condition is a function of the following

internal variables:  $F = F(\boldsymbol{\sigma}, \mathbf{q}) = F(\sigma, \varepsilon_{\text{eff}}^{\text{pl}}, \alpha_\sigma, \alpha_\tau)$ . The corresponding terms in the expression for the elasto-plastic modulus matrix for the two-component  $\sigma$ - $\tau/\varepsilon$ - $\gamma$  space are as follows:

$$\mathbf{C} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}, \quad (2.80)$$

$$\left( \frac{\partial F}{\partial \boldsymbol{\sigma}} \right) = \frac{1}{\sqrt{(\sigma - \alpha_\sigma)^2 + 3(\tau - \alpha_\tau)^2}} \begin{bmatrix} \sigma - \alpha_\sigma \\ 3(\tau - \alpha_\tau) \end{bmatrix}, \quad (2.81)$$

$$\mathbf{r} = \frac{1}{\sqrt{(\sigma - \alpha_\sigma)^2 + 3(\tau - \alpha_\tau)^2}} \begin{bmatrix} \sigma - \alpha_\sigma \\ 3(\tau - \alpha_\tau) \end{bmatrix}, \quad (2.82)$$

$$\left( \frac{\partial F}{\partial \mathbf{q}} \right) = \begin{bmatrix} -E^{\text{pl}} \\ -\frac{\sigma - \alpha_\sigma}{\sqrt{(\sigma - \alpha_\sigma)^2 + 3(\tau - \alpha_\tau)^2}} \\ -\frac{3(\tau - \alpha_\tau)}{\sqrt{(\sigma - \alpha_\sigma)^2 + 3(\tau - \alpha_\tau)^2}} \end{bmatrix}, \quad (2.83)$$

$$\mathbf{h} = \begin{bmatrix} 1 \\ \frac{H(\sigma - \alpha_\sigma)}{\sqrt{(\sigma - \alpha_\sigma)^2 + 3(\tau - \alpha_\tau)^2}} \\ \frac{H3(\tau - \alpha_\tau)}{\sqrt{(\sigma - \alpha_\sigma)^2 + 3(\tau - \alpha_\tau)^2}} \end{bmatrix}. \quad (2.84)$$

Introducing these five relationships in Eq. (2.78) gives finally the following specific expression for the elasto-plastic modulus matrix:

$$\mathbf{C}^{\text{elpl}} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} - \frac{1}{(E + H + E^{\text{pl}})(\sigma - \alpha_\sigma)^2 + (9G + 9H + 3E^{\text{pl}})(\tau - \alpha_\tau)^2} \times \begin{bmatrix} E^2(\sigma - \alpha_\sigma)^2 & 3EG(\sigma - \alpha_\sigma)(\tau - \alpha_\tau) \\ 3EG(\sigma - \alpha_\sigma)(\tau - \alpha_\tau) & 9G^2(\tau - \alpha_\tau)^2 \end{bmatrix}. \quad (2.85)$$

The last equation can be simplified to the special case of a one-dimensional stress and strain state by assigning  $G = 0$  and  $\tau = 0$ . Then one obtains again Eq. (2.73).

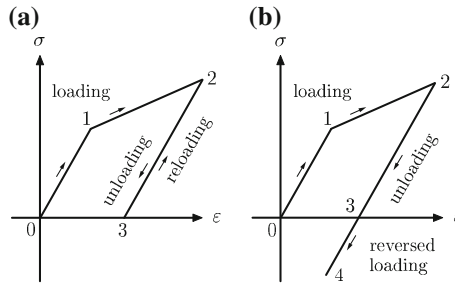
At the end of this section, Table 2.3 compares the different equations and formulations of one-dimensional plasticity with the general two-dimensional representations (see for example [10, 96]).

## 2.7 Consideration of Unloading, Reversed Loading and Cyclic Loading

The previous sections considered only monotonic loading either in the tensile or compressive regimes. We will now briefly look at the cases where the loading direction can change. Figure 2.9a shows the case of loading in the elastic ( $0 \rightarrow 1$ ) and

**Table 2.3** Comparison between general 2D plasticity and 1D plasticity with combined hardening ( $E^{\text{pl}}$  and  $H$  are assumed constant)

General 2D plasticity	1D <i>Linear</i> hardening plasticity
Yield condition $F(\boldsymbol{\sigma}, \mathbf{q}) \leq 0$	$F =  \sigma - \alpha  - (k^{\text{init}} + E^{\text{pl}} \varepsilon_{\text{eff}}^{\text{pl}}) \leq 0$
Flow rule $\varepsilon^{\text{pl}} = d\lambda \times \mathbf{r}(\boldsymbol{\sigma}, \mathbf{q})$	$d\varepsilon^{\text{pl}} = d\lambda \times \text{sgn}(\sigma - \alpha)$
Hardening law $\mathbf{q} = [\kappa, \boldsymbol{\alpha}]^T$	$\mathbf{q} = [d\varepsilon_{\text{eff}}^{\text{pl}}, \alpha]^T$
$d\mathbf{q} = d\lambda \times \mathbf{h}(\boldsymbol{\sigma}, \mathbf{q})$	$d\varepsilon_{\text{eff}}^{\text{pl}} = d\lambda, d\alpha = d\lambda H \text{sgn}(\sigma - \alpha)$
Elasto-plastic modulus matrix $\mathbf{C}^{\text{elpl}} = \left( \mathbf{C} - \frac{\left( \mathbf{C} \frac{\partial F}{\partial \boldsymbol{\sigma}} \right) \times (\mathbf{C} \mathbf{r})^T}{\left( \frac{\partial F}{\partial \boldsymbol{\sigma}} \right)^T \mathbf{C} \mathbf{r} - \left( \frac{\partial F}{\partial \mathbf{q}} \right)^T \mathbf{h}} \right)$	$E^{\text{elpl}} = \frac{E \times (H + E^{\text{pl}})}{E + (H + E^{\text{pl}})}$

**Fig. 2.9** Idealized stress-strain curve with **a** loading—unloading—reloading and **b** loading—unloading—reversed loading

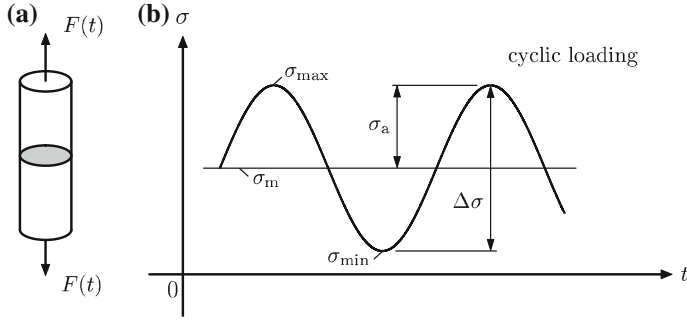
elasto-plastic (1  $\rightarrow$  2) range, followed by elastic unloading (2  $\rightarrow$  3) and elastic reloading (3  $\rightarrow$  2).

In the case of Fig. 2.9b, the elastic unloading (2  $\rightarrow$  3) is followed by reversed loading (3  $\rightarrow$  4). The important feature which should be highlighted here is that the unloading phase (2  $\rightarrow$  3) can be described based on HOOKE's law, cf. Eq. (2.3). Figure 2.10 shows the case of cyclic loading where a specimen is exposed to fluctuating loads  $F(t)$ .

Some characteristic stress quantities are indicated in Fig. 2.10b: The stress range  $\Delta\sigma$  is the difference between the maximum and minimum stress:

$$\Delta\sigma = \sigma_{\max} - \sigma_{\min} . \quad (2.86)$$

The stress amplitude  $\sigma_a$  is half the value of the stress range:



**Fig. 2.10** Cyclic loading: **a** idealized specimen and **b** stress-time curve at constant amplitude

$$\sigma_a = \frac{\Delta\sigma}{2} = \frac{\sigma_{\max} - \sigma_{\min}}{2}. \quad (2.87)$$

The so-called stress ratio  $R$  is often used to characterize the stress level in cyclic tests:

$$R = \frac{\sigma_{\min}}{\sigma_{\max}}, \quad (2.88)$$

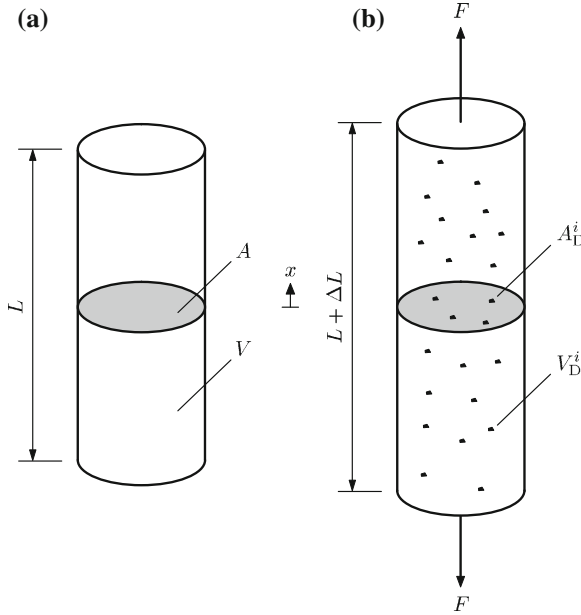
where a value  $R = -1$  characterizes a fully-reversed load cycle,  $R = 1$  stands for static loading and  $R = 0$  refers to the case where the mean stress is positive and equal to the stress amplitude. In materials testing cyclic tests are performed to determine the fatigue life of components and structures. Further details can be found in [78, 93].

## 2.8 Consideration of Damage

### 2.8.1 Lemaitre's Damage Model

The following section summarizes briefly the major ideas of ductile damage based on the concept given by LEMAITRE in [61, 62]. Let us consider an idealized uniaxial tensile sample as shown in Fig. 2.11 where the undamaged or initial tensile specimen is shown on the left-hand side and the damaged or deformed specimen on the right-hand side. A pure uniaxial tensile sample is assumed in the following which means in this context that the specimen deforms only in its longitudinal direction (i.e. the  $x$ -direction as shown in Fig. 2.11) and does not show any deformation perpendicular, i.e. contraction, to the loading direction. This corresponds to the assumption that Poisson's ratio is equal to zero.

It should be noted here that the size of the specimen must be in such a range that the considered volume represents a representative volume element (RVE) for the considered material. Some estimates for the minimum size of RVEs for different



**Fig. 2.11** Schematic representation of **a** an undamaged and **b** a damaged tensile specimen

materials are given in [63]. Let  $A$  be the overall cross-sectional area of the specimen (marked in grey in Fig. 2.11a) and  $A_D$  be the total area of the micro-cracks and voids in the considered area which is in Fig. 2.11b marked in black. The effective resisting area is denoted by  $\bar{A}$ . Based on these quantities, the damage variable  $D$  can be introduced as:

$$D = \frac{\sum_i A_D^i}{A} = \frac{A_D}{A} = \frac{A - \bar{A}}{A}. \quad (2.89)$$

If this definition is based on a RVE, then the same damage variable is obtained based on the volume of the micro-cracks and voids [111]:  $D = V_D/V$ . A state  $D = 0$  corresponds to the undamaged state,  $D = 1$  represents the rupture of the specimen into two parts and  $0 < D < 1$  characterizes the damaged state. In the scope of this chapter, an isotropic damage variable is assumed. This means that the defects are equally distributed in all directions of the specimen. Thus, a scalar description of the damage is sufficient under the *hypothesis of isotropy*. If the resisting area in Eq. (2.89) is used to calculate the stress in the specimens, the *concept of effective stress* is obtained which states that the effective stress in the specimen is given by:

$$\bar{\sigma} = \frac{\sigma}{1 - D}. \quad (2.90)$$

It must be stated here that this definition of the effective stress holds only in the tensile regime. Under compression, some defects may close again or in the limiting case, all defects can be closed again so that the effective stress is again equal to the usual stress  $\sigma$ . However, this effect in the compressive regime will be not considered within this chapter. In the case of the strain, the *hypothesis of strain equivalence* is applied which states that the strain behavior of a damaged material is represented by the virgin material:

$$\bar{\varepsilon} = \varepsilon. \quad (2.91)$$

Based on these assumptions and simplifications, HOOKE's law can be written with the effective stress  $\bar{\sigma}$  and elastic strain  $\varepsilon^{\text{el}}$  as:

$$\bar{\sigma} = E \varepsilon^{\text{el}}, \quad (2.92)$$

which can be expressed with the definition of the effective stress given in Eq. (2.90) as:

$$\sigma = \underbrace{(1 - D)E}_{\bar{E}} \varepsilon, \quad (2.93)$$

where  $E$  is the elastic modulus of the undamaged material (initial modulus) and  $\bar{E}$  is the modulus of the damaged material. The last equation offers an elegant way to experimentally determine the evolution of the damage variable  $D$ . Measuring during tests with reversed loading stress and strain based on the usual engineering definitions, i.e.  $\sigma = F/A$  and  $\varepsilon = \Delta L/L$ , the damage variable can be indirectly obtained from the variation of the elasticity modulus as:

$$D = 1 - \frac{\bar{E}}{E}. \quad (2.94)$$

The classical continuum theory of plasticity is based on three equations, i.e. the yield condition, the flow rule and the hardening law. For a one-dimensional stress state, the yield condition reads under consideration of the damage effects as

$$F = \frac{|\sigma|}{1 - D} - k(\kappa), \quad (2.95)$$

where  $|\sigma|/(1 - D)$  is the equivalent stress which is compared to the experimental value  $k$ . The flow rule and the evolution equation for the internal variable  $\kappa$  do not change and are given by Eqs. (2.19) and (2.28).

In the case of damage mechanics, there is in addition the evolution equation required for the damage variable. Following the notation in [74] and considering a one-dimensional stress state, the model for the ductile damage evolution can be expressed as

**Table 2.4** Basic equations of the LEMAITRE model (isotropic hardening) in the case of a uniaxial stress state with  $\sigma$  as acting stress

1D LEMAITRE model
HOOKE's law
$\sigma = (1 - D)E\varepsilon$
Yield condition
$F = \frac{ \sigma }{1 - D} - k(\kappa)$
Flow rule
$d\varepsilon^{\text{pl}} = d\lambda \frac{\text{sgn}(\sigma)}{1 - D}$
Evolution of hardening variable
$d\kappa = d\lambda$
Evolution of damage variable
$dD = \frac{d\lambda}{1 - D} \left( \frac{-Y}{r} \right)^s = d \varepsilon^{\text{pl}}  \left( \frac{-Y}{r} \right)^s$
with $Y = -\frac{\sigma^2}{2E(1 - D)^2}$

$$dD = \frac{d\lambda}{1 - D} \left( \frac{-Y}{r} \right)^s = d|\varepsilon^{\text{pl}}| \left( \frac{-Y}{r} \right)^s, \quad (2.96)$$

where  $Y$  is the so-called damage energy release rate which corresponds to the variation of internal energy density due to damage growth at constant stress, and  $r$  and  $s$  are damage evolution material parameters. For a one-dimensional stress state,  $Y$  takes the form:

$$Y = -\frac{\sigma^2}{2E(1 - D)^2} \stackrel{(2.90)}{=} -\frac{\bar{\sigma}^2}{2E}. \quad (2.97)$$

The basic equations for the one-dimensional model according to LEMAITRE are collected in the following Table 2.4.

An experimental strategy based on a simple tensile test and a fatigue test (WÖHLER curve) for the determination of the parameters  $r$  and  $s$  is given in [64]. A simpler form of Eq. (2.96) is given in [63] as:

$$dD = d|\varepsilon^{\text{pl}}| \times \frac{-Y}{r}, \quad (2.98)$$

or under consideration of Eq. (2.97) as:



$$dD = d|\varepsilon^{pl}| \times \frac{\sigma^2}{2Er(1-D)^2}. \quad (2.99)$$

The last equation can be rearranged to give

$$\frac{dD}{d|\varepsilon^{pl}|} = \frac{\sigma^2}{2Er(1-D)^2}, \quad (2.100)$$

or

$$r = \frac{\sigma^2}{2E(1-D)^2 \frac{dD}{d|\varepsilon^{pl}|}}, \quad (2.101)$$

which allows to determine the material parameter  $r$  from a tensile test with reversed loading in the following way:

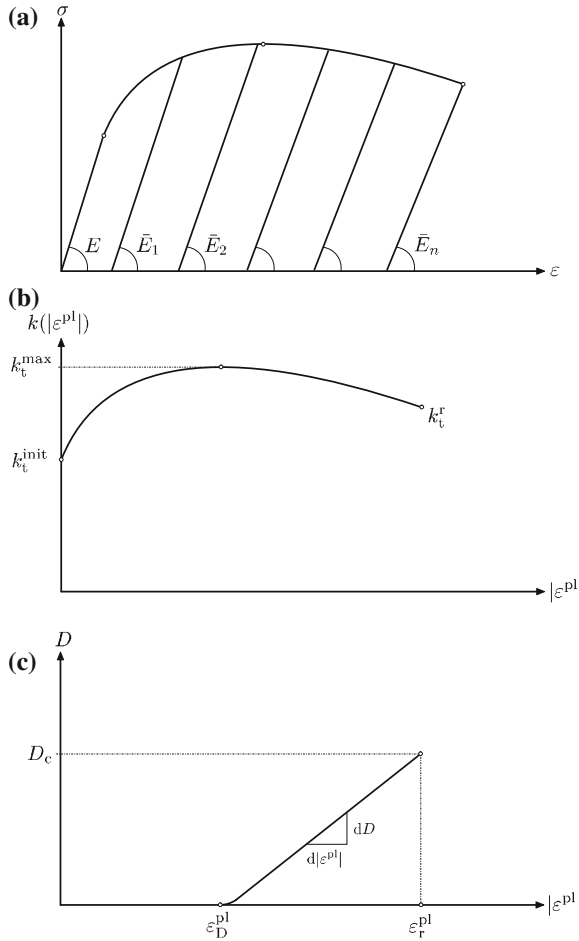
- Perform a uniaxial tensile test with reversed loading as shown in Fig. 2.12a. Determine the elastic modulus  $E$  of the undamaged material (initial modulus) and for each unloading-loading cycle  $n$  the modulus of the damaged material  $\bar{E}$ , the corresponding stress  $\sigma$  where the unloading starts and the plastic strain at  $\sigma = 0$ .
- Construct from the values of stress and corresponding plastic strain the flow curve as shown in Fig. 2.12b.
- Calculate from the modulus of the damaged material the damage variable  $D$  as given in Eq. (2.94). Plot the damage variable over the plastic strain as shown in Fig. 2.12c.
- Calculate the damage material parameter  $r$  according to Eq. (2.101) for given values of stress  $\sigma$ , elastic modulus  $E$ , damage variable  $D$  and slope  $\frac{dD}{d|\varepsilon^{pl}|}$ .
- Calculate an average (or interpolated) value of  $r$  by considering several evaluations as described in the previous step.

### 2.8.2 Gurson's Damage Model

To model the arbitrary distribution of voids in a matrix (cf. Fig. 2.13a), GURSON introduced the idealized models of two void geometries [48]. For the first model, a single long circular cylindrical void is considered in a similarly shaped matrix (cf. Fig. 2.13b) whereas the second model considers a spherical shape of void and matrix (cf. Fig. 2.13c). The matrix material is considered as a homogeneous, isotropic, rigid-plastic VON MISES material. It should be highlighted here that the approach neglected the elastic material response and assumed the matrix material to be incompressible.

Based on these assumptions, the following yield conditions could be derived

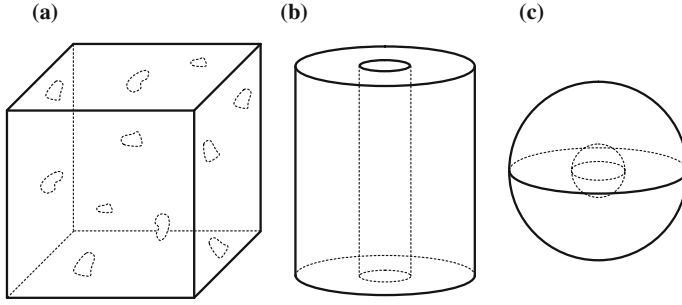
$$F = \left( \frac{\sigma_{\text{eff}}}{k_t} \right)^2 + 2D \cosh \left( \frac{3\sqrt{3}}{2} \frac{\sigma_m}{k_t} \right) - (1 + D^2) = 0, \text{ (cylindrical)} \quad (2.102)$$



**Fig. 2.12** **a** Uniaxial stress-strain diagram with reversed loading; **b** flow curve; **c** damage variable as a function of plastic strain

$$F = \left( \frac{\sigma_{eff}}{k_t} \right)^2 + 2D \cosh \left( \frac{3\sigma_m}{2k_t} \right) - (1 + D^2) = 0, \quad (\text{spherical}) \quad (2.103)$$

where  $\sigma_{eff}$  is the effective stress based on the VON MISES definition and  $\sigma_m$  is the mean stress (hydrostatic stress). The yield stress  $k_t$  is used to normalize the effective and mean stress. The damage variable  $D$  in this model is based on the void volume fraction, i.e.  $D = V_D/V$  where  $V_D$  is the volume of the void and  $V$  is the volume of the matrix. If the calculation of the void volume fraction is based on real distributions as schematically shown in Fig. 2.13, then some account is taken of the interaction of neighboring voids [48].



**Fig. 2.13** **a** Voids of random shape and orientation distributed in a matrix; **b** long circular cylindrical void; **c** spherical void. Adapted from [48]

**Table 2.5** Intersections of the GURSON flow curves with the coordinate axes

$D$	$\left(\frac{\tau}{k_t}\right)_{x=0}$	$\left(\frac{\sigma}{k_t}\right)_{y=0}$
0.00	$\pm 0.5774$	$\pm 1.000000$
0.01	$\pm 0.5715$	$\pm 0.988639$
0.10	$\pm 0.5132$	$\pm 0.877740$
0.30	$\pm 0.3215$	$\pm 0.536890$

Let us consider in the following the two-component  $\sigma$ - $\tau$  stress space. Thus, the yield conditions given in Eqs. (2.102) and (2.103) take the following form:

$$F = \left( \frac{\sqrt{\sigma^2 + 2\tau^2}}{k_t} \right)^2 + 2D \cosh \left( \frac{\sqrt{3}\sigma}{2k_t} \right) - (1 + D^2) = 0, \text{ (cylindrical)} \quad (2.104)$$

$$F = \left( \frac{\sqrt{\sigma^2 + 2\tau^2}}{k_t} \right)^2 + 2D \cosh \left( \frac{1\sigma}{2k_t} \right) - (1 + D^2) = 0. \quad \text{(spherical)} \quad (2.105)$$

A graphical representation of the GURSON yield condition based on Eq. (2.105) is shown in Fig. 2.14 where different values of the damage parameter have been assigned. A value of  $D = 0$  results in the classical VON MISES ellipse and values  $D > 0$  result in this graphical representation in similar looking shapes<sup>22</sup> but with a smaller elastic region ( $F < 0$ ), i.e. plastic yielding will start earlier compared to the classical VON MISES condition.

Abscissae and ordinates of the intersection of the flow curve with the coordinate axes can be obtained from Eq. (2.105) for  $\sigma = 0$  or  $\tau = 0$  and are summarized in Table 2.5.

<sup>22</sup> The yield surfaces for  $D > 0$  look like ellipses but because of the cosh function, they are not from a mathematical point of view classified as ellipses.

The derivative of the yield condition or the plastic strain increments with respect to the stresses are obtained in the case of a general stress state from Eq. (2.103)

$$d\epsilon^{\text{pl}} = d\lambda \frac{\partial F}{\partial \sigma} = d\lambda \left( \frac{3\mathbf{L}s}{k_t^2} + \frac{D}{k_t} \sinh\left(\frac{3\sigma_m}{2k_t}\right) \mathbf{1} \right), \quad (2.106)$$

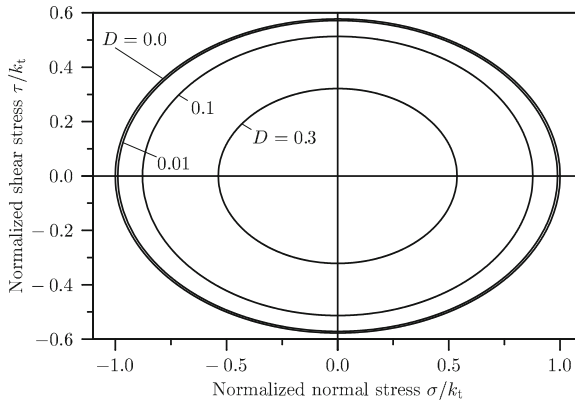
where  $s = [s_x \ s_y \ s_z \ \tau_{xy} \ \tau_{yz} \ \tau_{xz}]^T$  is the column matrix of the stress deviator components,<sup>23</sup>  $\mathbf{L} = [1 \ 1 \ 1 \ 0 \ 0 \ 0]$  is a diagonal scaling matrix and  $\mathbf{1} = [1 \ 1 \ 1 \ 0 \ 0 \ 0]^T$  is an identity column matrix. In the case of a  $\sigma$ - $\tau$  stress space, Eq. (2.105) gives under consideration of  $\frac{d}{dx}|x|^2 = 2|\sigma|\text{sgn}(\sigma) = 2\sigma$  the following expression for the plastic strain increments:

$$d\epsilon^{\text{pl}} = \begin{bmatrix} d\epsilon^{\text{pl}} \\ d\gamma^{\text{pl}} \end{bmatrix} = d\lambda \begin{bmatrix} \frac{2\sigma}{k_t^2} + \frac{D}{k_t} \sinh\left(\frac{\sigma}{2k_t}\right) \\ \frac{6\tau}{k_t^2} \end{bmatrix}. \quad (2.107)$$

At the end of this section, the evolution equation for the ductile damage should be given. According to [74], the evolution equation is given as

$$dD = (1 - D)(d\epsilon_x^{\text{pl}} + d\epsilon_y^{\text{pl}} + d\epsilon_z^{\text{pl}}) = (1 - D)d\epsilon_V^{\text{pl}}, \quad (2.108)$$

where  $\epsilon_V^{\text{pl}}$  is the volumetric plastic strain. Consideration of Eq. (2.106) and the fact that  $s_x + s_y + s_z = 0$  allows to reformulate the last equation to obtain:



**Fig. 2.14** Graphical representation of the yield condition according to GURSON in the two-component  $\sigma$ - $\tau$  space for different values of the damage variable (spherical voids assumed)

<sup>23</sup> The first component of the stress deviator is given by  $s_x = \frac{2}{3}\sigma_x - \frac{1}{3}(\sigma_y + \sigma_z)$ .

**Table 2.6** Basic equations of the original GURSON model (spherical inclusions, rigid-plastic material) in the case of a uniaxial stress state with  $\sigma$  as acting stress

1D GURSON model
Yield condition
$F = \left( \frac{ \sigma }{k_t} \right)^2 + 2D \cosh \left( \frac{\sigma}{2k_t} \right) - (1 + D^2) = 0$
Flow rule
$d\varepsilon^{\text{pl}} = d\lambda \left( \frac{2\sigma}{k_t^2} + \frac{D}{k_t} \sinh \left( \frac{\sigma}{2k_t} \right) \right)$
Damage evolution equation
$dD = d\lambda \frac{3(D - D^2)}{k_t} \sinh \left( \frac{\sigma}{2k_t} \right)$

$$dD = d\lambda \frac{3(D - D^2)}{k_t} \sinh \left( \frac{3\sigma_m}{2k_t} \right). \quad (2.109)$$

Considering a pure one-dimensional stress state where only the normal stress  $\sigma$  is acting, Eqs. (2.105), (2.107) and (2.109) can be simplified to the forms given in Table 2.6.

In many practical applications, the GURSON yield condition is applied to *elasto*-plastic material behavior, even under consideration of isotropic hardening ( $k_t = k_t(\kappa)$ ). In such a case it is necessary to indicate the evolution equation for the internal variable  $\kappa$ . Assuming that the effective plastic strain is assigned as the internal variable, i.e.  $\varepsilon_{\text{eff}}^{\text{pl}} = \kappa$ , and furthermore assuming that the increment of equivalent plastic work in the matrix material equals the macroscopic increment of plastic work [10, 109], i.e.

$$\sigma d\varepsilon^{\text{pl}} = (1 - D)k_t d\varepsilon_{\text{eff}}^{\text{pl}}, \quad (2.110)$$

the evolution of the internal variable is given in the one-dimensional case as:

$$d\kappa = \frac{d\lambda}{1 - D} \left( \frac{2\sigma^2}{k_t^3} + \frac{D\sigma}{k_t^2} \sinh \left( \frac{\sigma}{2k_t} \right) \right). \quad (2.111)$$

Thus, the equations presented in Table 2.6 can be extended to the case of an *elasto*-plastic material with isotropic hardening as summarized in Table 2.7.

**Table 2.7** Basic equations of the extended GURSON model (elastic range, isotropic hardening) in the case of a uniaxial stress state with  $\sigma$  as acting stress

1D GURSON model
HOOKE's law
$\sigma = E\varepsilon$
Yield condition
$F = \left( \frac{ \sigma }{k_t(\kappa)} \right)^2 + 2D \cosh \left( \frac{\sigma}{2k_t(\kappa)} \right) - (1 + D^2) = 0$
Flow rule
$d\varepsilon^{\text{pl}} = d\lambda \left( \frac{2\sigma}{(k_t(\kappa))^2} + \frac{D}{k_t(\kappa)} \sinh \left( \frac{\sigma}{2k_t(\kappa)} \right) \right)$
Evolution of hardening variable
$d\kappa = \frac{d\lambda}{1 - D} \left( \frac{2\sigma^2}{(k_t(\kappa))^3} + \frac{D\sigma}{(k_t(\kappa))^2} \sinh \left( \frac{\sigma}{2k_t(\kappa)} \right) \right)$
Evolution of damage variable
$dD = d\lambda \frac{3(D - D^2)}{k_t(\kappa)} \sinh \left( \frac{\sigma}{2k_t(\kappa)} \right)$

## 2.9 Supplementary Problems

### 2.1 Derivation of Basic Equations for von Mises' Yield Condition

Given are the following two forms of a one-dimensional VON MISES (or TRESCA) yield condition:

- (a)  $F = \left( \frac{|\sigma|}{k_t} \right)^2 - 1 = 0,$
- (b)  $F = \frac{|\sigma|}{k_t} - 1 = 0.$

Derive the expressions for the associated flow rule and the evolution equation for the internal hardening variable in the case of strain gardening.

### 2.2 Derivation of the Elasto-plastic Modulus for the One-Dimensional Case with Isotropic Hardening

The general expression for the elasto-plastic modulus for a one-dimensional stress and strain state is generally given in Eq. (2.68) as:

$$E^{\text{elpl}} = E - \frac{E \left( \frac{\partial F}{\partial \sigma} \right) Er}{\left( \frac{\partial F}{\partial \sigma} \right) Er - \left( \frac{\partial F}{\partial q} \right)^T \mathbf{h}}. \quad (2.112)$$

Simplify this equation for the case of pure isotropic hardening.

### 2.3 Derivation of the Elasto-plastic Modulus Matrix for Tresca Yield Condition

Given is the two-dimensional TRESCA yield condition in the following form

$$F_{\sigma-\tau} = \sqrt{(\sigma - \alpha_\sigma)^2 + 4(\tau - \alpha_\tau)^2} - (k^{\text{init}} + E^{\text{pl}} \varepsilon_{\text{eff}}^{\text{pl}}) = 0, \quad (2.113)$$

which considers combined linear kinematic and isotropic hardening where the kinematic hardening modulus  $H$  (PRAGER) and the plastic modulus  $E^{\text{pl}}$  are constant. Furthermore, the flow rule is assumed to be associated. Assume in the following that the yield condition is a function of the following internal variables:  $F = F(\boldsymbol{\sigma}, \boldsymbol{q}) = F(\sigma, \varepsilon_{\text{eff}}^{\text{pl}}, \alpha_\sigma, \alpha_\tau)$ . Derive the expression for the elasto-plastic modulus matrix  $\mathbf{C}^{\text{elpl}}$ .

### 2.4 Derivation of Basic Equations for Gurson's Yield Condition

Given is the following form of a one-dimensional GURSON's yield condition:

$$F = |\sigma|^2 + 2D \cosh\left(\frac{1}{2} \frac{\sigma}{k_t(\kappa)}\right) k_t(\kappa)^2 - (1 + D^2) k_t(\kappa)^2 = 0. \quad (2.114)$$

Derive the expressions for the flow rule, damage evolution equation, and evolution equation for the internal hardening variable. Assume for the derivation of the internal hardening variable that the increment of equivalent plastic work in the matrix material equals the macroscopic increment of plastic work.

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2014, XX, 596 p. 328 illus., Hardcover

ISBN: 978-3-662-44224-1